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The Dunkl-Williams constant related to the Singer orthogonality and red isosceles orthogonality in Banach spaces

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Abstract. In this paper, we shall consider two new constants $DW_S(X)$ and $DW_I(X)$, which are the Dunkl-Williams constant related to the Singer orthogonality and theisosceles orthogonality, respectively. We discuss the relationships between $DW_S(X)$ and some geometric properties of Banach spaces, including uniform non-squareness, uniform convexity. Furthermore, an equivalent form of $DW_S(X)$ in the symmetric Minkowski planes is given and used to compute the value of $DW_S((\mathbb{R}^2, \|\cdot\|_p))$, $1 < p < \infty$, and we also give a characterization of the Radon plane with affine regular hexagonal unit sphere in terms of $DW_S(X)$. Finally, we establish some estimates for $DW_I(X)$ and show that $DW_I(X)$ does not necessarily coincide with $DW_S(X)$.

1. Introduction

In 1964, Dunkl and Williams [1] showed that, in any Banach space X, the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\| + \|y\|}$$

holds for any nonzero elements $x, y \in X$. This inequality is called the Dunkl-Williams inequality. Actually, the Dunkl-Williams inequality gives the upper bound for the angular distance

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$$

between two nonzero elements $x, y \in X$. The angular distance, also called the Clarkson distance, was introduced by Clarkson [2], in order to make a detailed analysis of the triangle inequality in uniformly convex Banach spaces.

In the same year that the Dunkl-Williams inequality came out, Kirk and Smiley [3] found that the Dunkl-Williams inequality with 2 in place of 4 in fact characterizes the Hilbert space, that is, a Banach space X is a Hilbert space if and only if the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$$

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holds for any nonzero elements $x, y \in X$.

According to the above result, Jiménez-Melado et al. [4] pointed out that the smallest number which can replace 4 in Dunkl-Williams inequality measures "how much" this Banach space is close to be a Hilbert space. Thus, Jiménez-Melado et al. [4] introduced the Dunkl-Williams constant as following:

$$DW(X) = \sup\left\{\frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \;:\; x, y \in X\backslash\{0\},\ x \neq y\right\}, \tag{1}$$

and also obtained some conclusions about $DW(X)$:

(1) $2 \leq DW(X) \leq 4$ holds for any Banach space X.

(2) $DW(X) = 2$ if and only if X is a Hilbert space.

(3) $DW(X) < 4$ if and only if X is a uniformly non-square Banach space, that is, there exists $\delta > 0$ such that for any $x, y \in S_X$ we have $\min\{\|x-y\|, \|x+y\|\} \leq 2-\delta$.

(4) If $DW(X) < 4$, then X has the fixed point property, that is, every nonexpansive self mapping of any closed convex bounded subset of X has a fixed point in this subset.

For more results about the Dunkl-Williams constant $DW(X)$, we recommend [5–9] to interested readers.

Let x, y be two elements in a real Banach space X. Then x is said to be Birkhoff orthogonal to y and denoted by $x \perp_B y$ (cf. [10]), if

$$\|x+\lambda y\| \geq \|x\|,\ \lambda \in \mathbb{R}.$$

In addition, x is said to be isosceles orthogonal to y and denoted by $x \perp_I y$ (cf. [11]), if

$$\|x+y\| = \|x-y\|.$$

One can easily know that the Birkhoff orthogonality coincides with the isosceles orthogonality if X is Hilbert space. In fact, the Birkhoff orthogonality coincides with the isosceles orthogonality only if X is a Hilbert space (see [12], Theorem 5.1), that is, X is a Hilbert space if and only if the Birkhoff orthogonality coincides with the isosceles orthogonality in X. This conclusion also indicates that these two orthogonalities are different in Banach spaces. To quantify the difference between these two orthogonalities, some parameters are introduced (see [8], [13]):

$$D(X) = \inf\left\{\inf_{\lambda\in\mathbb{R}}\|x+\lambda y\| \;:\; x, y \in S_X,\ x \perp_I y\right\}.$$

$$IB(X) = \inf\left\{\frac{\inf_{\lambda\in\mathbb{R}}\|x+\lambda y\|}{\|x\|} \;:\; x, y \in X\backslash\{0\},\ x \perp_I y\right\}.$$

$$IB'(X) = \inf\left\{\frac{\inf_{\lambda\in\mathbb{R}}\|x+\lambda y\|}{\|x\|} \;:\; x, y \in X\backslash\{0\},\ \|y\| \leq \|x\|,\ x \perp_I y\right\}.$$

For more studies of the difference between these two orthogonalities can be found in [14], [15].

In [8], Mizuguchi gave an unexpected result which connects $IB(X)$ with $DW(X)$, that is,

$$IB(X)DW(X) = 2.$$

The proof of this result is based on the following equivalent form of $DW(X)$

$$DW(X) = \sup\left\{\frac{\|x+y\|}{\|(1-t)x+ty\|} \;:\; x, y \in S_X,\ x+y \neq 0,\ 0 \leq t \leq 1\right\}. \tag{2}$$

In [8], Mizuguchi also considered the following supremum which is similar to $DW(X)$

$$\sup\left\{\frac{\|x+y\|}{\|(1-t)x+ty\|} \;:\; x, y \in S_X,\ x \perp_I y,\ 0 \leq t \leq 1\right\}. \tag{3}$$

Recall that x is said to be Singer orthogonal to y and denoted by $x \perp_S y$ (cf. [16]), if either $\|x\| \cdot \|y\| = 0$ or

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|.$$

Obviously, the Singer orthogonality is defined by restricting isosceles orthogonality to the unit sphere (i.e. $x \perp_S y \Leftrightarrow \frac{x}{\|x\|} \perp_I \frac{y}{\|y\|}$). Thus, (3) can be rewritten as following

$$\sup\left\{ \frac{\|x+y\|}{\|(1-t)x+ty\|} \ : \ x, y \in S_X, \ x \perp_S y, \ 0 \le t \le 1 \right\}. \tag{4}$$

Now, take into account (1), (2), (3), and (4), it is natural for us to consider the following two parameters

$$DW_S(X) = \sup\left\{ \frac{\|x\| + \|y\|}{\|x-y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \ : \ x, y \in X \backslash \{0\}, \ x \perp_S y \right\}$$

and

$$DW_I(X) = \sup\left\{ \frac{\|x\| + \|y\|}{\|x-y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \ : \ x, y \in X \backslash \{0\}, \ x \perp_I y \right\}.$$

Actually, due to the following equality

$$\frac{\|x\| + \|y\|}{\|x-y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \frac{\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|}{\left\| \frac{x}{\|x\|+\|y\|} - \frac{y}{\|x\|+\|y\|} \right\|}.$$

$DW_S(X)$ and $DW_I(X)$ can be regarded as discussing the ratio of the following two dotted lines under different orthogonal conditions.

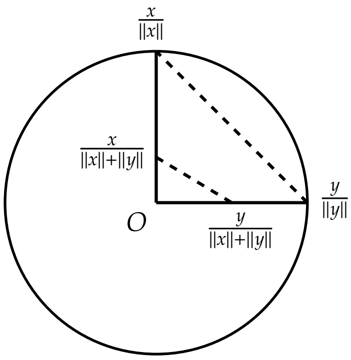

Figure 1. The graphic interpretation of $DW_S(X)$ and $DW_I(X)$.

Throughout this paper, we always assume that X is a real Banach space with $dimX \geq 2$. The arrangement of this paper is as follows:

In Section 2, we consider the constant $DW_S(X)$. First, we present the bounds of $DW_S(X)$ and show that the lower bound of $DW_S(X)$ can be used to characterize the Hilbert space. In addition, we also state the relationships between $DW_S(X)$ and some geometric properties of Banach spaces, including uniform non-squareness, uniform convexity. Second, we study $DW_S(X)$ in symmetric Minkowski planes. An equivalent form of $DW_S(X)$ in symmetric Minkowski planes is given and used to compute the value of $DW_S((\mathbb{R}^2, \|\cdot\|_p))$, $1 < p < \infty$. Finally, we also discuss $DW_S(X)$ in Radon planes. The bounds of $DW_S(X)$ in Radon planes are given. Moreover, we use the upper bound to characterize the Radon plane with affine regular hexagonal unit sphere.

In Section 3, we will be concerned with the constant $DW_I(X)$. First, we discuss the bounds of $DW_I(X)$ and give an example to illustrate that $DW_I(X)$, $DW_S(X)$ and $DW(X)$ do not necessarily coincide with each other. Second, we also give some estimates for $DW_I(X)$ in terms of other well-known constants.

2. The Dunkl-Williams constant related to the Singer orthogonality

2.1. Some geometric properties related to $DW_S(X)$

This section is devoted to the relationships between $DW_S(X)$ and some geometric properties of Banach spaces. The following result is the key to our subsequent discussion.

Proposition 2.1. *Let X be a Banach space. Then*

$$D(X)DW_S(X) = 2.$$

Proof. First, for any $x, y \in X\backslash\{0\}$ with $x \perp_S y$, let $u = \frac{x}{\|x\|}, v = -\frac{y}{\|y\|}$. Due to $x \perp_S y$, we obtain $\|u + v\| = \|u - v\|$, which means that $\frac{u+v}{\|u+v\|}, \frac{u-v}{\|u+v\|} \in S_X$. Moreover, since $u, v \in S_X$ and $\|u + v\| = \|u - v\|$, we get $\frac{u+v}{\|u+v\|} \perp_I \frac{u-v}{\|u+v\|}$. Thus, according to the definition of $D(X)$, we have

$$\begin{aligned}\frac{\|x\| + \|y\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| &= \frac{\|u + v\|}{\left\|\frac{\|x\|}{\|x\|+\|y\|}u + \frac{\|y\|}{\|x\|+\|y\|}v\right\|}\\ &= \frac{\|u + v\|}{\frac{1}{2}\left\|u + v + \frac{\|x\|-\|y\|}{\|x\|+\|y\|}(u - v)\right\|}\\ &= \frac{2}{\left\|\frac{u+v}{\|u+v\|} + \frac{\|x\|-\|y\|}{\|x\|+\|y\|}\frac{u-v}{\|u+v\|}\right\|}\\ &\leq \frac{2}{D(X)},\end{aligned}$$

which implies that

$$DW_S(X) \leq \frac{2}{D(X)}.$$

On the other hand, for any $x, y \in S_X$ with $x \perp_I y$, it is clear that there exists a $\lambda_1 \in [-1, 1]$ such that $\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| = \|x + \lambda_1 y\|$. Now, we consider the following two cases:

Case 1: $\lambda_1 \in [0, 1]$.

Let

$$u = \frac{1 + \lambda_1}{2}\frac{x + y}{\|x + y\|},\ v = -\frac{1 - \lambda_1}{2}\frac{x - y}{\|x + y\|}.$$

Since $x, y \in S_X$ and $x \perp_I y$, one can easily deduce that $u \perp_S v$. Thus, we have

$$\begin{aligned}\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| = \|x + \lambda_1 y\| &= 2\frac{\|x + y\|}{2}\frac{\|x + \lambda_1 y\|}{\|x + y\|}\\ &= 2\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\|^{-1}\frac{\|x + \lambda_1 y\|}{\|x + y\|}\\ &= 2\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\|^{-1}\left\|\frac{(1 + \lambda_1)(x + y)}{2\|x + y\|} + \frac{(1 - \lambda_1)(x - y)}{2\|x + y\|}\right\|\\ &= 2\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\|^{-1}\|u - v\|\\ &= \frac{2}{\frac{1}{\|u-v\|}\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\|}\\ &= \frac{2}{\frac{\|u\|+\|v\|}{\|u-v\|}\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\|}\\ &\geq \frac{2}{DW_S(X)}.\end{aligned}$$

Case 2: $\lambda_1 \in [-1, 0]$.
It is evident that $\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| = \inf_{\lambda\in\mathbb{R}} \|x + \lambda(-y)\|$, thus we obtain

$$\inf_{\lambda\in\mathbb{R}} \|x + \lambda(-y)\| = \|x + \lambda_1 y\| = \|x + (-\lambda_1)(-y)\|.$$

Since $-\lambda_1 \in [0, 1]$ and $x \perp_I -y$, then, due to the Case 1, we have

$$\inf_{\lambda\in\mathbb{R}} \|x + \lambda y\| = \inf_{\lambda\in\mathbb{R}} \|x + \lambda(-y)\| = \|x + (-\lambda_1)(-y)\| \geq \frac{2}{DW_S(X)}.$$

Consequently, we obtain

$$D(X) \geq \frac{2}{DW_S(X)}.$$

This completes the proof. □

Now, by virtue of Proposition 2.1, we can obtain the following results which give us the bounds of $DW_S(X)$, the relationship between $DW_S(X)$ and Hilbert spaces, and the relationship between $DW_S(X)$ and uniform non-squareness.

Corollary 2.2. *Let X be a Banach space. Then the following statements hold*
(1) $2 \leq DW_S(X) \leq \sqrt{2} + 1$.
(2) X is a Hilbert space if and only if $DW_S(X) = 2$.
(3) If $DW_S(X) < \sqrt{2} + 1$, *then X is uniformly non-square. The converse is not true.*

Proof. The above results can be easily deduced from Proposition 2.1 and following conclusions of $D(X)$.
(1) $2(\sqrt{2} - 1) \leq D(X) \leq 1$ (see [13], Theorem 1).
(2) X is a Hilbert space if and only if $D(X) = 1$ (see [13], Theorem 1).
(3) If $D(X) > 2(\sqrt{2} - 1)$, then X is uniformly non-square (see [14], Theorem 3.2). Furthermore, the converse is not true. Since, in Example 4 of [14], Papini and Wu pointed out that if we take X as $\mathbb{R}^2$ endowed with the norm whose unit sphere S_X is the hexagon with

$$p_1 = (1, 1),\ p_2 = (1 - \sqrt{2}, 1),\ p_3 = (-1, \sqrt{2} - 1),$$

and their opposites as vertices (see the following Figure), then X is uniformly non-square and such that $D(X) = 2(\sqrt{2} - 1)$.

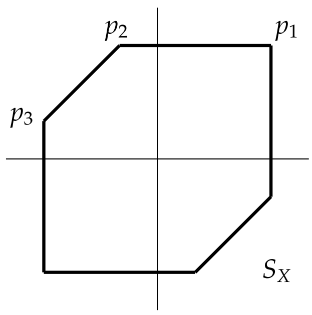

Figure 1. $D(X) = 2(\sqrt{2} - 1)$

□

Remark 2.3. *From Corollary 2.2 (1) and (3), we know that if X is not uniformly non-square, then* $DW_S(X) = \sqrt{2}+1$. *On the other hand, Jiménez-Melado et al. [4] proved that if X is not uniformly non-square, then* $DW(X) = 4$. *Thus,* $DW_S(X)$ *does not necessarily coincide with* $DW(X)$.

Since the converse of Corollary 2.2 (3) is not true, a natural question arises, that is, what condition does X satisfy to make $DW_S(X) < \sqrt{2}+1$ valid. To answer this question, we recall that a Banach space X is said to be uniformly convex, if, for any $0 < \varepsilon \leq 2$, there exists a $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq 1-\delta$$

holds for any $x, y \in S_X$ with $\|x-y\| \geq \varepsilon$.

The following result is a well-known characterization of uniform convexity.

Lemma 2.4. *[17] Let X be a Banach space. Then the following statements are equivalent*
(1) X is uniformly convex.
(2) If $x_n, y_n \in X$ such that $\|x_n\| \to 1, \|y_n\| \to 1$, and $\|x_n+y_n\| \to 2$, then $\|x_n-y_n\| \to 0$.

Now, it is time to answer the question, we assert that if X is uniformly convex, then $DW_S(X) < \sqrt{2}+1$.

Theorem 2.5. *Let X be a Banach space. If X is uniformly convex, then $DW_S(X) < \sqrt{2}+1$.*

Proof. According to Proposition 2.1, it is sufficient to prove that $D(X) > 2(\sqrt{2}-1)$. Now, we suppose conversely that $D(X) = 2(\sqrt{2}-1)$. Then there exist $x_n, y_n \in S_X$ with $x_n \perp_I y_n$ such that

$$\inf_{\lambda\in\mathbb{R}} \|x_n+\lambda y_n\| \to 2(\sqrt{2}-1).$$

For any $n \in \mathbb{N}$, since $x_n \perp_I y_n$, one can easily deduce that $\inf_{\lambda\in\mathbb{R}} \|x_n+\lambda y_n\|$ is attained at some $\lambda_n \in [-1,1]$. Without loss of generality, we can assume that $\lambda_n \in [0,1]$, since otherwise y_n could be replaced by $-y_n$. Therefore,

$$\|x_n+\lambda_n y_n\| \to 2(\sqrt{2}-1),\ \lambda_n \in [0,1]. \tag{5}$$

In addition, since $\lambda_n \in [0,1]$, we can also assume that $\lim_{n\to\infty} \lambda_n$ exists.

Due to $x_n \perp_I y_n$, we have

$$\begin{aligned}1+\lambda_n = (1+\lambda_n)\|x_n\| &\leq \|x_n+\lambda_n y_n\| + \lambda_n\|x_n-y_n\| \\ &= \|x_n+\lambda_n y_n\| + \lambda_n\|x_n+y_n\| \\ &\leq \|x_n+\lambda_n y_n\| + \lambda_n(\|x_n+\lambda_n y_n\| + (1-\lambda_n)\|y_n\|) \\ &= (1+\lambda_n)\|x_n+\lambda_n y_n\| + \lambda_n(1-\lambda_n),\end{aligned}$$

which implies that

$$\|x_n+\lambda_n y_n\| \geq \frac{1+\lambda_n^2}{1+\lambda_n} \geq \min_{0\leq k\leq 1} \frac{1+k^2}{1+k} = 2(\sqrt{2}-1).$$

Let $n \to \infty$, it follows that

$$\lambda_n \to \sqrt{2}-1. \tag{6}$$

Further, according (5), (6) and the following inequalities

$$\begin{aligned}\Big|\|x_n+(\sqrt{2}-1)y_n\| - 2(\sqrt{2}-1)\Big| &\leq \Big|\|x_n+(\sqrt{2}-1)y_n\| - \|x_n+\lambda_n y_n\|\Big| + \Big|\|x_n+\lambda_n y_n\| - 2(\sqrt{2}-1)\Big| \\ &\leq \Big|\lambda_n - (\sqrt{2}-1)\Big| + \Big|\|x_n+\lambda_n y_n\| - 2(\sqrt{2}-1)\Big|,\end{aligned}$$

we can easily deduce that

$$\|x_n+(\sqrt{2}-1)y_n\| \to 2(\sqrt{2}-1). \tag{7}$$

Now, let $u_n = \frac{1}{2}y_n + \frac{\sqrt{2}+1}{2}x_n$, $v_n = \frac{\sqrt{2}}{2}(x_n - y_n)$. Then we have

$$\|u_n\| = \left\|\frac{1}{2}y_n + \frac{\sqrt{2}+1}{2}x_n\right\| = \frac{1}{2(\sqrt{2}-1)}\|x_n + (\sqrt{2}-1)y_n\|.$$

Let $n \to \infty$, by (7), we obtain

$$\|u_n\| \to 1.$$

Further, since $x_n \perp_I y_n$, we get

$$\|v_n\| = \left\|\frac{\sqrt{2}}{2}(x_n - y_n)\right\| = \|(1+\sqrt{2})x_n - \sqrt{2}u_n\| \geq 1 + \sqrt{2} - \sqrt{2}\|u_n\|,$$

and

$$\|v_n\| = \left\|\frac{\sqrt{2}}{2}(x_n - y_n)\right\| = \left\|\frac{\sqrt{2}}{2}(x_n + y_n)\right\| = \|(2-\sqrt{2})u_n + (\sqrt{2}-1)y_n\| \leq \left(2-\sqrt{2}\right)\|u_n\| + \sqrt{2} - 1.$$

Let $n \to \infty$, from $\|u_n\| \to 1$, we also obtain

$$\|v_n\| \to 1.$$

In addition, it is clear that

$$\|(\sqrt{2}-1)v_n + (2-\sqrt{2})u_n\| = \|x_n\| = 1, \tag{8}$$

and

$$\|(2-\sqrt{2})u_n + (\sqrt{2}-1)y_n\| = \left\|\frac{\sqrt{2}}{2}(x_n + y_n)\right\| = \left\|\frac{\sqrt{2}}{2}(x_n - y_n)\right\| = \|v_n\| \to 1. \tag{9}$$

Notice that $\sqrt{2}-1, 2-\sqrt{2} \in (0,1)$ and $(\sqrt{2}-1) + (2-\sqrt{2}) = 1$, thus, according to $\|y_n\| = 1$, $\|u_n\| \to 1$, $\|v_n\| \to 1$, (8) and (9), it follows that

$$\|v_n + u_n\| \to 2,$$

and

$$\|u_n - v_n\| = \|(\sqrt{2}-1)u_n + y_n\| = \sqrt{2}\left\|\left(1 - \frac{\sqrt{2}}{2}\right)u_n + \frac{\sqrt{2}}{2}y_n\right\| \to \sqrt{2}.$$

Consequently, we have

$$\|u_n\| \to 1,\ \|v_n\| \to 1,\ \|u_n + v_n\| \to 2,\ \|u_n - v_n\| \to \sqrt{2} \neq 0,$$

which contradicts Lemma 2.4. This completes the proof. □

Remark 2.6. *In Corollary 2.13, we will prove that, for any $\varepsilon > 0$, there always exists a uniformly convex Banach space X such that $DW_S(X) > \sqrt{2}+1-\varepsilon$. This indicates that the number $\sqrt{2}+1$ in the above result cannot be replaced by a smaller number.*

2.2. *$DW_S(X)$ in symmetric Minkowski planes*

In this section, we will establish an equivalent form of $DW_S(X)$ in symmetric Minkowski planes, which gives us a way to compute the value of $DW_S((\mathbb{R}^2, \|\cdot\|_p))$, $1 < p < \infty$.

Recall that a Minkowski plane (i.e. two-dimensional normed linear space) X is called symmetric Minkowski plane, if there exist $e_1, e_2 \in S_X$ such that

$$\|e_1 + te_2\| = \|e_1 - te_2\| = \|e_2 + te_1\| = \|e_2 - te_1\|,\ t \in \mathbb{R},$$

where $\{e_1, e_2\}$ is called a pair of axes of X. In fact, $\{e_1, e_2\}$ is a basis of X. If not, there exists $\gamma \in \mathbb{R}$ such that $e_1 = \gamma e_2$. Since $e_1, e_2 \in S_X$, we obtain $\gamma = \pm 1$. Further, if $\gamma = 1$, then, from $\|e_1 + te_2\| = \|e_1 - te_2\|$, $t \in \mathbb{R}$, we obtain that $|1 + t| = |1 - t|$, $t \in \mathbb{R}$. This is impossible. Similarly, if $\gamma = -1$, the contradiction will also be derived. Thus, $\{e_1, e_2\}$ is a basis of X.

Here are some classic examples of symmetric Minkowski plane.

Example 2.7. *Let* $X = (\mathbb{R}^2, \|\cdot\|_p)$ $(1 \le p \le \infty)$, *then* X *is a symmetric Minkowski plane and* $\{(1,0),(0,1)\}$ *is a pair of axes of* X.

Example 2.8. *Let* $X = (\mathbb{R}^2, \|\cdot\|_{p_1+p_2})$ $(1 \le p_1 \le p_2 \le \infty)$, *where* $\|\cdot\|_{p_1+p_2} = \frac{\|\cdot\|_{p_1}+\|\cdot\|_{p_2}}{2}$, *then* X *is a symmetric Minkowski plane and* $\{(1,0),(0,1)\}$ *is a pair of axes of* X.

The following results are key to obtaining the equivalent form of $DW_S(X)$ in symmetric Minkowski planes.

Lemma 2.9. *[13] Let* X *be a symmetric Minkowski plane,* $\{e_1, e_2\}$ *be a pair of axes of* X. *Then* $x, y \in S_X$, $x = \alpha e_1 + \beta e_2$, $x \perp_I y$ *if and only if* $y = \pm(-\beta e_1 + \alpha e_2)$.

Proposition 2.10. *Let* X *be a Banach space. Then*

$$DW_S(X) = \sup\left\{\frac{\|x+y\|}{\|(1-t)x+ty\|} \;:\; x, y \in S_X,\ x \perp_I y,\ 0 \le t \le 1\right\}.$$

Proof. First, for any $x, y \in X\backslash\{0\}$ with $x \perp_S y$, let $u = \frac{x}{\|x\|}, v = -\frac{y}{\|y\|}$. Then, we have $u \perp_I v$ and

$$\begin{aligned}\frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| &= \frac{\|u+v\|}{\left\|\frac{\|x\|}{\|x\|+\|y\|}u+\frac{\|y\|}{\|x\|+\|y\|}v\right\|}\\ &\le \sup\left\{\frac{\|x+y\|}{\|(1-t)x+ty\|} \;:\; x, y \in S_X,\ x \perp_I y,\ 0 \le t \le 1\right\},\end{aligned}$$

which implies that

$$DW_S(X) \le \sup\left\{\frac{\|x+y\|}{\|(1-t)x+ty\|} \;:\; x, y \in S_X,\ x \perp_I y,\ 0 \le t \le 1\right\}.$$

On the other hand, let $u, v \in S_X$ with $u \perp_I v$. We consider the following two cases:
Case 1: $0 < t < 1$.
let $x = (1-t)u \ne 0$, $y = -tv \ne 0$. Then, $x \perp_S y$ and

$$\frac{\|u+v\|}{\|(1-t)u+tv\|} = \frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \le DW_S(X).$$

Case 2: $t = 0$ or $t = 1$.
Then, by Corollary 2.2 (1), we obtain

$$\frac{\|u+v\|}{\|(1-t)u+tv\|} \le \frac{\|u\|+\|v\|}{1} = 2 \le DW_S(X).$$

Consequently, we obtain

$$DW_S(X) \ge \sup\left\{\frac{\|x+y\|}{\|(1-t)x+ty\|} \;:\; x, y \in S_X,\ x \perp_I y,\ 0 \le t \le 1\right\}.$$

This completes the proof □

The ideas and techniques for the following proposition are based on literature [18].

Theorem 2.11. *Let X be a symmetric Minkowski plane, $\{e_1, e_2\}$ be a pair of axes of X. Then*

$$DW_S(X) = \sup\left\{ \frac{\|(1+\lambda)e_1 + (1-\lambda)e_2\|}{\|(1-t(1-\lambda))e_1 + (\lambda - t(\lambda+1))e_2\|} \ : \ -1 \le \lambda \le 1,\ 0 \le t \le 1 \right\}.$$

Proof. Let $x, y \in S_X$ with $x \perp_I y$. Since $\{e_1, e_2\}$ is a basis of X, thus we assume that $x = ae_1 + be_2$. Moreover, due to $x \in S_X$, we always have $a \neq 0$ or $b \neq 0$. Without loss of generality, we assume that $a \neq 0$. Then,

$$\left\| e_1 + \frac{b}{a} e_2 \right\| = \left\| \frac{x}{a} \right\| = \frac{1}{|a|},$$

which means that

$$x = \frac{e_1 + \frac{b}{a}e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} sgn(a).$$

Now, according to Lemma 2.9, we obtain

$$y = \pm \frac{\frac{b}{a}e_1 - e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|}.$$

For convenience, we denote $\frac{\frac{b}{a}e_1 - e_2}{\|e_1 + \frac{b}{a}e_2\|}$ and $-\frac{\frac{b}{a}e_1 - e_2}{\|e_1 + \frac{b}{a}e_2\|}$ by y_x and $\overline{y_x}$, respectively.

Step 1: We will calculate the values of $\|x + y_x\|$, $\|x + \overline{y_x}\|$, $\|(1-t)x + ty_x\|$ and $\|(1-t)x + t\overline{y_x}\|$, $0 \le t \le 1$, respectively.

First, since $x \perp_I y$, it follows that

$$\|x + y_x\| = \|x + \overline{y_x}\| = \left\| \frac{e_1 + \frac{b}{a}e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} + \frac{\frac{b}{a}e_1 - e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} \right\| = \frac{\left\|\left(1 + \frac{b}{a}\right)e_1 + \left(\frac{b}{a} - 1\right)e_2\right\|}{\left\|e_1 + \frac{b}{a}e_2\right\|}. \tag{10}$$

Second, to calculate $\|(1-t)x + ty_x\|$ and $\|(1-t)x + t\overline{y_x}\|$, $0 \le t \le 1$, we consider the following two cases:

Case 1: $a > 0$.

Then, for any $0 \le t \le 1$, we have

$$\|(1-t)x + ty_x\| = \left\| (1-t)\frac{e_1 + \frac{b}{a}e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} + t\frac{\frac{b}{a}e_1 - e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} \right\| = \frac{\left\|\left(1 - t\left(1 - \frac{b}{a}\right)\right)e_1 + \left(\frac{b}{a} - t\left(\frac{b}{a} + 1\right)\right)e_2\right\|}{\left\|e_1 + \frac{b}{a}e_2\right\|}, \tag{11}$$

and

$$\|(1-t)x + t\overline{y_x}\| = \left\| (1-t)\frac{e_1 + \frac{b}{a}e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} - t\frac{\frac{b}{a}e_1 - e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} \right\| = \frac{\left\|\left(1 - t\left(1 + \frac{b}{a}\right)\right)e_1 + \left(\frac{b}{a} - t\left(\frac{b}{a} - 1\right)\right)e_2\right\|}{\left\|e_1 + \frac{b}{a}e_2\right\|}. \tag{12}$$

Case 2: $a < 0$.

Then, for any $0 \le t \le 1$, we have

$$\|(1-t)x + ty_x\| = \left\| -(1-t)\frac{e_1 + \frac{b}{a}e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} + t\frac{\frac{b}{a}e_1 - e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} \right\| = \frac{\left\|\left(1 - t\left(1 + \frac{b}{a}\right)\right)e_1 + \left(\frac{b}{a} - t\left(\frac{b}{a} - 1\right)\right)e_2\right\|}{\left\|e_1 + \frac{b}{a}e_2\right\|}, \tag{13}$$

and

$$\|(1-t)x + t\overline{y_x}\| = \left\| -(1-t)\frac{e_1 + \frac{b}{a}e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} - t\frac{\frac{b}{a}e_1 - e_2}{\left\|e_1 + \frac{b}{a}e_2\right\|} \right\| = \frac{\left\|\left(1 - t\left(1 - \frac{b}{a}\right)\right)e_1 + \left(\frac{b}{a} - t\left(\frac{b}{a} + 1\right)\right)e_2\right\|}{\left\|e_1 + \frac{b}{a}e_2\right\|}. \tag{14}$$

Step 2: We will show that

$$\begin{aligned}&\left\{\frac{\|(1+\lambda)e_1+(\lambda-1)e_2\|}{\|(1-t(1-\lambda))e_1+(\lambda-t(\lambda+1))e_2\|}\ :\ -\infty<\lambda<+\infty,\ 0\le t\le 1\right\}\\ =&\left\{\frac{\|(1+\lambda)e_1+(\lambda-1)e_2\|}{\|(1-t(1+\lambda))e_1+(\lambda-t(\lambda-1))e_2\|}\ :\ -\infty<\lambda<+\infty,\ 0\le t\le 1\right\}.\end{aligned} \tag{15}$$

Now, for any $\lambda\in(-\infty,+\infty)$ and $t\in[0,1]$, we have

$$\begin{aligned}&\frac{\|(1+\lambda)e_1+(\lambda-1)e_2\|}{\|(1-t(1-\lambda))e_1+(\lambda-t(\lambda+1))e_2\|}\\ =&\frac{\|(1+\lambda)e_1+(1-\lambda)e_2\|}{\|(1-t(1-\lambda))e_1+(\lambda-t(\lambda+1))e_2\|}\\ =&\frac{\|(1-\lambda)e_1+(1+\lambda)e_2\|}{\|(1-t(1-\lambda))e_1+(\lambda-t(\lambda+1))e_2\|}\\ =&\frac{\|(1+(-\lambda))e_1+(-\lambda-1)e_2\|}{\|(1-t(1+(-\lambda)))e_1+(-\lambda-t(-\lambda-1))e_2\|},\end{aligned}$$

which indicates that

$$\begin{aligned}&\left\{\frac{\|(1+\lambda)e_1+(\lambda-1)e_2\|}{\|(1-t(1-\lambda))e_1+(\lambda-t(\lambda+1))e_2\|}\ :\ -\infty<\lambda<+\infty,\ 0\le t\le 1\right\}\\ =&\left\{\frac{\|(1+\lambda)e_1+(\lambda-1)e_2\|}{\|(1-t(1+\lambda))e_1+(\lambda-t(\lambda-1))e_2\|}\ :\ -\infty<\lambda<+\infty,\ 0\le t\le 1\right\}.\end{aligned}$$

Step 4: For any $t\in[0,1]$, let

$$f_t(\lambda)=\frac{\|(1+\lambda)e_1+(\lambda-1)e_2\|}{\|(1-t(1-\lambda))e_1+(\lambda-t(\lambda+1))e_2\|},\ \lambda\in(-\infty,+\infty).$$

We will prove that

$$f_t\left(\frac{1}{\lambda}\right)=f_t\left(-\lambda\right),\ t\in[0,1]. \tag{16}$$

For any $t\in[0,1]$, it is easily seen that

$$\begin{aligned}f_t\left(\frac{1}{\lambda}\right)=&\frac{\left\|(1+\frac{1}{\lambda})e_1+(\frac{1}{\lambda}-1)e_2\right\|}{\left\|\left(1-t\left(1-\frac{1}{\lambda}\right)\right)e_1+\left(\frac{1}{\lambda}-t\left(\frac{1}{\lambda}+1\right)\right)e_2\right\|}\\ =&\frac{\|(\lambda+1)e_1+(1-\lambda)e_2\|}{\|(\lambda-t(\lambda-1))e_1+(1-t(1+\lambda))e_2\|}\\ =&\frac{\|(1-\lambda)e_1+(\lambda+1)e_2\|}{\|(1-t(1+\lambda))e_1+(\lambda-t(\lambda-1))e_2\|}\\ =&\frac{\|(1-\lambda)e_1+(-\lambda-1)e_2\|}{\|(1-t(1+\lambda))e_1+(-\lambda-t(-\lambda+1))e_2\|}\\ =&f_t\left(-\lambda\right).\end{aligned}$$

Step 5: We will prove that

$$DW_S(X)=\sup\left\{\frac{\|(1+\lambda)e_1+(1-\lambda)e_2\|}{\|(1-t(1-\lambda))e_1+(\lambda-t(\lambda+1))e_2\|}\ :\ -1\le\lambda\le 1,\ 0\le t\le 1\right\}.$$

Denote $\frac{b}{a}$ by λ, then, for any $x = ae_1 + be_2 \in S_X$, we have $\lambda = \frac{b}{a} \in (-\infty, +\infty)$. Conversely, for any $\lambda \in (-\infty, +\infty)$, let $y = e_1 + \lambda e_2$ and $x =: \frac{y}{\|y\|} = \frac{1}{\|y\|}e_1 + \frac{\lambda}{\|y\|}e_2$. Then, we have $x \in S_X$ and $\frac{\frac{\lambda}{\|y\|}}{\frac{1}{\|y\|}} = \lambda$, which shows that, for any $\lambda \in (-\infty, +\infty)$, we can always find a $x_\lambda = a_\lambda e_1 + b_\lambda e_2 \in S_X$ such that $\frac{b_\lambda}{a_\lambda} = \lambda$. Therefore, taking into account (10), (11), (12), (13), (14), (15), and (16), we obtain

$$\begin{aligned} DW_S(X) &= \sup\left\{\frac{\|x+y\|}{\|(1-t)x+ty\|} \;:\; x, y \in S_X,\ x \perp_I y,\ 0 \le t \le 1\right\} \\ &= \sup\left\{\left\{\frac{\|(1+\lambda)e_1 + (\lambda-1)e_2\|}{\|(1-t(1-\lambda))e_1 + (\lambda - t(\lambda+1))e_2\|} \;:\; -\infty < \lambda < +\infty,\ 0 \le t \le 1\right\}\right. \\ &\quad \left.\cup \left\{\frac{\|(1+\lambda)e_1 + (\lambda-1)e_2\|}{\|(1-t(1+\lambda))e_1 + (\lambda - t(\lambda-1))e_2\|} \;:\; -\infty < \lambda < +\infty,\ 0 \le t \le 1\right\}\right\} \\ &= \sup\left\{\frac{\|(1+\lambda)e_1 + (\lambda-1)e_2\|}{\|(1-t(1-\lambda))e_1 + (\lambda - t(\lambda+1))e_2\|} \;:\; -\infty < \lambda < +\infty,\ 0 \le t \le 1\right\} \\ &= \sup\left\{\frac{\|(1+\lambda)e_1 + (\lambda-1)e_2\|}{\|(1-t(1-\lambda))e_1 + (\lambda - t(\lambda+1))e_2\|} \;:\; -1 \le \lambda \le 1,\ 0 \le t \le 1\right\}. \end{aligned}$$

□

As an application of Theorem 2.11, we give the value of $DW_S((\mathbb{R}^2, \|\cdot\|_p))$, $1 < p < \infty$.

Theorem 2.12. *Let $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $X = (\mathbb{R}^2, \|\cdot\|_p)$. Then*

$$DW_S(X) = \sup\left\{\frac{2\|(1,t)\|_p \|(1,t)\|_q}{1+t^2} \;:\; 0 \le t \le 1\right\}.$$

Proof. Let $e_1 = (1, 0), e_2 = (0, 1)$, it is immediate that $\{(1, 0), (0, 1)\}$ is a pair of axes of X. Now, for any $\lambda \in [-1, 1]$, let

$$f_\lambda(t) = |1 - t(1-\lambda)|^p + |\lambda - t(\lambda+1)|^p,\ t \in [0, 1].$$

Step 1: We will show that

$$\min_{0 \le t \le 1} f_\lambda(t) = \frac{(1+\lambda^2)^p}{\left((1-\lambda)^{\frac{p}{p-1}} + (1+\lambda)^{\frac{p}{p-1}}\right)^{p-1}},\ \lambda \in [-1, 1]. \tag{17}$$

We only prove the above equality for $\lambda \in [0, 1]$, since the proof for $\lambda \in [-1, 0]$ is similar. To obtain $\min_{0 \le t \le 1} f_\lambda(t)$, we consider the following two cases:

Case 1: $0 \le t \le \frac{\lambda}{\lambda+1}$.

Then $f_\lambda(t) = (1 - t(1-\lambda))^p + (\lambda - t(\lambda+1))^p$. Obviously, $f_\lambda(t)$ is a decreasing function of t in $\left[0, \frac{\lambda}{\lambda+1}\right]$. Thus, we obtain

$$\min_{0 \le t \le \frac{\lambda}{\lambda+1}} f_\lambda(t) = f_\lambda\left(\frac{\lambda}{\lambda+1}\right) = \left(\frac{1+\lambda^2}{\lambda+1}\right)^p.$$

Case 2: $\frac{\lambda}{\lambda+1} \le t \le 1$.

Then $f_\lambda(t) = (1 - t(1-\lambda))^p + (t(\lambda+1) - \lambda)^p$. Further, one can easily obtain that $f'_\lambda(t) \ge 0$ if and only if

$$t \ge \frac{(1-\lambda)^{\frac{1}{p-1}} + \lambda(1+\lambda)^{\frac{1}{p-1}}}{(1-\lambda)^{\frac{p}{p-1}} + (1+\lambda)^{\frac{p}{p-1}}}.$$

Moreover, through some calculations, we can get that the following inequalities

$$\frac{(1-\lambda)^{\frac{1}{p-1}}+\lambda(1+\lambda)^{\frac{1}{p-1}}}{(1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}}\leq 1$$

and

$$\frac{(1-\lambda)^{\frac{1}{p-1}}+\lambda(1+\lambda)^{\frac{1}{p-1}}}{(1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}}\geq \frac{\lambda}{1+\lambda}$$

are equivalent to

$$\lambda^{p-1}-\lambda^{p}\leq 1+\lambda \tag{18}$$

and

$$\left(1+\frac{1}{\lambda}\right)^{p-1}\geq (1-\lambda)^{p-1}, \tag{19}$$

respectively. Since $\lambda\in[0,1]$, the inequalities (18) and (19) obviously hold. Thus, we obtain

$$\frac{(1-\lambda)^{\frac{1}{p-1}}+\lambda(1+\lambda)^{\frac{1}{p-1}}}{(1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}}\in\left[\frac{\lambda}{\lambda+1},1\right].$$

This means that

$$\begin{aligned}
&\min_{\frac{\lambda}{\lambda+1}\leq t\leq 1} f_\lambda(t)\\
=&f_\lambda\left(\frac{(1-\lambda)^{\frac{1}{p-1}}+\lambda(1+\lambda)^{\frac{1}{p-1}}}{(1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}}\right)\\
=&\left(1-\frac{(1-\lambda)^{\frac{1}{p-1}}+\lambda(1+\lambda)^{\frac{1}{p-1}}}{(1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}}(1-\lambda)\right)^p+\left(\frac{(1-\lambda)^{\frac{1}{p-1}}+\lambda(1+\lambda)^{\frac{1}{p-1}}}{(1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}}(\lambda+1)-\lambda\right)^p\\
=&\frac{\left((1+\lambda)^{\frac{p}{p-1}}-\lambda(1-\lambda)(1+\lambda)^{\frac{1}{p-1}}\right)^p+\left((1+\lambda)(1-\lambda)^{\frac{1}{p-1}}-\lambda(1-\lambda)^{\frac{p}{p-1}}\right)^p}{\left((1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}\right)^p}\\
=&\frac{(1+\lambda)^{\frac{p}{p-1}}\left(1+\lambda^2\right)^p+(1-\lambda)^{\frac{p}{p-1}}\left(1+\lambda^2\right)^p}{\left((1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}\right)^p}\\
=&\frac{\left(1+\lambda^2\right)^p}{\left((1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}\right)^{p-1}}.
\end{aligned}$$

It is obvious that

$$\left((1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}\right)^{p-1}\geq(1+\lambda)^p,$$

thus

$$\min_{0\leq t\leq 1}f_\lambda(t)=\min\left\{\min_{0\leq t\leq\frac{\lambda}{\lambda+1}}f_\lambda(t),\ \min_{\frac{\lambda}{\lambda+1}\leq t\leq 1}f_\lambda(t)\right\}=\min_{\frac{\lambda}{\lambda+1}\leq t\leq 1}f_\lambda(t)=\frac{\left(1+\lambda^2\right)^p}{\left((1-\lambda)^{\frac{p}{p-1}}+(1+\lambda)^{\frac{p}{p-1}}\right)^{p-1}}.$$

Step 2: We will show that

$$DW_S(X)=\sup\left\{\frac{\|(1+\lambda,1-\lambda)\|_p\|(1+\lambda,1-\lambda)\|_q}{1+\lambda^2}\ :\ 0\leq\lambda\leq 1\right\},$$

Notice that, by (17), we have

$$\min_{0\le t\le 1} f_\lambda(t) = \min_{0\le t\le 1} f_{-\lambda}(t),\ \lambda \in [-1,1].$$

Thus, according to Theorem 2.11, we obtain

$$\begin{aligned}
DW_S(X) &= \sup\left\{\frac{\|(1+\lambda)e_1 + (1-\lambda)e_2\|}{\|(1-t(1-\lambda))e_1 + (\lambda - t(\lambda+1))e_2\|} \ :\ -1\le\lambda\le 1,\ 0\le t\le 1\right\} \\
&= \sup\left\{\frac{\|(1+\lambda)e_1 + (1-\lambda)e_2\|}{\min_{0\le t\le 1}\|(1-t(1-\lambda))e_1 + (\lambda - t(\lambda+1))e_2\|} \ :\ -1\le\lambda\le 1\right\} \\
&= \sup\left\{\frac{((1+\lambda)^p + (1-\lambda)^p)^{\frac{1}{p}}}{(\min_{0\le t\le 1} f_\lambda(t))^{\frac{1}{p}}} \ :\ -1\le\lambda\le 1\right\} \\
&= \sup\left\{\frac{((1+\lambda)^p + (1-\lambda)^p)^{\frac{1}{p}}}{(\min_{0\le t\le 1} f_\lambda(t))^{\frac{1}{p}}} \ :\ 0\le\lambda\le 1\right\} \\
&= \sup\left\{\frac{((1+\lambda)^p + (1-\lambda)^p)^{\frac{1}{p}}((1+\lambda)^q + (1-\lambda)^q)^{\frac{1}{q}}}{1+\lambda^2} \ :\ 0\le\lambda\le 1\right\} \\
&= \sup\left\{\frac{\|(1+\lambda, 1-\lambda)\|_p\|(1+\lambda, 1-\lambda)\|_q}{1+\lambda^2} \ :\ 0\le\lambda\le 1\right\}.
\end{aligned}$$

Step 3: We will prove that

$$DW_S(X) = \sup\left\{\frac{2\|(1,t)\|_p\|(1,t)\|_q}{1+t^2} \ :\ 0\le t\le 1\right\}.$$

For any $\lambda\in[0,1]$, let $t = \frac{1-\lambda}{1+\lambda}\in[0,1]$. Then we have $\lambda = \frac{1-t}{1+t}$. Moreover, by the Step 2, we obtain

$$\begin{aligned}
DW_S(X) &= \sup\left\{\frac{\|(1+\lambda, 1-\lambda)\|_p\|(1+\lambda, 1-\lambda)\|_q}{1+\lambda^2} \ :\ 0\le\lambda\le 1\right\} \\
&= \sup\left\{\frac{(1+\lambda)^2\left\|\left(1, \frac{1-\lambda}{1+\lambda}\right)\right\|_p\left\|\left(1, \frac{1-\lambda}{1+\lambda}\right)\right\|_q}{1+\lambda^2} \ :\ 0\le\lambda\le 1\right\} \\
&= \sup\left\{\frac{2\|(1,t)\|_p\|(1,t)\|_q}{1+t^2} \ :\ 0\le t\le 1\right\}.
\end{aligned}$$

□

By applying Theorem 2.12, we can obtain the following result which implies that the number $\sqrt{2}+1$ in the Theorem 2.5 cannot be replaced by a smaller number.

Corollary 2.13. *For any $\varepsilon > 0$, there exists a uniformly convex Banach space X such that $DW_S(X) > \sqrt{2}+1-\varepsilon$.*

Proof. Let $p, q\in(1,+\infty)$ with $\frac{1}{p}+\frac{1}{q} = 1$. Now, by Theorem 2.12 and Corollary 2.2 (1), we obtain

$$\sqrt{2}+1 \ge DW_S((\mathbb{R}^2, \|\cdot\|_p)) \ge \frac{2\|(1, \sqrt{2}-1)\|_p\|(1, \sqrt{2}-1)\|_q}{1+(\sqrt{2}-1)^2} = \frac{(1+(\sqrt{2}-1)^p)^{\frac{1}{p}}(1+(\sqrt{2}-1)^{\frac{p}{p-1}})^{\frac{p-1}{p}}}{2-\sqrt{2}}.$$

Let $p\to\infty$, we obtain $DW_S((\mathbb{R}^2, \|\cdot\|_p)) \to \sqrt{2}+1$. Thus, for any $\varepsilon > 0$, there exists a p large enough, such that $DW_S((\mathbb{R}^2, \|\cdot\|_p)) > \sqrt{2}+1-\varepsilon$. Moreover, it is well-known that $(\mathbb{R}^2, \|\cdot\|_p)$ is uniformly convex, so we obtain the desired result. □

2.3. $DW_S(X)$ in Radon planes

The usual orthogonality in the Hilbert space is always symmetric, that is, $x \perp y$ implies $y \perp x$. However, the Birkhoff orthogonality in the Banach space is not symmetric in general. In [19], James proved the following result.

Theorem 2.14. *[19] A normed linear space X whose dimension is at least three is an inner product space if and only if the Birkhoff orthogonality is symmetric in X.*

The assumption of the dimension of the space X in the above theorem cannot be omitted. James [19] provided examples of two-dimensional normed linear spaces, in which the Birkhoff orthogonality is symmetric, that is, the space $l_p - l_q$ is defined for $1 \leq p, q \leq \infty$ as the space $\mathbb{R}^2$ with the norm

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_p & (x_1 x_2 \geq 0)\,; \\ \|(x_1, x_2)\|_q & (x_1 x_2 \leq 0)\,, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.15. *[20] A two-dimensional normed linear space in which the Birkhoff orthogonality is symmetric is called Radon plane.*

Radon planes have many remarkable, almost-Euclidean properties. For example, the radial projection on the Radon plane X is non-expansive, that is, the map $R : X \to X$, defined by

$$R(x) = \begin{cases} x & \|x\| \leq 1; \\ \frac{x}{\|x\|} & \|x\| > 1, \end{cases}$$

such that $\|R(x) - R(y)\| \leq \|x - y\|,\ x, y \in X$. However, in higher dimensions only Euclidean space has this property. For a survey on Radon planes, including further results, see [20].

In [9], Mizuguchi considered the Dunkl-Williams constant $DW(X)$ in Radon planes and gave the following results:

(1) Let X be a Radon plane. Then $2 \leq DW(X) \leq \frac{9}{4}$.

(2) Let X be a Radon plane. Then $DW(X) = \frac{9}{4}$ if and only if its unit sphere is an affine regular hexagon (see [9], Preliminaries).

In this section, we will study the constant $DW_S(X)$ in Radon planes and show that the above two results still hold for $DW_S(X)$.

Since $2 \leq DW(X) \leq \frac{9}{4}$ holds for any Radon plane, we can get the following result easily by Corollary 2.2 (1) and the fact that $DW_S(X) \leq DW(X)$.

Proposition 2.16. *Let X be a Radon plane. Then* $2 \leq DW_S(X) \leq \frac{9}{4}$.

Since two-dimensional Hilbert spaces are Radon planes, by Corollary 2.2 (2), we can know that the lower bound shown in the above result is sharp. Moreover, the following example will indicate that the upper bound shown in the above result is also sharp.

Example 2.17. *Let X be a Radon plane $l_\infty - l_1$, that is, the space $\mathbb{R}^2$ with the norm defined by*

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_\infty & (x_1 x_2 \geq 0)\,; \\ \|(x_1, x_2)\|_1 & (x_1 x_2 \leq 0)\,. \end{cases}$$

Then $DW_S(X) = \frac{9}{4}$.

Proof. Take $x = (-1, 0)$ and $y = \left(\frac{1}{2}, 1\right)$, it is clear that $x, y \in S_X$. Further, since $\|x + y\| = \|x - y\| = \frac{3}{2}$, we have $x \perp_I y$. Now, by Proposition 2.10 and Proposition 2.16, we have

$$\frac{9}{4} \geq DW_s(X) \geq \frac{\|x + y\|}{\left\|\frac{1}{3}x + \frac{2}{3}y\right\|} = \frac{9}{4}.$$

This completes the proof. □

Notice that the unit sphere of $l_\infty - l_1$ is actually an affine regular hexagon, hence $l_\infty - l_1$ is a Radon plane with $DW_S(X) = \frac{9}{4}$ and such that its unit sphere is an affine regular hexagon. In fact, this is also true in general, see the following result.

Theorem 2.18. *Let X be a Radon plane. Then $DW_S(X) = \frac{9}{4}$ if and only if its unit sphere is an affine regular hexagon.*

Proof. If $DW_S(X) = \frac{9}{4}$, from the inequalities $DW_S(X) \le DW(X) \le \frac{9}{4}$ and the fact the $DW(X) = \frac{9}{4}$ if and only if unit sphere is an affine regular hexagon, we obtain that S_X is an affine regular hexagon.

Conversely, if S_X is an affine regular hexagon, then there exist $u, v \in S_X$ such that $\pm u$, $\pm v$, $\pm(u+v)$ are the vertices of S_X.

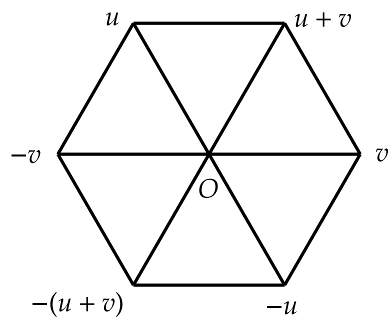

Figure 2. Affine regular hexagonal unit sphere.

Let $x = -v$ and $y = u + \frac{1}{2}v$. Then, we have $x \in S_X$ and $y = \frac{1}{2}u + \frac{1}{2}(u+v) \in S_X$. Moreover, we also have

$$\|x+y\| = \left\|u - \frac{1}{2}v\right\| = \frac{3}{2}\left\|\frac{2}{3}u + \frac{1}{3}(-v)\right\| = \frac{3}{2}$$

and

$$\|x-y\| = \left\|u + \frac{3}{2}v\right\| = \frac{3}{2}\left\|\frac{2}{3}u + v\right\| = \frac{3}{2}\left\|\frac{2}{3}(u+v) + \frac{1}{3}v\right\| = \frac{3}{2},$$

which indicate that $x \perp_I y$. Consequently, by Proposition 2.10 and Proposition 2.16, we have

$$\frac{9}{4} \ge DW_S(X) \ge \frac{\|x+y\|}{\left\|\frac{1}{3}x + \frac{2}{3}y\right\|} = \frac{\frac{3}{2}}{\left\|-\frac{1}{3}v + \frac{2}{3}u + \frac{1}{3}v\right\|} = \frac{\frac{3}{2}}{\frac{2}{3}} = \frac{9}{4}.$$

This completes the proof. □

3. The Dunkl-Williams constant related to the isosceles orthogonality

3.1. The difference between $DW_I(X)$, $DW_S(X)$ and $DW(X)$

In this section, we will discuss the bounds of $DW_I(X)$ and give a example to show that $DW_I(X)$, $DW_S(X)$ and $DW(X)$ do not necessarily coincide with each other. To obtain the bounds of $DW_I(X)$, we need the following result.

Lemma 3.1. *[11] If x and y are isosceles orthogonal elements in a Banach space, and $\|y\| \le \|x\|$, then $\|x+ky\| \ge 2(\sqrt{2}-1)\|x\|$, $k \in \mathbb{R}$.*

Proposition 3.2. *Let X be a Banach space. Then $2 \le DW_I(X) \le \sqrt{2}+1$.*

Proof. First, take $x, y \in S_X$ with $x \perp_I y$, then,

$$DW_I(X) \geq \frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = 2.$$

Second, for any $x, y \in X \backslash \{0\}$ with $x \perp_I y$, we consider the following two cases:

Case 1: $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geq \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|$.

Let $u = \frac{x}{\|x\|}$ and $v = -\frac{y}{\|y\|}$. Then, we have $\|u + v\| \geq \|u - v\|$ and $u + v \perp_I u - v$. Further, by Lemma 3.1, we have

$$\begin{aligned}
\frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &= \frac{\|u + v\|}{\left\| \frac{\|x\|}{\|x\|+\|y\|} u + \frac{\|y\|}{\|x\|+\|y\|} v \right\|} \\
&= \frac{\|u + v\|}{\frac{1}{2} \left\| u + v + \frac{\|x\|-\|y\|}{\|x\|+\|y\|} (u - v) \right\|} \\
&= \frac{2}{\frac{\left\| u+v+\frac{\|x\|-\|y\|}{\|x\|+\|y\|}(u-v) \right\|}{\|u+v\|}} \\
&\leq \frac{2}{2(\sqrt{2} - 1)} = \sqrt{2} + 1.
\end{aligned}$$

Case 2: $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| < \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|$.

Let $u = \frac{x}{\|x\|}$ and $v = \frac{y}{\|y\|}$. Then, we have $\|u + v\| > \|u - v\|$ and $u + v \perp_I u - v$. Notice that $x \perp_I y$, then, by Lemma 3.1, it follows that

$$\begin{aligned}
\frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &< \frac{\|x\| + \|y\|}{\|x - y\|} \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \\
&= \frac{\|x\| + \|y\|}{\|x + y\|} \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \\
&= \frac{\|u + v\|}{\left\| \frac{\|x\|}{\|x\|+\|y\|} u + \frac{\|y\|}{\|x\|+\|y\|} v \right\|} \\
&= \frac{\|u + v\|}{\frac{1}{2} \left\| u + v + \frac{\|x\|-\|y\|}{\|x\|+\|y\|} (u - v) \right\|} \\
&= \frac{2}{\frac{\left\| u+v+\frac{\|x\|-\|y\|}{\|x\|+\|y\|}(u-v) \right\|}{\|u+v\|}} \\
&\leq \frac{2}{2(\sqrt{2} - 1)} = \sqrt{2} + 1.
\end{aligned}$$

Consequently, we obtain $DW_I(X) \leq \sqrt{2} + 1$. □

Remark 3.3. *If X is a Hilbert space, then $DW(X) = 2$ (see [1]). So, by the inequalities $2 \leq DW_I(X) \leq DW(X) = 2$, we obtain $DW_I(X) = 2$, which shows that the lower bound given in the above proposition is sharp. But for the upper bound $\sqrt{2} + 1$, we don't know if there's any space attains it.*

Next, we will give an example to show that $DW_I(X)$, $DW_S(X)$ and $DW(X)$ do not necessarily coincide with each other. The following result is needed.

Proposition 3.4. *Let X be a Banach space. Then*

$$DW_I(X) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|}-\frac{x-y}{\|x-y\|}\right\| \ :\ x,y\in S_X, x\neq \pm y\right\}.$$

Proof. First, for any $x, y \in S_X$ with $x \neq \pm y$, let

$$u = \frac{x+y}{2},\ v = \frac{x-y}{2}.$$

It is clearly that $u, v \neq 0$ and $u \perp_I v$. Then we get

$$DW_I(X) \geq \frac{\|u\|+\|v\|}{\|u-v\|}\left\|\frac{u}{\|u\|}-\frac{v}{\|v\|}\right\| = \frac{\|x+y\|+\|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|}-\frac{x-y}{\|x-y\|}\right\|,$$

which means that

$$DW_I(X) \geq \sup\left\{\frac{\|x+y\|+\|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|}-\frac{x-y}{\|x-y\|}\right\| \ :\ x,y\in S_X, x\neq \pm y\right\}.$$

On the other hand, for any $u, v \in X \setminus \{0\}$ such that $u \perp_I v$, taking

$$x = \frac{u+v}{\|u+v\|},\ y = \frac{u-v}{\|u-v\|}.$$

It is easy to see that $x, y \in S_X$ and $x \neq \pm y$. Thus we have

$$\begin{aligned}\frac{\|u\|+\|v\|}{\|u-v\|}\left\|\frac{u}{\|u\|}-\frac{v}{\|v\|}\right\| &= \frac{\|x+y\|+\|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|}-\frac{x-y}{\|x-y\|}\right\| \\ &\leq \sup\left\{\frac{\|x+y\|+\|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|}-\frac{x-y}{\|x-y\|}\right\| \ :\ x,y\in S_X, x\neq \pm y\right\},\end{aligned}$$

which shows that

$$DW_I(X) \leq \sup\left\{\frac{\|x+y\|+\|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|}-\frac{x-y}{\|x-y\|}\right\| \ :\ x,y\in S_X, x\neq \pm y\right\}.$$

This completes the proof. □

By applying the above equivalent form of $DW_I(X)$, we only need to consider $x, y \in S_X$ and x, y are not required to be isosceles orthogonal, when we are calculating the value $DW_I(X)$ for some space X. This is much more convenient than using the definition of $DW_I(X)$ to calculate the value of $DW_I(X)$, since the definition of $DW_I(X)$ requires us to consider $x, y \in X\setminus\{0\}$ with $x \perp_I y$.

Example 3.5. *Let* $X = (\mathbb{R}^2, \|\cdot\|_\infty)$*. Then* $DW_I(X) = \frac{9}{4}$.

Proof. Next, we will prove that

$$\frac{\|x+y\|+\|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|}-\frac{x-y}{\|x-y\|}\right\| \leq \frac{9}{4} \tag{20}$$

holds for any $x, y \in S_X$ with $x \neq \pm y$.

Obviously, if x is replaced by $-x$ or y is replaced by $-y$, the value on the left side of (20) will not change. Thus, we only need to prove the inequality (20), for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $x_1 \geq 0, y_1 \geq 0$. Now, for x, we consider the following four cases, that is

$$x = (a, 1),\ a \in [0, 1],\ x = (1, b),\ b \in [0, 1],$$

$$x = (1, c),\ c \in [-1, 0],\ x = (d, -1),\ d \in [0, 1].$$

Further, for any of the above four cases, we always divide y into the following four cases, that is

$$y = (\bar{a}, 1),\ \bar{a} \in [0, 1],\ y = (1, \bar{b}),\ \bar{b} \in [0, 1],$$

$$y = (1, \bar{c}),\ \bar{c} \in [-1, 0],\ y = (\bar{d}, -1),\ \bar{d} \in [0, 1].$$

According to the above method, there are a total of 16 cases to be considered. However, since each case is very similar, we will only take the case of $x = (a, 1),\ a \in [0, 1],\ y = (1, b),\ b \in [0, 1]$ as an example to prove that the inequality (20) is true.

Case 1: $a \leq b$.

Then, we have

$$\begin{aligned}
&\frac{\|x+y\| + \|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|} - \frac{x-y}{\|x-y\|}\right\| \\
=&\frac{2+b-a}{2}\frac{2+a+b}{1+b} \\
=&\frac{(2+b)^2 - a^2}{2(1+b)} \leq \frac{(2+b)^2}{2(1+b)} \leq \sup_{0\leq b\leq 1}\frac{(2+b)^2}{2(1+b)} = \frac{9}{4}.
\end{aligned}$$

Case 2: $b < a$.

Then, we have

$$\begin{aligned}
&\frac{\|x+y\| + \|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|} - \frac{x-y}{\|x-y\|}\right\| \\
=&\frac{2+a-b}{2}\frac{2+a+b}{1+a} \\
=&\frac{(2+a)^2 - b^2}{2(1+a)} \leq \frac{(2-a)^2}{2(1+a)} \leq \sup_{0\leq a\leq 1}\frac{(2+a)^2}{2(1+a)} = \frac{9}{4}.
\end{aligned}$$

Thus, we have (20) holds for any $x, y \in S_X$ with $x \neq \pm y$.

On the other hand, we take $x = (0, 1)$ and $y = (1, 1)$. Then, due to Proposition 3.4, we have

$$\frac{9}{4} \geq DW_I(X) \geq \frac{\|x+y\| + \|x-y\|}{2}\left\|\frac{x+y}{\|x+y\|} - \frac{x-y}{\|x-y\|}\right\| = \frac{9}{4},$$

which shows that $DW_I(X) = \frac{9}{4}$. □

Since $X = (\mathbb{R}^2, \|\cdot\|_\infty)$ is not uniformly non-square, by Remark 2.3 and above result, we can assert that $DW_I(X)$, $DW_S(X)$ and $DW(X)$ do not necessarily coincide with each other.

Additionally, there's another thing we can see from Example 3.5. We know that $DW_S(X)$ and $DW(X)$ both attain their upper bounds $\sqrt{2} + 1$ and 4 in $(\mathbb{R}^2, \|\cdot\|_\infty)$, but $DW_I(X)$ does not attain its upper bound $\sqrt{2} + 1$ in $(\mathbb{R}^2, \|\cdot\|_\infty)$. Thus, we suspect that $\sqrt{2} + 1$ may not be the best upper bound of $DW_I(X)$, but $\frac{9}{4}$ is. Although we cannot prove that our conjecture is correct, we can prove the following conclusion, which can be regarded as a necessary condition for "$DW_I(X) \leq \frac{9}{4}$".

Proposition 3.6. *Let X be a Banach space. Then, for any $x, y \in X\backslash\{0\}$ with $x \perp_I y$, the following inequality holds*

$$\min\left\{\frac{\|x\| + \|y\|}{\|x-y\|}, \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right\} \leq \frac{3}{2}.$$

Proof. Suppose conversely that there exist $x, y \in X\backslash\{0\}$ with $x \perp_I y$ such that

$$\frac{\|x\|+\|y\|}{\|x-y\|} > \frac{3}{2}, \ \left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| > \frac{3}{2}.$$

Without loss of generality, we suppose that $\|x\| \le \|y\|$. Then, it follows that

$$\begin{aligned}
\frac{3}{2} &< \left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \\
&= 2\left\|\frac{\|y\|-\|x\|}{2\|y\|}\frac{x}{\|x\|}+\frac{\|x\|+\|y\|}{2\|y\|}\left(\frac{\|x\|}{\|x\|+\|y\|}\frac{x}{\|x\|}-\frac{\|y\|}{\|x\|+\|y\|}\frac{y}{\|y\|}\right)\right\| \\
&\le \frac{\|y\|-\|x\|}{\|y\|}+\frac{\|x\|+\|y\|}{\|y\|}\left\|\frac{\|x\|}{\|x\|+\|y\|}\frac{x}{\|x\|}-\frac{\|y\|}{\|x\|+\|y\|}\frac{y}{\|y\|}\right\| \\
&= \frac{\|y\|-\|x\|}{\|y\|}+\frac{\|x\|+\|y\|}{\|y\|}\frac{1}{\frac{\|x\|+\|y\|}{\|x-y\|}} \\
&< \frac{\|y\|-\|x\|}{\|y\|}+\frac{\|x\|+\|y\|}{\|y\|}\frac{2}{3} \\
&= \frac{5\|y\|-\|x\|}{3\|y\|},
\end{aligned}$$

which implies that

$$\|x\| < \frac{1}{2}\|y\|. \tag{21}$$

Then, from (21) and $x \perp_I y$, we have

$$\frac{3}{2} < \frac{\|x\|+\|y\|}{\|x-y\|} < \frac{3}{2}\frac{\|y\|}{\|x-y\|} = \frac{3}{4}\frac{\|2y\|}{\|x-y\|} \le \frac{3}{4}\frac{\|x+y\|+\|x-y\|}{\|x-y\|} = \frac{3}{2}.$$

This is impossible. Thus, we obtain the desired result. □

Remark 3.7. *It is possible to give an example of a Banach space X where $x, y \in X\backslash\{0\}$ with $x \perp_I y$ and*

$$\min\left\{\frac{\|x\|+\|y\|}{\|x-y\|}, \left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\right\} = \frac{3}{2}.$$

Actually, we only need to consider the Banach space $(\mathbb{R}^2, \|\cdot\|_\infty)$, and $x = \left(\frac{1}{2}, 1\right), y = \left(-\frac{1}{2}, 0\right)$.

3.2. *Some estimates for $DW_I(X)$ in terms of other well-known constants*

In this section, we will give some estimates for $DW_I(X)$ in terms of other well-known constants. Recall that the modulus of convexity of X introduced by Clarkson [2], which can be used to characterize the uniform convexity is the function $\delta_X(\varepsilon) : [0, 2] \to [0, 1]$ given by

$$\delta_X(\varepsilon) = \inf\left\{1-\frac{\|x+y\|}{2} \ : \ x, y \in B_X, \ \|x-y\| \ge \varepsilon\right\}.$$

In [4], Jiménez-Melado et al. gave a nice relationship between $DW(X)$ and $\delta_X(\varepsilon)$, that is,

$$DW(X) \le \sup_{0\le\varepsilon\le 2}\{\min\{4-2\delta_X(\varepsilon), \ 2+\varepsilon\}\}.$$

Thus, it is natural for us to consider whether there exists a relationship between $DW_I(X)$ and $\delta_X(\varepsilon)$.

Proposition 3.8. *Let X be a Banach space. Then*

$$DW_I(X) \leq \sup_{0\leq\varepsilon\leq 1}\{3 - 2\delta_X(\varepsilon)\}.$$

Proof. Without loss of generality, for any $x, y \in X\backslash\{0\}$ with $x \perp_I y$, we assume that $\|x\| \leq \|y\|$. Then,

$$\begin{aligned}\frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| &= \left\|x - y + \frac{\|y\|}{\|x\|}x - \frac{\|x\|}{\|y\|}y\right\| \|x-y\|^{-1}\\ &\leq \left(\|x-y\| + \left\|\frac{\|y\|}{\|x\|}x - \frac{\|x\|}{\|y\|}y\right\|\right)\|x-y\|^{-1}\\ &= 1 + \left\|\frac{\frac{\|y\|}{\|x\|}x}{\|x-y\|} - \frac{\frac{\|x\|}{\|y\|}y}{\|x-y\|}\right\|.\end{aligned}$$

Now, since $x \perp_I y$, one can easily deduce that $\max\{\|x\|, \|y\|\} \leq \|x-y\|$, which means that $\frac{\frac{\|y\|}{\|x\|}x}{\|x-y\|}, \frac{\frac{\|x\|}{\|y\|}y}{\|x-y\|} \in B_X$. Thus, by the definition of $\delta_X(\varepsilon)$, we get

$$\frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leq 1 + 2\left(1 - \delta_X\left(\left\|\frac{\frac{\|y\|}{\|x\|}x}{\|x-y\|} + \frac{\frac{\|x\|}{\|y\|}y}{\|x-y\|}\right\|\right)\right). \tag{22}$$

In addition, from $\|x\| \leq \|y\|$ and $x \perp_I y$, we also have

$$\begin{aligned}\left\|\frac{\|y\|}{\|x\|}x + \frac{\|x\|}{\|y\|}y\right\| &= \left\|\frac{1}{2}\left(\frac{\|y\|}{\|x\|}+\frac{\|x\|}{\|y\|}\right)(x+y) + \frac{1}{2}\left(\frac{\|y\|}{\|x\|}-\frac{\|x\|}{\|y\|}\right)(x-y)\right\|\\ &\geq \frac{1}{2}\left(\frac{\|y\|}{\|x\|}+\frac{\|x\|}{\|y\|}\right)\|x+y\| - \frac{1}{2}\left(\frac{\|y\|}{\|x\|}-\frac{\|x\|}{\|y\|}\right)\|x-y\|\\ &= \frac{\|x\|}{\|y\|}\|x-y\|,\end{aligned}$$

which shows that

$$\left\|\frac{\frac{\|y\|}{\|x\|}x}{\|x-y\|} + \frac{\frac{\|x\|}{\|y\|}y}{\|x-y\|}\right\| \geq \frac{\|x\|}{\|y\|}.$$

Then, since $\delta_X(\varepsilon)$ is a nondecreasing function, we have the following inequality from (22),

$$\frac{\|x\|+\|y\|}{\|x-y\|}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leq 1 + 2\left(1 - \delta_X\left(\frac{\|x\|}{\|y\|}\right)\right) \leq \sup_{0\leq\varepsilon\leq 1}\{3 - 2\delta_X(\varepsilon)\}.$$

which implies that

$$DW_I(X) \leq \sup_{0\leq\varepsilon\leq 1}\{3 - 2\delta_X(\varepsilon)\}.$$

□

In Proposition 2.1, we prove the equality $D(X)DW_S(X) = 2$, which leads us to think that $DW_I(X)$ may also be related to some constant which measures difference between Birkhoff orthogonality and isosceles orthogonality. Next, we give the relationship between $DW_I(X)$ and $IB'(X)$. Its ideas and techniques come from Proposition 3.2.

Proposition 3.9. *Let X be a Banach space. Then*

$$DW_I(X) \leq \frac{2}{IB'(X)}.$$

Proof. For any $x, y \in X\backslash\{0\}$ with $x \perp_I y$, we consider the following two cases:

Case 1: $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \geq \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|$.

Let $u = \frac{x}{\|x\|}$ and $v = -\frac{y}{\|y\|}$. Then, we have $\|u + v\| \geq \|u - v\|$ and $u + v \perp_I u - v$. Further,

$$\begin{aligned}\frac{\|x\| + \|y\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| &= \frac{\|u + v\|}{\left\|\frac{\|x\|}{\|x\|+\|y\|}u + \frac{\|y\|}{\|x\|+\|y\|}v\right\|}\\ &= \frac{\|u + v\|}{\frac{1}{2}\left\|u + v + \frac{\|x\|-\|y\|}{\|x\|+\|y\|}(u - v)\right\|}\\ &= \frac{2}{\frac{\left\|u+v+\frac{\|x\|-\|y\|}{\|x\|+\|y\|}(u-v)\right\|}{\|u+v\|}}\\ &\leq \frac{2}{IB'(X)}.\end{aligned}$$

Case 2: $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| < \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|$.

Let $u = \frac{x}{\|x\|}$ and $v = \frac{y}{\|y\|}$. Then, we have $\|u + v\| > \|u - v\|$ and $u + v \perp_I u - v$. Notice that $x \perp_I y$, then we obtain

$$\begin{aligned}\frac{\|x\| + \|y\|}{\|x - y\|}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| &< \frac{\|x\| + \|y\|}{\|x - y\|}\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\\ &= \frac{\|x\| + \|y\|}{\|x + y\|}\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\\ &= \frac{\|u + v\|}{\left\|\frac{\|x\|}{\|x\|+\|y\|}u + \frac{\|y\|}{\|x\|+\|y\|}v\right\|}\\ &= \frac{\|u + v\|}{\frac{1}{2}\left\|u + v + \frac{\|x\|-\|y\|}{\|x\|+\|y\|}(u - v)\right\|}\\ &= \frac{2}{\frac{\left\|u+v+\frac{\|x\|-\|y\|}{\|x\|+\|y\|}(u-v)\right\|}{\|u+v\|}}\\ &\leq \frac{2}{IB'(X)}.\end{aligned}$$

Consequently, we obtain $DW_I(X) \leq \frac{2}{IB'(X)}$. □

4. Conclusions

In this paper, we introduce two new constants $DW_S(X)$ and $DW_I(X)$, which are the Dunkl-Williams constant related to theSinger orthogonality and the isosceles orthogonality, respectively. It is of interest to investigate their relationships with other well-known constants. Of course, it is also interesting and meaningful to explore the relationships between $DW_S(X)$ and some geometric properties, such as uniform non-squareness and uniform convexity. Moreover, we study $DW_S(X)$ in symmetric Minkowski planes and Radon planes, respectively. An equivalent form of $DW_S(X)$ in symmetric Minkowski planes is given and used to compute the value of $DW_S((\mathbb{R}^2, \|\cdot\|_p))$, $1 < p < \infty$, and a characterization of the Radon plane with affine regular hexagonal unit sphere in terms of $DW_S(X)$ is also given. However, there are still plenty of interesting problems that await discussion. Are $DW(X)$ and $DW_S(X)$ necessarily equal in Radon planes? What is the best upper bound of $DW_I(X)$? Are these two constants also related to other geometric properties? Henceforth, more results about the Dunkl-Williams constant will be presented in future research for the reader who are interested in the theory of geometric constants in Banach spaces.

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