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# Nonlinear bi-skew Jordan-type derivations on factor von Neumann algebras

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**Abstract.** Let $\mathfrak{T}$ be a factor von Neumann algebra acting on complex Hilbert space with $\dim(\mathfrak{T}) \geq 2$. For any $T, T_1, T_2, \ldots, T_n \in \mathfrak{T}$, define $q_1(T) = T$, $q_2(T_1, T_2) = T_1 \diamond T_2 = T_1T_2^* + T_2T_1^*$ and $q_n(T_1, \ldots, T_n) = q_{n-1}(T_1, \ldots, T_{n-1}) \diamond T_n$ for all integers $n \geq 2$. In this article, we prove that a map $\zeta : \mathfrak{T} \to \mathfrak{T}$ satisfies $\zeta(q_n(T_1, \ldots, T_n)) = \sum_{i=1}^{n} q_n(T_1, \ldots, T_{i-1}, \zeta(T_i), T_{i+1}, \ldots, T_n)$ for all $T_1, \ldots, T_n \in \mathfrak{T}$ if and only if $\zeta$ is an additive $*$-derivation.

## 1. Introduction and Preliminaries

Let $\mathfrak{T}$ be a $*$-algebra over the complex field $\mathbb{C}$. A map $\zeta : \mathfrak{T} \to \mathfrak{T}$ is said to be an additive derivation if $\zeta(T_1 + T_2) = \zeta(T_1) + \zeta(T_2)$ and $\zeta(T_1T_2) = \zeta(T_1)T_2 + T_1\zeta(T_2)$ for all $T_1, T_2 \in \mathfrak{T}$. It is said to be an additive $*$-derivation if it is an additive derivation and $\zeta(T^*) = \zeta(T)^*$ for all $T \in \mathfrak{T}$. The products $T_1 \bullet T_2 = T_1T_2 + T_2T_1^*$ and $[T_1, T_2]_\bullet = T_1T_2 - T_2T_1^*$ denote the usual skew Jordan product and skew Lie product of elements $T_1, T_2 \in \mathfrak{T}$, respectively. In recent years, many mathematicians devoted themselves to the study of these type of new products. These new products are found playing an important role in some research topics, and their study has attracted many authors' attention (see [2–4, 12–16, 18–20] and references therein). A map $\zeta : \mathfrak{T} \to \mathfrak{T}$ (not necessarily linear) is said to be a nonlinear skew Jordan derivation (resp. nonlinear skew Lie derivation) if $\zeta(T_1 \bullet T_2) = \zeta(T_1) \bullet T_2 + T_1 \bullet \zeta(T_2)$ (resp. $\zeta([T_1, T_2]_\bullet) = [\zeta(T_1), T_2]_\bullet + [T_1, \zeta(T_2)]_\bullet$) for all $T_1, T_2 \in \mathfrak{T}$. In [20], Zhang proved that every nonlinear skew Jordan derivation on a factor von Neumann algebra is an additive $*$-derivation. Yu and Zhang [18] proved that every nonlinear skew Lie derivation on a factor von Neumann algbera is an additive $*$-derivation. Recently, Kong and Zhang [8] introduced the concept of bi-skew Lie product of elements $T_1, T_2 \in \mathfrak{T}$ defined as $[T_1, T_2]_\diamond = T_1T_2^* - T_2T_1^*$ and proved that a map $\zeta$ (not necessarily linear) on a factor von Neumann algebra $\mathfrak{T}$ satisfies $\zeta([T_1, T_2]_\diamond) = [\zeta(T_1), T_2]_\diamond + [T_1, \zeta(T_2)]_\diamond$ for all $T_1, T_2 \in \mathfrak{T}$ if and only if $\zeta$ is an additive $*$-derivation. Motivated by the aforementioned works, we define a new product called bi-skew Jordan product by $T_1 \diamond T_2 = T_1T_2^* + T_2T_1^*$ of any two elements $T_1, T_2 \in \mathfrak{T}$. A map $\zeta : \mathfrak{T} \to \mathfrak{T}$ (not necessarily linear) is said to be a nonlinear bi-skew Jordan derivation if it satisfies $\zeta(T_1 \diamond T_2) = \zeta(T_1) \diamond T_2 + T_1 \diamond \zeta(T_2)$ for all $T_1, T_2 \in \mathfrak{T}$. The notion of bi-skew Jordan product can be extended in a more natural way. For $T, T_1, T_2, \ldots, T_n \in \mathfrak{T}$, set $q_1(T) = T$ and $q_n(T_1, T_2, \ldots, T_n) = q_{n-1}(T_1, T_2, \ldots, T_{n-1}) \diamond T_n$

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for all integers $n \geq 2$. The polynomial $q_n(T_1, T_2, \ldots, T_n)$ is called a bi-skew Jordan $n$-product, where $n \geq 2$. A map $\zeta : \mathfrak{T} \to \mathfrak{T}$ (not necessarily linear) is said to be a nonlinear bi-skew Jordan $n$-derivation if it satisfies $\zeta(q_n(T_1, \ldots, T_n)) = \sum_{i=1}^{n} q_n(T_1, \ldots, T_{i-1}, \zeta(T_i), T_{i+1}, \ldots, T_n)$ for all $T_1, T_2, \ldots T_n \in \mathfrak{T}$. A nonlinear bi-skew Jordan 2-derivation is called a nonlinear bi-skew Jordan derivation and a nonlinear bi-skew Jordan 3-derivation is called a nonlinear bi-skew Jordan triple derivation. Nonlinear bi-skew Jordan 2-derivations, nonlinear bi-skew Jordan 3-derivations and nonlinear bi-skew Jordan $n$-derivations are collectively known as nonlinear bi-skew Jordan-type derivations.

The aim of this article is to describe the form of nonlinear bi-skew Jordan-type derivations on factor von Neumann algebras. Several authors have made important contributions to the related topics (see [1, 5–7, 9–11, 17] and references therein). The main motivation of our study actually comes from the papers [1, 10]. Lin [10] investigated nonlinear skew Lie-type derivations on von Neumann algebras. Recently, in [1], the authors proved that every nonlinear bi-skew Lie-type derivation on a factor von Neumann algebra $\mathfrak{T}$ with $\dim(\mathfrak{T}) \geq 2$ is an additive $*$-derivation. Motivated by these results, in this paper, we prove that a map on a factor von Neumann algebra $\mathfrak{T}$ acting on a complex Hilbert space with $\dim(\mathfrak{T}) \geq 2$ is a bi-skew Jordan-type derivation if and only it is an additive $*$-derivation (Theorem 2.1).

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real numbers and complex numbers, respectively. A von Neumann algebra $\mathfrak{T}$ is a weakly closed, self adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator $I$. A von Neumann algebra $\mathfrak{T}$ is said to be factor if its center is trivial. It is well known that a factor von Neumann algebra is prime, that is, for any $T_1, T_2 \in \mathfrak{T}$, $T_1\mathfrak{T}T_2 = 0$ implies either $T_1 = 0$ or $T_2 = 0$. Let $P_1$ be a nontrivial projection in $\mathfrak{T}$ and write $P_2 = I - P_1$. The following elementary lemma shall be used frequently throughout the paper without further mentioning.

**Lemma 1.1.** *For any $T \in \mathfrak{T}$, we have*

(*i*) $q_n(T, P_1, \ldots, P_1) = 2^{n-2}(P_1TP_1 + P_1T^*P_1) + P_1T^*P_2 + P_2TP_1$,

(*ii*) $q_n(T, P_2, \ldots, P_2) = 2^{n-2}(P_2TP_2 + P_2T^*P_2) + P_1TP_2 + P_2T^*P_1$,

(*iii*) $q_n(T, I, \ldots, I) = 2^{n-2}(T + T^*)$,

(*iv*) $q_n(T, \frac{I}{2}, \ldots, \frac{I}{2}) = \frac{I}{2}(T + T^*)$.

## 2. The Main Theorem

The main theorem of this article reads as follows.

**Theorem 2.1.** *Let $\mathfrak{T}$ be a factor von Neumann algebra acting on complex Hilbert space with dim($\mathfrak{T}$) $\geq 2$. A map $\zeta : \mathfrak{T} \to \mathfrak{T}$ satisfies*

$$\zeta(q_n(T_1, \ldots, T_n)) = \sum_{i=1}^{n} q_n(T_1, \ldots, T_{i-1}, \zeta(T_i), T_{i+1}, \ldots, T_n) \tag{1}$$

*for all $T_1, \ldots, T_n \in \mathfrak{T}$ if and only if $\zeta$ is an additive $*$-derivation.*

*Proof.* Let us choose an arbitrary nontrivial projection $P_1$ and write $P_2 = I - P_1$. Then $\mathfrak{T}$ can be written as $\mathfrak{T} = P_1\mathfrak{T}P_1 + P_1\mathfrak{T}P_2 + P_2\mathfrak{T}P_1 + P_2\mathfrak{T}P_2$. Let $\mathfrak{T}^+ = \{M \in \mathfrak{T} : M^* = M\}$ and $\mathfrak{T}^- = \{N \in \mathfrak{T} : N^* = -N\}$. Further, write $\mathfrak{T}^+_{ii} = P_i\mathfrak{T}^+P_i$, $(i = 1, 2)$ and $\mathfrak{T}^+_{12} = \{P_1MP_2 + P_2MP_1 : M \in \mathfrak{T}^+\}$. Then, any $M \in \mathfrak{T}^+$ can be written as $M = M_{11} + M_{12} + M_{22}$, where $M_{ii} \in \mathfrak{T}^+_{ii}, M_{12} \in \mathfrak{T}^+_{12}$.

Obviously, if $\zeta$ is an additive $*$-derivation, then it satisfies (1). Here we only need to prove the necessity part which shall be established by checking the following series of claims. Taking $T_1 = T_2 = \cdots = T_n = 0$ in (1), the following claim is easy to obtain.

**Claim 2.2.** $\zeta(0) = 0$.

**Claim 2.3.** *For any $M \in \mathfrak{T}^+$, we have $\zeta(M)^* = \zeta(M)$.*

For any $M \in \mathfrak{T}^+$, it follows from $q_n(M, \frac{I}{2}, \ldots, \frac{I}{2}) = M$ that

$$\begin{aligned}\zeta(M) &= \zeta\left(q_n\left(M, \frac{I}{2}, \ldots, \frac{I}{2}\right)\right) \\ &= q_n\left(\zeta(M), \frac{I}{2}, \ldots, \frac{I}{2}\right) + q_n\left(M, \zeta\left(\frac{I}{2}\right), \ldots, \frac{I}{2}\right) + \cdots + q_n\left(M, \frac{I}{2}, \ldots, \zeta\left(\frac{I}{2}\right)\right) \\ &= \left(\frac{I}{2}\right)\left\{\zeta(M)\left(\frac{I}{2}\right) + \left(\frac{I}{2}\right)\zeta(M)^*\right\} + (n-1)\left\{M\zeta\left(\frac{I}{2}\right)^* + \zeta\left(\frac{I}{2}\right)M\right\}.\end{aligned}$$

Therefore, $\zeta(M)^* = \zeta(M)$.

**Claim 2.4.** *For any $M_{ii} \in \mathfrak{T}^+_{ii}, M_{12} \in \mathfrak{T}^+_{12}$ $(i = 1, 2)$, we have $\zeta(M_{ii} + M_{12}) = \zeta(M_{ii}) + \zeta(M_{12})$.*

For any $M_{11} \in \mathfrak{T}^+_{11}, M_{12} \in \mathfrak{T}^+_{12}$, assume that $S = \zeta(M_{11} + M_{12}) - \zeta(M_{11}) - \zeta(M_{12})$. Our aim is to show $S = 0$. It is easy to see that $S^* = S$ by Claim 2.3. On the one hand, we have

$$\begin{aligned}\zeta(q_n(M_{11} + M_{12}, P_2, \ldots, P_2)) &= q_n(\zeta(M_{11} + M_{12}), P_2, \ldots, P_2) + q_n(M_{11} + M_{12}, \zeta(P_2), \ldots, P_2) \\ &\quad + \cdots + q_n(M_{11} + M_{12}, P_2, \ldots, \zeta(P_2)).\end{aligned}$$

On the other hand, using the fact that $q_n(M_{11}, P_2, \ldots, P_2) = 0$ and Claim 2.2, we get

$$\begin{aligned}\zeta(q_n(M_{11} + M_{12}, P_2, \ldots, P_2)) &= \zeta(q_n(M_{11}, P_2, \ldots, P_2)) + \zeta(q_n(M_{12}, P_2, \ldots, P_2)) \\ &= q_n(\zeta(M_{11}) + \zeta(M_{12}), P_2, \ldots, P_2) + q_n(M_{11} + M_{12}, \zeta(P_2), \ldots, P_2) \\ &\quad + \cdots + q_n(M_{11} + M_{12}, P_2, \ldots, \zeta(P_2)).\end{aligned}$$

Comparing the above expressions for $\zeta(q_n(M_{11} + M_{12}, P_2, \ldots, P_2)$, it follows that $q_n(S, P_2, \ldots, P_2) = 0$. This together with the fact that $S^* = S$ leads us to $S_{12} = S_{21} = S_{22} = 0$.
Again, on the one hand,

$$\begin{aligned}&\zeta(q_n(M_{11} + M_{12}, P_2 - P_1, P_1, \ldots, P_1)) \\ &= q_n(\zeta(M_{11} + M_{12}), P_2 - P_1, P_1, \ldots, P_1) + q_n(M_{11} + M_{12}, \zeta(P_2 - P_1), P_1, \ldots, P_1) \\ &\quad + \cdots + q_n(M_{11} + M_{12}, P_2 - P_1, P_1, \ldots, \zeta(P_1)).\end{aligned}$$

On the other hand, using the fact that $q_n(M_{12}, P_2 - P_1, P_1, \ldots, P_1) = 0$ and Claim 2.2, we obtain

$$\begin{aligned}&\zeta(q_n(M_{11} + M_{12}, P_2 - P_1, P_1, \ldots, P_1)) \\ &= \zeta(q_n(M_{11}, P_2 - P_1, P_1, \ldots, P_1)) + \zeta(q_n(M_{12}, P_2 - P_1, P_1, \ldots, P_1)) \\ &= q_n(\zeta(M_{11}) + \zeta(M_{12}), P_2 - P_1, P_1, \ldots, P_1) + q_n(M_{11} + M_{12}, \zeta(P_2 - P_1), P_1, \ldots, P_1) \\ &\quad + \cdots + q_n(M_{11} + M_{12}, P_2 - P_1, P_1, \ldots, \zeta(P_1)).\end{aligned}$$

The last two expressions for $\zeta(q_n(M_{11} + M_{12}, P_2 - P_1, P_1, \ldots, P_1))$ imply that $q_n(S, P_2 - P_1, P_1, \ldots, P_1) = 0$. This together with the fact $S^* = S$ gives $S_{11} = 0$. Hence, $S = 0$, that is, $\zeta(M_{11} + M_{12}) = \zeta(M_{11}) + \zeta(M_{12})$.

Symmetrically, we can prove the other cases.

**Claim 2.5.** *For any $M_{ii} \in \mathfrak{T}^+_{ii}, M_{12} \in \mathfrak{T}^+_{12}$ $(i = 1, 2)$, we have $\zeta(M_{11} + M_{12} + M_{22}) = \zeta(M_{11}) + \zeta(M_{12}) + \zeta(M_{22})$.*

Set $S = \zeta(M_{11} + M_{12} + M_{22}) - \zeta(M_{11}) - \zeta(M_{12}) - \zeta(M_{22})$ for $M_{11} \in \mathfrak{T}^+_{11}, M_{12} \in \mathfrak{T}^+_{12}, M_{22} \in \mathfrak{T}^+_{22}$. One the one hand, we have

$$\begin{aligned}&\zeta(q_n(M_{11} + M_{12} + M_{22}, P_2, \ldots, P_2)) \\ &= q_n(\zeta(M_{11} + M_{12} + M_{22}), P_2, \ldots, P_2) + q_n(M_{11} + M_{12} + M_{22}, \zeta(P_2), P_2, \ldots, P_2) \\ &\quad + \cdots + q_n(M_{11} + M_{12} + M_{22}, \zeta(P_2), P_2, \ldots, \zeta(P_2)).\end{aligned}$$

On the other hand, using the fact that $q_n(M_{11}, P_2, \ldots, P_2) = 0$ and Claims 2.2, 2.4, we find that

$$\begin{aligned}
&\zeta(q_n(M_{11}+M_{12}+M_{22}, P_2, \ldots, P_2)) \\
&= \zeta(q_n(M_{11}, P_2, \ldots, P_2)) + \zeta(q_n(M_{12}+M_{22}, P_2, \ldots, P_2)) \\
&= q_n(\zeta(M_{11})+\zeta(M_{12})+\zeta(M_{22}), P_2, \ldots, P_2) + q_n(M_{11}+M_{12}+M_{22}, \zeta(P_2), \ldots, P_2) \\
&\quad + \cdots + q_n(M_{11}+M_{12}+M_{22}, P_2, \ldots, \zeta(P_2)).
\end{aligned}$$

Comparing these two expressions for $\zeta(q_n(M_{11}+M_{12}+M_{22}, P_2, \ldots, P_2))$, we get $q_n(S, P_2, \ldots, P_2) = 0$. This together with the fact that $S^* = S$ yields $S_{12} = S_{21} = S_{22} = 0$.
Again, on the one hand

$$\begin{aligned}
&\zeta(q_n(M_{11}+M_{12}+M_{22}, P_1, \ldots, P_1)) \\
&= q_n(\zeta(M_{11}+M_{12}+M_{22}), P_1, \ldots, P_1) + q_n(M_{11}+M_{12}+M_{22}, \zeta(P_1), P_1, \ldots, P_1) \\
&\quad + \cdots + q_n(M_{11}+M_{12}+M_{22}, P_1, \ldots, \zeta(P_1)).
\end{aligned}$$

On the other hand, using the fact that $q_n(M_{22}, P_1, \ldots, P_1) = 0$ and Claims 2.2, 2.4, we obtain

$$\begin{aligned}
&\zeta(q_n(M_{11}+M_{12}+M_{22}, P_1, \ldots, P_1)) \\
&= \zeta(q_n(M_{11}+M_{12}, P_1, \ldots, P_1)) + \zeta(q_n(M_{22}, P_1, \ldots, P_1)) \\
&= q_n(\zeta(M_{11})+\zeta(M_{12})+\zeta(M_{22}), P_1, \ldots, P_1) + q_n(M_{11}+M_{12}+M_{22}, \zeta(P_1), \ldots, P_1) \\
&\quad + \cdots + q_n(M_{11}+M_{12}+M_{22}, P_1, \ldots, \zeta(P_1)).
\end{aligned}$$

Using the two expressions for $\zeta(q_n(M_{11}+M_{12}+M_{22}, P_1, \ldots, P_1))$, we get $q_n(S, P_1, \ldots, P_1) = 0$. This together with the fact that $S^* = S$ implies that $S_{11} = 0$. Hence, $S = 0$, that is, $\zeta(M_{11}+M_{12}+M_{22}) = \zeta(M_{11}) + \zeta(M_{12}) + \zeta(M_{22})$.

**Claim 2.6.** *For any $M_{12}, M'_{12} \in \mathfrak{T}^+_{12}$, we have $\zeta(M_{12}+M'_{12}) = \zeta(M_{12}) + \zeta(M'_{12})$.*

For any $M_{12}, M'_{12} \in \mathfrak{T}^+_{12}$, observe that

$$q_n\left(P_1+M_{12}, P_2+M'_{12}, \frac{I}{2}, \ldots, \frac{I}{2}\right) = M_{12}+M'_{12}+M_{12}M'_{12}+M'_{12}M_{12}.$$

Since $M_{12}M'_{12}+M'_{12}M_{12} \in \mathfrak{T}^+_{11}+\mathfrak{T}^+_{22}$, write, $M_{12}M'_{12}+M'_{12}M_{12} = M_{11}+M_{22}$ for some $M_{11} \in \mathfrak{T}^+_{11}$ and $M_{22} \in \mathfrak{T}^+_{22}$. Applying Claims 2.4 and 2.5, we have

$$\begin{aligned}
&\zeta(M_{12}+M'_{12}) + \zeta(M_{11}) + \zeta(M_{22}) \\
&= \zeta(M_{12}+M'_{12}+M_{11}+M_{22}) \\
&= \zeta\left(q_n\left(P_1+M_{12}, P_2+M'_{12}, \frac{I}{2}, \ldots, \frac{I}{2}\right)\right) \\
&= q_n\left(\zeta(P_1+M_{12}), P_2+M'_{12}, \frac{I}{2}, \ldots, \frac{I}{2}\right) + q_n\left(P_1+M_{12}, \zeta(P_2+M'_{12}), \frac{I}{2}, \ldots, \frac{I}{2}\right) \\
&\quad + q_n\left(P_1+M_{12}, P_2+M'_{12}, \zeta\left(\frac{I}{2}\right), \ldots, \frac{I}{2}\right) + \cdots + q_n\left(P_1+M_{12}, P_2+M'_{12}, \frac{I}{2}, \ldots, \zeta\left(\frac{I}{2}\right)\right) \\
&= q_n\left(\zeta(P_1)+\zeta(M_{12}), P_2+M'_{12}, \frac{I}{2}, \ldots, \frac{I}{2}\right) + q_n\left(P_1+M_{12}, \zeta(P_2)+\zeta(M'_{12}), \frac{I}{2}, \ldots, \frac{I}{2}\right) \\
&\quad + q_n\left(P_1+M_{12}, P_2+M'_{12}, \zeta\left(\frac{I}{2}\right), \ldots, \frac{I}{2}\right) + \cdots + q_n\left(P_1+M_{12}, P_2+M'_{12}, \frac{I}{2}, \ldots, \zeta\left(\frac{I}{2}\right)\right) \\
&= \zeta\left(q_n\left(P_1, P_2, \frac{I}{2}, \ldots, \frac{I}{2}\right)\right) + \zeta\left(q_n\left(M_{12}, P_2, \frac{I}{2}, \ldots, \frac{I}{2}\right)\right) + \zeta\left(q_n\left(P_1, M'_{12}, \frac{I}{2}, \ldots, \frac{I}{2}\right)\right) \\
&\quad + \zeta\left(q_n\left(M_{12}, M'_{12}, \frac{I}{2}, \ldots, \frac{I}{2}\right)\right) \\
&= \zeta(M_{12}) + \zeta(M'_{12}) + \zeta(M_{12}M'_{12}+M'_{12}M_{12}) \\
&= \zeta(M_{12}) + \zeta(M'_{12}) + \zeta(M_{11}+M_{22}) \\
&= \zeta(M_{12}) + \zeta(M'_{12}) + \zeta(M_{11}) + \zeta(M_{22}).
\end{aligned}$$

Thus, $\zeta(M_{12}+M'_{12})=\zeta(M_{12})+\zeta(M'_{12})$ for any $M_{12},M'_{12}\in\mathfrak{T}^+$.

**Claim 2.7.** *For any $M_{ii},M'_{ii}\in\mathfrak{T}^+_{11}(i=1,2)$, we have $\zeta(M_{ii}+M'_{ii})=\zeta(M_{ii})+\zeta(M'_{ii})$.*

Let $S=\zeta(M_{11}+M'_{11})-\zeta(M_{11})-\zeta(M'_{11})$ for $M_{11},M'_{11}\in\mathfrak{T}^+_{11}$. On the one hand, we have

$$\begin{aligned}\zeta(q_n(M_{11}+M'_{11},P_2,\dots,P_2)) &= q_n(\zeta(M_{11}+M'_{11}),P_2,\dots,P_2))+q_n(M_{11}+M'_{11},\zeta(P_2),P_2,\dots,P_2)\\ &\quad +\cdots+q_n(M_{11}+M'_{11},P_2,\dots,\zeta(P_2)).\end{aligned}$$

On the other hand, using the fact that $q_n(M_{11},P_2,\dots,P_2)=q_n(M'_{11},P_2,\dots,P_2)=0$ and Claim 2.2, we obtain

$$\begin{aligned}\zeta(q_n(M_{11}+M'_{11},P_2,\dots,P_2)) &= \zeta(q_n(M_{11},P_2,\dots,P_2))+\zeta(q_n(M'_{11},P_2,\dots,P_2))\\ &= q_n(\zeta(M_{11})+\zeta(M'_{11}),P_2,\dots,P_2)+q_n(M_{11}+M'_{11},\zeta(P_2),\dots,P_2)\\ &\quad +\cdots+q_n(M_{11}+M'_{11},P_2,\dots,\zeta(P_2)).\end{aligned}$$

Comparing these two expressions for $\zeta(q_n(M_{11}+M'_{11},P_2,\dots,P_2))$, we have that $q_n(S,P_2,\dots,P_2)=0$. This together with the fact that $S^*=S$ leads us to $S_{12}=S_{21}=S_{22}=0$. Next we show that $S_{11}=0$. Let $M_{12}=T_{12}+T^*_{12}$ for any $T_{12}\in\mathfrak{T}_{12}$. Then $q_n\left(M_{11},M_{12},\frac{I}{2},\dots,\frac{I}{2}\right)$, $q_n\left(M'_{11},M_{12},\frac{I}{2},\dots,\frac{I}{2}\right)\in\mathfrak{T}^+_{12}$. On the one hand, we have

$$\begin{aligned}&\zeta\left(q_n\left(M_{11}+M'_{11},M_{12},\frac{I}{2},\dots,\frac{I}{2}\right)\right)\\ &= q_n\left(\zeta(M_{11}+M'_{11}),M_{12},\frac{I}{2},\dots,\frac{I}{2}\right)+q_n\left(M_{11}+M'_{11},\zeta(M_{12}),\frac{I}{2},\dots,\frac{I}{2}\right)\\ &\quad +q_n\left(M_{11}+M'_{11},M_{12},\zeta\left(\frac{I}{2}\right),\dots,\frac{I}{2}\right)+\cdots+q_n\left(M_{11}+M'_{11},M_{12},\frac{I}{2},\dots,\zeta\left(\frac{I}{2}\right)\right).\end{aligned}$$

On the other hand, Claim 2.6 gives

$$\begin{aligned}&\zeta\left(q_n\left(M_{11}+M'_{11},M_{12},\frac{I}{2},\dots,\frac{I}{2}\right)\right)\\ &= \zeta\left(q_n\left(M_{11},M_{12},\frac{I}{2},\dots,\frac{I}{2}\right)\right)+\zeta\left(q_n\left(M'_{11},M_{12},\frac{I}{2},\dots,\frac{I}{2}\right)\right)\\ &= q_n\left(\zeta(M_{11})+\zeta(M'_{11}),M_{12},\frac{I}{2},\dots,\frac{I}{2}\right)+q_n\left(M_{11}+M'_{11},\zeta(M_{12}),\frac{I}{2},\dots,\frac{I}{2}\right)\\ &\quad +q_n\left(M_{11}+M'_{11},M_{12},\zeta\left(\frac{I}{2}\right),\dots,\frac{I}{2}\right)+\cdots+q_n\left(M_{11}+M'_{11},M_{12},\frac{I}{2},\dots,\zeta\left(\frac{I}{2}\right)\right).\end{aligned}$$

The last two expressions for $\zeta(q_n(M_{11}+M'_{11},M_{12},\frac{I}{2},\dots,\frac{I}{2}))$ imply that

$$q_n(S,M_{12},\frac{I}{2},\dots,\frac{I}{2})=0.$$

Simplifying the last equation, we get $ST_{12}+ST^*_{12}+T_{12}S+T^*_{12}S=0$. Multiplying the above equation by $P_1$ on the left and by $P_2$ on the right, we get $P_1ST_{12}+T_{12}SP_2=0$. Since $S_{22}=0$, we obtain $S_{11}=0$ by using the primeness of $\mathfrak{T}$. Thus $S=0$, that is, $\zeta(M_{11}+M'_{11})=\zeta(M_{11})+\zeta(M'_{11})$. Symmetrically, we can prove that $\zeta(M_{22}+M'_{22})=\zeta(M_{22})+\zeta(M'_{22})$ for any $M_{12},M'_{12}\in\mathfrak{A}^+_{12}$.

Using Claims 2.5, 2.6 and 2.7, the following claim is easy to obtain.

**Claim 2.8.** *$\zeta$ is additive on $\mathfrak{T}^+$.*

**Claim 2.9.** *For any $N\in\mathfrak{T}^-$, we have $\zeta(N)^*=-\zeta(N)$.*

Let us first show that $\zeta\left(\frac{I}{2}\right) = 0$. Since $q_n\left(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}\right) = \frac{I}{2}$ and $\zeta\left(\frac{I}{2}\right)^* = \zeta\left(\frac{I}{2}\right)$ by Claim 2.3, we have

$$\begin{aligned} \zeta\left(\frac{I}{2}\right) &= q_n\left(\zeta\left(\frac{I}{2}\right), \frac{I}{2}, \ldots, \frac{I}{2}\right) + q_n\left(\frac{I}{2}, \zeta\left(\frac{I}{2}\right), \ldots, \frac{I}{2}\right) + \cdots + q_n\left(\frac{I}{2}, \frac{I}{2}, \ldots, \zeta\left(\frac{I}{2}\right)\right) \\ &= n\zeta\left(\frac{I}{2}\right). \end{aligned}$$

Thus, $\zeta\left(\frac{I}{2}\right) = 0$. Now using Claim 2.2 and the fact that $q_n\left(N, \frac{I}{2}, \ldots, \frac{I}{2}\right) = 0$ for any $N \in \mathfrak{T}^-$, we obtain

$$\begin{aligned} 0 &= \zeta\left(q_n\left(N, \frac{I}{2}, \ldots, \frac{I}{2}\right)\right) \\ &= q_n\left(\zeta(N), \frac{I}{2}, \ldots, \frac{I}{2}\right) + q_n\left(N, \zeta\left(\frac{I}{2}\right), \ldots, \frac{I}{2}\right) + \cdots + q_n\left(N, \frac{I}{2}, \ldots, \zeta\left(\frac{I}{2}\right)\right) \\ &= \frac{I}{2}\left\{\zeta(N)\frac{I}{2} + \frac{I}{2}\zeta(N)^*\right\}. \end{aligned}$$

Thus, we have $\zeta(N)^* = -\zeta(N)$ for any $N \in \mathfrak{T}^-$.

**Claim 2.10.** $\zeta(I) = 0$, $\zeta(iI) = 0$.

Since $\zeta\left(\frac{1}{2}\right) = 0$ and $\zeta$ is additive on $\mathfrak{T}^+$, we get $\zeta(I) = \zeta\left(\frac{I}{2}\right) + \zeta\left(\frac{I}{2}\right) = 0$.

Next we show that $\zeta(\mathrm{i}I) = 0$. Since $\zeta(I) = 0$, using Claim 2.8, we find that

$$\begin{aligned} 0 &= 2^{n-1}\zeta(I) \\ &= \zeta(2^{n-1}I) \\ &= \zeta(q_n(\mathrm{i}I, \mathrm{i}I, I, \ldots, I)) \\ &= q_n(\zeta(\mathrm{i}I), \mathrm{i}I, I, \ldots, I) + q_n(\mathrm{i}I, \zeta(\mathrm{i}I), I, \ldots, I) \\ &= q_{n-1}(-2\mathrm{i}\zeta(\mathrm{i}I) + 2\mathrm{i}\zeta(\mathrm{i}I)^*, I, \ldots, I) + q_{n-1}(2\mathrm{i}\zeta(\mathrm{i}I)^* - 2\mathrm{i}\zeta(\mathrm{i}I), I, \ldots, I) \\ &= -2^{n-1}\mathrm{i}\{\zeta(\mathrm{i}I)^* - \zeta(\mathrm{i}I)\}. \end{aligned}$$

Thus $\zeta(\mathrm{i}I)^* = \zeta(\mathrm{i}I)$. Moreover, by Claim 2.9, we have $\zeta(\mathrm{i}I)^* = -\zeta(\mathrm{i}I)$. Therefore, $\zeta(\mathrm{i}I) = 0$.

**Claim 2.11.** *For any $M \in \mathfrak{T}^+$, we have $\zeta(iM) = i\zeta(M)$.*

Applying Claim 2.8, 2.9 and 2.10, we obtain

$$\begin{aligned} 2^{n-1}\zeta(M) &= \zeta(2^{n-1}M) \\ &= \zeta(q_n(\mathrm{i}M, \mathrm{i}I, I, \ldots, I)) \\ &= q_n(\zeta(\mathrm{i}M), \mathrm{i}I, I, \ldots, I) \\ &= q_{n-1}(-2\mathrm{i}\zeta(\mathrm{i}M), I, \ldots, I) \\ &= -2^{n-1}\mathrm{i}\zeta(\mathrm{i}M). \end{aligned}$$

That is, $\zeta(\mathrm{i}M) = \mathrm{i}\zeta(M)$ for any $M \in \mathfrak{T}^+$.

**Claim 2.12.** *For any $T \in \mathfrak{T}$, we have $\zeta(T^*) = \zeta(T)^*$*

Assume that $T = M_1 + \mathrm{i}M_2$ for $M_1, M_2 \in \mathfrak{T}^+$. As $\zeta(I) = 0$, we have

$$\begin{aligned} \zeta(q_n(M_1 + \mathrm{i}M_2, I, \ldots, I)) &= q_n(\zeta(M_1 + \mathrm{i}M_2), I, \ldots, I) \\ &= 2^{n-2}\{\zeta(M_1 + \mathrm{i}M_2) + \zeta(M_1 + \mathrm{i}M_2)^*\}. \end{aligned}$$

On the other hand, using the fact that $q_n(\mathrm{i}M, I, \ldots, I) = 0$ and Claims 2.2, 2.11, we obtain

$$\begin{aligned}\zeta(q_n(M_1 + \mathrm{i}M_2, I, \ldots, I)) &= \zeta(q_n(M_1, I, \ldots, I)) + \zeta(q_n(\mathrm{i}M_2, I, \ldots, I)) \\ &= q_n(\zeta(M_1), I, \ldots, I)) + q_n(\mathrm{i}\zeta(M_2), I, \ldots, I) \\ &= q_n(\zeta(M_1) + \mathrm{i}\zeta(M_2), I, \ldots, I) \\ &= 2^{n-1}\zeta(M_1).\end{aligned}$$

From the above two expressions for $\zeta(q_n(M_1 + \mathrm{i}M_2, I, \ldots, I))$, we get

$$\zeta(M_1 + \mathrm{i}M_2) + \zeta(M_1 + \mathrm{i}M_2)^* = 2\zeta(M_1) \tag{2}$$

for all $M_1, M_2 \in \mathfrak{T}^+$. Again invoking Claims 2.10, we have

$$\begin{aligned}\zeta(q_n(M_1 + \mathrm{i}M_2, \mathrm{i}I, I, \ldots, I)) &= q_n(\zeta(M_1 + \mathrm{i}M_2), \mathrm{i}I, I, \ldots, I) \\ &= q_{n-1}(-\mathrm{i}\zeta(M_1 + \mathrm{i}M_2) + \mathrm{i}\zeta(M_1 + \mathrm{i}M_2)^*, I, \ldots, I) \\ &= -2^{n-2}\mathrm{i}\{\zeta(M_1 + \mathrm{i}M_2) - \zeta(M_1 + \mathrm{i}M_2)^*\}.\end{aligned}$$

On the other hand, using the fact $q_n(M, \mathrm{i}I, I, \ldots, I) = 0$ and Claims 2.2, 2.10 and 2.11, we have

$$\begin{aligned}\zeta(q_n(M_1 + \mathrm{i}M_2, \mathrm{i}I, I, \ldots, I)) &= \zeta(q_n(M_1, \mathrm{i}I, I, \ldots, I)) + \zeta(q_n(\mathrm{i}M_2, \mathrm{i}I, I, \ldots, I)) \\ &= q_n(\zeta(M_1) + \mathrm{i}\zeta(M_2), \mathrm{i}I, I, \ldots, I) \\ &= q_{n-1}(2\zeta(M_2), I, \ldots, I) \\ &= 2^{n-1}\zeta(M_2).\end{aligned}$$

Comparing the last two expressions for $\zeta(q_n(M_1 + \mathrm{i}M_2, \mathrm{i}I, I, \ldots, I))$, it follows that

$$\zeta(M_1 + \mathrm{i}M_2) - \zeta(M_1 + \mathrm{i}M_2)^* = 2\mathrm{i}\zeta(M_2) \tag{3}$$

for all $M_1, M_2 \in \mathfrak{T}^+$. Addition of (2) and (3) yields

$$\zeta(M_1 + \mathrm{i}M_2) = \zeta(M_1) + \mathrm{i}\zeta(M_2) \tag{4}$$

for all $M_1, M_2 \in \mathfrak{T}^+$.

Now, we show that $\zeta(T^*) = \zeta(T)^*$ for any $T \in \mathfrak{T}$. Since $\zeta$ is additive on $\mathfrak{T}^+$ and $\zeta(0) = 0$, we have $\zeta(-M) = -\zeta(M)$ for any $M \in \mathfrak{T}^+$. Using (4) and Claims 2.3, 2.8, we find that

$$\zeta(T)^* = \zeta(M_1)^* - \mathrm{i}\zeta(M_2)^* = \zeta(M_1^*) + \zeta(-\mathrm{i}M_2^*) = \zeta((M_1 - \mathrm{i}M_2)^*) = \zeta(T^*)$$

for all $T \in \mathfrak{T}$.

**Claim 2.13.** *$\zeta$ is additive on $\mathfrak{T}$.*

Let $T = M_1 + \mathrm{i}M_2, T' = M_1' + \mathrm{i}M_2' \in \mathfrak{T}$ for $M_1, M_2, M_1', M_2' \in \mathfrak{T}^+$. Using (4) and Claim 2.8, we have

$$\begin{aligned}\zeta(T + T') &= \zeta((M_1 + M_1') + \mathrm{i}(M_2 + M_2')) \\ &= \zeta(M_1 + M_1') + \mathrm{i}\zeta(M_2 + M_2') \\ &= \zeta(M_1) + \mathrm{i}\zeta(M_2) + \zeta(M_1') + \mathrm{i}\zeta(M_2') \\ &= \zeta(M_1 + \mathrm{i}M_2) + \zeta(M_1' + \mathrm{i}M_2') \\ &= \zeta(T) + \zeta(T').\end{aligned}$$

**Claim 2.14.** *For any $T \in \mathfrak{T}$, we have $\zeta(iT) = i\zeta(T)$.*

Let $T = M_1 + \mathrm{i}M_2$ for $M_1, M_2 \in \mathfrak{T}^+$. Using Claims 2.11 and 2.13, we obtain

$$\begin{aligned}\zeta(\mathrm{i}T) &= \zeta(\mathrm{i}M_1 - M_2)\\ &= \zeta(\mathrm{i}M_1) - \zeta(M_2)\\ &= \mathrm{i}\zeta(M_1) - \zeta(M_2)\\ &= \mathrm{i}\{\zeta(M_1) + \mathrm{i}\zeta(M_2)\}\\ &= \mathrm{i}\zeta(M_1 + \mathrm{i}M_2)\\ &= \mathrm{i}\zeta(T).\end{aligned}$$

**Claim 2.15.** *$\zeta$ is an additive $*$-derivation on $\mathfrak{T}$.*

In view of Claims 2.12 and 2.13, it suffices to show that $\zeta$ is a derivation on $\mathfrak{T}$. First, we show that $\zeta$ is a derivation on $\mathfrak{T}^+$, that is,

$$\zeta(MM') = \zeta(M)M' + M\zeta(M') \ \text{ for all } \ M, M' \in \mathfrak{T}^+.$$

Using Claims 2.3, 2.10, 2.12 and 2.14, we see that

$$\begin{aligned}&2^{n-2}\zeta(MM' + M'M)\\ &= \zeta(2^{n-2}(MM' + M'M))\\ &= \zeta(q_n(M, M', I, \ldots, I))\\ &= q_n(\zeta(M), M', I, \ldots, I) + q_n(M, \zeta(M'), I, \ldots, I)\\ &= q_{n-1}(\zeta(M)M' + M'\zeta(M), I, \ldots, I) + q_{n-1}(M\zeta(M') + \zeta(M')M, I, \ldots, I)\\ &= 2^{n-2}\{\zeta(M)M' + M'\zeta(M)\} + 2^{n-2}\{M\zeta(M') + \zeta(M')M\}\end{aligned}$$

and

$$\begin{aligned}&2^{n-2}\mathrm{i}\zeta(MM' - M'M)\\ &= \zeta(2^{n-2}(\mathrm{i}MM' - \mathrm{i}M'M))\\ &= \zeta(q_n(\mathrm{i}M, M', I, \ldots, I))\\ &= q_n(\zeta(\mathrm{i}M), M', I, \ldots, I) + q_n(\mathrm{i}M, \zeta(M'), I, \ldots, I)\\ &= q_{n-1}(\mathrm{i}\zeta(M)M' - \mathrm{i}M'\zeta(M), I, \ldots, I) + q_{n-1}(\mathrm{i}M\zeta(M') - \mathrm{i}\zeta(M')M, I, \ldots, I)\\ &= 2^{n-2}\mathrm{i}\{\zeta(M)M' - M'\zeta(M)\} + 2^{n-2}\mathrm{i}\{M\zeta(M') - \zeta(M')M\}.\end{aligned}$$

It follows from the last two equations that

$$\zeta(MM') = \zeta(M)M' + M\zeta(M') \ \text{ for all } \ M, M' \in \mathfrak{T}^+.$$

Take any two arbitrary elements $T = M_1 + \mathrm{i}M_2, T' = M_1' + \mathrm{i}M_2' \in \mathfrak{T}$, where $M_1, M_2, M_1', M_2' \in \mathfrak{T}^+$. Using Claims 2.13 and 2.14, we get

$$\begin{aligned}\zeta(TT') &= \zeta((M_1M_1' - M_2M_2') + \mathrm{i}(M_1M_2' + M_2M_1'))\\ &= \zeta(M_1M_1' - M_2M_2') + \mathrm{i}\zeta(M_1M_2' + M_2M_1')\\ &= \zeta(M_1M_1') - \zeta(M_2M_2') + \mathrm{i}\zeta(M_1M_2') + \mathrm{i}\zeta(M_2M_1')\\ &= \zeta(M_1)M_1' + M_1\zeta(M_1') - \zeta(M_2)M_2' - M_2\zeta(M_2') + \mathrm{i}\zeta(M_1)M_2' + \mathrm{i}M_1\zeta(M_2') + \mathrm{i}\zeta(M_2)M_1'\\ &\quad + \mathrm{i}M_2\zeta(M_1')\end{aligned}$$

and

$$\begin{aligned}\zeta(T)T' + T\zeta(T') &= \zeta(M_1 + \mathrm{i}M_2)(M_1' + \mathrm{i}M_2') + (M_1 + \mathrm{i}M_2)\zeta(M_1' + \mathrm{i}M_2')\\ &= \zeta(M_1)M_1' + M_1\zeta(M_1') - \zeta(M_2)M_2' - M_2\zeta(M_2') + \mathrm{i}\zeta(M_1)M_2' + \mathrm{i}M_1\zeta(M_2') + \mathrm{i}\zeta(M_2)M_1'\\ &\quad + \mathrm{i}M_2\zeta(M_1').\end{aligned}$$

Thus, $\zeta(TT') = \zeta(T)T' + T\zeta(T')$ for all $T, T' \in \mathfrak{T}$. Consequently, $\zeta$ is an additive $*$-derivation and the proof of the theorem is completed. □

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