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On some inequalities for metallic Riemannian space forms

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Abstract. The metallic Riemannian space forms are the subject of this article, specifically their slant submanifolds. Several basic inequalities have been derived for these submanifolds and particular conclusions are examined.

1. Introduction

After being first described in [10], the golden structure on a Riemannian manifold has since worked out by several scholars using various theories. The discovery that on a Riemannian manifold, the structure of metallic type may be extended from golden type was initially made in [16]. Recent publications include a thorough analysis of Norden manifolds in [3] and several findings on the curvature of generalised metallic pseudo-Riemannian structures in [4]. Mention that substantial research has been done for various slant cases in ([2], [18], [17] and [19]). Metallic warped product manifolds were explored in 2018 with some noteworthy findings. The same authors ([20] and [19]) also looked at warped product submanifolds in Riemannian manifolds equipped with metallic structure.

The theory of slant submanifolds was proposed in [5]. Later, Vilcu [24] and Shukla and Rao [23] explored this theory. Recently, slant cases have also been explored in different space forms [1], [9].

As an alternative, Chen examined real space form submanifolds in his 1993 article [6], where he also developed the notion of sharp relations involving intrinsic and extrinsic invariants. Following that, many other ambient spaces were reviewed for Chen-like inequalities, including [7],[21],[22],[11],[12] and their references.

As a result of all these advancements in the area, we are able to prove sharp inequality for metallic Riemannian space forms.

2. Preliminaries

Let M^n isometrically immerses in $(\overline{M}^m, g)$, where M and $\overline{M}$ are Riemannian manifolds. Identification of covariant differentiation can be made with ∇ on M and normal connection can be made with $\nabla^\perp$ on $TM^\perp$.

It can also be noted that following link also holds when η describes the second fundamental form

$$\tilde{\nabla}_{\mathcal{U}_1}\mathcal{U}_2 = \nabla_{\mathcal{U}_1}\mathcal{U}_2 + \eta(\mathcal{U}_1, \mathcal{U}_2),$$
$$\tilde{\nabla}_{\mathcal{U}_1}\mathcal{V} = -S_{\mathcal{V}}\mathcal{U}_1 + \nabla^\perp_{\mathcal{U}_1}\mathcal{V}, \quad \forall \mathcal{U}_1, \mathcal{U}_2 \in \Gamma(TM), \forall \mathcal{V} \in \Gamma(TM^\perp).$$

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It is also worth noting that

$$g(S_{\mathcal{V}}\mathcal{U}_1, \mathcal{U}_2) = g(\eta(\mathcal{U}_1, \mathcal{U}_2), \mathcal{V}).$$

As a reminder, the equations of Gauss can be formulated like this

$$\tilde{R}(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4) = R(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4) - g(\eta(\mathcal{U}_1, \mathcal{U}_4), \eta(\mathcal{U}_2, \mathcal{U}_3)) + g(\eta(\mathcal{U}_1, \mathcal{U}_3), \eta(\mathcal{U}_2, \mathcal{U}_4)). \tag{1}$$

Consider any local orthonormal frame field $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ on M. Compute

$$\mathcal{G}(p) = \sum_{i=1}^{n} \frac{1}{n} \eta(e_i, e_i), \quad 1 \le i, j \le n;$$

also

$$\eta_{ij}^{s} = g(\eta(e_i, e_j), e_s), \quad n+1 \le s \le m.$$

Additionally, take note

$$\|\mathcal{G}\|^2 = \frac{1}{n^2} \sum_{s=n+1}^{m} \left\{ \sum_{i=1}^{n} \eta_{ii}^{s} \right\}^2,$$

also

$$\|\eta\|^2 = \sum_{s=n+1}^{m} \sum_{i,j=1}^{n} g(\eta(e_i, e_j), \eta(e_i, e_j)).$$

One recalls.

Lemma 2.1. *[6] In case of $(t+1)$ real numbers $u_1, \ldots, u_t, v$ obeying*

$$\Big(\sum_{k=1}^{t} u_k\Big)^2 = (t-1)\Big(\sum_{k=1}^{t} u_k^2 + v\Big), t \ge 2$$

one obtains $2u_1u_2 \ge v$. Equivalence is true if and only if $u_1 + u_2 = u_3 = \cdots = u_t$.

In order to describe the relative null space of M, we have

$$\mathcal{L}_p = \{\mathcal{U}_1 \in T_pM | \eta(\mathcal{U}_1, \mathcal{U}_2) = 0, \ \forall\ \mathcal{U}_2 \in T_pM\}, p \in M.$$

We now have $K(\pi)$, which serves the sectional curvature of the manifold M, and $\pi \subset T_pM$ reflects plane section. Write

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_i).$$

One can notice

$$(\inf K)(p) = \inf\{K(\pi) | \pi \subset T_pM, \dim \pi = 2\},$$

$$\delta_M(p) = \tau(p) - (\inf K)(p),$$

wherein $\delta_M(p)$ specifies first invariant of Chen.

Recognize the q-dimension subspace of T_pM by L' ($q \geq 2$) and write

$$\tau(L') = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta).$$

We visualize μ-tuples of integers ≥ 2 $(\zeta'_1 \dots \zeta'_\mu)$ like a set $S(\zeta', \mu)$ satisfying

$$\zeta'_1 < \zeta', \zeta'_1 + \cdots + \zeta'_\mu \leq \zeta',$$

$\mu \geq 0, \mu \in \mathbb{Z}$. Additionally, adjust a ζ' and take unordered μ-tuples into account, like a set $S(\zeta')$. This provides

$$\delta(\zeta'_1 \dots \zeta'_\mu)(p) = \tau(p) - S(\zeta'_1 \dots \zeta'_\mu)(p), \quad \forall(\zeta'_1 \dots \zeta'_\mu) \in S(\zeta'),$$

in this case

$$S(\zeta'_1 \dots \zeta'_\mu)(p) = \inf\{\tau(L'_1) + \cdots + \tau(L'_\mu)\}.$$

Assuming $\dim L'_i = \zeta'_i, i \in \{1, \dots, \mu\}$ and fix

$$d(\zeta'_1, \dots, \zeta'_\mu) = \frac{1}{2} \frac{(\zeta' + \mu - 1 - \sum_{i=1}^{\mu} \zeta'_i)}{(\zeta' + \mu - \sum_{i=1}^{\mu} \zeta'_i)} \zeta'^2$$

and

$$b(\zeta'_1, \dots, \zeta'_\mu) = \frac{1}{2}[\zeta'(\zeta' - 1) - \sum_{i=1}^{k} \zeta'_i(\zeta'_i - 1)].$$

3. Metallic Riemannian manifolds

On Riemannian manifold $(\overline{M}^m, g)$, $(1,1)$-tensor field F corresponds to a polynomial structure whenever $P(F) = 0$, here

$$P(\mathcal{U}) := \mathcal{U}^n + a_n \mathcal{U}^{n-1} + \cdots + a_2 \mathcal{U} + a_1 I, \tag{2}$$

I being the identity transformation on $\Gamma(T\overline{M})$ and $a_1, \dots, a_n$ standing for real numbers. When $P(\mathcal{U}) = \mathcal{U}^2 + I$, F forms an almost complex structure and for $P(\mathcal{U}) = \mathcal{U}^2 - I$, F gives rise to an almost product structure ([10], [13], [1]).

Again, on $(\overline{M}^m, g)$, $(1,1)$-tensor field φ defines a metallic structure [14] when

$$\varphi^2 = p\varphi + qI, \qquad \forall p, q \in \mathbb{N}^*.$$

We describe g as φ-compatible when

$$g(\mathcal{V}, \varphi\mathcal{U}) = g(\varphi\mathcal{V}, \mathcal{U}), \forall \mathcal{V}, \mathcal{U} \in \Gamma(T\overline{M}).$$

$(\overline{M}, g, \varphi)$ describes a metallic Riemannian manifold, where g and φ satisfy above equations.

Putting $\varphi\mathcal{V}$ for $\mathcal{V}$, one derives

$$g(\varphi\mathcal{V}, \varphi\mathcal{U}) = pg(\mathcal{V}, \varphi\mathcal{U}) + qg(\mathcal{V}, \mathcal{U}).$$

According to ([10], [15]), this simplifies to golden case with $p = 1 = q$.

Whenever $F^2 = I, F \neq \pm I$, F defines an almost product structure on $(\overline{M}^m, g)$ [1]. Additionally, $(\overline{M}, g)$ claims to be a product Riemannian manifold if

$$g(F\mathcal{V}, \mathcal{U}) = g(\mathcal{V}, F\mathcal{U}), \forall \mathcal{V}, \mathcal{U} \in \Gamma(T\overline{M}).$$

One may generate two almost product structures on $\overline{M}$ using the metallic structure φ, according to [14]

$$F_1 = \frac{2}{2\eta_{p,q} - p}\varphi - \frac{p}{2\eta_{p,q} - p}I,$$
$$F_2 = -\frac{2}{2\eta_{p,q} - p}\varphi + \frac{p}{2\eta_{p,q} - p}I, \tag{3}$$

$\eta_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ indicate the metallic proportions. Further, F on $\overline{M}$ generates

$$\varphi_1 = \frac{p}{2}I + \frac{2\eta_{p,q} - p}{2}F,$$
$$\varphi_2 = \frac{p}{2}I - \frac{2\eta_{p,q} - p}{2}F. \tag{4}$$

Definition 3.1. *[2] (i) The linear connection ∇ on metallic Riemannian manifold $(\overline{M}, g, \varphi)$ refers to φ-connection provided $\nabla\varphi = 0$.*
(ii) If $\overline{\nabla}$ (Levi-Civita connection) of g is φ-connection, $(\overline{M}, g, \varphi)$ specifies a locally metallic Riemannian manifold.

Let M^n be isometrically immersed in $(\overline{M}^m, g, \varphi)$ and $\theta(\mathcal{V})$ indicates the angle between $\varphi\mathcal{V}$ and T_xM, $x \in M$. Here, $0 \neq \mathcal{V}$ denotes vector tangent to M.

Definition 3.2. *If $\theta(\mathcal{V})$ does not rely on x or $\mathcal{V}$ options, M is slant (or θ-slant, if θ is given). M is invariant when $\theta = 0$ and anti-invariant when $\theta = \frac{\pi}{2}$. M that is neither invariant nor anti-invariant is called as proper slant.*
Further, $T_x\overline{M}$ of $\overline{M}$ can be viewed as

$$T_x\overline{M} = T_xM \oplus T_x^{\perp}M, \quad x \in M.$$

Next, we have

Lemma 3.3. *([18], [2]) For any metallic Riemannian manifold $(\overline{M}, g, \varphi)$ and isometrically immersed slant submanifold M of $\overline{M}$, one has*

$$g(T\mathcal{V}, T\mathcal{U}) = \cos^2\theta[pg(\mathcal{V}, \varphi\mathcal{U}) + qg(\mathcal{V}, \mathcal{U})]$$

$\forall\mathcal{V}, \mathcal{U} \in \Gamma(TM)$.

Additionally, I being identity on $\Gamma(TM)$ helps us to write

$$T^2 = \cos^2\theta(pT + qI)$$

and

$$\nabla T^2 = p\cos^2\theta \cdot \nabla T.$$

Here T denotes a tensor field of $(1,1)$-type on M (associating the tangent vector field $\mathcal{U}$ on M the tangential part of $\varphi\mathcal{U}$).

Example.[8] If we examine the Euclidean 7-space E^7 and the canonical coordinates $(\mho_1, \ldots, \mho_7)$. Then we may consider $f : M \to E^7$ with natural Euclidean metric $\langle\cdot,\cdot\rangle$ by:

$$f(\eth_1, \eth_2) := (\sin\eth_1 + 2, \cos\eth_1, 2\sin\eth_2 + 1, 2\cos\eth_2, \eth_2, 2\eth_1, 3),$$

$M := \{(\eth_1, \eth_2) | \eth_1, \eth_2 \in (0, \frac{\pi}{2})\}$. It results a local orthonormal frame on TM represented as

$$\mathcal{V}_1 = \cos\eth_1\frac{\partial}{\partial\mho_1} - \sin\eth_1\frac{\partial}{\partial\mho_2} + 2\frac{\partial}{\partial\mho_6}$$
$$\mathcal{V}_2 = 2\cos\eth_2\frac{\partial}{\partial\mho_3} - 2\sin\eth_2\frac{\partial}{\partial\mho_4} + \frac{\partial}{\partial\mho_5}.$$

Look at $\varphi : E^7 \to E^7$:

$$\varphi\left(\frac{\partial}{\partial \mho_i}\right) = \eta \frac{\partial}{\partial \mho_i}, \ \ i \in \{1,2,5\}; \ \ \varphi\left(\frac{\partial}{\partial \mho_j}\right) = \overline{\eta} \frac{\partial}{\partial \mho_j}, \ \ j \in \{3,4,6,7\},$$

$\eta := \eta_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ being metallic number, $p, q \in N^*$, $\overline{\eta} = p - \eta$. This shows that $(E^7, \langle \cdot, \cdot \rangle, \varphi)$ is a Riemannian manifold equipped with metallic structure and M is slant with

$$\theta = \arccos \frac{|\eta + 4\overline{\eta}|}{\sqrt{5(\eta^2 + 4\overline{\eta}^2)}}.$$

Let's now find real-space forms M_p and M_q with sectional curvatures c_p and c_q, respectively. Then for a locally metallic product space form $\overline{M}(= M_p(c_p) \times M_q(c_q), g, \varphi)$ [8]:

$$\begin{aligned} R(\mathcal{V},\mathcal{U})\mathcal{W} \ = \ & \frac{1}{4}(c_p + c_q)[g(\mathcal{U},\mathcal{W})\mathcal{V} - g(\mathcal{V},\mathcal{W})\mathcal{U}] \\ & + \frac{1}{4}(c_p + c_q)\Big\{\frac{4}{(2\eta_{p,q} - p)^2}[g(\varphi\mathcal{U},\mathcal{W})\varphi\mathcal{V} - g(\varphi\mathcal{V},\mathcal{W})\varphi\mathcal{U}] \\ & + \frac{p^2}{(2\eta_{p,q} - p)^2}[g(\mathcal{U},\mathcal{W})\mathcal{V} - g(\mathcal{V},\mathcal{W})\mathcal{U}] \\ & + \frac{2p}{(2\eta_{p,q} - p)^2}[g(\varphi\mathcal{V},\mathcal{W})\mathcal{U} + g(\mathcal{V},\mathcal{W})\varphi\mathcal{U} - g(\varphi\mathcal{U},\mathcal{W})\mathcal{V} - g(\mathcal{U},\mathcal{W})\varphi\mathcal{V}]\Big\} \\ & \pm \frac{1}{2}(c_p - c_q)\Big\{\frac{1}{2\eta_{p,q} - p}[g(\mathcal{U},\mathcal{W})\varphi\mathcal{V} - g(\mathcal{V},\mathcal{W})\varphi\mathcal{U}] \\ & + \frac{1}{2\eta_{p,q} - p}[g(\varphi\mathcal{U},\mathcal{W})\mathcal{V} - g(\varphi\mathcal{V},\mathcal{W})\mathcal{U}] \\ & + \frac{p}{2\eta_{p,q} - p}[g(\mathcal{V},\mathcal{W})\mathcal{U} - g(\mathcal{U},\mathcal{W})\mathcal{V}]\Big\}. \end{aligned} \tag{5}$$

4. Inequalities of the Chen type on metallic Riemannian manifolds

We fix $\overline{M}^m = (M_p(c_p) \times M_q(c_q), g, \varphi)$ for locally metallic product manifold.

Theorem 4.1. *The following inequality follows for any proper θ-slant submanifold M^n isometrically immersed in $\overline{M}^m$:*

$$\begin{aligned} \delta_M(p) \leq & \frac{(n-2)}{2}\Big[\frac{4}{(n-1)}\mathbb{A}_1 + \frac{1}{2}\mathbb{A}_2\{\mathbb{A}_3(n+1) - \mathbb{E}_1\}\Big] \\ & + \frac{1}{2}\mathbb{A}_2\Big[(\mathbb{D}_2 - 4q)cos^2\theta - \mathbb{D}_1\Big] + \frac{1}{4}\mathbb{A}_4(n-2)\Big[\mathbb{D}_3 + p\Big], \ \ p \in M, \end{aligned} \tag{6}$$

wherein
$\mathbb{A}_1 = \frac{n^2}{4}\|\mathcal{G}\|^2$, $\mathbb{A}_2 = \frac{1}{(p^2+4q)}(c_p + c_q)$, $\mathbb{A}_3 = p^2 + 2q$, $\mathbb{A}_4 = \frac{1}{\sqrt{p^2+4q}}(c_p - c_q)$, $\mathbb{D}_1 = Trace^2(\varphi)$, $\mathbb{D}_2 = pTrace(T) - nq$, $\mathbb{D}_3 = 2\mathrm{Trace}(\varphi) - \mathrm{pn}$, $\mathbb{E}_1 = 2pTrace(\varphi)$.

Proof. Using (1), we have

$$\begin{aligned}
R(\mathcal{V},\mathcal{U},\mathcal{W},\mathcal{E}) &= \frac{1}{4}(c_p+c_q)[g(\mathcal{U},\mathcal{W})g(\mathcal{V},\mathcal{E})-g(\mathcal{V},\mathcal{W})g(\mathcal{U},\mathcal{E})] \\
&+\frac{1}{4}(c_p+c_q)\Big\{\frac{4}{(2\eta_{p,q}-p)^2}[g(\varphi\mathcal{U},\mathcal{W})g(\varphi\mathcal{V},\mathcal{E})-g(\varphi\mathcal{V},\mathcal{W})g(\varphi\mathcal{U},\mathcal{E})] \\
&+\frac{p^2}{(2\eta_{p,q}-p)^2}[g(\mathcal{U},\mathcal{W})g(\mathcal{V},\mathcal{E})-g(\mathcal{V},\mathcal{W})g(\mathcal{U},\mathcal{E})] \\
&+\frac{2p}{(2\eta_{p,q}-p)^2}[g(\varphi\mathcal{V},\mathcal{W})g(\mathcal{U},\mathcal{E})+g(\mathcal{V},\mathcal{W})g(\varphi\mathcal{U},\mathcal{E}) \\
&-g(\varphi\mathcal{U},\mathcal{W})g(\mathcal{V},\mathcal{E})-g(\mathcal{U},\mathcal{W})g(\varphi\mathcal{V},\mathcal{E})]\Big\} \\
&\pm\frac{1}{2}(c_p-c_q)\Big\{\frac{1}{2\eta_{p,q}-p}[g(\mathcal{U},\mathcal{W})g(\varphi\mathcal{V},\mathcal{E})-g(\mathcal{V},\mathcal{W})g(\varphi\mathcal{U},\mathcal{E})] \\
&+\frac{1}{2\eta_{p,q}-p}[g(\varphi\mathcal{U},\mathcal{W})g(\mathcal{V},\mathcal{E})-g(\varphi\mathcal{V},\mathcal{W})g(\mathcal{U},\mathcal{E})] \\
&+\frac{p}{2\eta_{p,q}-p}[g(\mathcal{V},\mathcal{W})g(\mathcal{U},\mathcal{E})-g(\mathcal{U},\mathcal{W})g(\mathcal{V},\mathcal{E})]\Big\} \qquad (7)\\
&+g(\eta(\mathcal{V},\mathcal{E}),\eta(\mathcal{U},\mathcal{W}))-g(\eta(\mathcal{V},\mathcal{W}),\eta(\mathcal{U},\mathcal{E})),
\end{aligned}$$

$\forall\mathcal{V},\mathcal{U},\mathcal{W},\mathcal{E}\in\Gamma(TM)$. Assume $\pi=\mathrm{Span}\{e_1,e_2\}$ and let e_{n+1} be parallel to $\mathcal{G}(p)$. Lemma 3.3 implies

$$2\tau(p)=\frac{1}{4}\mathbb{A}_2(n^2-n)\Big\{2\mathbb{A}_3-\frac{2\mathbb{E}_1}{n}+\frac{4}{n(n-1)}\Big[\mathbb{D}_1-\mathbb{D}_4cos^2\theta\Big]\Big\}+\frac{1}{2}\mathbb{A}_4\mathbb{D}_3(n-1)+4\mathbb{A}_1-||\eta||^2, \tag{8}$$

where $\mathbb{D}_4=pTrace(T)+nq$.

Using

$$\varsigma=2\tau(p)-\frac{4(n-2)}{n-1}\mathbb{A}_1-\frac{1}{2}\mathbb{A}_4\mathbb{D}_3-\frac{1}{4}\mathbb{A}_2(n^2-n)\Big\{2\mathbb{A}_3-\frac{2\mathbb{E}_1}{n}+\frac{4}{n(n-1)}\Big[\mathbb{D}_1-\mathbb{D}_4cos^2\theta\Big]\Big\}. \tag{9}$$

(8) and (9) give

$$\varsigma+||\eta||^2=\frac{4\mathbb{A}_1}{n-1}. \tag{10}$$

On simplification, one gets

$$\Big(\sum_{j=1}^{n}\eta_{jj}^{n+1}\Big)^2=(n-1)\Big\{\varsigma+\sum_{j=1}^{n}(\eta_{jj}^{n+1})^2+\mathbb{F}_1+\mathbb{F}_2\Big\}, \tag{11}$$

wherein $\mathbb{F}_1=\sum_{i\neq j}(\eta_{ij}^{n+1})^2$, $\mathbb{F}_2=\sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(\eta_{ij}^{s})^2$.

Setting

$$a_1=\eta_{11}^{n+1},a_2=\eta_{22}^{n+1},\dots,a_n=\eta_{nn}^{n+1},$$
$$b=\varsigma+\mathbb{F}_1+\mathbb{F}_2,$$

one obtains

$$\eta_{11}^{n+1}\eta_{22}^{n+1}\geq\frac{1}{2}[\varsigma+\mathbb{F}_1+\mathbb{F}_2], \tag{12}$$

wherein Lemma 2.1 was taken into use. Moreover, (1) and (5) produce

$$K(\pi)=\frac{1}{4}\mathbb{A}_2\Big\{2\mathbb{A}_3-\mathbb{E}_1+4\Big[\mathbb{D}_1-\mathbb{E}_2cos^2\theta\Big]\Big\}+\frac{1}{4}\mathbb{A}_4\mathbb{E}_3+\sum_{s=n+1}^{m}[\eta_{11}^{s}\eta_{22}^{s}-(\eta_{12}^{s})^2], \tag{13}$$

here $\mathbb{E}_2 = p.Trace(T) + 2q$, $\mathbb{E}_3 = 2Trace(\varphi) - 2p$.

Thus, (12) and (13) help us to get

$$\begin{aligned} K(\pi) &\geq \frac{1}{4}\mathbb{A}_2\Big\{2\mathbb{A}_3 - \mathbb{E}_1 + 4\Big[\mathbb{D}_1 - \mathbb{E}_2 cos^2\theta\Big]\Big\} + \frac{1}{2}\mathbb{F}_1 + \sum_{s=n+2}^{m} \eta_{11}^s\eta_{22}^s - \sum_{s=n+1}^{m}(\eta_{12}^s)^2 + \frac{1}{2}\mathbb{F}_2 + \frac{1}{2}\varsigma + \frac{1}{4}\mathbb{A}_4\mathbb{E}_3 \\ &= \frac{1}{4}\mathbb{A}_2\Big\{2\mathbb{A}_3 - \mathbb{E}_1 + 4\Big[\mathbb{D}_1 - (\mathbb{E}_2) cos^2\theta\Big]\Big\} + \frac{1}{4}\mathbb{A}_4\mathbb{E}_3 + \frac{1}{2}\varsigma + \sum_{s=n+1}^{m}[\eta_{11}^s\eta_{22}^s - (\eta_{12}^s)^2] \\ &+ \frac{1}{2}\sum_{i\neq j>2}(\eta_{ij}^{n+1})^2 + \frac{1}{2}\sum_{s=n+2}^{m}\sum_{i,j>2}(\eta_{ij}^s)^2 + \sum_{s=n+1}^{m}\sum_{i>2}[(\eta_{1i}^s)^2 + (\eta_{2i}^s)^2] + \frac{1}{2}\sum_{s=n+2}^{m}(\eta_{11}^s + \eta_{22}^s)^2, \end{aligned} \tag{14}$$

i.e.,

$$K(\pi) \geq \frac{1}{4}\mathbb{A}_2\Big\{2\mathbb{A}_3 - \mathbb{E}_1 + 4\Big[\mathbb{D}_1 - \mathbb{E}_2 cos^2\theta\Big]\Big\} + \frac{1}{4}\mathbb{A}_4\mathbb{E}_3 + \sum_{s=n+1}^{m}[\eta_{11}^s\eta_{22}^s - (\eta_{12}^s)^2] + \frac{1}{2}\varsigma. \tag{15}$$

proving the desired inequality. □

Next, we have

Theorem 4.2. *Assuming the aforementioned considerations for Theorem 4.1 are true, equality in (6) is satisfied at $p \in M$ if and only if S may be written as follows for an orthonormal frame $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$:*

$$S_{n+1} = \begin{pmatrix} y & 0 & 0 & \dots & 0 \\ 0 & y' & 0 & \dots & 0 \\ 0 & 0 & y+y' & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & y+y' \end{pmatrix}, \tag{16}$$

and

$$S_s = \begin{pmatrix} y_s & y'_s & 0 & \dots & 0 \\ y'_s & -y_s & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad s \in \{n+2, \dots, m\}. \tag{17}$$

Proof. (6) satisfies equality if and only if equality holds in every previous unequal condition as well as in Lemma 2.1:

$$\begin{aligned} \eta_{ij}^{n+1} &= 0, i \neq j > 2, \\ \eta_{1i}^s = \eta_{2i}^s = \eta_{ij}^s &= 0, s \geq n+2, i, j > 2, \\ \eta_{1i}^{n+1} = \eta_{2i}^{n+1} &= 0, i > 2, \\ \eta_{11}^s + \eta_{22}^s &= 0, s \geq n+2, \\ \eta_{11}^{n+1} + \eta_{22}^{n+1} &= \eta_{33}^{n+1} = \dots = \eta_{nn}^{n+1}. \end{aligned}$$

Lastly, $S_s, s \in \{n+1, \dots, m\}$ look like (16) and (17) since $\{e_1, e_2\}$ could be chosen to get $\eta_{12}^{n+1} = 0$. □

Now let's discuss the inequality concerning $\delta(n_1,\dots,n_\mu)$.

Theorem 4.3. *Let M^n stands for proper θ-slant submanifold immersing in $\overline{M}$ and $(n_1,\dots,n_\mu)\in S(n)$ be μ-tuple. Then*

$$\begin{aligned}\delta(n_1,\dots,n_\mu) \quad \le \quad & d(n_1,\dots,n_\mu)\|\mathcal{G}\|^2+\frac{1}{2}\mathbb{A}_2\mathbb{A}_3 b(n_1,\dots,n_k)\\ & -\frac{1}{2}\mathbb{A}_2\{qcos^2\theta+\frac{1}{2}\mathbb{E}_1\}\mathbb{E}_4-\frac{1}{4}\mathbb{A}_4\{p(2n-\mathbb{E}_4)-\mathbb{E}_1-p\}\mathbb{E}_4,\end{aligned} \tag{18}$$

where, $\mathbb{E}_4=n-\sum_{j=1}^{k}n_j$.

Things considered, (18) is valid for equality if and only if for any orthonormal frame field, S emerges as follows:

$$S_{n+1}=\begin{pmatrix} a_1 & 0 & 0 & \dots & 0\\ 0 & a_2 & 0 & \dots & 0\\ 0 & 0 & a_3 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0\\ 0 & 0 & 0 & \dots & a_n\end{pmatrix}, \tag{19}$$

$$S_s=\begin{pmatrix} B_1^s & \dots & 0 & 0 & \dots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & B_\mu^s & 0 & \dots & 0\\ 0 & \dots & 0 & c_s & \dots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & 0 & \dots & c_s\end{pmatrix}, \tag{20}$$

and $a_1,\dots,a_n$ meets

$$a_1+\dots+a_{n_1}=\dots=a_{n_1+\dots n_{\mu-1}+1}+\dots+a_{n_1+\dots n_\mu}=a_{n_1+\dots n_\mu+1}=\dots=a_n$$

wherein $n_i\times n_i$ symmetric submatrix B_i^s supports

$$\mathrm{Trace(B_1^s)}=\dots=\mathrm{Trace(B_\mu^s)}=\mathrm{c_s}.$$

Proof. Adjust e_{n+1} parallel to $\mathcal{G}(p)$. Secondly, assume $\dim L_i=n_i,\forall i\in\{1,\dots,\mu\}$ and select μ mutually orthogonal subspaces $L_1,\dots,L_\mu$ of T_pM holding

$$L_1=\mathrm{Span}\{e_1,\dots,e_{n+1}\},L_2=\mathrm{Span}\{e_{n_1+1},\dots,e_{n_1+n_2}\},\dots,L_\mu=\mathrm{Span}\{e_{n_1+\dots+n_{\mu-1}+1},\dots,e_{n_1+\dots+n_\mu}\}.$$

Gauss equation produces

$$\begin{aligned}\tau(L_i)=&\frac{1}{8}\mathbb{A}_2n_i(n_i-1)\Big\{2\mathbb{A}_3-\frac{2}{n_i}\mathbb{E}_1+\frac{4}{n_i(n_i-1)}\Big[\mathbb{D}_1-\mathbb{D}_5cos^2\theta\Big]\Big\}\\ &+\frac{1}{4}\mathbb{A}_4(n_i-1)\mathbb{D}_6+\sum_{s=n+1}^{m}\sum_{\alpha_i<\beta_i}[\eta^s_{\alpha_i\alpha_i}\eta^s_{\beta_i\beta_i}-(\eta_{\alpha_i\beta_i})^2],\end{aligned} \tag{21}$$

where $\mathbb{D}_5=p.Trace(T)+q.n_i$.

Let's set

$$\eta = 2\tau(p) - 2d(n_1, \ldots, n_\mu)\|\mathcal{G}\|^2 - \frac{1}{4}n(n-1)\mathbb{A}_2\Big\{2\mathbb{A}_3 - \frac{2}{n}\mathbb{E}_1 + \frac{4}{n(n-1)}\Big[\mathbb{D}_1 - \mathbb{D}_4 cos^2\theta\Big]\Big\} - \frac{1}{2}\mathbb{A}_4\mathbb{D}_3(n-1) \tag{22}$$

and

$$\vartheta = n + \mu - \sum_{i=1}^{\mu} n_i. \tag{23}$$

Consequently, there is

$$\eta + \|\eta\|^2 = \frac{4\mathbb{A}_1}{\vartheta}, \tag{24}$$

and as a result, one gets

$$\Big(\sum_{j=1}^{n}\eta_{jj}^{n+1}\Big)^2 = \vartheta\Big\{\eta + \sum_{j=1}^{n}(\eta_{jj}^{n+1})^2 + \mathbb{F}_1 + \mathbb{F}_2\Big\}, \tag{25}$$

which shrinks to

$$\Big(\sum_{j=1}^{\vartheta+1} b_j\Big)^2 = \vartheta\Big\{\eta + \sum_{j=1}^{\vartheta+1}(b_j)^2 + \mathbb{F}_1 + \mathbb{F}_2 - 2\sum_{i=1}^{\mu}\sum_{\alpha_i<\beta_i} a_{\alpha_i}a_{\beta_i}\Big\}, \tag{26}$$

in this case

$$a_j = \eta_{jj}^{n+1}, \forall j \in \{1, \ldots, n\},$$
$$b_1 = a_1, b_2 = a_2 + \cdots + a_{n_1}, b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2,\ldots,}$$
$$b_{\mu+1} = a_{n_1+\cdots+n_{\mu-1}+1} + \cdots + a_{n_1+n_2\cdots+n_\mu}, b_{\mu+2} = a_{n_1+\cdots+n_\mu+1}, \ldots, b_{\vartheta+1} = a_n.$$

As a result, we approach

$$\sum_{i=1}^{\mu}\sum_{\alpha_i<\beta_i} a_{\alpha_i}a_{\beta_i} \geq \frac{1}{2}[\eta + \mathbb{F}_1 + \mathbb{F}_2], \tag{27}$$

Lemma 2.1 has been used in the preceding arguments.

Additionally, consider the sets $e_1, \ldots, e_\mu, e$

$$e_1 = \{1, \ldots, n_1\}, e_2 = \{n_1 + 1, \ldots, n_1 + n_2\}, \ldots, e_\mu = \{n_1 + \cdots + n_{\mu-1} + 1, \ldots, n_1 + \cdots + n_\mu\},$$
$$e^2 = (e_1 \times e_1) \cup \cdots \cup (e_\mu \times e_\mu),$$

ultimately, to find

$$\sum_{i=1}^{\mu}\sum_{s=n+1}^{m}\sum_{\alpha_i<\beta_i}[\eta_{\alpha_i\alpha_i}^s\eta_{\beta_i\beta_i}^s - (\eta_{\alpha_i\beta_i}^s)^2] \geq \frac{1}{2}\eta + \frac{1}{2}\sum_{s=n+1}^{m}\sum_{(\alpha,\beta)\notin e^2}(\eta_{\alpha\beta}^s)^2 + \sum_{s=n+2}^{m}\sum_{\alpha_i\in e_i}(\eta_{\alpha_i\alpha_i}^s)^2, \tag{28}$$

which further generates

$$\sum_{i=1}^{\mu}\sum_{s=n+1}^{m}\sum_{\alpha_i<\beta_i}[\eta_{\alpha_i\alpha_i}^s\eta_{\beta_i\beta_i}^s - (\eta_{\alpha_i\beta_i}^s)^2] \geq \frac{1}{2}\eta. \tag{29}$$

Consequently, based on (21), receives

$$\tau(L_i) \geq \sum_{i=1}^{\mu} \frac{1}{8}\mathbb{A}_2 n_i(n_i-1)\Big\{2\mathbb{A}_3 - \frac{2}{n_i}\mathbb{E}_1 + \frac{4}{n_i(n_i-1)}\Big[\mathbb{D}_1 - \mathbb{D}_5 cos^2\theta\Big]\Big\} + \sum_{i=1}^{\mu}\frac{1}{4}\mathbb{A}_4\mathbb{D}_6(n_i-1) + \frac{1}{2}\eta. \tag{30}$$

Thus, we have the needed inequality when accounting for (22) and (30). Further to that, (18) is true for equality if and only if equality holds in Lemma 2.1 and in all previous inequalities. Likewise, $S_s, s \in \{n+1,\dots,m\}$ corresponds to (19) and (20). □

As a specific instance of Theorems 4.1 and 4.3, we construct

Corollary 4.4. *The following inequality is true for φ-invariant submanifold M^n immersed in $\overline{M}$*

$$\delta_M(p) \leq \frac{(n-2)}{2}\Big[\frac{4}{(n-1)}\mathbb{A}_1 + \frac{1}{2}\mathbb{A}_2\{\mathbb{A}_3(n+1) - \mathbb{E}_1\}\Big] + \frac{1}{2}\mathbb{A}_2\Big[\mathbb{D}_2 + 4q - \mathbb{D}_1\Big] + \frac{1}{4}\mathbb{A}_4(n-2)\Big[\mathbb{D}_3 - p\Big], \quad p \in M. \tag{31}$$

Corollary 4.5. *The following inequality exists for φ-anti-invariant submanifold M^n immersed in $\overline{M}$*

$$\delta_M(p) \leq \frac{(n-2)}{2}\Big[\frac{4}{(n-1)}\mathbb{A}_1 + \frac{1}{2}\mathbb{A}_2\{\mathbb{A}_3(n+1) - \mathbb{E}_1\} - \mathbb{D}_1\Big] + \frac{1}{4}\mathbb{A}_4(n-2)\Big[\mathbb{D}_3 - p\Big]. \tag{32}$$

Additionally, equality holds in (31) and (32) if and only if for $\{e_1,\dots,e_n,e_{n+1},\dots,e_m\}$, S take the shape of (16) and (17).

Corollary 4.6. *For μ-tuples $(n_1,\dots,n_\mu) \in S(n)$, the following inequality follows for every φ-invariant submanifold M^n immersed in $\overline{M}$*

$$\delta(n_1,\dots,n_\mu) \leq d(n_1,\dots,n_\mu)\|\mathcal{G}\|^2 + \frac{1}{2}\mathbb{A}_2\mathbb{A}_3 b(n_1,\dots,n_k) - \frac{1}{2}\mathbb{A}_2\{q + \frac{1}{2}\mathbb{E}_1\}\mathbb{E}_4 - \frac{1}{4}\mathbb{A}_4\{p(2n - \mathbb{E}_4) - \mathbb{E}_1 - p\}\mathbb{E}_4. \tag{33}$$

Additionally, (33) satisfies equality if and only if there exists any orthonormal frame and S take the shape of (19) and (20).

Corollary 4.7. *For μ-tuples $(n_1,\dots,n_\mu) \in S(n)$, the following inequality follows for every φ-anti-invariant submanifold M^n immersed in $\overline{M}$*

$$\delta(n_1,\dots,n_\mu) \leq d(n_1,\dots,n_\mu)\|\mathcal{G}\|^2 + \frac{1}{2}\mathbb{A}_2\mathbb{A}_3 b(n_1,\dots,n_k) - \frac{1}{4}\mathbb{A}_2\mathbb{E}_1\mathbb{E}_4 - \frac{1}{4}\mathbb{A}_4\{p(2n - \mathbb{E}_4) - \mathbb{E}_1 - p\}\mathbb{E}_4. \tag{34}$$

Further, equality holds in (34) if and only if for any orthonormal frame $\{e_1,\dots,e_n,e_{n+1},\dots,e_m\}$, S takes the shape of (19) and (20).

5. Inequalities for Ricci curvature tensor

Let be a proper θ-slant submanifold immersing in $\overline{M}^m$. Next, locate the unit tangent vector $\mathcal{U} \in T_tM, \forall t \in M$ and on M, let $\{e_1, \ldots, e_n\}$ be the local orthonormal frame so that $e_1 = \mathcal{U}$. Utilizing (8) and employing lemma 3.3, we obtain

$$2\tau(t) = \frac{1}{4}\mathbb{A}_2\Big\{2\mathbb{A}_3 n(n-1) - 4p(n-1)Trace(\varphi) + 4\Big[\mathbb{D}_1 - \mathbb{D}_4 cos^2\theta\Big]\Big\} + \frac{1}{2}\mathbb{A}_4\mathbb{D}_3 + 4\mathbb{A}_1 - \frac{1}{2}\sum_{s=n+1}^{m}[\sum_{j=1}^{n}(\eta^s_{jj})^2 + (\eta^s_{11} - \sum_{j=2}^{n}\eta^s_{jj})^2] - 2\sum_{s=n+1}^{m}[\sum_{i<j}(\eta^s_{ij})^2 - \sum_{2\le j<i\le n}\eta^s_{jj}\eta^s_{ii}]. \tag{35}$$

Then again, the Gauss equation yields

$$\sum_{2\le j<i\le n} K(e_j \wedge e_i) = \frac{1}{8}\mathbb{A}_2\Big\{2(n-1)(n-2)\mathbb{A}_3 - 4p(n-2)\mathrm{Trace}(\varphi) + 4[\mathbb{D}_1 - (\mathbb{D}_4 - \mathrm{q})\cos^2\theta]\Big\} - \frac{1}{4}(n-2)\mathbb{A}_4[\mathbb{B}_1 - \mathrm{Trace}(\varphi)] + \sum_{\mathrm{s=n+1}}^{\mathrm{m}}\sum_{2\le \mathrm{i}<\mathrm{j}\le \mathrm{n}}[\eta^{\mathrm{s}}_{\mathrm{jj}}\eta^{\mathrm{s}}_{\mathrm{ii}} - (\eta^{\mathrm{s}}_{\mathrm{ij}})^2], \tag{36}$$

here $\mathbb{B}_1 = (n-1)p - Trace(\varphi)$.

Let $\mathbb{B}_2 = pTrace(\varphi) - (p^2 + 2q)(n-1)$. Then, (35) and (36) produce

$$4\mathbb{A}_1 \ge 2\tau(t) + 2\sum_{s=n+1}^{m}\sum_{i=2}^{n}(\eta^s_{1i})^2 - 2\sum_{2\le j<i\le n}K(e_j \wedge e_i) + \mathbb{A}_2[\mathbb{B}_2 + q\cos^2\theta] + \mathbb{A}_4\mathbb{B}_1, \tag{37}$$

which demonstrates

$$Ric(\mathcal{U}) \le 4\mathbb{A}_1 - \frac{1}{2}\mathbb{A}_4\mathbb{B}_1 - \frac{1}{2}\mathbb{A}_2[\mathbb{B}_2 + q\cos^2\theta]. \tag{38}$$

Likewise, the equality case for (38) is true with $\mathcal{G}(t) = 0$ if and only if

$$\eta^s_{1i} = 0, \qquad i \in \{2, \ldots, n\}, \tag{39}$$

$$\eta^s_{11} = \sum_{j=2}^{n}\eta^s_{jj}, \qquad s \in \{n+1, \ldots, m\} \tag{40}$$

revealing that $\mathcal{U}$ falls in the relative null space L_t. Moreover, equality is true in (38) if and only if

$$\eta^s_{ij} = 0, \quad n+1 \le s \le m, i \ne j, \tag{41}$$

$$\sum_{j=1}^{n}\eta^s_{jj} = 2\eta^s_{ii}, \quad s \in \{n+1, \ldots, m\}, i \in \{1, \ldots, n\}. \tag{42}$$

Implying t to be

- totally geodesic when $n \ne 2$
- totally umbilical in case $n = 2$.

The converse portion is evident, as may be seen.

Consequently, it may be summed up as

Theorem 5.1. *For submanifold M^n of $\overline{M}^m$, we have the following relations*

submanifold type	
θ-slant	$Ric(\mathcal{U}) \leq 4\mathbb{A}_1 - \frac{1}{2}\mathbb{A}_4\mathbb{B}_1 - \frac{1}{2}\mathbb{A}_2[\mathbb{B}_2 + q\cos^2\theta]$
φ-invariant	$Ric(\mathcal{U}) \leq 4\mathbb{A}_1 - \frac{1}{2}\mathbb{A}_4\mathbb{B}_1 - \frac{1}{2}\mathbb{A}_2[\mathbb{B}_2 + q]$
φ-anti-invariant	$Ric(\mathcal{U}) \leq 4\mathbb{A}_1 - \frac{1}{2}\mathbb{A}_4\mathbb{B}_1 - \frac{1}{2}\mathbb{A}_2\mathbb{B}_2$

In this instance, $\mathcal{U} \in T_tM, t \in M$ indicates unit vector.

Note: $\mathcal{U} \in L_t$ if and only if inequalities in above table agree at equality for $\mathcal{G}(t) = 0$. Furthermore, t is totally geodesic point in M or $n = 2$ and with totally umbilical point t if and only if above inequalities satisfy equality.

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