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Contact screen transversal Cauchy-Riemann lightlike submanifolds of indefinite Sasakian manifolds

Nergiz (Önen) Poyraz[a]

[a] *Çukurova University, Faculty of Arts and Sciences, Department of Mathematics, Adana, Turkey*

Abstract. We study contact screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds of indefinite Sasakian manifolds. We prove existence and non-existence theorems and find the integrability conditions of integrability of various distributions. We derive some characterization theorems for a contact STCR-lightlike submanifold to be a STCR-lightlike product. Moreover, we find results for minimal contact STCR-lightlike submanifolds of indefinite Sasakian manifolds. We also give examples.

1. Introduction

Since the intersection of normal vector bundle and the tangent bundle is non-trivial, then in the study of lightlike submanifolds is more interesting and remarkably different from the study of non-degenerate submanifolds. Lightlike submanifolds have been developed in [5, 10].

Duggal and Bejancu [5] introduced CR-lightlike submanifolds of indefinite Kaehler manifolds and Duggal and Şahin [8] introduced contact CR-lightlike submanifolds of indefinite Sasakian manifolds. But CR-lightlike submanifolds exclude the complex and totally real submanifolds as subcases. Then, screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds [6] and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds [8] were presented by Duggal and Şahin. But there is no inclusion relation between screen Cauchy-Riemann and CR submanifolds, so Duggal and Şahin [7] presented a new class named generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Kaehler manifolds and GCR-lightlike submanifolds of indefinite Sasakian manifolds [9] which is an umbrella for all these types of submanifolds. These types of submanifolds have been studied by many authors [11, 13, 15, 17].

But CR-lightlike, screen CR-lightlike and generalized CR-lightlike do not contain real lightlike curves. For this reason, Şahin presented screen transversal lightlike submanifolds of indefinite Kaehler manifolds and show that such submanifolds contain lightlike real curves [18]. Screen transversal lightlike submanifolds of indefinite almost contact manifolds were introduced in [19]. Such submanifolds have been studied in [12, 14, 20]. On the other hand, as a generalization of CR-lightlike submanifolds and screen transversal lightlike submanifolds, in [3], Doğan, Şahin and Yaşar introduced screen transversal CR-lightlike submanifolds.

In this paper, we study contact screen transversal Cauchy-Riemann (STCR)-lightlike submanifolds of indefinite Sasakian manifolds. We prove existence and non-existence theorems and find the integrability

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Email address: nonen@cu.edu.tr (Nergiz (Önen) Poyraz)

conditions of various distributions. We derive some characterization theorems for a contact STCR-lightlike submanifold to be a STCR-lightlike product. Moreover, we find results for minimal contact STCR-lightlike submanifolds of indefinite Sasakian manifolds. We also give examples.

2. Preliminaries

Let $(\bar{M}, \bar{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold of constant index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an $m-$dimensional submanifold of $(\bar{M}, \bar{g})$, where g is the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is named a lightlike submanifold of $(\bar{M}, \bar{g})$. For a degenerate metric g on M

$$TM^{\perp} = \cup \{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_x\bar{M}, x \in M\} \tag{1}$$

is a degenerate $n-$dimensional subspace of $T_x\bar{M}$. Hence, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. Thus, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) space. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution, named radical distribution on M of rank $r > 0$ then the submanifold M of $(\bar{M}, \bar{g})$ is named an $r-$lightlike submanifold.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM. This means that

$$TM = S(TM) \perp Rad(TM) \tag{2}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}_{|M}$ and $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then, we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{3}$$

$$T\bar{M}\,|_M = TM \oplus tr(TM) = \{Rad(TM) \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{4}$$

Theorem 2.1. *[5] Let $(M, g, S(TM), S(TM^{\perp}))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Suppose U is a coordinate neighbourhood of M and $\{\xi_i\}$, $i \in \{1, .., r\}$ is a basis of $\Gamma(Rad(TM)_{|u})$. Then, there exist a complementary vector subbundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}_{|u}$ and a basis $\{N_i\}$, $i \in \{1, .., r\}$ of $\Gamma(ltr(TM)_{|u})$ such that*

$$\bar{g}\left(N_i, \xi_j\right) = \delta_{ij}, \quad \bar{g}\left(N_i, N_j\right) = 0 \tag{5}$$

for any $i, j \in \{1, .., r\}$.

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $(\bar{M}, \bar{g})$ is
Case 1: $r-$lightlike if $r < min\{m, n\}$,
Case 2: Coisotropic if $r = n < m$, $S(TM^{\perp}) = \{0\}$,
Case 3: Isotropic if $r = m < n$, $S(TM) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$.
Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, using (4) we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{6}$$
$$\bar{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{7}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\left\{h(X, Y), \nabla^t_X U\right\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. According to (2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (6) and (7) become

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + h^l(X,Y) + h^s(X,Y), &(8)\\ \bar{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X,N), &(9)\\ \bar{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X,W), &(10)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$, where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$, $\nabla^l_X N, D^l(X,W) \in \Gamma(ltr(TM))$ and $\nabla^s_X W, D^s(X,N) \in \Gamma(S(TM^\perp))$. Hence, using (8)-(10) and letting into account that $\bar{\nabla}$ is a metric connection we derive

$$\begin{aligned}g(h^s(X,Y),W) + g(Y, D^l(X,W)) &= g(A_W X, Y), &(11)\\ g(D^s(X,N),W) &= g(A_W X, N), &(12)\\ g(h^l(X,Y),\xi) + g(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) &= 0. &(13)\end{aligned}$$

Let Q be a projection of TM on $S(TM)$. Thus, using (2) we obtain

$$\begin{aligned}\nabla_X QY &= \nabla^*_X QY + h^*(X,QY)\xi, &(14)\\ \nabla_X \xi &= -A^*_\xi X + \nabla^{*t}_X \xi, &(15)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X QY, A^*_\xi X\}$ and $\left\{h^*(X,QY), \nabla^{*t}_X \xi\right\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively.

Using the equations given above, we derive

$$\begin{aligned}g(h^l(X,QY),\xi) &= g(A^*_\xi X, QY), &(16)\\ g(h^*(X,QY),N) &= g(A_N X, QY), &(17)\\ g(h^l(X,\xi),\xi) &= 0, \quad A^*_\xi \xi = 0. &(18)\end{aligned}$$

Generally, ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, from (8) we obtain

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y).$$

But, ∇^* is a metric connection on $S(TM)$.

Definition 2.2. *A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$ is said to be an irrotational submanifold if $\tilde{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$ [16]. Thus M is an irrotational lightlike submanifold iff $h^l(X,\xi) = 0$, $h^s(X,\xi) = 0$.*

Theorem 2.3. *Let M be an $r-$lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then ∇ is a metric connection iff $Rad(TM)$ is a parallel distribution with respect to ∇ [5].*

An odd dimensional semi-Riemannian manifolds $(\bar{M},\bar{g})$ is named a contact metric manifold [4] if there is a $(1,1)$ tensor field ϕ, a vector field V named characteristic vector field, and a 1-form η such that

$$\begin{aligned}\bar{g}(\phi X, \phi Y) &= \bar{g}(X,Y) - \epsilon\eta(X)\eta(Y), \bar{g}(V,V) = \epsilon, &(19)\\ \phi^2 X &= -X + \eta(X)V, \bar{g}(X,V) = \epsilon\eta(X), &(20)\\ d\eta(X,Y) &= \bar{g}(X,\phi Y), \epsilon = \pm 1 &(21)\end{aligned}$$

for any $X, Y \in \Gamma(T\bar{M})$.

It follows that

$$\phi V = 0, \phi \circ \eta = 0, \eta(V) = \epsilon. \tag{22}$$

Then $(\phi, V, \eta, \bar{g})$ is named contact metric structure of $(\bar{M}, \bar{g})$. We say that $(\bar{M}, \bar{g})$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ [23]. A normal contact metric manifold is named an indefinite Sasakian manifold [21, 22] for which we have

$$\nabla_X V = \phi X, \tag{23}$$

$$(\nabla_X \phi)Y = -\bar{g}(X, Y)V + \epsilon\eta(Y)X. \tag{24}$$

$(\bar{M}, \bar{g})$ is named indefinite Sasakian space form, denoted by $\bar{M}(c)$, if it has the constant $\phi-$sectional curvature c [22]. The curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}(c)$ is given by

$$\begin{aligned}\bar{R}(X, Y)Z = {} & \frac{(c+3)}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} + \frac{(c-1)}{4}\{\eta(X)\eta(Z)Y \\ & -\eta(Y)\eta(Z)X + \bar{g}(X, Z)\eta(Y)V - \bar{g}(Y, Z)\eta(X)V \\ & +\bar{g}(\phi Y, Z)\phi X + \bar{g}(\phi Z, X)\phi Y - 2\bar{g}(\phi X, Y)\phi Z\}\end{aligned} \tag{25}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$.

3. Contact Screen transversal Cauchy-Riemann (STCR)-Lightlike Submanifolds

Definition 3.1. *Let M be a real r-lightlike submanifold of an indefinite Sasakian manifold manifold $(\bar{M}, \bar{g})$. Then we say that M is a contact screen transversal Cauchy-Riemann (STCR)-lightlike submanifold if the condition (A) and (B) are holded:*

(A) There exist two subbundles σ_1 and σ_2 of $Rad(TM)$ such that

$$Rad(TM) = \sigma_1 \oplus \sigma_2, \ \ \phi(\sigma_1) \subset S(TM), \ \ \phi(\sigma_2) \subset S(TM^\perp). \tag{26}$$

(B) There exist two subbundles σ_0 and σ' of $S(TM)$ such that

$$S(TM) = \left\{\phi(\sigma_1) \oplus \sigma'\right\} \perp \sigma_0, \ \phi(\sigma_0) = \sigma_0, \ \phi(\sigma') = L_1 \perp S, \tag{27}$$

where σ_0 is a non-degenerate distribution on M, L_1 and S are vector subbundles of $ltr(TM)$ and $S(TM^\perp)$, respectively.

Then TM of M is decomposed as

$$TM = \sigma \oplus \bar{\sigma} \perp \{V\} \tag{28}$$

where

$$\sigma = \sigma_0 \oplus \sigma_1 \oplus \phi(\sigma_1) \tag{29}$$

and

$$\bar{\sigma} = \sigma_2 \oplus \phi(L_1) \oplus \phi(S). \tag{30}$$

It is clear that σ is invariant and $\bar{\sigma}$ is anti-invariant. Besides, we have

$$ltr(TM) = L_1 \oplus L_2, \phi(L_1) \subset S(TM), \phi(L_2) \subset S(TM^\perp) \tag{31}$$

and

$$S(TM^\perp) = \left\{\phi(\sigma_2) \oplus \phi(L_2)\right\} \perp S. \tag{32}$$

If $\sigma_1 \neq \{0\}$, $\sigma_2 \neq \{0\}$, $\sigma_0 \neq \{0\}$ and $S \neq \{0\}$, then M is called a proper contact STCR-lightlike submanifold of an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. For proper contact STCR-lightlike submanifold we note that the following features:

1. The condition (A) implies that $dim(Rad(TM)) \geq 2$.
2. The condition (B) implies $dim(\sigma) = 2s \geq 4$, $dim(\sigma') \geq 2$ and $dim(\sigma_2) = dim(L_2)$. Thus $dim(M) \geq 8$ and $dim(\bar{M}) \geq 13$.
3. Any proper 8-dimensional contact STCR-lightlike submanifold must be 2-lightlike.
4. (A) and contact distribution ($\eta = 0$) imply that index$(\bar{M}) \geq 2$.

Proposition 3.2. *A contact STCR-lightlike submanifold M of an indefinite Sasakian manifold $(\bar{M}, \bar{g})$ is contact CR-lightlike submanifold (respectively, contact screen transversal lightlike submanifold) iff $\sigma_2 = \{0\}$ (respectively, $\sigma_1 = \{0\}$).*

Proof. Suppose that M is a contact CR-lightlike submanifold of an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. Then $\phi(Rad(TM))$ is a distribution on M such that $\phi(Rad(TM)) \cap Rad(TM) = \{0\}$. Therefore we get $\sigma_1 = Rad(TM)$ and $\sigma_2 = \{0\}$. Thus we conclude that $\phi(ltr(TM)) \cap ltr(TM) = \{0\}$. Then it follows that $\phi(ltr(TM)) \subset S(TM)$. Conversely, suppose that M is a contact STCR-lightlike submanifold such that $\sigma_2 = \{0\}$. Then we have $\sigma_1 = Rad(TM)$. Therefore $\phi(Rad(TM)) \cap Rad(TM) = \{0\}$, that is, $\phi(Rad(TM))$ is a vector subbundle of $S(TM)$. Hence M is a contact CR-lightlike submanifold. Similarly one can obtain the other assertion. □

Proposition 3.3. *There exist no coisotropic, isotropic or totally lightlike proper contact STCR-lightlike submanifolds M of an indefinite Sasakian manifold. Any isotropic contact STCR-lightlike submanifold is a screen transversal lightlike submanifold. Besides, a coisotropic contact STCR-lightlike submanifold is a contact CR-lightlike submanifold.*

Proof. Suppose that M is a proper contact STCR-lightlike submanifold. From definition of proper contact STCR-lightlike submanifold, we know that $\sigma_1 \neq \{0\}$, $\sigma_2 \neq \{0\}$, $\sigma_0 \neq \{0\}$ and $S \neq \{0\}$, that is both $S(TM)$ and $S(TM^\perp)$ are non-zero. Hence, M can not be a coisotropic, isotropic or totally lightlike submanifold. On the other hand, if M be a isotropic contact STCR-lightlike submanifold, then $S(TM) = \{0\}$, i.e., $\phi(\sigma_1) = \{0\}$ and $Rad(TM) = \sigma_2$. Hence, we obtain $\phi(Rad(TM)) = \phi(\sigma_2) \subset \Gamma(S(TM^\perp))$ and M is a contact screen transversal lightlike submanifold. Similarly, if M is a coisotropic contact STCR-lightlike submanifold, then $S(TM^\perp) = \{0\}$, i.e., $\phi(\sigma_2) = \{0\}$ and $Rad(TM) = \sigma_1$. Since, $\phi(Rad(TM)) = \phi(\sigma_1) \subset \Gamma(S(TM))$ then M is a contact CR-lightlike submanifold. □

The following construction will help in understanding the examples of this paper. Consider $(R_q^{2m+1}, \phi_0, V, \eta, g)$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{j=1}^{m} y^j dx^j), V = 2\partial z,$$

$$\bar{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{j=1}^{\frac{q}{2}} dx^j \otimes dx^j + dy^j \otimes dy^j + \sum_{i=q+1}^{m} dx^j \otimes dx^j + dy^j \otimes dy^j),$$

$$\phi_0(\sum_{j=1}^{m}(X_j \partial x^j + Y_j \partial y^j)) + Z\partial z) = \sum_{j=1}^{m}(Y_j \partial x^j - X_j \partial y^j) + Y_j y^j \partial z$$

where (x_j, y_j, z) are the Cartesian coordinates.

Example 3.4. *Let $(\bar{M} = \mathbb{R}_4^{13}, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,-,+,+,+,+,-,-,+,+,+,+,+)$ with respect to canonical basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z)$. Suppose M is a submanifold of $\mathbb{R}_4^{13}$ defined by*

$$x^1 = y^4, x^3 = \cos\theta x^2, y^3 = \sin\theta x^2, x_5 = \sqrt{1 + (y^5)^2}.$$

A local frame of TM is given by

$$\begin{aligned} \xi_1 &= \partial x_1 + \partial y_4 + y^1 \partial z, \xi_2 = \partial x_2 + \cos\theta \partial x_3 + \sin\theta \partial y_3 + (y^2 + \cos\theta y^3)\partial z, \\ Z_1 &= \partial x_4 - \partial y_1 + y^4 \partial z, Z_2 = 2(\partial x_4 + \partial y_1 + y^1 \partial z), \\ Z_3 &= 2(y^5 \partial x_5 + x^5 \partial y_5 + y^5 \partial z), Z_4 = 2\partial x_6 + y^6 \partial z, Z_5 = -2\partial y_6, Z = 2\partial z = V. \end{aligned}$$

Hence M is a 2- lightlike submanifold of $\mathbb{R}_4^{13}$ with $Rad(TM) = Span\{\xi_1, \xi_2\}$. It is easy to see $\phi_0(\xi_1) = Z_1 \in \Gamma(S(TM))$, hence $\sigma_1 = Span\{\xi_1\}$ and $\sigma_2 = Span\{\xi_2\}$. On the other hand, since $\phi_0(Z_4) = Z_5 \in \Gamma(S(TM))$, we derive $\sigma_0 = Span\{Z_4, Z_5\}$ and by direct calculations, we derive the lightlike transversal bundle spanned by

$$N_1 = 2(-\partial x_1 + \partial y_4 + y^1 \partial z), N_2 = 2(-\partial x_2 + \cos\theta \partial x_3 + \sin\theta \partial y_3 + (y^2 + \cos\theta y^3)\partial z).$$

Then we see that $L_1 = Span\{N_1\}$, $L_2 = Span\{N_2\}$, $S(TM^\perp) = Span\{\phi_0(\xi_2), \phi_0(N_2), \phi_0(Z_3)\}$ and $S = Span\{\phi_0(Z_3) = W\}$. Thus, $\sigma' = Span\{\phi_0(N_1) = Z_2, \phi_0(W) = -Z_3\}$ and M is a proper contact STCR-lightlike submanifold of $\mathbb{R}_4^{13}$.

We indicate the projections from $\Gamma(TM)$ to $\Gamma(\sigma_0)$, $\Gamma(\phi(\sigma_1))$, $\Gamma(\phi(L_1))$, $\Gamma(\phi(S))$, $\Gamma(\sigma_1)$ and $\Gamma(\sigma_2)$ by P_0, P_1, P_2, P_3, R_1 and R_2, respectively. We also indicate the projections from $\Gamma(tr(TM))$ to $\Gamma(\phi(\sigma_2))$, $\Gamma(\phi(L_2))$, $\Gamma(S)$, $\Gamma(L_1)$ and $\Gamma(L_2)$ by S_1, S_2, S_3, Q_1 and Q_2, respectively. Hence, we write

$$X = PX + RX + \eta(X)V = P_0X + P_1X + P_2X + P_3X + R_1X + R_2X + \eta(X)V \tag{33}$$

and

$$\phi X = TX + \omega X \tag{34}$$

for any $X \in \Gamma(TM)$, where $PX \in \Gamma(\sigma)$, $RX \in \Gamma(\bar{\sigma})$ and TX and ωX are the tangential parts and the transversal parts of ϕX, respectively. Applying ϕ to (33) and denoting ϕP_0, ϕP_1, ϕP_2, ϕP_3, ϕR_1, ϕR_2 by T_0, T_1, ω_L, ω_S, $T_{\bar{1}}$, $\omega_{\bar{2}}$, respectively, we derive

$$\phi X = T_0X + T_1X + T_{\bar{1}}X + \omega_LX + \omega_SX + \omega_{\bar{2}}X \tag{35}$$

for any $X \in \Gamma(TM)$, where $T_0X \in \Gamma(\sigma_0)$, $T_1X \in \Gamma(\sigma_1)$, $T_{\bar{1}}X \in \Gamma(\phi(\sigma_1))$, $\omega_LX \in \Gamma(L_1)$, $\omega_SX \in \Gamma(S)$ and $\omega_{\bar{2}}X \in \Gamma(\phi(\sigma_2))$. Similarly we write

$$U = S_1U + S_2U + S_3U + Q_1U + Q_2U \tag{36}$$

for any $U \in \Gamma(tr(TM))$ and we denote ϕS_1, ϕS_2, ϕS_3, ϕQ_1, ϕQ_2 by B_2, C_L, $B_{\bar{S}}$, $B_{\bar{L}}$, $C_{\bar{L}}$, respectively. Thus we write

$$\phi U = B_2U + B_{\bar{S}}U + B_{\bar{L}}U + C_LU + C_{\bar{L}}U \tag{37}$$

and

$$\phi U = BU + CU \tag{38}$$

where BU and CU are sections of TM and $tr(TM)$, respectively. Now, differentiating (35) and using (8)-(10), (24), (35) and (38), we derive

$$\begin{aligned} & \nabla_XTY + h^l(X,TY) + h^s(X,TY) + \{-A_{\omega_LY}X + \nabla^l_X(\omega_LY) + D^s(X,\omega_LY)\} \\ & +\{-A_{\omega_SY}X + \nabla^s_X(\omega_SY) + D^l(X,\omega_SY)\} \\ & +\{-A_{\omega_{\bar{2}}Y}X + \nabla^s_X(\omega_{\bar{2}}Y) + D^l(X,\omega_{\bar{2}}Y)\} \\ = \ & T\nabla_XY + \omega_L\nabla_XY + \omega_S\nabla_XY + \omega_{\bar{2}}\nabla_XY + Bh^l(X,Y) + Ch^l(X,Y) \\ & +Bh^s(X,Y) + Ch^s(X,Y) - g(X,Y)V + \eta(Y)X \end{aligned} \tag{39}$$

for any $X, Y \in \Gamma(TM)$. Taking the tangential, lightlike transversal and screen transversal parts of (39) we derive

$$\begin{aligned} (\nabla_XT)Y &= \nabla_XTY - T\nabla_XY = A_{\omega_LY}X + A_{\omega_SY}X + A_{\omega_{\bar{2}}Y}X \\ & \quad +Bh(X,Y) - g(X,Y)V + \eta(Y)X, \end{aligned} \tag{40}$$

$$\begin{aligned} & D^l(X,\omega_SY) + D^l(X,\omega_{\bar{2}}Y) \\ = \ & \omega_L\nabla_XY - \nabla^l_X(\omega_LY) - h^l(X,TY) + Ch^l(X,Y) \end{aligned} \tag{41}$$

and

$$\begin{aligned} D^s(X,\omega_LY) &= \omega_S\nabla_XY + \omega_{\bar{2}}\nabla_XY - \nabla^s_X(\omega_SY) \\ & \quad -\nabla^s_X(\omega_{\bar{2}}Y) - h^s(X,TY) + Ch^s(X,Y) \end{aligned} \tag{42}$$

respectively.

Theorem 3.5. *There does not exist an induced metric connection of a proper contact STCR-lightlike submanifold of an indefinite Sasakian manifold* $(\bar{M}, \bar{g})$.

Proof. Assume that ∇ is a metric connection. Then from Theorem 2.3, $Rad(TM)$ is parallel with respect to ∇, i.e., $\nabla_X \xi \in \Gamma(Rad(TM))$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. From (24) we obtain

$$\bar{\nabla}_X \phi \xi = \phi \bar{\nabla}_X \xi \tag{43}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Applying ϕ to (43) and using (20) and (24), we get

$$\phi \bar{\nabla}_X \phi \xi = -\bar{\nabla}_X \xi - \bar{g}(\xi, \bar{\nabla}_X V)V. \tag{44}$$

Then from (23) and (44) we derive

$$\phi \bar{\nabla}_X \phi \xi = -\bar{\nabla}_X \xi - g(\xi, \phi X)V. \tag{45}$$

Choose $X \in \Gamma(\phi(L_1))$ and $\xi \in \Gamma(\sigma_1)$ such that $g(\phi X, \xi) \neq 0$ (since $\sigma_1 \oplus \phi(L_1)$ is a non-degenerate distribution on M, so we can choose such vector fields). Hence from (6), (14), (38) and (45) we obtain

$$\begin{aligned} -\nabla_X \xi - h(X, \xi) - g(\xi, \phi X)V \quad &= \quad T\nabla^*_X \phi \xi + \omega \nabla^*_u \phi \xi + Th^*(X, \phi \xi) \\ &\quad +\omega h^*(X, \phi \xi) + Bh(X, \phi \xi) + Ch(X, \phi \xi), \end{aligned} \tag{46}$$

for any $X \in \Gamma(\phi(L_1))$ and $\xi \in \Gamma(\sigma_1)$. Then taking tangential parts of (46) we derive

$$T\nabla^*_X \phi \xi + \nabla_X \xi + Th^*(X, \phi \xi) + Bh(X, \phi \xi) = -\bar{g}(\xi, \phi X)V. \tag{47}$$

Since $Rad(TM)$ is parallel, $\nabla_X \xi \in \Gamma(Rad(TM))$. On the other hand, $T\nabla^*_X \phi \xi + Th^*(X, \phi \xi) \in \Gamma(\sigma_1 \perp \phi(\sigma_1) \perp \sigma_0)$ and $Bh(X, \phi \xi) \in \Gamma(\bar{\sigma})$, thus we obtain $\bar{g}(\xi, \phi X)V = 0$. Since $V \neq 0$ and $\bar{g}(\xi, \phi X) \neq 0$ we have a contradiction so $Rad(TM)$ is not parallel. Hence ∇ is not a metric connection. □

Theorem 3.6. *Let M be a lightlike submanifold tangent to the structure vector field V in an indefinite Sasakian $\bar{M}(c)$ with $c \neq 1$ Then, M is a contact STCR-lightlike submanifold of $\bar{M}(c)$ iff:*

(a) The maximal invariant subspaces of TpM, $p \in M$, define a distribution

$$\sigma = \sigma_0 \oplus \sigma_1 \oplus \phi(\sigma_1)$$

where $Rad(TM) = \sigma_1 \perp \sigma_2$ and σ_0 is a non-degenerate invariant distribution.

(b) There exists a lightlike transversal vector bundle $ltr(TM)$ such that

$$\bar{g}(\bar{R}(X, Y)\xi, N) = 0$$

for any $X, Y \in \Gamma(\sigma)$, $\xi \in \Gamma(Rad(TM)), N \in \Gamma(ltr(TM))$.

(c) There exists a vector subbundle M_2 on M such that

$$\bar{g}(\bar{R}(X, Y)W_1, W_2) = 0$$

for any $X, Y \in \Gamma(\sigma)$, $W_1, W_2 \in \Gamma(M_2)$, where M_2 is orthogonal to σ and $\bar{R}$ is the curvature tensor of $\bar{M}(c)$.

Proof. Let M be a contact STCR-lightlike submanifold of $\bar{M}(c)$, $c \neq 1$. From (a), $\sigma = \sigma_0 \oplus \sigma_1 \oplus \phi(\sigma_1)$ is maximal invariant subspaces. Next from (25), we have

$$\bar{g}(\bar{R}(X, Y)\xi, N) = \frac{-c+1}{2}\{g(\phi X, Y)\bar{g}(\phi \xi, N)\}$$

for any $X, Y \in \Gamma(\sigma)$, $\xi \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$. Since $g(\phi X, Y) \neq 0$ and $\bar{g}(\phi \xi, N) = 0$, we get $\bar{g}(\bar{R}(X, Y)\xi, N) = 0$. Thus (b) holds. Similarly, from (25) we get

$$\bar{g}(\bar{R}(X, Y)W_1, W_2) = \frac{-c+1}{2}\{g(\phi X, Y)\bar{g}(\phi W_1, W_2)\}$$

for any $X, Y \in \Gamma(\sigma)$, $W_1, W_2 \in \Gamma(M_2)$. Then (c) satisfies.

$\Longleftarrow$) : Conversely, we suppose that (a), (b) and (c) are holded. From (a), $\sigma = \sigma_0 \oplus \sigma_1 \oplus \phi(\sigma_1)$ is maximal invariant subspaces and $Rad(TM) = \sigma_1 \perp \sigma_2$, while $\phi(\sigma_1)$ is an invariant distribution on TM, σ_2 isn't invariant on TM with respect to ϕ. For this reason, $\phi(\sigma_2) \subset \Gamma(tr(TM))$. Hence, it is easy to see that $\phi(\sigma_1) \neq \sigma_2$ and $\phi(\sigma_1)$ is a distribution on $S(TM)$. Besides, for $ltr(TM) = L_1 \oplus L_2$ and $\xi_1 \in \Gamma(\sigma_1)$, $N_1 \in \Gamma(L_1)$ from (b) and (25) we get

$$\bar{g}(\phi\xi_1, N_1) = -\bar{g}(\xi_1, \phi N_1) = 0$$

which implies $\phi(L_1)$ is a distribution on $S(TM)$. It is easy to see that $\phi(\sigma_2) \neq L_1$ or $\phi(\sigma_2) \neq L_2$. Thus $\phi(\sigma_2)$ is a distribution on $S(TM^{\perp})$. Similarly, for any $\xi_2 \in \Gamma(\sigma_2)$ and $N_2 \in \Gamma(L_2)$, since $\bar{g}(\phi\xi_2, N_2) = -\bar{g}(\xi_2, \phi N_2) = 0$, then $\phi(L_2)$ is a distribution on $S(TM^{\perp})$, too. From (c), there exists a non-degenerate distribution M_2 such that $M_2 \perp \sigma$ and for any $X, Y \in \Gamma(\sigma)$, $W_1, W_2 \in \Gamma(M_2)$, we have

$$\bar{g}(\phi W_1, W_2) = 0.$$

This implies that $\phi(M_2) \perp M_2$. Also $\bar{g}(\phi\xi, W) = -\bar{g}(\xi, \phi W) = 0$ implies that $\phi(M_2) \perp Rad(TM)$. Furthermore, this say that $\phi(M_2)$ does not belong to $ltr(TM)$. Besides, since $\phi(M_2) \perp \sigma$ and σ is invariant, we write

$$\bar{g}(X, W) = \bar{g}(\phi X, W) = -\bar{g}(X, \phi W) = 0.$$

for any $X \in \Gamma(\sigma)$ and $W \in \Gamma(M_2)$, that is, $\phi(M_2)$ is orthogonal to σ, too. Hence, M_2 and $\phi(M_2)$ are distributions on $S(TM)$ and $S(TM^{\perp})$, respectively. Moreover, from a result in [2], we know that the structure vector field V belongs to $S(TM)$. Then summing up the above arguments, we conclude that

$$S(TM) = \{\phi(\sigma_1) \oplus \phi(L_1)\} \perp M_2 \perp \sigma_o \perp \{V\}.$$

Thus, M is a contact STCR-lightlike submanifold of $\bar{M}$. □

Theorem 3.7. *Let M be a contact STCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then*

(1) $\bar{\sigma}$ is integrable iff

$$A_{\phi X}Y = A_{\phi Y}X.$$

(2) $\sigma \perp \{V\}$ is integrable iff

$$h(X, \phi Y) = h(\phi X, Y).$$

(3) σ is not integrable.

Proof. From (40) we derive

$$-T\nabla_X Y = A_{\omega_L Y}X + A_{\omega_S Y}X + A_{\omega_{\bar{2}} Y}X + Bh(X, Y) - g(X, Y)V$$

for any $X, Y \in \Gamma(\bar{\sigma})$. Hence we have

$$T\,[X, Y] = -A_{\omega_L Y}X + A_{\omega_L X}Y - A_{\omega_S Y}X + A_{\omega_S X}Y - A_{\omega_{\bar{2}} Y}X + A_{\omega_{\bar{2}} X}Y$$

which proves assertion (1). From (41) and (42) we get

$$h(X, TY) = \omega_L \nabla_X Y + \omega_S \nabla_X Y + \omega_{\bar{2}} \nabla_X Y + Ch(X, Y)$$

for any $X, Y \in \Gamma(\sigma \perp \{V\})$. Hence we derive

$$h(X, TY) - h(Y, TX) = \omega_L\,[X, Y] + \omega_S\,[X, Y] + \omega_{\bar{2}}\,[X, Y]$$

which proves the assertion (2). Assume that σ is integrable. Then, we have $\bar{g}([X, Y], V) = 0$, for any $X, Y \in \Gamma(\sigma_0)$. Using that $\bar{\nabla}$ is metric connection and (23) we derive $g([X, Y], V) = 2g(\phi Y, X)$. Hence we have $\bar{g}(\phi Y, X) = 0$. Since σ_0 is non-degenerate, this is a contradiction. Thus σ is not integrable. □

Theorem 3.8. *Let M be a contact STCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, $\sigma\perp\{V\}$ is integrable iff the followings are holded:*

$$h^s(X,\phi Y) - h^s(Y,\phi X) \in \Gamma(\phi(L_2))$$

and

$$h^l(X,\phi Y) - h^l(Y,\phi X) \in \Gamma(L_2)$$

for any $X, Y \in \Gamma(\sigma\perp\{V\})$.

Proof. From definition of contact STCR-lightlike submanifolds, σ is integrable iff for any $X, Y \in \Gamma(\sigma\perp\{V\})$, $[X,Y] \in \Gamma(\sigma\perp\{V\})$,

$$\bar{g}([X,Y], N_2) = \bar{g}([X,Y], \phi\xi_1) = \bar{g}([X,Y], \phi W) = 0,$$

for any $X, Y \in \Gamma(\sigma\perp\{V\})$, $\xi_1 \in \Gamma(\sigma_1)$, $N_2 \in \Gamma(L_2)$ and $W \in \Gamma(S)$. Thus, for any $X, Y \in \Gamma(\sigma\perp\{V\})$, $\xi_1 \in \Gamma(\sigma_1)$, $N_2 \in \Gamma(L_2)$ and $W \in \Gamma(S)$, using (8), (19) and (24) we have

$$\bar{g}([X,Y], N_2) = \bar{g}(h^s(X,\phi Y) - h^s(Y,\phi X), \phi N_2), \tag{48}$$

$$\bar{g}([X,Y], \phi\xi_1) = \bar{g}(h^l(Y,\phi X) - h^l(X,\phi Y), \xi_1), \tag{49}$$

$$\bar{g}([X,Y], \phi W) = \bar{g}(h^s(Y,\phi X) - h^s(X,\phi Y), W). \tag{50}$$

Hence, the proof comes from (48)-(50). □

Theorem 3.9. *Let M be a contact STCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, $\bar{\sigma}$ is integrable iff*

$$A_{\phi X}Y - A_{\phi Y}X \in \Gamma(\tilde{\sigma})$$

for any $X, Y \in \Gamma(\bar{\sigma})$.

Proof. $\bar{\sigma}$ is integrable iff for any $X, Y \in \Gamma(\bar{\sigma})$, $[X,Y] \in \Gamma(\bar{\sigma})$, i.e.,

$$\bar{g}([X,Y], N_1) = \bar{g}([X,Y], \phi N_1) = \bar{g}([X,Y], Z) = \bar{g}([X,Y], V) = 0,$$

for any $X, Y \in \Gamma(\bar{\sigma})$, $Z \in \Gamma(\sigma_0)$ and $N_1 \in \Gamma(L_1)$. Thus, using (7), (19) and (24) we have

$$\bar{g}([X,Y], N_1) = \bar{g}(A_{\phi X}Y - A_{\phi Y}X, \phi N_1) \tag{51}$$

for any $X, Y \in \Gamma(\bar{\sigma})$ and $N_1 \in \Gamma(L_1)$. Similarly, using again (7), (19), (23) and (24) we derive

$$\bar{g}([X,Y], \phi N_1) = \bar{g}(A_{\phi Y}X - A_{\phi X}Y, N_1), \tag{52}$$

$$\bar{g}([X,Y], Z) = \bar{g}(A_{\phi X}Y - A_{\phi Y}X, \phi Z), \tag{53}$$

$$\bar{g}([X,Y], V) = 2\bar{g}(\phi Y, X) = 0 \tag{54}$$

for any $X, Y \in \Gamma(\bar{\sigma})$, $Z \in \Gamma(\sigma_0)$ and $N_1 \in \Gamma(L_1)$. Thus the proof follows from (51)-(54). □

4. STCR-Lightlike Product

Definition 4.1. *A STCR-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ is named STCR-lightlike product if both the distributions $\sigma \oplus \{V\}$ and $\bar{\sigma}$ define totally geodesic foliation in M.*

Theorem 4.2. *Let M be a contact STCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, $\sigma \perp \{V\}$ defines a totally geodesic foliation in M iff*

$$Bh(X, \phi Y) = 0$$

for any $X, Y \in \Gamma(\sigma \perp \{V\})$.

Proof. $\sigma \perp \{V\}$ defines a totally geodesic foliation in M iff

$$g(\nabla_X Y, \phi \xi_1) = g(\nabla_X Y, N_2) = g(\nabla_X Y, \phi W) = 0$$

for any $X, Y \in \Gamma(\sigma \perp \{V\})$, $\xi_1 \in \Gamma(\sigma_1)$, $N_2 \in \Gamma(L_2)$ and $W \in \Gamma(S)$. From (8), (19) and (24) we derive

$$g(\nabla_X Y, \phi \xi_1) = -\bar{g}(h^l(X, \phi Y), \xi_1), \tag{55}$$
$$g(\nabla_X Y, N_2) = \bar{g}(h^s(X, \phi Y), \phi N_2), \tag{56}$$
$$g(\nabla_X Y, \phi W) = -\bar{g}(h^s(X, \phi Y), W). \tag{57}$$

Thus from (55) we see that $h^l(X, \phi Y)$ has no components in L_1 and from (56) and (57) we see that $h^s(X, \phi Y)$ has no components in $\phi(\sigma_2) \perp S$, i.e., $Bh(X, \phi Y) = 0$. This completes the proof. □

Theorem 4.3. *Let M be a contact STCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, $\bar{\sigma}$ defines a totally geodesic foliation in M iff*

(i) $A_{N_1} X$ has no components in $\phi(\sigma_1) \perp \phi(S)$.
(ii) $A_{\phi Y} X$ has no components in $\sigma_o \perp \sigma_1$,
for any $X, Y \in \Gamma(\bar{\sigma})$ and $N_1 \in \Gamma(L_1)$.

Proof. $\bar{\sigma}$ defines a totally geodesic foliation in M iff

$$\bar{g}(\nabla_X Y, N_1) = g(\nabla_X Y, \phi N_1) = g(\nabla_X Y, Z) = g(\nabla_X Y, V) = 0$$

for any $X, Y \in \Gamma(\bar{\sigma})$, $N_1 \in \Gamma(L_1)$ and $Z \in \Gamma(\sigma_0)$. Since $\bar{\nabla}$ is a metric connection, (6), (9) and (24) imply

$$\bar{g}(\nabla_X Y, N_1) = g(A_{N_1} X, Y). \tag{58}$$

Using (6), (7), (19) and (24) we obtain

$$g(\nabla_X Y, \phi N_1) = g(A_{\phi Y} X, N_1), \tag{59}$$

$$g(\nabla_X Y, Z) = -g(A_{\phi Y} X, \phi Z). \tag{60}$$

Similarly, since $\bar{\nabla}$ is a metric connection and from (6) and (23), we derive

$$g(\nabla_X Y, V) = -\bar{g}(Y, \phi X) = 0. \tag{61}$$

Thus the proof comes from (58)-(61). □

Theorem 4.4. *Let M be a contact STCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. If $(\nabla_X T)Y = 0$, then M is a STCR lightlike product.*

Proof. Let $X, Y \in \Gamma(\bar{\sigma})$, hence $TY = 0$. Then using (40) with the hypothesis, we get $T\nabla_X Y = 0$. Thus $\nabla_X Y \in \Gamma(\bar{\sigma})$ i.e. $\bar{\sigma}$ defines a totally geodesic foliation in M. Let $X, Y \in \Gamma(\sigma \perp \{V\})$; hence $\omega Y = 0$. Then using (40), we derive $Bh(X, \phi Y) = 0$. From Theorem 4.2, $\sigma \perp \{V\}$ defines a totally geodesic foliation in M. Therefore, M is a STCR lightlike product. This completes the proof. □

Theorem 4.5. *Let M be an irrotational contact STCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, M is a STCR lightlike product if the following conditions are holded:*
i) $\nabla_X U \in \Gamma(S(TM^{\perp}))$, for any $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$.
ii) $A^*_{\xi} Y \in \Gamma(\phi(\sigma_1) \perp \phi(S))$, for any $Y \in \Gamma(\sigma \perp \{V\})$ and $\xi \in \Gamma(Rad(TM))$.

Proof. Let (i) holds, then using (9) and (10) we get $A_N X = 0$, $A_W X = 0$, $D^l(X, W) = 0$ and $\nabla^l_X N = 0$ for any $X \in \Gamma(TM)$, $N \in \Gamma(ltr\,(TM))$ and $W \in \Gamma(S\,(TM^{\perp}))$. Therefore for any $X, Y \in \Gamma(\sigma \perp \{V\})$ and $W \in \Gamma(S(TM^{\perp}))$ and using (11), we derive $\bar{g}(h^s(X, Y), W) = 0$. Since $S(TM^{\perp})$ is non-degenerate, $h^s(X, Y) = 0$. Therefore, $Bh^s(X, Y) = 0$. Since M is irrotational, using (13) and (ii) we derive $\bar{g}(h^l(X, Y), \xi) = \bar{g}(Y, A^*_{\xi} X) = 0$ for any $X, Y \in \Gamma(\sigma \perp \{V\})$ and $\xi \in \Gamma(Rad(TM))$. Thus, we derive $h^l(X, Y) = 0$. Hence $Bh^l(X, Y) = 0$. Then, from Theorem 4.2 the distribution $\sigma \perp \{V\}$ defines a totally geodesic foliation in M.

Next, for any $X, Y \in \Gamma(\bar{\sigma})$, then $\phi Y = \omega Y \in \Gamma(L_1 \perp S \perp \phi(\sigma_2)) \subset tr(TM)$. Using (40) we derive $T\nabla_X Y = -Bh(X, Y) + g(X, Y)V$, comparing the components along $\bar{\sigma}$, we get $T\nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(\bar{\sigma})$. Thus $\bar{\sigma}$ defines a totally geodesic foliation in M and M is a STCR-lightlike product. □

Definition 4.6. *[23] If the second fundamental form h of a submanifold tangent to characteristic vector field V, of an indefinite Sasakian manifold $\bar{M}$ is of the form*

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(X)\}\beta + \eta(X)h(Y, V) + \eta(Y)h(X, V) \tag{62}$$

for any $X, Y \in \Gamma(TM)$, where β is a vector field transversal to M, then M is named a totally contact umbilical submanifold and totally contact geodesic if $\beta = 0$.

Theorem 4.7. *Let M be a totally contact umbilical contact STCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is a STCR-lightlike product if $Bh(X, \phi Y) = 0$, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(\sigma \perp \{V\})$.*

Proof. Assume that $Bh(X, \phi Y) = 0$. Then $\sigma \perp \{V\}$ defines totally geodesic foliation in M for any $X, Y \in \Gamma(\sigma \perp \{V\})$. Using (40) we have

$$-T\nabla_X Y = A_{\omega Y} X + Bh(X, Y) - g(X, Y)V, \tag{63}$$

for any $X, Y \in \Gamma(\bar{\sigma})$. Using (7), (19), (24), (34) and (38) then equation (63) becomes

$$\begin{aligned} -g(T\nabla_X Y, Z) &= g(A_{\omega Y} X + Bh(X, Y) - g(X, Y)V, Z) \\ &= \bar{g}(\bar{\nabla}_X \phi Y, Z) \\ &= -\bar{g}(\bar{\nabla}_X Y, \phi Z) \\ &= \bar{g}(Y, \nabla_X Z') \end{aligned} \tag{64}$$

for any $Z \in \Gamma(\sigma_0)$, where $\phi Z = Z' \in \Gamma(\sigma_0)$. From (24), we obtain

$$\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z \tag{65}$$

for any $X, Y \in \Gamma(\bar{\sigma})$ and $Z \in \Gamma(\sigma_0)$. Using (6), (34), (38) and taking transversal part of resulting equation we derive

$$\omega Q \nabla_X Z = h(X, TZ) - Ch(X, Z). \tag{66}$$

Using (62), we derive $\omega Q \nabla_X Z = 0$, this implies $\nabla_X Z \in \Gamma(\sigma)$. Hence, (64) becomes $g(T\nabla_X Y, Z) = 0$. Since σ_0 is non-degenerate, $\bar{\sigma}$ defines a totally geodesic foliation in M. Hence the proof is proved. □

5. Minimal STCR-lightlike submanifolds

Definition 5.1. *We say that a lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if:*

(i) $h^s = 0$ on $Rad(TM)$ and
(ii) $trh = 0$, where trace is written with respect to g restricted to $S(TM)$.

It has been proved in [1] that the above definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on $tr(TM)$.

Example 5.2. *Consider a semi-Euclidean space $(\bar{M} = \mathbb{R}^{15}_4, \bar{g})$, where $\bar{g}$ is of signature $(-,-,+,+,+,+,+,-,-,+,+,+,+,+,+)$ with respect to canonical basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial z)$. Let M be a 9-dimensional submanifold of $\mathbb{R}^{15}_4$ given by*

$$\begin{aligned}
x^1 &= u^1, x^2 = u^2\cosh\beta, x^3 = u^1, x^4 = u^2\sinh\beta,\\
x^5 &= \cos u^3\cosh u^4, x^6 = \cos u^5\sinh u^6, x^7 = \sin u^5\sinh u^6,\\
y^1 &= u^7, y^2 = u^2\sinh\beta, y^3 = u^8, y^4 = u^2\cosh\beta,\\
y^5 &= \sin u^3\sinh u^4, y^6 = \cos u^5\cosh u^6, y^7 = \sin u^5\cosh u^6,\\
z &= u^9.
\end{aligned}$$

Then a local frame of TM is given by

$$\begin{aligned}
Z_1 &= \partial x_1 + \partial x_3,\\
Z_2 &= \cosh\beta\partial x_2 + \sinh\beta\partial x_4 + \sinh\beta\partial y_2 + \cosh\beta\partial y_4 + (y^2\cosh\beta + y^4\sinh\beta)\partial z,\\
Z_3 &= -\sin u^3\cosh u^4\partial x_5 + \cos u^3\sinh u^4\partial y_5 + (-y^5\sin u^3\cosh u^4)\partial z,\\
Z_4 &= \cos u^3\sinh u^4\partial x_5 + \sin u^3\cosh u^4\partial y_5 + (y^5\cos u^3\sinh u^4)\partial z,\\
Z_5 &= -\sin u^5\sinh u^6\partial x_6 + \cos u^5\sinh u^6\partial x_7 - \sin u^5\cosh u^6\partial y_6 + \cos u^5\cosh u^6\partial y_7\\
&\quad +(-y^5\sin u^5\sinh u^6 + y^6\cos u^5\sinh u^6)\partial z,\\
Z_6 &= \cos u^5\cosh u^6\partial x_6 + \sin u^5\cosh u^6\partial x_7 + \cos u^5\sinh u^6\partial y_6 + \sin u^5\sinh u^6\partial y_7\\
&\quad +(y^5\cos u^5\cosh u^6 + y^6\sin u^5\cosh u^6)\partial z,\\
Z_7 &= \partial y_1, Z_8 = \partial y_3, Z = 2\partial z = V.
\end{aligned}$$

Thus M is a 2-lightlike submanifold with $Rad(TM) = Span\{Z_1, Z_2\}$, $\phi_0(\sigma_1) = Span\{\phi_0(Z_1) = Z_7 + Z_8\}$, $\sigma_0 = Span\{Z_3, Z_4\}$ and it is easy to say that

$$\begin{aligned}
ltr(TM) &= Span\{N_1 = 2(-\partial x_1 + \partial x_3),\\
N_2 &= 2(-\cosh\beta\partial x_2 - \sinh\beta\partial x_4 + \sinh\beta\partial y_2 + \cosh\beta\partial y_4 + (-y^2\cosh\beta - y^4\sinh\beta)\partial z)\},\\
\phi_0(N_1) &= 2(Z_7 - Z_8), S(TM^{\perp}) = Span\{\phi_0(Z_2), \phi_0(N_2), \phi_0(Z_5), \phi_0(Z_6)\}.
\end{aligned}$$

Hence, M is a proper contact STCR-lightlike submanifold of $\mathbb{R}^{15}_4$, with a quasi-orthonormal basis of $\bar{M}$ along M is

$$\begin{aligned}
\{\xi_1 &= Z_1, \xi_2 = Z_2, \phi_0(\xi_1) = -Z_7 - Z_8, \phi_0(N_1) = 2(Z_7 - Z_8),\\
e_1 &= \frac{1}{\sqrt{\cosh^2 u^4 - \cos^2 u^3}} Z_3, e_2 = \frac{1}{\sqrt{\cosh^2 u^4 - \cos^2 u^3}} Z_4,\\
e_3 &= \frac{1}{\sqrt{\sinh^2 u^6 + \cosh^2 u^6}} Z_5, e_4 = \frac{1}{\sqrt{\sinh^2 u^6 + \cosh^2 u^6}} Z_6, V = Z_{10},\\
W_1 &= \phi_0(\xi_2), W_2 = \phi_0(N_2), W_3 = \frac{1}{\sqrt{\sinh^2 u^6 + \cosh^2 u^6}}\phi_0(Z_5),\\
W_4 &= \frac{1}{\sqrt{\sinh^2 u^6 + \cosh^2 u^6}}\phi_0(Z_6), N_1, N_2,
\end{aligned}$$

where $\varepsilon_1 = g(e_1, e_1) = 1$, $\varepsilon_2 = g(e_2, e_2) = 1$, $\varepsilon_3 = g(e_3, e_3) = 1$ *and* $\varepsilon_4 = g(e_4, e_4) = 1$. *Using (8), we get*

$$\begin{aligned} h(\xi_1, \xi_1) &= h(\xi_2, \xi_2) = h(e_1, e_1) = h(e_2, e_2) = 0, \\ h(\phi_0(\xi_1), \phi_0(\xi_1)) &= h(\phi_0(N_1), \phi_0(N_1)) = h^l(e_3, e_3) = h^l(e_4, e_4) = 0, \\ h^s(e_3, e_3) &= \frac{1}{\sinh^2 u^6 + \cosh^2 u^6} Z_4, \; h^s(e_4, e_4) = -\frac{1}{\sinh^2 u^6 + \cosh^2 u^6} Z_4. \end{aligned}$$

Thus

$$trace h_{g|S(TM)} = \epsilon_3 h^s(e_3, e_3) + \epsilon_4 h^s(e_4, e_4) = h^s(e_3, e_3) + h^s(e_4, e_4) = 0.$$

Hence M is a minimal proper contact STCR-lightlike submanifold of $\mathbb{R}^{15}_4$.

Let take a quasi-orthonormal frame

$$\{\xi_1, ..., \xi_q, e_1, ..., e_m, V, W_1, ..., \dot{W}_n, N_1, ..., N_q\}$$

such that $(\xi_1, ..., \xi_q, e_1, ..., e_m, V)$ belongs to $\Gamma(TM)$. Then take $(\xi_1, ..., \xi_q, e_1, ..., e_m)$ such that $\{\xi_1, ..., \xi_p\}$ form a basis of σ_1, $\{\xi_{p+1}, ..., \xi_q\}$ form a basis of σ_2 and $\{e_1, ..., e_{2s}\}$ form a basis of σ_0. Besides, we take $\{W_1, ..., W_k\}$ a basis of S, $\{N_1, ..., N_p\}$ a basis of L_1 and $\{N_{p+1}, ..., N_q\}$ a basis of L_2. Hence we have a quasi-orthonormal basis of M as follows:

$$\{\xi_1, ..., \xi_p, \xi_{p+1}, ..., \xi_r, e_1, ..., e_l, \phi e_1, ..., \phi e_l, \phi\xi_1, ..., \phi\xi_p, \phi N_1, ..., \phi N_p, \phi W_1, ..., \phi W_k\}.$$

Theorem 5.3. *Let M be a proper contact STCR-lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$. *Then M is minimal iff*

$$trace A_{W_j|S(TM)} = 0, \; trace A^*_{\xi_q|S(TM)} = 0 \tag{67}$$

and $\bar{g}(Y, D^l(X, W)) = 0$ *for any* $X, Y \in \Gamma(Rad(TM))$ *and* $W \in \Gamma(S(TM^\perp))$.

Proof. We know that $h^l = 0$ on $Rad(TM)$ [1]. Definition of a contact STCR-lightlike submanifold, M is minimal iff

$$\sum_{j=1}^{2s} \epsilon_j h(e_j, e_j) + \sum_{j=1}^{p} h(\phi\xi_j, \phi\xi_j) + \sum_{j=1}^{p} h(\phi N_j, \phi N_j) + \sum_{\alpha=1}^{k} \epsilon_\alpha h(\phi W_\alpha, \phi W_\alpha) = 0.$$

Now from (11), we have $h^s = 0$ on $Rad(TM)$ iff $\bar{g}(Y, D^l(X, W)) = 0$, for any $X, Y \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$. Besides, we derive

$$\begin{aligned} trace h \;|_{S(TM)} &= \frac{1}{r} \sum_{q=1}^{r} \sum_{j=1}^{p} \bar{g}(h^l(\phi\xi_j, \phi\xi_j), \xi_q) N_q + \bar{g}(h^l(\phi N_j, \phi N_j), \xi_q) N_q \\ &+ \frac{1}{n-r} \sum_{j=1}^{p} \sum_{\beta=1}^{n-r} \epsilon_\beta \{\bar{g}(h^s(\phi\xi_j, \phi\xi_j), W_\beta) W_\beta + \bar{g}(h^s(\phi N_j, \phi N_j), W_\beta) W_\beta\} \\ &+ \sum_{\beta=1}^{n-r} \epsilon_\beta \frac{1}{n-r} \{\sum_{j=1}^{2s} \bar{g}(h^s(e_j, e_j), W_\beta) W_\beta + \sum_{\alpha=1}^{k} \bar{g}(h^s(\phi W_\alpha, \phi W_\alpha), W_\beta) W_\beta\} \\ &+ \sum_{q=1}^{r} \frac{1}{r} \{\sum_{j=1}^{2s} \bar{g}(h^l(e_j, e_j), \xi_q) N_q + \sum_{\alpha=1}^{k} \bar{g}(h^l(\phi W_\alpha, \phi W_\alpha), \xi_q) N_q\}. \end{aligned} \tag{68}$$

Using (11) and (16) in (68), we get

$$\begin{aligned} traceh \mid_{S(TM)} &= \frac{1}{r}\sum_{q=1}^{r}\sum_{j=1}^{p} g(A^*_{\xi_q}\phi\xi_j,\phi\xi_j)N_q + g(A^*_{\xi_q}\phi N_j,\phi N_j)N_q \\ &+\frac{1}{n-r}\sum_{j=1}^{p}\sum_{\beta=1}^{n-r}\epsilon_\beta\{g(A_{W_\beta}\phi\xi_j,\phi\xi_j)W_\beta + g(A_{W_\beta}\phi N_j,\phi N_j)W_\beta\} \\ &+\sum_{\beta=1}^{n-r}\epsilon_\beta\frac{1}{n-r}\{\sum_{j=1}^{2s} g(A_{W_\beta}e_j,e_j)W_\beta + \sum_{\alpha=1}^{k} g(A_{W_\beta}\phi W_\alpha,\phi W_\alpha)W_\beta\} \\ &+\sum_{q=1}^{r}\frac{1}{r}\{\sum_{j=1}^{2s} g(A^*_{\xi_q}e_j,e_j)N_q + \sum_{\alpha=1}^{k} g(A^*_{\xi_q}\phi W_\alpha,\phi W_\alpha)N_q\}. \end{aligned} \tag{69}$$

Equation (69) completes the proof. □

Theorem 5.4. *A totally umbilical STCR-lightlike submanifold M is minimal iff*

$$traceA_{W_\beta}\mid_{\sigma_0\perp\phi(S)} = traceA^*_{\xi_q}\mid_{\sigma_0\perp\phi(S)} = 0 \tag{70}$$

for any $\xi_q \in \Gamma(Rad(TM))$ *and* $W_\beta \in \Gamma(S(TM^\perp))$*, where* $k \in \{1,2,...,r\}$ *and* $\beta \in \{1,2,...,n-r\}$*.*

Proof. M is minimal iff $h^s = 0$ on $Rad(TM)$ and $traceh = 0$ on $S(TM)$, i.e.

$$\begin{aligned} traceh \mid_{S(TM)} &= traceh\mid_{\sigma_0} + traceh\mid_{\phi(\sigma_1)} + traceh\mid_{\phi(L_1)} + traceh\mid_{\phi(S)} \\ &= \sum_{j=1}^{2s}\epsilon_j h(e_j,e_j) + \sum_{j=1}^{p} h(\phi\xi_j,\phi\xi_j) + \sum_{j=1}^{p} h(\phi N_j,\phi N_j) + \sum_{\alpha=1}^{k}\epsilon_\alpha h(\phi W_\alpha,\phi W_\alpha). \end{aligned} \tag{71}$$

Using (62) in (71) we derive

$$\begin{aligned} traceh \mid_{S(TM)} &= traceh\mid_{\sigma_0} + traceh\mid_{\phi(S)} \\ &= \sum_{j=1}^{2s}\epsilon_j h(e_j,e_j) + \sum_{\alpha=1}^{k}\epsilon_l h(\phi W_\alpha,\phi W_\alpha) \\ &= \sum_{j=1}^{2s}\epsilon_j (h^l(e_j,e_j) + h^s(e_j,e_j)) + \sum_{\alpha=1}^{k}\epsilon_l (h^l(\phi\dot{W}_\alpha,\phi W_\alpha) + h^s(\phi W_\alpha,\phi W_\alpha)) \\ &= \sum_{q=1}^{r}\frac{1}{r}\{\sum_{j=1}^{2s}\bar{g}(h^l(e_j,e_j),\xi_q)N_q + \sum_{\alpha=1}^{k}\bar{g}(h^l(\phi W_\alpha,\phi W_\alpha),\xi_q)N_q\} \\ &+\sum_{\beta=1}^{n-r}\epsilon_\beta\frac{1}{n-r}\{\sum_{j=1}^{2s}\bar{g}(h^s(e_j,e_j),W_\beta)W_\beta + \sum_{\alpha=1}^{k}\bar{g}(h^s(\phi W_\alpha,\phi W_\alpha),W_\beta)W_\beta\} \end{aligned} \tag{72}$$

Besides, if we consider (11) and (16) in (72), we obtain

$$\begin{aligned} traceh \mid_{S(TM)} &= \sum_{q=1}^{r}\frac{1}{r}\{\sum_{j=1}^{2s} g(A^*_{\xi_q}e_j,e_j)N_q + \sum_{\alpha=1}^{k} g(A^*_{\xi_q}\phi W_\alpha,\phi W_\alpha)N_q\} \\ &+\sum_{\beta=1}^{n-r}\epsilon_\beta\frac{1}{n-r}\{\sum_{j=1}^{2s} g(A_{W_\beta}e_j,e_j)W_\beta + \sum_{\alpha=1}^{k} g(A_{W_\beta}\phi W_\alpha,\phi W_\alpha)W_\beta\} \\ &= 0 \end{aligned}$$

which completes the proof. □

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