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The maximum and minimum value of homogeneous polynomial under different norms via tensors

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Abstract. For any homogeneous polynomial, it can be expressed as the product of a tensor $\mathcal{A}$ and a vector x, we denote it by $P_{\mathcal{A}}(x)$. With the change of the norm of x, the maximum value (resp. the minimum value) of $P_{\mathcal{A}}(x)$ is changed. In this paper, by the properties of tensor $\mathcal{A}$, we study the relationships between the maximum values (resp. minimum values) of $P_{\mathcal{A}}(x)$ under different norms of x. We present that the maximum values (resp. the minimum values) of $P_{\mathcal{A}}(x)$ at different norms of x always have the same sign. Moreover, the relationship between the magnitudes of the maximum values (resp. the minimum values) of $P_{\mathcal{A}}(x)$ at different norms of x are characterized. Further, some inequalities on H-eigenvalues and Z-eigenvalues of tensor $\mathcal{A}$ are obtained directly. And some applications on definite positive of tensors and hypergraphs are given.

1. Introduction

Tensors have been widely applied to many fields, such as signal processing [1], computing vision [2] and statistical data analysis [3] etc. Let $[n] = \{1, 2, \ldots, n\}$, n is a positive integer. A k-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_k})$ $(i_j \in [n],\ j \in [k])$ is a multidimensional array with n^k entries. If $a_{i_1 i_2 \cdots i_k} \geq 0$ for all $i_j \in [n],\ j \in [k]$, then $\mathcal{A}$ is called *nonnegative*. If $a_{i_1 i_2 \cdots i_k} = a_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)}$, where σ is any permutation of the indices $i_1, \ldots, i_k$, then $\mathcal{A}$ is called *symmetric*. Let $\mathbb{C}^{[k,n]}$ and $\mathbb{R}^{[k,n]}$ denote the set of k-order n-dimensional complex tensors and real tensors, respectively.

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_k}) \in \mathbb{C}^{[k,n]}$ and $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$. $\mathcal{A}x^{k-1}$ is an n-dimensional vector (see [4]), whose i-th component is

$$(\mathcal{A}x^{k-1})_i = \sum_{i_2,\ldots,i_k=1}^{n} a_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k},\ i \in [n].$$

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For any kth degree real coefficient homogeneous polynomial, it can be expressed as

$$P_{\mathcal{A}}(x) := x^{\mathrm{T}}(\mathcal{A}x^{k-1}) = \sum_{i_1,i_2,\ldots,i_k=1}^{n} a_{i_1i_2\cdots i_k}x_{i_1}x_{i_2}\cdots x_{i_k},$$

where $\mathcal{A} = (a_{i_1i_2\cdots i_k}) \in \mathbb{R}^{[k,n]}$ and $x = (x_1,\ldots,x_n)^{\mathrm{T}}$. Note that the tensor $\mathcal{A}$ isn't unique, it maybe symmetric or maybe not. By analyzing the critical value and positive definite of $P_{\mathcal{A}}(x)$ under different norms of real vector x, the (H/Z-)eigenvalues of tensor $\mathcal{A}$ were proposed and studied [4, 5]. After that, eigenvalues of tensors have attracted much attention and have been found wide applications in quantum physics, higher order Markov chains, spectral hypergraph theory and automatic control etc (see [6]-[19]).

For tensor $\mathcal{A} \in \mathbb{R}^{[k,n]}$ and $x = (x_1,\ldots,x_n)^{\mathrm{T}} \in \mathbb{R}^n$, let

$$\lambda^{(p)}_{\max}(\mathcal{A}) = \max\{P_{\mathcal{A}}(x) : x \in \mathbb{R}^n \textit{ and } \|x\|_p = 1\},$$

$$\lambda^{(p)}_{\min}(\mathcal{A}) = \min\{P_{\mathcal{A}}(x) : x \in \mathbb{R}^n \textit{ and } \|x\|_p = 1\},$$

where $\|x\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$ is the p-norm of x. Thus, $\lambda^{(p)}_{\max}(\mathcal{A})$ and $\lambda^{(p)}_{\min}(\mathcal{A})$ denote the maximum and minimum value of $P_{\mathcal{A}}(x)$ under p-norm of real vector x, respectively. For some tensors, $\lambda^{(k)}_{\max}(\mathcal{A})$ and $\lambda^{(k)}_{\min}(\mathcal{A})$ (resp. $\lambda^{(2)}_{\max}(\mathcal{A})$ and $\lambda^{(2)}_{\min}(\mathcal{A})$) are exactly the largest and smallest H-eigenvalue (resp. Z-eigenvalue) of $\mathcal{A}$, respectively, such as the even order real symmetric tensors [4].

In this paper, we study the maximum and minimum value of $P_{\mathcal{A}}(x)$ under different norms of real vector x. We present that $\lambda^{(p)}_{\max}(\mathcal{A})$ and $\lambda^{(q)}_{\max}(\mathcal{A})$ (resp. $\lambda^{(p)}_{\min}(\mathcal{A})$ and $\lambda^{(q)}_{\min}(\mathcal{A})$) always have the same sign for any $p, q \geq 1$. In other words, for any $p, q \geq 1$, $\lambda^{(p)}_{\max}(\mathcal{A})$ and $\lambda^{(q)}_{\max}(\mathcal{A})$ (resp. $\lambda^{(p)}_{\min}(\mathcal{A})$ and $\lambda^{(q)}_{\min}(\mathcal{A})$) are either both greater than zero or both less than zero, or both equal to zero. Furthermore, when $1 \leq q \leq p$, we obtain that

$$n^{\frac{k}{p}-\frac{k}{q}} \leq \frac{\lambda^{(q)}_{\max}(\mathcal{A})}{\lambda^{(p)}_{\max}(\mathcal{A})} \leq 1 \ \text{ and } \ n^{\frac{k}{p}-\frac{k}{q}} \leq \frac{\lambda^{(q)}_{\min}(\mathcal{A})}{\lambda^{(p)}_{\min}(\mathcal{A})} \leq 1$$

if $\lambda^{(p)}_{\max}(\mathcal{A}) \neq 0$, $\lambda^{(p)}_{\min}(\mathcal{A}) \neq 0$. By the above inequalities, some relationships between H-eigenvalues and Z-eigenvalues of tensors are obtained. Thus, some results on Z-eigenvalus can be got directly from the results on the H-eigenvalues, and the vice verse. Moreover, some applications on definite positive of tensors and hypergraphs are given.

2. Preliminaries

In this section, we list some helpful notions and lemmas.

Let $\mathcal{A} \in \mathbb{C}^{[k,n]}$. If there exist $\lambda \in \mathbb{C}$ and a nonzero vector $x = (x_1,\ldots,x_n)^{\mathrm{T}} \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{k-1} = \lambda x^{[k-1]},$$

then λ is called an *eigenvalue* of $\mathcal{A}$ and x is called an *eigenvector* of $\mathcal{A}$ corresponding to λ, where $x^{[k-1]} = (x_1^{k-1},\ldots,x_n^{k-1})^{\mathrm{T}}$. Further, if $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then λ is called an *H-eigenvalue* of $\mathcal{A}$ and x is called the corresponding *H-eigenvector*. Let $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$ be the *spectral radius* of $\mathcal{A}$. If there exist $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$\mathcal{A}x^{k-1} = \lambda x \ \text{ and } \ x^{\mathrm{T}}x = 1,$$

then λ is called a *Z-eigenvalue* of $\mathcal{A}$ and x is called the corresponding *Z-eigenvector*. Let $\rho_Z(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is a Z-eigenvalue of } \mathcal{A}\}$ denote the *Z-spectral radius* of $\mathcal{A}$ if $\mathcal{A}$ has Z-eigenvalues.

For a tensor $\mathcal{A} \in \mathbb{C}^{[k,n]}$, it maybe doesn't have the H-eigenvalues and Z-eigenvalues. But in the following cases, it always has the H-eigenvalues and Z-eigenvalues.

Lemma 2.1. *[4, 20] Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ be an even order symmetric tensor. Then $\mathcal{A}$ always has H-eigenvalues and Z-eigenvalues, and $\lambda^{(k)}_{\max}(\mathcal{A})$, $\lambda^{(k)}_{\min}(\mathcal{A})$, $\lambda^{(2)}_{\max}(\mathcal{A})$ and $\lambda^{(2)}_{\min}(\mathcal{A})$ are exactly the largest and the smallest H-eigenvalue, the largest and the smallest Z-eigenvalue, respectively.*

Lemma 2.2. *[21–23] Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ be a nonnegative symmetric tensor. Then $\rho(\mathcal{A})$ and $\rho_Z(\mathcal{A})$ are the H-eigenvalue and Z-eigenvalue of $\mathcal{A}$, respectively. And*

$$\rho_Z(\mathcal{A}) = \lambda^{(2)}_{\max}(\mathcal{A}),\ \ \rho(\mathcal{A}) = \lambda^{(k)}_{\max}(\mathcal{A}).$$

A real symmetric tensor $\mathcal{A} \in \mathbb{R}^{[k,n]}$ is called *positive (semi-)definite* if $x^{\mathrm{T}}(\mathcal{A}x^{k-1}) > 0\ (\geq 0)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ [4]. Note that there are no positive (semi-)definite tensors when k is odd.

Lemma 2.3. *[4] Let $\mathcal{A}$ be an even order real symmetric tensor. Then $\mathcal{A}$ is positive definite (resp. positive semi-definite) if and only if its all H-eigenvalues or all Z-eigenvalues are positive (resp. nonnegative).*

Lemma 2.4. *Let $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $1 \leq q \leq p$. Then*

$$\left(\frac{\sum\limits_{i=1}^{n} |x_i|^q}{n}\right)^{\frac{1}{q}} \leq \left(\frac{\sum\limits_{i=1}^{n} |x_i|^p}{n}\right)^{\frac{1}{p}}.$$

Proof. Without loss of generality, suppose that $x_1, x_2, \dots, x_t \neq 0$ and $x_{t+1} = \cdots = x_n = 0$. Then the Power Mean Inequality implies that

$$\left(\frac{\sum\limits_{i=1}^{t} |x_i|^q}{t}\right)^{\frac{1}{q}} \leq \left(\frac{\sum\limits_{i=1}^{t} |x_i|^p}{t}\right)^{\frac{1}{p}}.$$

Since $\left(\frac{t}{n}\right)^{\frac{1}{q}} \leq \left(\frac{t}{n}\right)^{\frac{1}{p}}$, we get $\left(\frac{\sum_{i=1}^{t} |x_i|^q}{t}\right)^{\frac{1}{q}} \left(\frac{t}{n}\right)^{\frac{1}{q}} \leq \left(\frac{\sum_{i=1}^{t} |x_i|^p}{t}\right)^{\frac{1}{p}} \left(\frac{t}{n}\right)^{\frac{1}{p}}$. Thus,

$$\left(\frac{(\sum\limits_{i=1}^{t} |x_i|^q) + x_{t+1}^q + \cdots + x_n^q}{n}\right)^{\frac{1}{q}} \leq \left(\frac{(\sum\limits_{i=1}^{t} |x_i|^p) + x_{t+1}^q + \cdots + x_n^q}{n}\right)^{\frac{1}{p}}.$$

This finishes the proof. □

3. The maximum value of homogeneous polynomials under the different norms

In this section, we give some relationships on the $\lambda^{(p)}_{\max}(\mathcal{A})$ and $\lambda^{(q)}_{\max}(\mathcal{A})$ when $1 \leq q \leq p$. Furthermore, we get the conclusions on the largest H-eigenvalue and the largest Z-eigenvalue of tensors.

Theorem 3.1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_k}) \in \mathbb{R}^{[k,n]}$ and $1 \leq q \leq p$. Then the following cases hold.*
(1) If $\lambda^{(q)}_{\max}(\mathcal{A}) \geq 0$, then $0 \leq n^{\frac{k}{p} - \frac{k}{q}} \lambda^{(p)}_{\max}(\mathcal{A}) \leq \lambda^{(q)}_{\max}(\mathcal{A}) \leq \lambda^{(p)}_{\max}(\mathcal{A})$.
(2) If $\lambda^{(q)}_{\max}(\mathcal{A}) < 0$, then $0 > n^{\frac{k}{p} - \frac{k}{q}} \lambda^{(p)}_{\max}(\mathcal{A}) \geq \lambda^{(q)}_{\max}(\mathcal{A}) \geq \lambda^{(p)}_{\max}(\mathcal{A})$.

Proof. (1) *a*) When $\lambda_{\max}^{(q)}(\mathcal{A}) \geq 0$, we firstly prove that $\lambda_{\max}^{(q)}(\mathcal{A}) \leq \lambda_{\max}^{(p)}(\mathcal{A})$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be the vector such that $\lambda_{\max}^{(q)}(\mathcal{A}) = x^{\mathrm{T}}(\mathcal{A}x^{k-1})$. Then we have $\|x\|_q = 1$ and $\|x\|_p^p = \sum_{i=1}^{n} |x_i|^p \leq \sum_{i=1}^{n} |x_i|^q = 1$. Thus,

$$\begin{aligned} 0 \leq \lambda_{\max}^{(q)}(\mathcal{A}) &= x^{\mathrm{T}}(\mathcal{A}x^{k-1}) \\ &= \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} x_{i_1} \cdots x_{i_k} \\ &= \|x\|_p^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} \frac{x_{i_1}}{\|x\|_p} \cdots \frac{x_{i_k}}{\|x\|_p}. \end{aligned} \tag{3.1}$$

Let $y = (y_1, \ldots, y_n)^{\mathrm{T}} = (\frac{x_1}{\|x\|_p}, \ldots, \frac{x_n}{\|x\|_p})^{\mathrm{T}}$. So, $y \in \mathbb{R}^n$ and $\|y\|_p = 1$. By (3.1), we get

$$0 \leq \lambda_{\max}^{(q)}(\mathcal{A}) = \|x\|_p^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} y_{i_1} \cdots y_{i_k} = \|x\|_p^k y^{\mathrm{T}}(\mathcal{A}y^{k-1}) \leq \|x\|_p^k \lambda_{\max}^{(p)}(\mathcal{A}).$$

From $\|x\|_p^k \leq 1$, so $0 \leq \lambda_{\max}^{(q)}(\mathcal{A}) \leq \lambda_{\max}^{(p)}(\mathcal{A})$.

b) Next, we prove that $0 \leq n^{\frac{k}{p}-\frac{k}{q}} \lambda_{\max}^{(p)}(\mathcal{A}) \leq \lambda_{\max}^{(q)}(\mathcal{A})$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be the vector such that $\lambda_{\max}^{(p)}(\mathcal{A}) = x^{\mathrm{T}}(\mathcal{A}x^{k-1})$. Then $\|x\|_p = 1$. And Lemma 2.4 implies that $\left(\frac{\sum_{i=1}^{n} |x_i|^q}{n}\right)^{\frac{1}{q}} \leq \left(\frac{\sum_{i=1}^{n} |x_i|^p}{n}\right)^{\frac{1}{p}}$, i.e., $n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q \leq \|x\|_p = 1$. Let $y = n^{\frac{1}{p}-\frac{1}{q}} x$, so $\|y\|_q = n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q \leq 1$. By *a*), we get that if $\lambda_{\max}^{(q)}(\mathcal{A}) \geq 0$, then $\lambda_{\max}^{(p)}(\mathcal{A}) \geq 0$. Thus,

$$\begin{aligned} 0 \leq \lambda_{\max}^{(p)}(\mathcal{A}) &= \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} x_{i_1} \cdots x_{i_k} \\ &= (n^{\frac{1}{q}-\frac{1}{p}})^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} (n^{\frac{1}{p}-\frac{1}{q}} x_{i_1}) \cdots (n^{\frac{1}{p}-\frac{1}{q}} x_{i_k}) \\ &= (n^{\frac{1}{q}-\frac{1}{p}})^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} y_{i_1} \cdots y_{i_k} \\ &= \|y\|_q^k (n^{\frac{1}{q}-\frac{1}{p}})^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} \frac{y_{i_1}}{\|y\|_q} \cdots \frac{y_{i_k}}{\|y\|_q}. \end{aligned} \tag{3.2}$$

Let $z = (z_1, \ldots, z_n)^{\mathrm{T}} = (\frac{y_1}{\|y\|_q}, \ldots, \frac{y_n}{\|y\|_q})^{\mathrm{T}}$. So, $z \in \mathbb{R}^n$ and $\|z\|_q = 1$. By (3.2), we get

$$0 \leq \lambda_{\max}^{(p)}(\mathcal{A}) = \|y\|_q^k (n^{\frac{1}{q}-\frac{1}{p}})^k z^{\mathrm{T}}(\mathcal{A}z^{k-1}) \leq \|y\|_q^k (n^{\frac{1}{q}-\frac{1}{p}})^k \lambda_{\max}^{(q)}(\mathcal{A}).$$

From $\|y\|_q^k \leq 1$, so $0 \leq n^{\frac{k}{p}-\frac{k}{q}} \lambda_{\max}^{(p)}(\mathcal{A}) \leq \lambda_{\max}^{(q)}(\mathcal{A})$.

It follows from *a*) and *b*) that the statement (1) holds.

(2) *c*) When $\lambda_{\max}^{(q)}(\mathcal{A}) < 0$, we firstly prove that $\lambda_{\max}^{(q)}(\mathcal{A}) \geq \lambda_{\max}^{(p)}(\mathcal{A})$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be the vector such that $\lambda_{\max}^{(p)}(\mathcal{A}) = x^{\mathrm{T}}(\mathcal{A}x^{k-1})$. Then we have $\|x\|_p = 1$ and $\|x\|_q^q = \sum_{i=1}^{n} |x_i|^q \geq \sum_{i=1}^{n} |x_i|^p = 1$. Thus,

$$\lambda_{\max}^{(p)}(\mathcal{A}) = \|x\|_q^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} \frac{x_{i_1}}{\|x\|_q} \cdots \frac{x_{i_k}}{\|x\|_q}. \tag{3.3}$$

Let $y = (y_1, \ldots, y_n)^{\mathrm{T}} = (\frac{x_1}{\|x\|_q}, \ldots, \frac{x_n}{\|x\|_q})^{\mathrm{T}}$. So, $y \in \mathbb{R}^n$ and $\|y\|_q = 1$. By (3.3), we get

$$\lambda_{\max}^{(p)}(\mathcal{A}) = \|x\|_q^k y^{\mathrm{T}}(\mathcal{A}y^{k-1}) \leq \|x\|_q^k \lambda_{\max}^{(q)}(\mathcal{A}) < 0.$$

From $\|x\|_q^k \geq 1$, we have $\lambda_{\max}^{(p)}(\mathcal{A}) \leq \lambda_{\max}^{(q)}(\mathcal{A}) < 0$.

d) Next, we prove that $0 > n^{\frac{k}{p}-\frac{k}{q}} \lambda_{\max}^{(p)}(\mathcal{A}) \geq \lambda_{\max}^{(q)}(\mathcal{A})$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be the vector such that $\lambda_{\max}^{(q)}(\mathcal{A}) = x^{\mathrm{T}}(\mathcal{A}x^{k-1})$. Then $\|x\|_q = 1$. And Lemma 2.4 implies that $1 = \|x\|_q \leq n^{\frac{1}{q}-\frac{1}{p}}\|x\|_p$. Let $y = n^{\frac{1}{q}-\frac{1}{p}}x$, so $\|y\|_p = n^{\frac{1}{q}-\frac{1}{p}}\|x\|_p \geq 1$. Similar to the (3.2), we have

$$\lambda_{\max}^{(q)}(\mathcal{A}) = \|y\|_p^k (n^{\frac{1}{p}-\frac{1}{q}})^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} \frac{y_{i_1}}{\|y\|_p} \cdots \frac{y_{i_k}}{\|y\|_p}. \tag{3.4}$$

Let $z = (z_1, \ldots, z_n)^{\mathrm{T}} = (\frac{y_1}{\|y\|_p}, \ldots, \frac{y_n}{\|y\|_p})^{\mathrm{T}}$. So, $z \in \mathbb{R}^n$ and $\|z\|_p = 1$. By *c*), we get that if $\lambda_{\max}^{(q)}(\mathcal{A}) < 0$, then $\lambda_{\max}^{(p)}(\mathcal{A}) < 0$. By (3.4), we get

$$\lambda_{\max}^{(q)}(\mathcal{A}) = \|y\|_p^k (n^{\frac{1}{p}-\frac{1}{q}})^k z^{\mathrm{T}}(\mathcal{A}z^{k-1}) \leq \|y\|_p^k (n^{\frac{1}{p}-\frac{1}{q}})^k \lambda_{\max}^{(p)}(\mathcal{A}) < 0.$$

From $\|y\|_p^k \geq 1$, we have $0 > n^{\frac{k}{p}-\frac{k}{q}} \lambda_{\max}^{(p)}(\mathcal{A}) \geq \lambda_{\max}^{(q)}(\mathcal{A})$.

It follows from *c*) and *d*) the statement (2) holds. □

Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ and $1 \leq q \leq p$. From Theorem 3.1 (1), we directly get that if $\lambda_{\max}^{(q)}(\mathcal{A}) > 0$, then $\lambda_{\max}^{(p)}(\mathcal{A}) > 0$, and if $\lambda_{\max}^{(p)}(\mathcal{A}) > 0$, then $\lambda_{\max}^{(q)}(\mathcal{A}) > 0$. From Theorem 3.1 (2), we get that if $\lambda_{\max}^{(q)}(\mathcal{A}) < 0$, then $\lambda_{\max}^{(p)}(\mathcal{A}) < 0$, and if $\lambda_{\max}^{(p)}(\mathcal{A}) < 0$, then $\lambda_{\max}^{(q)}(\mathcal{A}) < 0$. Thus, we have the following result.

Theorem 3.2. *Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ and $1 \leq q \leq p$. Then $\lambda_{\max}^{(p)}(\mathcal{A})$ and $\lambda_{\max}^{(q)}(\mathcal{A})$ always have the same sign for any $p, q \geq 1$. In other words, for any $p, q \geq 1$, $\lambda_{\max}^{(p)}(\mathcal{A})$ and $\lambda_{\max}^{(q)}(\mathcal{A})$ are either both greater than zero or both less than zero, or both equal to zero.*

By Theorem 3.1 and 3.2, we have the following corollary.

Corollary 3.3. *For real tensor $\mathcal{A} \in \mathbb{R}^{[k,n]}$ and $1 \leq q \leq p$, $\lambda_{\max}^{(p)}(\mathcal{A})$ and $\lambda_{\max}^{(q)}(\mathcal{A})$ always have the same sign for any $p, q \geq 1$. And the following inequalities hold*

$$n^{\frac{k}{p}-\frac{k}{q}} \leq \frac{\lambda_{\max}^{(q)}(\mathcal{A})}{\lambda_{\max}^{(p)}(\mathcal{A})} \leq 1,$$

if $\lambda_{\max}^{(p)}(\mathcal{A}) \neq 0$.

For the even order symmetric tensor $\mathcal{A}$, by Lemma 2.1, $\lambda_{\max}^{(2)}(\mathcal{A})$ and $\lambda_{\max}^{(k)}(\mathcal{A})$ are exactly the largest Z-eigenvalue and the largest H-eigenvalue of $\mathcal{A}$, respectively. Thus, by Corollary 3.3, we directly get the following relationships between the largest Z-eigenvalue and the largest H-eigenvalue of tensors.

Theorem 3.4. *Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ be an even order real symmetric tensor. Then $\lambda_{\max}^{(2)}(\mathcal{A})$ and $\lambda_{\max}^{(k)}(\mathcal{A})$ always have the same sign. And*

$$n^{\frac{2-k}{2}} \leq \frac{\lambda_{\max}^{(2)}(\mathcal{A})}{\lambda_{\max}^{(k)}(\mathcal{A})} \leq 1,$$

if $\lambda_{\max}^{(k)}(\mathcal{A}) \neq 0$.

For the k-order nonnegative symmetric tensor $\mathcal{A}$, by Lemma 2.2, $\lambda_{\max}^{(2)}(\mathcal{A})$ and $\lambda_{\max}^{(k)}(\mathcal{A})$ are exactly the Z-spectral radius $\rho_Z(\mathcal{A})$ and spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$, respectively. And clearly,

$$\rho(\mathcal{A}) = \lambda_{\max}^{(k)}(\mathcal{A}) = \max\{x^{\mathrm{T}}(\mathcal{A}x^{k-1}) : \sum_{i=1}^{n} x_i^k = 1, x \in \mathbb{R}^n\} \geq 0.$$

Thus, by Corollary 3.3, we directly get the following relationships between the Z-spectral radius and spectral radius.

Theorem 3.5. *Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ be a nonnegative symmetric tensor. Then $\rho_Z(\mathcal{A})$ and $\rho(\mathcal{A})$ always have the same sign. And*

$$n^{\frac{2-k}{2}} \leq \frac{\rho_Z(\mathcal{A})}{\rho(\mathcal{A})} \leq 1,$$

if $\rho(\mathcal{A}) > 0$.

Next, we give some examples to show the Theorem 3.5.

Example 3.6. *Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,3]}$ be a nonnegative symmetric tensor, where $a_{1111} = 4$, $a_{1112} = 1$, $a_{1121} = 1$, $a_{1211} = 1$, $a_{2111} = 1$, $a_{2222} = 1$, $a_{3333} = 6$ and others are zero. By calculation, we get the all H-eigenvalues (not counting multiplicity) are*

$$\lambda_1 = 1,\ \lambda_2 = 6,\ \lambda_3 \approx 0.4143,\ \lambda_4 \approx 5.781.$$

The all Z-eigenvalues (not counting multiplicity) are

$$\lambda_1 = 1, \lambda_2 = 6,\ \lambda_3 \approx 0.2127,\ \lambda_4 \approx 0.955,\ \lambda_5 \approx 4.442,$$

$$\lambda_6 \approx 0.2054,\ \lambda_7 \approx 0.8239,\ \lambda_8 \approx 2.552,\ \lambda_9 \approx 0.8571.$$

The calculation results show that

$$\frac{1}{3} < \frac{\rho_Z(\mathcal{A})}{\rho(\mathcal{A})} = 1.$$

Example 3.7. *Let $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,2]}$ be a nonnegative symmetric tensor, where $a_{1111} = 3$, $a_{1112} = 1$, $a_{1121} = 1$, $a_{1211} = 1$, $a_{2111} = 1$, $a_{2222} = 1$ and others are zero. By calculation, we get the all H-eigenvalues (not counting multiplicity) are*

$$\lambda_1 = 1,\ \lambda_2 \approx 4.905,\ \lambda_3 = 0.$$

The all Z-eigenvalues (not counting multiplicity) are

$$\lambda_1 = 1,\ \lambda_2 \approx 0.951,\ \lambda_3 \approx 3.549,\ \lambda_4 = 0.$$

The calculation results show that

$$\frac{1}{2} < \frac{\rho_Z(\mathcal{A})}{\rho(\mathcal{A})} < 1.$$

4. The minimum value of homogeneous polynomials under the different norms

In this section, we give some relationships on the $\lambda_{\min}^{(p)}(\mathcal{A})$ and $\lambda_{\min}^{(q)}(\mathcal{A})$ when $1 \leq q \leq p$. Furthermore, we get the conclusions on the smallest H-eigenvalue and the smallest Z-eigenvalue of tensors. And the inequalities on p-spectral radius are obtained.

Theorem 4.1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_k}) \in \mathbb{R}^{[k,n]}$ and $1 \leq q \leq p$. Then the following cases hold.*
(1) *If $\lambda_{\min}^{(q)}(\mathcal{A}) \geq 0$, then $0 \leq n^{\frac{k}{p}-\frac{k}{q}} \lambda_{\min}^{(p)}(\mathcal{A}) \leq \lambda_{\min}^{(q)}(\mathcal{A}) \leq \lambda_{\min}^{(p)}(\mathcal{A})$.*
(2) *If $\lambda_{\min}^{(q)}(\mathcal{A}) < 0$, then $0 > n^{\frac{k}{p}-\frac{k}{q}} \lambda_{\min}^{(p)}(\mathcal{A}) \geq \lambda_{\min}^{(q)}(\mathcal{A}) \geq \lambda_{\min}^{(p)}(\mathcal{A})$.*

Proof. (1) *a*) When $\lambda^{(q)}_{\min}(\mathcal{A}) \geq 0$, we firstly prove that $\lambda^{(q)}_{\min}(\mathcal{A}) \leq \lambda^{(p)}_{\min}(\mathcal{A})$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be the vector such that $\lambda^{(p)}_{\min}(\mathcal{A}) = x^{\mathrm{T}}(\mathcal{A}x^{k-1})$. Then $\|x\|_p = 1$ and $\|x\|_q \geq 1$. Thus, we get

$$\lambda^{(p)}_{\min}(\mathcal{A}) = \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} x_{i_1}\cdots x_{i_k} = \|x\|_q^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} \frac{x_{i_1}}{\|x\|_q}\cdots\frac{x_{i_k}}{\|x\|_q}.$$

Let $y = (y_1, \ldots, y_n)^{\mathrm{T}} = (\frac{x_1}{\|x\|_q}, \ldots, \frac{x_n}{\|x\|_q})^{\mathrm{T}}$. So, $y \in \mathbb{R}^n$ and $\|y\|_q = 1$. Then

$$\lambda^{(p)}_{\min}(\mathcal{A}) = \|x\|_q^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} y_{i_1}\cdots y_{i_k} = \|x\|_q^k y^{\mathrm{T}}(\mathcal{A}y^{m-1}) \geq \|x\|_q^k \lambda^{(q)}_{\min}(\mathcal{A}) \geq 0.$$

From $\|x\|_q^k \geq 1$, so $\lambda^{(p)}_{\min}(\mathcal{A}) \geq \lambda^{(q)}_{\min}(\mathcal{A}) \geq 0$.

b) Next, we prove that $0 \leq n^{\frac{k}{p}-\frac{k}{q}}\lambda^{(p)}_{\min}(\mathcal{A}) \leq \lambda^{(q)}_{\min}(\mathcal{A})$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be the vector such that $\lambda^{(q)}_{\min}(\mathcal{A}) = x^{\mathrm{T}}(\mathcal{A}x^{k-1})$. Then $\|x\|_q = 1$. And Lemma 2.4 implies that $1 = \|x\|_q \leq n^{\frac{1}{q}-\frac{1}{p}}\|x\|_p$. Let $y = n^{\frac{1}{q}-\frac{1}{p}}x$, so $\|y\|_p = n^{\frac{1}{q}-\frac{1}{p}}\|x\|_p \geq 1$. Thus,

$$\begin{aligned}
\lambda^{(q)}_{\min}(\mathcal{A}) &= \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} x_{i_1}\cdots x_{i_k} \\
&= (n^{\frac{1}{p}-\frac{1}{q}})^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} (n^{\frac{1}{q}-\frac{1}{p}}x_{i_1})\cdots(n^{\frac{1}{q}-\frac{1}{p}}x_{i_k}) \\
&= (n^{\frac{1}{p}-\frac{1}{q}})^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} y_{i_1}\cdots y_{i_k} \\
&= \|y\|_p^k (n^{\frac{1}{p}-\frac{1}{q}})^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} \frac{y_{i_1}}{\|y\|_p}\cdots\frac{y_{i_k}}{\|y\|_p}. \qquad (4.1)
\end{aligned}$$

Let $z = (z_1, \ldots, z_n)^{\mathrm{T}} = (\frac{y_1}{\|y\|_p}, \ldots, \frac{y_n}{\|y\|_p})^{\mathrm{T}}$. So, $z \in \mathbb{R}^n$ and $\|z\|_p = 1$. By *a*), we get that if $\lambda^{(q)}_{\min}(\mathcal{A}) \geq 0$, then $\lambda^{(p)}_{\min}(\mathcal{A}) \geq 0$. Hence,

$$\lambda^{(q)}_{\min}(\mathcal{A}) = \|y\|_p^k (n^{\frac{1}{p}-\frac{1}{q}})^k z^{\mathrm{T}}(\mathcal{A}z^{k-1}) \geq \|y\|_p^k (n^{\frac{1}{p}-\frac{1}{q}})^k \lambda^{(p)}_{\min}(\mathcal{A}) \geq 0.$$

From $\|y\|_p^k \geq 1$, so $\lambda^{(q)}_{\min}(\mathcal{A}) \geq n^{\frac{k}{p}-\frac{k}{q}}\lambda^{(p)}_{\min}(\mathcal{A}) \geq 0$.

It follows from *a*) and *b*) that the statement (1) holds.

(2) *c*) When $\lambda^{(q)}_{\min}(\mathcal{A}) < 0$, we firstly prove that $\lambda^{(q)}_{\min}(\mathcal{A}) \geq \lambda^{(p)}_{\min}(\mathcal{A})$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be the vector such that $\lambda^{(q)}_{\min}(\mathcal{A}) = x^{\mathrm{T}}(\mathcal{A}x^{k-1})$. Then $\|x\|_q = 1$ and $\|x\|_p \leq 1$. Thus, we get

$$0 > \lambda^{(q)}_{\min}(\mathcal{A}) = \|x\|_p^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} \frac{x_{i_1}}{\|x\|_p}\cdots\frac{x_{i_k}}{\|x\|_p}.$$

Let $y = (y_1, \ldots, y_n)^{\mathrm{T}} = (\frac{x_1}{\|x\|_p}, \ldots, \frac{x_n}{\|x\|_p})^{\mathrm{T}}$. So, $y \in \mathbb{R}^n$ and $\|y\|_p = 1$. Then

$$0 > \lambda^{(q)}_{\min}(\mathcal{A}) = \|x\|_p^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} y_{i_1}\cdots y_{i_k} = \|x\|_p^k y^{\mathrm{T}}(\mathcal{A}y^{m-1}) \geq \|x\|_p^k \lambda^{(p)}_{\min}(\mathcal{A}).$$

From $\|x\|_p^k \leq 1$, so $0 > \lambda^{(q)}_{\min}(\mathcal{A}) \geq \lambda^{(p)}_{\min}(\mathcal{A})$.

d) Next, we prove that $0 > n^{\frac{k}{p}-\frac{k}{q}}\lambda^{(p)}_{\min}(\mathcal{A}) \geq \lambda^{(q)}_{\min}(\mathcal{A})$. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be the vector such that $\lambda^{(p)}_{\min}(\mathcal{A}) = x^{\mathrm{T}}(\mathcal{A}x^{k-1})$. Then $\|x\|_p = 1$. And Lemma 2.4 implies that $n^{\frac{1}{p}-\frac{1}{q}}\|x\|_q \leq \|x\|_p = 1$. Let $y = n^{\frac{1}{p}-\frac{1}{q}}x$, so $\|y\|_q = n^{\frac{1}{p}-\frac{1}{q}}\|x\|_q \leq 1$. By c), we get that if $\lambda^{(q)}_{\min}(\mathcal{A}) < 0$, then $\lambda^{(p)}_{\min}(\mathcal{A}) < 0$. Thus, similar to the (4.1), we have

$$0 > \lambda^{(p)}_{\min}(\mathcal{A}) = \|y\|_q^k (n^{\frac{1}{q}-\frac{1}{p}})^k \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1\cdots i_k} \frac{y_{i_1}}{\|y\|_q} \cdots \frac{y_{i_k}}{\|y\|_q}.$$

Let $z = (z_1, \ldots, z_n)^{\mathrm{T}} = (\frac{y_1}{\|y\|_q}, \ldots, \frac{y_n}{\|y\|_q})^{\mathrm{T}}$. So, $z \in \mathbb{R}^n$ and $\|z\|_q = 1$. Hence,

$$0 > \lambda^{(p)}_{\min}(\mathcal{A}) = \|y\|_q^k (n^{\frac{1}{q}-\frac{1}{p}})^k z^{\mathrm{T}}(\mathcal{A}z^{k-1}) \geq \|y\|_q^k (n^{\frac{1}{q}-\frac{1}{p}})^k \lambda^{(q)}_{\min}(\mathcal{A}).$$

From $\|y\|_q^k \leq 1$, so $0 > n^{\frac{k}{p}-\frac{k}{q}}\lambda^{(p)}_{\min}(\mathcal{A}) \geq \lambda^{(q)}_{\min}(\mathcal{A})$.

It follows from c) and d) that the statement (2) holds. □

Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ and $1 \leq q \leq p$. From Theorem 4.1 (1), we directly get that if $\lambda^{(q)}_{\min}(\mathcal{A}) > 0$, then $\lambda^{(p)}_{\min}(\mathcal{A}) > 0$, and if $\lambda^{(p)}_{\min}(\mathcal{A}) > 0$, then $\lambda^{(q)}_{\min}(\mathcal{A}) > 0$. By (2), we get that if $\lambda^{(q)}_{\min}(\mathcal{A}) < 0$, then $\lambda^{(p)}_{\min}(\mathcal{A}) < 0$, and if $\lambda^{(p)}_{\min}(\mathcal{A}) < 0$, then $\lambda^{(q)}_{\min}(\mathcal{A}) < 0$. Thus, we have the following result.

Theorem 4.2. *Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ and $1 \leq q \leq p$. Then $\lambda^{(p)}_{\min}(\mathcal{A})$ and $\lambda^{(q)}_{\min}(\mathcal{A})$ always have the same sign for any $p, q \geq 1$. In other words, for any $p, q \geq 1$, $\lambda^{(p)}_{\min}(\mathcal{A})$ and $\lambda^{(q)}_{\min}(\mathcal{A})$ are either both greater than zero or both less than zero, or both equal to zero.*

By Theorem 4.1 and 4.2, we have the following corollary.

Corollary 4.3. *For real tensor $\mathcal{A} \in \mathbb{R}^{[k,n]}$ and $1 \leq q \leq p$, $\lambda^{(p)}_{\min}(\mathcal{A})$ and $\lambda^{(q)}_{\min}(\mathcal{A})$ always have the same sign for any $p, q \geq 1$. And the following inequalities hold*

$$n^{\frac{k}{p}-\frac{k}{q}} \leq \frac{\lambda^{(q)}_{\min}(\mathcal{A})}{\lambda^{(p)}_{\min}(\mathcal{A})} \leq 1,$$

if $\lambda^{(p)}_{\min}(\mathcal{A}) \neq 0$.

For even order real symmetric tensor $\mathcal{A}$, by Lemma 2.1, $\lambda^{(2)}_{\min}(\mathcal{A})$ and $\lambda^{(k)}_{\min}(\mathcal{A})$ are exactly the smallest Z-eigenvalue and the smallest H-eigenvalue of $\mathcal{A}$, respectively. Thus, by Corollary 4.3, we directly get the following relationships between the smallest H-eigenvalue and the smallest Z-eigenvalue of tensors.

Theorem 4.4. *Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ be an even order real symmetric tensor. Then $\lambda^{(2)}_{\min}(\mathcal{A})$ and $\lambda^{(k)}_{\min}(\mathcal{A})$ always have the same sign. And*

$$n^{\frac{2-k}{2}} \leq \frac{\lambda^{(2)}_{\min}(\mathcal{A})}{\lambda^{(k)}_{\min}(\mathcal{A})} \leq 1,$$

if $\lambda^{(k)}_{\min}(\mathcal{A}) \neq 0$.

Next, we give some examples to show the Theorem 4.4.

Example 4.5. *Let tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,3]}$ coincide with the tensor in Example 3.6. Thus, the calculation results show that*

$$\frac{1}{3} < \frac{\lambda^{(2)}_{\min}(\mathcal{A})}{\lambda^{(4)}_{\min}(\mathcal{A})} < 1.$$

Example 4.6. *Let tensor $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4,2]}$ coincide with the tensor in Example 3.7. Thus, the calculation results show that*

$$\lambda^{(2)}_{\min}(\mathcal{A}) = \lambda^{(4)}_{\min}(\mathcal{A}) = 0.$$

The *p-spectral radius* [24] $\rho^{(p)}(\mathcal{A})$ of tensor $\mathcal{A}$ is defined as

$$\rho^{(p)}(\mathcal{A}) = \max\{|P_{\mathcal{A}}(x)|\} = \max\{|\lambda^{(p)}_{\max}(\mathcal{A})|,\ |\lambda^{(p)}_{\min}(\mathcal{A})|\}.$$

For nonnegative symmetric tensor $\mathcal{A}$, Nikiforov proposed the inequalities

$$n^{\frac{k}{p}-\frac{k}{q}}\rho^{(p)}(\mathcal{A}) \leq \rho^{(q)}(\mathcal{A}) \leq \rho^{(p)}(\mathcal{A}) \tag{4.2}$$

if $1 \leq q \leq p$.

By Theorem 3.1 and 4.1, we can also get the conclusion (4.2) as follows. But we cannot get the Theorem 3.1 and 4.1 from the conclusion (4.2) directly.

Theorem 4.7. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_k}) \in \mathbb{R}^{[k,n]}$ be a real tensor and $1 \leq q \leq p$. Then*

$$n^{\frac{k}{p}-\frac{k}{q}}\rho^{(p)}(\mathcal{A}) \leq \rho^{(q)}(\mathcal{A}) \leq \rho^{(p)}(\mathcal{A}).$$

Proof. If $\lambda^{(q)}_{\min}(\mathcal{A}) \geq 0$, so $\lambda^{(q)}_{\max}(\mathcal{A}) \geq 0$. Thus, Theorem 3.1 (1) yields that $0 \leq \lambda^{(q)}_{\max}(\mathcal{A}) \leq \lambda^{(p)}_{\max}(\mathcal{A})$. And Theorem 4.1 (1) yields that $0 \leq \lambda^{(q)}_{\min}(\mathcal{A}) \leq \lambda^{(p)}_{\min}(\mathcal{A})$. Hence, $\rho^{(q)}(\mathcal{A}) \leq \rho^{(p)}(\mathcal{A})$.

If $\lambda^{(q)}_{\min}(\mathcal{A}) < 0$. Then we consider the following cases.

Case 1. When $\lambda^{(q)}_{\max}(\mathcal{A}) \geq 0$. The Theorem 3.1 (1) yields that $0 \leq \lambda^{(q)}_{\max}(\mathcal{A}) \leq \lambda^{(p)}_{\max}(\mathcal{A})$. Since $\lambda^{(q)}_{\min}(\mathcal{A}) < 0$, Theorem 4.1 (2) yields that $0 > \lambda^{(q)}_{\min}(\mathcal{A}) \geq \lambda^{(p)}_{\min}(\mathcal{A})$. Hence, $\rho^{(q)}(\mathcal{A}) \leq \rho^{(p)}(\mathcal{A})$.

Case 2. When $\lambda^{(q)}_{\max}(\mathcal{A}) < 0$. The Theorem 3.1 (2) yields that $0 > \lambda^{(q)}_{\max}(\mathcal{A}) \geq \lambda^{(p)}_{\max}(\mathcal{A})$. Since $\lambda^{(q)}_{\min}(\mathcal{A}) < 0$, Theorem 4.1 (2) yields that $0 > \lambda^{(q)}_{\min}(\mathcal{A}) \geq \lambda^{(p)}_{\min}(\mathcal{A})$. Hence, $\rho^{(q)}(\mathcal{A}) \leq \rho^{(p)}(\mathcal{A})$.

In conclusion, $\rho^{(q)}(\mathcal{A}) \leq \rho^{(p)}(\mathcal{A})$.

Similarly, we can prove that $n^{\frac{k}{p}-\frac{k}{q}}\rho^{(p)}(\mathcal{A}) \leq \rho^{(q)}(\mathcal{A})$. □

5. Some applications on definite positive of tensors and hypergraphs

In this section, by the above conclusions, we obtain some applications on definite positive of tensors and hypergraphs.

5.1. The applications on definite positive of tensors

For even order real symmetric tensor $\mathcal{A}$, by Lemma 2.3, we know that $\mathcal{A}$ is positive (semi-)definite if and only if $\lambda^{(2)}_{\min}(\mathcal{A}) > (\geq)0$. Then it follows from Theorem 4.2 that $\lambda^{(m)}_{\min}(\mathcal{A}) > (\geq)0$ for any $m \geq 2$. Thus, by Theorem 4.1 and 4.2, we get the following result.

Theorem 5.1. *Let $\mathcal{A} \in \mathbb{R}^{[k,n]}$ be an even order real symmetric tensor. Then the following cases hold.*
(*i*) *If $\mathcal{A}$ is positive (semi-)definite, then*

$$\lambda^{(m)}_{\min}(\mathcal{A}) \geq \lambda^{(2)}_{\min}(\mathcal{A}) > (\geq)0,$$

for any $m \geq 2$.
(*ii*) *If there is a $m \geq 2$ such that $\lambda^{(m)}_{\min}(\mathcal{A}) > (\geq)0$, then $\mathcal{A}$ is positive (semi-)definite.*

5.2. The applications on hypergraphs

Let a hypergraph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, where $V(\mathcal{G}) = \{1, 2, \ldots, n\}$ and $E(\mathcal{G}) = \{e_1, e_2, \ldots, e_m\}$ are the vertex set and edge set of $\mathcal{G}$, respectively. If each edge of $\mathcal{G}$ contains k vertices, then $\mathcal{G}$ is called a *k-uniform hypergraph*. Clearly, 2-uniform hypergraphs are exactly the ordinary graphs. The degree of a vertex i of $\mathcal{G}$ is denoted by d_i, where $d_i = |\{e_j : i \in e_j, j = 1, \ldots, m\}|$, $i \in [n]$. The *adjacency tensor* [25] of k-uniform hypergraph $\mathcal{G}$, denoted by $\mathcal{A}_\mathcal{G}$, is a k-order n-dimensional nonnegative symmetric tensor with entries

$$a_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(\mathcal{G}); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}_\mathcal{G} = \mathcal{D}_\mathcal{G} - \mathcal{A}_\mathcal{G}$ and $\mathcal{Q}_\mathcal{G} = \mathcal{D}_\mathcal{G} + \mathcal{A}_\mathcal{G}$ be the *Laplacian tensor* and *signless Laplacian tensor* of $\mathcal{G}$ [26], where $\mathcal{D}_\mathcal{G}$ is a diagonal tensor, whose diagonal entries are $d_1, \ldots, d_n$, respectively. Clearly, $\mathcal{L}_\mathcal{G}$ and $\mathcal{Q}_\mathcal{G}$ are both symmetric, and $\mathcal{Q}_\mathcal{G}$ is nonnegative.

It's easy to check that $\mathcal{L}_\mathcal{G}$ is a diagonally dominated tensor, so $\mathcal{L}_\mathcal{G}$ is a positive semi-definite when k is even, see [20]. It follows Lemma 2.1 and 2.3 that $\lambda^{(k)}_{\min}(\mathcal{L}_\mathcal{G}) = \lambda^{(2)}_{\min}(\mathcal{L}_\mathcal{G}) = 0$. Thus, $\lambda^{(k)}_{\max}(\mathcal{L}_\mathcal{G}) \geq 0$ and $\lambda^{(2)}_{\max}(\mathcal{L}_\mathcal{G}) \geq 0$. Hence, we can get the following results for k-uniform hypergraph $\mathcal{G}$ by Theorem 3.4 and 3.5.

Theorem 5.2. *Let $\mathcal{G}$ be a k-uniform hypergraph, then $\rho_Z(\mathcal{A}_\mathcal{G})$ and $\rho(\mathcal{A}_\mathcal{G})$ (resp. $\rho_Z(\mathcal{Q}_\mathcal{G})$ and $\rho(\mathcal{Q}_\mathcal{G})$, $\lambda^{(2)}_{\max}(\mathcal{L}_\mathcal{G})$ and $\lambda^{(k)}_{\max}(\mathcal{L}_\mathcal{G})$) always have the same sign. And*

$$n^{\frac{2-k}{2}} \leq \frac{\rho_Z(\mathcal{A}_\mathcal{G})}{\rho(\mathcal{A}_\mathcal{G})} \leq 1,$$

$$n^{\frac{2-k}{2}} \leq \frac{\rho_Z(\mathcal{Q}_\mathcal{G})}{\rho(\mathcal{Q}_\mathcal{G})} \leq 1,$$

if $\rho(\mathcal{A}_\mathcal{G}) > 0$, $\rho(\mathcal{Q}_\mathcal{G}) > 0$. And when k is even, we have

$$n^{\frac{2-k}{2}} \leq \frac{\lambda^{(2)}_{\max}(\mathcal{L}_\mathcal{G})}{\lambda^{(k)}_{\max}(\mathcal{L}_\mathcal{G})} \leq 1,$$

if $\lambda^{(k)}_{\max}(\mathcal{L}_\mathcal{G}) > 0$. The $\lambda^{(2)}_{\max}(\mathcal{L}_\mathcal{G})$ and $\lambda^{(k)}_{\max}(\mathcal{L}_\mathcal{G})$ are exactly the the largest the Z-eigenvalue and H-eigenvalue of $\mathcal{L}_\mathcal{G}$, respectively.

Remark 5.3. *In [10], Lin et al. gave the inequalities of $\rho_Z(\mathcal{A}_\mathcal{G})$ and $\rho(\mathcal{A}_\mathcal{G})$ as follows*

$$\rho_Z(\mathcal{A}_\mathcal{G}) \leq \rho(\mathcal{A}_\mathcal{G}).$$

In this paper, the Theorem 5.2 generalizes this result.

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