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Subnormal n-th roots of matricially and spherically quasinormal pairs

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Abstract. In a recent paper, Curto et al. [4] asked the following question: *"Let T be a subnormal operator, and assume that T^2 is quasinormal. Does it follow that T is quasinormal?"*. Pietrzycki and Stochel have answered this question in the affirmative [18] and proved an even stronger result. Namely, the authors have showed that the subnormal n-th roots of a quasinormal operator must be quasinormal. In the present paper, using an elementary technique, we present a much simpler proof of this result and generalize some other results from [4]. We also show that we can relax a condition in the definition of matricially quasinormal n-tuples and we give a correction for one of the results from [4]. Finally, we give sufficient conditions for the equivalence of matricial and spherical quasinormality of $\mathbf{T}^{(n,n)} := (T_1^n, T_2^n)$ and matricial and spherical quasinormality of $\mathbf{T} = (T_1, T_2)$, respectively.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathfrak{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. An operator T is said to be *normal* if $T^*T = TT^*$, *quasinormal* if T commutes with T^*T, i.e., $TT^*T = T^*T^2$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$. It is well known that

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{subnormal} \Rightarrow \text{hyponormal}.$$

For $S, T \in \mathfrak{B}(\mathcal{H})$ let $[S, T] = ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{bmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{bmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [1], [5], [6]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. For $i, j, k \in \{1, 2, \dots, n\}$, $\mathbf{T}$ is called *matricially quasinormal* if each T_i commutes with each $T_j^* T_k$, $\mathbf{T}$ is (jointly) *quasinormal* if each T_i commutes with each $T_j^* T_j$, and *spherically quasinormal* if each T_i commutes with $\sum_{j=1}^n T_j^* T_j$ (see [14]). As shown in [2] and [13], we have

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$$\begin{aligned} \text{normal} &\Rightarrow \text{matricially quasinormal} \Rightarrow \text{(jointly) quasinormal} \\ &\Rightarrow \text{spherically quasinormal} \Rightarrow \text{subnormal}. \end{aligned}$$

On the other hand, the results in [7] and [13] show that the inverse implications do not hold.

In a recent paper [4], R. E. Curto, S. H. Lee and J. Yoon, partially motivated by the results of their previous articles [8, 9], asked the following question:

Problem 1.1. *Let T be a subnormal operator, and assume that T^2 is quasinormal. Does it follow that T is quasinormal?*

With the additional assumption of left invertibility they showed that a left invertible subnormal operator T whose square T^2 is quasinormal must be quasinormal (see [4, Theorem 2.3]). It remained an open question whether this is true in general without any assumption about left invertibility until the paper [18] was published. Moreover, the authors proved even stronger result:

Theorem 1.2. *[18] Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator such that T^n is quasinormal for some $n \in \mathbb{N}$. Then T is quasinormal.*

The proof is based on the theory of operator monotone functions and Hansen's inequality. More precisely, the following theorems were crucial for the proof:

Theorem 1.3. *(see [19, Theorem 12.12]) If $n \in \mathbb{N}$, then the commutants of a positive operator and it's n-th root coincide.*

Theorem 1.4. *[11] Let T be a bounded operator on $\mathcal{H}$. Then the following conditions are equivalent:*

(i) *T is quasinormal;*
(ii) $(T^*)^n T^n = (T^*T)^n$, $n \in \mathbb{N}$;
(iii) *there exists a (unique) spectral Borel measure E on $\mathbb{R}_+$ such that*

$$(T^*)^n T^n = \int_{\mathbb{R}_+} x^n \, E(dx), \quad n \in \mathbb{Z}_+.$$

Theorem 1.5. *(Löwner-Heinz inequality [16, 17]). If $A, B \in \mathfrak{B}_+(\mathcal{H})$ are such that $B \le A$ and $p \in [0,1]$, then $B^p \le A^p$.*

Theorem 1.6. *(Hansen inequality [15, 22]) Let $A \in \mathfrak{B}(\mathcal{H})$ be a positive operator, $T \in \mathfrak{B}(\mathcal{H})$ be a contraction and $f : [0, \infty) \to \mathbb{R}$ be a continuous operator monotone function such that $f(0) \ge 0$. Then*

$$T^* f(A) T \le f(T^*AT).$$

Moreover, if f is not an affine function and T is an orthogonal projection such that $T \neq I_{\mathcal{H}}$, then the equality holds if and only if $TA = AT$ and $f(0) = 0$.

In Section 3, using an elementary technique, we provide a much simpler proof of Theorem 1.2.

In the literature, similar properties to Problem 1.1 for other classes of operators are known. Namely, hyponormal nth-roots of normal operators are normal (see [20, Theorem 5]). In turn, if T is hyponormal operator and T^n is subnormal then T doesn't have to be subnormal (see [21]).

Motivated by these type of problems, in this paper we consider a problem of when matricial (spherical) quasinormality of $\mathbf{T}^{(n,n)} := (T_1^n, T_2^n)$ implies matricial (spherical) quasinormality of $\mathbf{T} = (T_1, T_2)$.

The starting point in [4] is the following lemma:

Lemma 1.7. *[4] Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator with normal extension*

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix}.$$

Then T is quasinormal if and only if $A^*T = 0$.

Although it may seem as a practical tool for determining whether some operator is quasinormal or not, this approach fails to give an answer to Problem 1.1 without imposing additional assumptions on T and becomes even more impractical if we replace the square of an operator with its arbitrary power.

Nevertheless, using Lemma 1.7, authors in [4] proved the following result:

Theorem 1.8. *[4] Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator and assume that T^2 is quasinormal. If T is bounded below, then T is quasinormal.*

Theorem 1.8 and Lemma 1.7 provided a foundation for proving multivariable analogues of these results. Namely, for a subnormal pair $\mathbf{T} = (T_1, T_2)$ with the normal extension $\mathbf{N} = (N_1, N_2)$, where

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

the authors proved the following results:

Corollary 1.9. *[4] Let $\mathbf{T}$ be a subnormal and assume that T_i is bounded below and T_i^2 is quasinormal. Then $\mathbf{T}$ is spherically quasinormal.*

Theorem 1.10. *[4] Let $\mathbf{T}$ be subnormal, with normal extension $\mathbf{N}$. Then $\mathbf{T}$ is spherically quasinormal if and only if $A_1^*T_1 + A_2^*T_2 = 0$.*

Theorem 1.11. *[4] Let $\mathbf{T}$ be subnormal, with normal extension $\mathbf{N}$. Then $\mathbf{T}$ is (jointly) quasinormal if and only if $A_i^*T_j = 0, i, j = 1, 2$.*

Corollary 1.12. *[4] Let $\mathbf{T}$ be a subnormal pair with normal extension $\mathbf{N}$. Then $\mathbf{T}$ is matricially quasinormal if and only if $A_iA_j^*T_k = 0, i, j, k = 1, 2$.*

Here we use the opportunity to state that Theorem 1.11 is actually false. Namely, if $A_j^*T_k = 0$, $j, k = 1, 2$, then obviously, $A_iA_j^*T_k = 0, i, j, k = 1, 2$. If Theorem 1.11 is true, then this implies that every (jointly) quasinormal n-tuple must be matricially quasinormal. But as mentioned earlier, the results in [23] and [28] show that this is not the case. The correction and other "unexpected" implications of this mistake will be presented later in Section 2.

Through the rest of the paper, we are going to use extensively the celebrated Fuglede-Putnam Theorem, and especially its most famous corollary:

Theorem 1.13. *[12] Let T and N be bounded operators on a complex Hilbert space with N being normal. If $TN = NT$, then $TN^* = N^*T$.*

Corollary 1.14. *[12] If M and N are commuting normal operators, then MN is also normal.*

The next simple, but useful observation, will be used in the several proofs, as well.

Lemma 1.15. *Let $A, P \in \mathfrak{B}(\mathcal{H})$ such that A is self-adjoint and P is an orthogonal projection. Then $\mathcal{R}(P)$ is invariant for A if and only if A and P commute.*

Proof. If $\mathcal{R}(P)$ is invariant for A, then, obviously, $PAP = AP$. By taking adjoints in the last equality, we have $PAP = PA$, and so $AP = PA$.

Conversely, if $AP = PA$, then $PAP = AP$, which implies that $\mathcal{R}(P)$ is invariant for A. □

2. Preliminary Results

The following lemma, due to Conway [19], turned out to be much more useful for proving the generalized version of Theorem 1.8. We present it here in a slightly different form:

Lemma 2.1. *[3, Lemma 3.1] Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator. If N is a normal extension for T, then T is quasinormal if and only if $\mathcal{H}$ is invariant for N^*N.*

Proof. Let N be the normal extension of T on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ given by

$$N = \begin{bmatrix} T & A \\ 0 & B^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^{\perp} \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Note that $\mathcal{H}$ is invariant for N^*N if and only if $PN^*NP = N^*NP$. A direct computation shows that

$$N^*NP = \begin{bmatrix} T^*T & 0 \\ A^*T & 0 \end{bmatrix} \quad \text{and} \quad PN^*NP = \begin{bmatrix} T^*T & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $PN^*NP = N^*NP$ if and only if $A^*T = 0$. The conclusion now follows from Lemma 1.7. □

We now give a complete answer to Problem 1.1:

Theorem 2.2. *Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator such that T^n is quasinormal for some $n \in \mathbb{N}$. Then T is quasinormal.*

Proof. Let $N \in \mathfrak{B}(\mathcal{K})$ be a normal extension for T, where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Then, N^n is a normal extension for T^n and since $\mathcal{H}$ is invariant for $(N^n)^*N^n = (N^*N)^n$ (Lemma 2.1), it follows that P commutes with $(N^*N)^n$ (Lemma 1.15). Hence, P also commutes with N^*N, by Theorem 1.3. Therefore, $\mathcal{H}$ is invariant for N^*N and by applying Lemma 2.1 again, we conclude that T is quasinormal. □

A subnormal operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be *pure* if it has no nonzero normal orthogonal summands; that is, if there exists no nonzero subspace $\mathcal{M}$ on $\mathcal{H}$ invariant under T such that $T|_{\mathcal{M}}$ is normal, where $T|_{\mathcal{M}}$ denotes the restriction of T on $\mathcal{M}$. Since quasinormal operators are subnormal, it makes sense to speak of pure quasinormal operators. The following corollary is a generalization of [4, Corollary 2.4]:

Corollary 2.3. *Let $T \in \mathfrak{B}(\mathcal{H})$ be a subnormal operator such that T^n is pure quasinormal for some $n \in \mathbb{N}$. Then T is pure quasinormal.*

Proof. Quasinormality follows from Theorem 2.2. If T is not pure, then there is a nonzero reducing subspace $\mathcal{M}$ of $\mathcal{H}$ such that $T|_{\mathcal{M}}$ is normal, where $T|_{\mathcal{M}}$ denotes the restriction of T on $\mathcal{M}$. Since $T^n|_{\mathcal{M}} = (T|_{\mathcal{M}})^n$ is also normal, T^n is not pure, which is a contradiction. Therefore, T is pure. □

Now we can shift the focus to the multivariable case. Although we present our results for commuting pairs of operators, the reader will easily see that the same (or analogous) statements work well for commuting n-tuples of operators, when $n > 2$.

Theorem 2.2 allows us to remove left invertibility assumption from Corollary 1.9. Moreover, we can prove even stronger result:

Corollary 2.4. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair and assume that T_1^k and T_2^l are quasinormal for some $k, l \in \mathbb{N}$. Then $\mathbf{T}$ is spherically quasinormal.*

Proof. Since T_i, $i = 1, 2$ are subnormal and T_1^k and T_2^l are quasinormal, Theorem 2.2 implies that T_i, $i = 1, 2$ are quasinormal. Therefore, $\mathbf{T}$ is spherically quasinormal ([4, Remark 2.6]). □

The following lemma can be considered as a multivariable analogue of Lemma 2.1 (see Remark 2.6 below):

Lemma 2.5. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair, with a normal extension* $\mathbf{N} = (N_1, N_2)$*. Then* $\mathbf{T}$ *is spherically quasinormal if and only if* $\mathcal{H}$ *is invariant for* $N_1^*N_1 + N_2^*N_2$*.*

Proof. Let $N_i, i = 1, 2,$ be the normal extensions of T_i on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ given by

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Note that $\mathcal{H}$ is invariant for $N_1^*N_1 + N_2^*N_2$ if and only if $P(N_1^*N_1 + N_2^*N_2)P = (N_1^*N_1 + N_2^*N_2)P$. By direct computation,

$$(N_1^*N_1 + N_2^*N_2)P = \begin{bmatrix} T_1^*T_1 + T_2^*T_2 & 0 \\ A_1^*T_1 + A_2^*T_2 & 0 \end{bmatrix}$$

and

$$P(N_1^*N_1 + N_2^*N_2)P = \begin{bmatrix} T_1^*T_1 + T_2^*T_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $P(N_1^*N_1 + N_2^*N_2)P = (N_1^*N_1 + N_2^*N_2)P$ if and only if $A_1^*T_1 + A_2^*T_2 = 0$. Now it only remains to apply Theorem 1.10. □

Remark 2.6. *If we treat* $\mathbf{N} = (N_1, N_2)$ *as a column vector on* $\mathcal{K} \oplus \mathcal{K}$*, we may use the notation* $\mathbf{N}^*\mathbf{N} = N_1^*N_1 + N_2^*N_2$*, which gives us the following analogue of Lemma 2.1:*

Lemma 2.7. *Let* $\mathbf{T}$ *be a subnormal, with a normal extension* $\mathbf{N}$*. Then* $\mathbf{T}$ *is spherically quasinormal if and only if* $\mathcal{H}$ *is invariant for* $\mathbf{N}^*\mathbf{N}$*.*

As shown in [4, Example 3.6], there exists a spherically quasinormal 2-variable weighted shift $W_{(\alpha,\beta)}$ such that $W_{(\alpha,\beta)}^{(2,1)}$ is not spherically quasinormal (for basic properties for 2-variable weighted shift $W_{(\alpha,\beta)}$ we refer to [9] and [10]). In other words, if $\mathbf{T} = (T_1, T_2)$ is a spherically quasinormal pair, then $\mathbf{T}^{(m,n)} = (T_1^m, T_2^n)$ may not be spherically quasinormal.

The following theorem gives a sufficient condition for the equivalence of spherical quasinormality of $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ and spherical quasinormality of $\mathbf{T} = (T_1, T_2)$:

Theorem 2.8. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair with the normal extension* $\mathbf{N} = (N_1, N_2)$ *such that* $N_1N_2 = 0$*. Then* $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ *is spherically quasinormal for some* $n \in \mathbb{N}$ *if and only if* $\mathbf{T}$ *is spherically quasinormal.*

Proof. Let $\mathbf{N} = (N_1, N_2) \in \mathfrak{B}(\mathcal{K})^2$ be a normal extension for $\mathbf{T}$, where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Then, $\mathbf{N}^{(n,n)} = (N_1^n, N_2^n)$ is a normal extension for $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$, and using the fact that $N_1N_2 = 0$ and Fuglede-Putnam Theorem, we have that

$$(N_1^*N_1 + N_2^*N_2)^n = (N_1^*N_1)^n + (N_2^*N_2)^n.$$

($\Rightarrow$:) Assume that $\mathbf{T}^{(n,n)}$ is spherically quasinormal. Then $\mathcal{H}$ is invariant for $(N_1^n)^*N_1^n + (N_2^n)^*N_2^n = (N_1^*N_1)^n + (N_2^*N_2)^n = (N_1^*N_1 + N_2^*N_2)^n$ (Lemma 2.5), and so P commutes with $(N_1^*N_1 + N_2^*N_2)^n$ (Lemma 1.15). It now follows that P also commutes with $N_1^*N_1 + N_2^*N_2$, by Theorem 1.3. Hence, $\mathcal{H}$ is invariant for $N_1^*N_1 + N_2^*N_2$. Lemma 2.5 now implies that $\mathbf{T}$ is spherically quasinormal.

($\Leftarrow$) : The converse can be proved in a similar manner. □

We now give another characterization of matricially quasinormal n-tuples and correct the mistake in [9]:

Lemma 2.9. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair, with a normal extension* $\mathbf{N} = (N_1, N_2)$*. Then* $\mathbf{T}$ *is matricially quasinormal if and only if* $A_i^* T_j = 0$, $i, j = 1, 2$.

Proof. Let $\mathbf{T} = (T_1, T_2)$ be a subnormal, with a normal extension $\mathbf{N} = (N_1, N_2)$. As shown in the proof of Theorem 1.11 (see [4, Theorem 2.9]),

$$T_i T_j^* T_k + A_i A_j^* T_k = T_j^* T_k T_i,$$

i.e., $[T_i, T_j^* T_k] = -A_i A_j^* T_k$.

If $\mathbf{T}$ is matricially quasinormal, then $A_i A_j^* T_k = 0$ for all $i, j, k = 1, 2$, and thus for $i = j$, we have $A_j A_j^* T_k = 0$. Since $\mathcal{N}(A_j A_j^*) = \mathcal{N}(A_j^*)$ it follows that $A_j^* T_k = 0$.

Now, assume that $A_j^* T_k = 0$ for all $j, k = 1, 2$. Then for all $i = 1, 2$, we have that $A_i A_j^* T_k = 0$, which means that $[T_i, T_j^* T_k] = 0$, $i, j, k = 1, 2$. By definition, $\mathbf{T}$ is matricially quasinormal. □

As a consequence of the previous result, we observe that we can relax a condition in the definition of matricial quasinormality:

Corollary 2.10. $\mathbf{T} = (T_1, T_2)$ *is matricially quasinormal if and only if* T_i *commutes with* $T_i^* T_j$, $i, j = 1, 2$.

Here is the correction of Theorem 1.11:

Corollary 2.11. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair, with a normal extension* $\mathbf{N} = (N_1, N_2)$*. Then* $\mathbf{T}$ *is (jointly) quasinormal if and only if* $A_i A_j^* T_j = 0$, $i, j = 1, 2$.

Proof. It follows from the proof of Lemma 2.9, by taking $j = k$. □

Based on Lemma 2.1, we give another analogue result in multivariable case:

Lemma 2.12. *Let* $\mathbf{T} = (T_1, T_2)$ *be a subnormal pair, with a normal extension* $\mathbf{N} = (N_1, N_2)$*. Then* $\mathbf{T}$ *is matricially quasinormal if and only if* $\mathcal{H}$ *is invariant for* $N_i^* N_j$, $i, j = 1, 2$.

Proof. Let $N_i, i = 1, 2$ be the normal extensions of T_i on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ given by

$$N_i = \begin{bmatrix} T_i & A_i \\ 0 & B_i^* \end{bmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{pmatrix},$$

and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Note that $\mathcal{H}$ is invariant for $N_i^* N_j$ if and only if $P N_i^* N_j P = N_i^* N_j P$. Since

$$N_i^* N_j P = \begin{bmatrix} T_i^* T_j & 0 \\ A_i^* T_j & 0 \end{bmatrix} \quad \text{and} \quad PN^*NP = \begin{bmatrix} T_i^* T_j & 0 \\ 0 & 0 \end{bmatrix}.$$

if follows that $P N_i^* N_j P = N_i^* N_j P$ if and only if $A_i^* T_j = 0$, $i, j = 1, 2$. By Lemma 2.9, this is further equivalent with matricial quasinormality of $\mathbf{T}$. □

3. Proof of Theorem 1.2

The following theorem gives sufficient conditions for the equivalence of matricial quasinormality of $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ and matricial quasinormality of $\mathbf{T} = (T_1, T_2)$:

Theorem 3.1. *Let $\mathbf{T} = (T_1, T_2)$ be a subnormal pair with the normal extension $\mathbf{N} = (N_1, N_2)$, such that $N_1^* N_2 \geq 0$. Then $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ is matricially quasinormal for some $n \in \mathbb{N}$ if and only if $\mathbf{T}$ is matricially quasinormal.*

Proof. Let $\mathbf{N} = (N_1, N_2) \in \mathfrak{B}(\mathcal{K})^2$ be a normal extension for $\mathbf{T}$, where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ and let $P \in \mathfrak{B}(\mathcal{K})$ be the orthogonal projection onto $\mathcal{H}$. Then, $\mathbf{N}^{(n,n)} = (N_1^n, N_2^n)$ is a normal extension for $\mathbf{T}^{(n,n)} = (T_1^n, T_2^n)$ and using the fact that N_1 and N_2 commute and Fuglede-Putnam Theorem, we have that

$$(N_i^* N_j)^n = (N_i^n)^* N_j^n.$$

Also, $N_1^* N_2 \geq 0$ implies $N_2^* N_1 = (N_1^* N_2)^* = N_1^* N_2 \geq 0$.

($\Rightarrow$:) Assume that $\mathbf{T}^{(n,n)}$ is matricially quasinormal and let $(i, j) \in \{1, 2\} \times \{1, 2\}$ be arbitrary. Then $\mathcal{H}$ is invariant for $(N_i^n)^* N_j^n = (N_i^* N_j)^n$ (Lemma 2.12), and so P commutes with $(N_i^* N_j)^n$ (Lemma 1.15). By assumption, $N_i^* N_j$ is positive, and thus P also commutes with $N_i^* N_j$ (Theorem 1.3). Hence, $\mathcal{H}$ is invariant for $N_i^* N_j$. Lemma 2.12 now implies that $\mathbf{T}$ is matricially quasinormal.

($\Leftarrow$) : The converse can be proved in a similar manner. $\square$

Remark 3.2. *We observe that Theorem 3.1 is a generalization of Theorem 1.2. Namely, we get Theorem 1.2 as a corollary, by taking $T_1 = T_2 = T$ and $N_1 = N_2$ in Theorem 3.1.*

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