



## Comparative index and Hörmander index in finite dimension and their connections

Julia Elyseeva<sup>a</sup>, Peter Šepitka<sup>b</sup>, Roman Šimon Hilscher<sup>b</sup>

<sup>a</sup>Department of Applied Mathematics, Moscow State University of Technology, Vadkovskii per. 3a, 101472, Moscow, Russia  
<sup>b</sup>Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, CZ-61137 Brno, Czech Republic

**Abstract.** In this paper we prove new relations between the comparative index and the Hörmander index (and the Maslov index) in the finite dimensional case. As a main result we derive an algebraic formula for calculating the Hörmander index of four given Lagrangian planes as a difference of two comparative indices involving certain transformed Lagrangian planes, or as a combination of four comparative indices. This result is based on a generalization of the comparison theorem for the Maslov index involving three Lagrangian paths. In this way we contribute to the recent efforts in the literature (by Zhou, Wu, Zhu in 2018 and by Howard in 2021) devoted to an efficient calculation of the Hörmander index in this finite dimensional case.

### 1. Introduction

Recently, there has been an intensive research activity in the study of the Maslov index  $\text{Mas}(Y, \hat{Y}, [a, b])$  of two Lagrangian paths  $Y$  and  $\hat{Y}$  or in the study of the Hörmander index  $s(Y_1, Y_2, \hat{Y}_1, \hat{Y}_2)$  associated with four Lagrangian planes  $Y_1, Y_2, \hat{Y}_1, \hat{Y}_2 \in \Lambda(n)$ , see [2, 16, 17, 19, 20, 23, 30] and the references given therein. Recall that for a fixed dimension  $n \in \mathbb{N}$  the space of *Lagrangian planes* is defined as

$$\Lambda(n) := \{Y \in \mathbb{R}^{2n \times n}, W(Y, Y) = 0, \text{rank } Y = n\},$$

where  $W(Y, \hat{Y}) \in \mathbb{R}^{n \times n}$  denotes the Wronskian of the two Lagrangian planes  $Y$  and  $\hat{Y}$ , i.e.,

$$W(Y, \hat{Y}) := Y^T \mathcal{J} \hat{Y}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (1.1)$$

Here  $I$  and  $0$  denote the identity and zero matrices and  $\mathcal{J} \in \mathbb{R}^{2n \times 2n}$  is the canonical skew-symmetric matrix. Each matrix  $Y \in \Lambda(n)$  can be identified via its image with a Lagrangian subspace of  $\mathbb{R}^{2n}$ , which is spanned by the columns of  $Y$ . The matrix  $Y$  is then sometimes referred to as a *frame* of the Lagrangian subspace generated by  $Y$ , or as a *conjoined* or *isotropic* basis, see [5, 24]. Furthermore, a continuous function  $Y : [a, b] \rightarrow \Lambda(n)$  is called a *Lagrangian path*.

---

2020 *Mathematics Subject Classification*. Primary 53D12; Secondary 34C10.

*Keywords*. Comparative index; Maslov index; Hörmander index; Lagrangian plane; Lagrangian path; Triple index; Wronskian.

Received: 04 October 2022; Accepted: 11 November 2022

Communicated by Dragan S. Djordjević

Research supported by the Czech Science Foundation under grant GA19-01246S

*Email addresses*: [elyseeva@gmail.com](mailto:elyseeva@gmail.com) (Julia Elyseeva), [sepitkap@math.muni.cz](mailto:sepitkap@math.muni.cz) (Peter Šepitka), [hilscher@math.muni.cz](mailto:hilscher@math.muni.cz) (Roman Šimon Hilscher)

The Maslov index for the Lagrangian paths and the Hörmander index  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$  are defined in a geometric way (see below), with the Hörmander index given as a difference of two Maslov indices involving a Lagrangian path  $Y(t)$  connecting  $Y_1$  and  $Y_2$ , which intersects with  $\tilde{Y}_1$  or with  $\tilde{Y}_2$ . The efforts in the recent papers [16, 30] are directed to an efficient calculation of the Hörmander index in this finite dimensional case. In particular, in [30, Theorem 1.1] the authors present the Hörmander index, denoted here by  $s_Z(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ , in terms of the triple index defined by [6, Eq. (5)], as

$$s_Z(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = i(Y_1, Y_2, \tilde{Y}_2) - i(Y_1, Y_2, \tilde{Y}_1) = i(Y_1, \tilde{Y}_1, \tilde{Y}_2) - i(Y_2, \tilde{Y}_1, \tilde{Y}_2). \tag{1.2}$$

Recall (see [6]) that the definition of the triple index  $i(\alpha, \beta, \gamma)$  with  $\alpha, \beta, \gamma \subseteq \mathbb{R}^{2n}$  being Lagrangian subspaces uses the bilinear form  $Q(\alpha, \beta, \gamma)$  defined on the subspace  $\alpha \cap (\beta + \gamma)$ , as well as it uses the information about the dimensions of the intersections  $\alpha \cap \gamma$  and  $\alpha \cap \beta \cap \gamma$ , see [30, Lemma 3.13] for more details. However, taking in mind the recent applications of the Maslov index in the oscillation and spectral theory of linear Hamiltonian differential systems (see [11, 15, 16, 18–20]), where the main results are formulated in terms of the frames of Lagrangian paths, it seems natural to present connections between the Maslov indices for different paths, and in particular the Hörmander index, in terms of the frames  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$ . According to our knowledge, the representations of the Hörmander index in terms of the frames  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$  are known in this situation only for special cases associated with different transversality conditions for the Lagrangian planes, meaning that some blocks of  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$  or/and their Wronskians (1.1) are nonsingular, see [20, Lemma 2.3, Corollary 1] and [16, Section 3].

The aim of this paper is to offer a convenient algebraic tool, which we call the *comparative index* (see [9] or [7, Chapter 3]), presenting connections between the Maslov indices for three Lagrangian paths  $Y_1, Y_2, Y_3$  in terms of the frames  $Y_1(a), Y_2(a), Y_3(a)$  and  $Y_1(b), Y_2(b), Y_3(b)$  defined by their endpoint values. More precisely, for the Lagrangian paths  $Y_1, Y_2, Y_3$  on  $[a, b]$  we consider a continuous symplectic matrix  $Z_1(t)$  satisfying  $Y_1(t) = Z_1(t)(0 \ I)^T$  on  $[a, b]$ . Then we prove the formula, see Theorem 2.2,

$$\begin{aligned} & \text{Mas}(Y_1, Y_2, [a, b]) + \text{Mas}(Y_2, Y_3, [a, b]) - \text{Mas}(Y_1, Y_3, [a, b]) \\ &= \mu(Z_1^{-1}(a)Y_3(a), Z_1^{-1}(a)Y_2(a)) - \mu(Z_1^{-1}(b)Y_3(b), Z_1^{-1}(b)Y_2(b)). \end{aligned}$$

The numbers  $\mu(Z_1^{-1}(t)Y_3(t), Z_1^{-1}(t)Y_2(t))$  for  $t \in \{a, b\}$  are defined by the Wronskians involving  $Y_1(t), Y_2(t), Y_3(t)$  for  $t \in \{a, b\}$  and they do not depend on the choice of the matrix  $Z_1(t)$ , for which  $Y_1(t)$  forms its second block column according to the above definition. The number  $\mu(Y, \hat{Y})$  is defined for arbitrary Lagrangian planes  $Y$  and  $\hat{Y}$  and it is called the *comparative index*, see [9] or [7, Chapter 3] and Section 2.1 for more details. It has useful applications in the oscillation and spectral theory of linear Hamiltonian systems and their discrete analogs – symplectic difference systems, see [7, 10–12, 26, 27] and the references therein.

For the special case when  $Y_1(t) := Y(t), Y_2(t) := \tilde{Y}(b), Y_3(t) := \tilde{Y}(a)$  and using the formula for the Hörmander index  $s(Y(a), Y(b), \tilde{Y}(a), \tilde{Y}(b))$  as the difference of the Maslov indices  $\text{Mas}(Y, \tilde{Y}(b), [a, b])$  and  $\text{Mas}(Y, \tilde{Y}(a), [a, b])$  (see [16, Section 3], [30, Section 3], and equation (3.5) below) we derive from Theorem 2.2 the representation of the Hörmander index in terms of the frames  $Y(a), Y(b), \tilde{Y}(a), \tilde{Y}(b)$  as

$$s(Y(a), Y(b), \tilde{Y}(a), \tilde{Y}(b)) = \mu(Z^{-1}(a)\tilde{Y}(a), Z^{-1}(a)\tilde{Y}(b)) - \mu(Z^{-1}(b)\tilde{Y}(a), Z^{-1}(b)\tilde{Y}(b)),$$

where  $Z(t)$  is a continuous symplectic matrix on  $[a, b]$  having  $Y(t)$  as its second block column (see Theorem 3.2). Alternatively, we obtain the formula

$$s(Y(a), Y(b), \tilde{Y}(a), \tilde{Y}(b)) = \mu(\tilde{Y}(b), Y(a)) - \mu(\tilde{Y}(b), Y(b)) - \mu(\tilde{Y}(a), Y(a)) + \mu(\tilde{Y}(a), Y(b)).$$

We also consider several applications of Theorem 3.2, which are based on the properties of the comparative index. These include general estimates for the Hörmander index, explicit conditions for its extreme values or its sign, or connections of the comparative index with the triple index (Corollaries 3.7, 3.9, and 3.13). Therefore, we trust that these results are a useful complement to the geometric approach to the Hörmander index in [6, 16, 30].

The organization of the paper is the following. In Section 2 we present the connections of the comparative index with the Maslov index, including the proof of the above mentioned Theorem 2.2. In Section 3 we study the relations between the comparative index and the Hörmander index and present several applications of Theorem 3.2, including a connection between the comparative index and the triple index. Finally, in Section 4 we make some additional comments about the results of this paper and their further development.

## 2. Comparative index and Maslov index in finite dimension

### 2.1. Comparative index

By [8, 9] or [7, Chapter 3], for two Lagrangian planes  $Y_1, Y_2 \in \Lambda(n)$  we define their *comparative index*  $\mu(Y_1, Y_2)$  and the *dual comparative index*  $\mu^*(Y_1, Y_2)$  by

$$\mu(Y_1, Y_2) := \text{rank } \mathcal{M} + \text{ind } \mathcal{P}, \quad 0 \leq \mu(Y_1, Y_2) \leq n, \tag{2.1}$$

$$\mu^*(Y_1, Y_2) := \text{rank } \mathcal{M} + \text{ind } (-\mathcal{P}), \quad 0 \leq \mu^*(Y_1, Y_2) \leq n, \tag{2.2}$$

where the matrices  $\mathcal{M}$  and  $\mathcal{P}$  are defined by

$$\mathcal{M} := (I - X_1^\dagger X_1) W(Y_1, Y_2), \quad \mathcal{P} := V [W(Y_1, Y_2)]^T X_1^\dagger X_2 V, \quad V := I - \mathcal{M}^\dagger \mathcal{M}, \tag{2.3}$$

and where  $X_1$  and  $X_2$  are the upper  $n \times n$  blocks of  $Y_1$  and  $Y_2$ . Here we use the partitions

$$Y_1 = (X_1^T, U_1^T)^T, \quad Y_2 = (X_2^T, U_2^T)^T. \tag{2.4}$$

Note that the matrix  $\mathcal{P}$  is symmetric according to [9, Theorem 2.1] or [7, Theorem 3.2(iii)]. The dagger in (2.3) denotes the Moore–Penrose pseudoinverse, see e.g. [1, 3].

The comparative index and the dual comparative index defined in (2.1) and (2.2) satisfy, among other properties, the relations

$$\mu(Y_1, Y_2) + \mu(Y_2, Y_1) = \text{rank } W(Y_1, Y_2) = \mu^*(Y_1, Y_2) + \mu^*(Y_2, Y_1), \tag{2.5}$$

$$\mu(Y_1, Y_2) + \mu^*(Y_1, Y_2) = \text{rank } W(Y_1, Y_2) - \text{rank } X_1 + \text{rank } X_2, \tag{2.6}$$

$$\mu(Z_1(0 I)^T, Z_2(0 I)^T) = \mu^*(Z_1^{-1}(0 I)^T, Z_1^{-1}Z_2(0 I)^T), \tag{2.7}$$

where  $Z_1$  and  $Z_2$  are arbitrary  $2n \times 2n$  symplectic matrices. These properties are proven in [9, pg. 448] or in [7, Theorem 3.5]. In addition, if the upper blocks in (2.4) are invertible, then the comparative index of  $Y_1$  and  $Y_2$  reduces to the index of the difference of the associated Riccati quotients, i.e.,

$$\mu(Y_1, Y_2) = \text{ind}(Q_2 - Q_1), \quad \mu^*(Y_1, Y_2) = \text{ind}(Q_1 - Q_2), \quad Q_j := U_j X_j^{-1}. \tag{2.8}$$

These formulas are easily obtained from (2.1), (2.2), and (2.3).

If we denote the vertical Lagrangian plane (also called the Dirichlet Lagrangian plane) and the horizontal Lagrangian plane (also called the Neuman Lagrangian plane) by

$$E := \begin{pmatrix} 0 & I \end{pmatrix}^T, \quad N := \begin{pmatrix} I & 0 \end{pmatrix}^T, \tag{2.9}$$

then we obtain for every Lagrangian plane  $Y \in \Lambda(n)$  the expressions

$$\mu(Y, E) = 0, \quad \mu^*(Y, E) = 0, \tag{2.10}$$

$$\mu(Y, N) = n - \text{rank } X + \text{ind}(-X^T U), \quad \mu^*(Y, N) = n - \text{rank } X + \text{ind}(X^T U), \tag{2.11}$$

which are easily obtained from (2.1), (2.2), (2.5), and (2.6).

2.2. Maslov index

For the definition of the Maslov index of two Lagrangian paths  $Y$  and  $\hat{Y}$  on  $[a, b]$  we will utilize the continuous angles  $\varphi_j(t)$  of the eigenvalues  $\gamma_j(t) = \exp(i\varphi_j(t))$  for  $j \in \{1, \dots, n\}$  of the unitary matrix

$$\Gamma(t) := [X(t) + iU(t)][X(t) - iU(t)]^{-1} [\hat{X}(t) - i\hat{U}(t)][\hat{X}(t) + i\hat{U}(t)]^{-1}$$

for  $t \in [a, b]$ . The matrices  $X, U$  and  $\hat{X}, \hat{U}$  are the  $n \times n$  blocks of  $Y$  and  $\hat{Y}$ , which are defined on  $[a, b]$  according to the notation introduced in (2.4). This approach to the Maslov index is known e.g. in [2, 16, 18, 19, 30], see also [22, 23, 25]. Equivalently we may use the Lidskii angles of the symplectic orthogonal matrix  $S(t) := Z_Y^T(t)Z_{\hat{Y}}(t)$ , where  $Z_Y(t)$  and  $Z_{\hat{Y}}(t)$  are the symplectic and orthogonal matrices associated with  $Y(t)$  and  $\hat{Y}(t)$  through the formula

$$Z_Y(t) := \begin{pmatrix} \mathcal{J}Y(t)K_Y(t) & Y(t)K_Y(t) \end{pmatrix}, \quad K_Y(t) := [Y^T(t)Y(t)]^{-1/2}, \quad t \in [a, b], \tag{2.12}$$

see [28, 29] for the notion of Lidskii angles of a symplectic matrix. In particular, we know that the numbers  $\gamma_j(t) = \exp(i\varphi_j(t))$  are equal to the eigenvalues of the unitary matrix

$$W(t) := K_{\hat{Y}}^{-1}(t)[Y^T(t)\hat{Y}(t) + iW(Y(t), \hat{Y}(t))]^{-1}[Y^T(t)\hat{Y}(t) - iW(Y(t), \hat{Y}(t))]K_{\hat{Y}}(t),$$

as the matrices  $\Gamma(t)$  and  $W(t)$  are similar by [15, Lemma 4.1]. Then the Maslov index of the Lagrangian paths  $Y$  and  $\hat{Y}$  is defined by

$$\text{Mas}(Y, \hat{Y}, [a, b]) := \sum_{j=1}^n \left( \left\lfloor \frac{\varphi_j(b)}{2\pi} \right\rfloor - \left\lfloor \frac{\varphi_j(a)}{2\pi} \right\rfloor \right), \tag{2.13}$$

where for  $x \in \mathbb{R}$  the notation  $\lfloor x \rfloor$  stands for the greatest integer which is smaller or equal to  $x$  (the floor function), see [2, Section 2.2], [30, Definition 2.2], and [15, Theorem 4.2]. Similarly, the dual Maslov index of the Lagrangian paths  $Y$  and  $\hat{Y}$  is defined by

$$\text{Mas}^*(Y, \hat{Y}, [a, b]) := \sum_{j=1}^n \left( \left\lceil \frac{\varphi_j(b)}{2\pi} \right\rceil - \left\lceil \frac{\varphi_j(a)}{2\pi} \right\rceil \right), \tag{2.14}$$

where  $\lceil x \rceil$  stands for the smallest integer which is greater or equal to  $x$  (the ceiling function), see also [15, Remark 4.5]. Moreover, by [15, Eq. (4.21)] we have the duality relation

$$\text{Mas}^*(Y, \hat{Y}, [a, b]) = -\text{Mas}(\hat{Y}, Y, [a, b]). \tag{2.15}$$

We note that the Maslov indices  $\text{Mas}$  and  $\text{Mas}^*$  in (2.13) and (2.14) coincide, respectively, with the Maslov indices  $\text{Mas}_-$  and  $\text{Mas}_+$  considered in [30, Definition 2.2] and [2, Section 2.2].

In the special case, when the Lagrangian path  $Y$  is constant and equal to the vertical plane  $E$  defined in (2.9), the above Maslov index  $\text{Mas}(E, \hat{Y}, [a, b])$  and the dual Maslov index  $\text{Mas}^*(E, \hat{Y}, [a, b])$  reduce respectively to the oscillation number  $\mathcal{N}(\hat{Y}, [a, b])$  and the dual oscillation number  $\mathcal{N}^*(\hat{Y}, [a, b])$  of the Lagrangian path  $\hat{Y}$ . These notions are developed in [11–13] and in [15, Sections 3 and 4] by means of the comparative index theory. Consequently, we derived in [15, Corollary 5.4 and Remark 5.5] the following comparison results for the Maslov index of two Lagrangian paths  $Y$  and  $\hat{Y}$ , which involve the comparative index of  $Y$  and  $\hat{Y}$  evaluated at the endpoints of the interval  $[a, b]$ .

**Proposition 2.1.** *Let  $Y$  and  $\hat{Y}$  be given Lagrangian paths on  $[a, b]$ . Then we have*

$$\text{Mas}(Y, \hat{Y}, [a, b]) = \text{Mas}(E, \hat{Y}, [a, b]) - \text{Mas}(E, Y, [a, b]) \left. \begin{array}{l} \\ + \mu(\hat{Y}(a), Y(a)) - \mu(\hat{Y}(b), Y(b)), \end{array} \right\} \tag{2.16}$$

$$\text{Mas}^*(Y, \hat{Y}, [a, b]) = \text{Mas}^*(E, \hat{Y}, [a, b]) - \text{Mas}^*(E, Y, [a, b]) \left. \begin{array}{l} \\ + \mu^*(\hat{Y}(b), Y(b)) - \mu^*(\hat{Y}(a), Y(a)). \end{array} \right\} \tag{2.17}$$

By the symplectic invariance of the Maslov index, see e.g. [4, Property V in Section 1], for an arbitrary continuous symplectic matrix-valued function  $S$  on  $[a, b]$  we have the relation

$$\left. \begin{aligned} \text{Mas}(SY, S\hat{Y}, [a, b]) &= \text{Mas}(Y, \hat{Y}, [a, b]), \\ \text{Mas}^*(SY, S\hat{Y}, [a, b]) &= \text{Mas}^*(Y, \hat{Y}, [a, b]). \end{aligned} \right\} \tag{2.18}$$

By using identity (2.18) we derive the following generalization of Proposition 2.1 to three Lagrangian paths  $Y_1, Y_2, Y_3$  on  $[a, b]$ . For a given Lagrangian path  $Y_j$  on  $[a, b]$  we consider a continuous symplectic matrix  $Z_j$  defined on  $[a, b]$  with the property  $Y_j(t) = Z_j(t)E$  on  $[a, b]$ , i.e., the matrix  $Y_j(t)$  forms the second blocks column of  $Z_j(t)$ . Such a matrix function  $Z = (\tilde{Y}, Y)$  always exists, in particular we can complete any Lagrangian path  $Y$  by another Lagrangian path  $\tilde{Y} := \mathcal{J}YK_Y^2$  to the so-called normalized pair of Lagrangian paths satisfying  $W(\tilde{Y}(t), Y(t)) = I$  on  $[a, b]$ , where the invertible matrix  $K_Y(t)$  is defined in (2.12). In this way the continuous symplectic matrix

$$Z(t) = \left( \mathcal{J}Y(t)K_Y^2(t), Y(t) \right), \quad t \in [a, b], \tag{2.19}$$

can be determined only by the Lagrangian path  $Y$ , see e.g. [21, Corollary 3.3.9]. Moreover, by (2.19) and the formula for the inverse of a symplectic matrix we obtain for another Lagrangian path  $\hat{Y}$  that

$$Z^{-1}(t)\hat{Y}(t) = -\mathcal{J}Z^T(t)\mathcal{J}\hat{Y}(t) \stackrel{(2.19)}{=} \begin{pmatrix} -W(Y(t), \hat{Y}(t)) \\ K_Y^2(t)Y^T(t)\hat{Y}(t) \end{pmatrix}, \quad t \in [a, b]. \tag{2.20}$$

It can be shown (see [15, Remark 2.2 and Theorem 3.12]) that the results below are invariant with respect to the choices of the matrices  $Z_j(t)$  satisfying  $Y_j(t) = Z_j(t)E$  for  $j \in \{1, 2, 3\}$ . In particular, the matrices  $Z_j(t)$  can be chosen in the form of (2.19). Expressions of the form (2.20) can be used in the following comparison result.

**Proposition 2.2.** *Let  $Y_1, Y_2, Y_3$  be given Lagrangian paths on  $[a, b]$ , with their associated continuous symplectic matrices  $Z_j$  satisfying  $Y_j(t) = Z_j(t)E$  on  $[a, b]$  for  $j \in \{1, 2, 3\}$ . Then*

$$\left. \begin{aligned} \text{Mas}(Y_1, Y_2, [a, b]) + \text{Mas}(Y_2, Y_3, [a, b]) - \text{Mas}(Y_1, Y_3, [a, b]) \\ = \mu\left(Z_1^{-1}(a)Y_3(a), Z_1^{-1}(a)Y_2(a)\right) - \mu\left(Z_1^{-1}(b)Y_3(b), Z_1^{-1}(b)Y_2(b)\right), \end{aligned} \right\} \tag{2.21}$$

$$\left. \begin{aligned} \text{Mas}^*(Y_1, Y_2, [a, b]) + \text{Mas}^*(Y_2, Y_3, [a, b]) - \text{Mas}^*(Y_1, Y_3, [a, b]) \\ = \mu^*\left(Z_1^{-1}(b)Y_3(b), Z_1^{-1}(b)Y_2(b)\right) - \mu^*\left(Z_1^{-1}(a)Y_3(a), Z_1^{-1}(a)Y_2(a)\right). \end{aligned} \right\} \tag{2.22}$$

*Proof.* We consider the Lagrangian paths  $Y := Z_1^{-1}Y_2$  and  $\hat{Y} := Z_1^{-1}Y_3$  on  $[a, b]$ . Then by formula (2.16) in Proposition 2.1 we have

$$\begin{aligned} \text{Mas}(Z_1^{-1}Y_2, Z_1^{-1}Y_3, [a, b]) &= \text{Mas}(E, Z_1^{-1}Y_3, [a, b]) - \text{Mas}(E, Z_1^{-1}Y_2, [a, b]) \\ &\quad + \mu\left(Z_1^{-1}(a)Y_3(a), Z_1^{-1}(a)Y_2(a)\right) - \mu\left(Z_1^{-1}(b)Y_3(b), Z_1^{-1}(b)Y_2(b)\right). \end{aligned}$$

Applying the symplectic invariance, i.e., formula (2.18) with the matrix  $S := Z_1^{-1}$ , to all Maslov indices in the above formula and incorporating that  $Y_1(t) = Z_1(t)E$  on  $[a, b]$  we derive equation (2.21). The proof of equation (2.22) is based on formulas (2.17) and (2.18) in a similar way.  $\square$

**Remark 2.3.** (i) The comparative indices on the right-hand sides of (2.21) and (2.22) are uniquely defined by the Wronskians  $W(Y_1, Y_2)$ ,  $W(Y_1, Y_3)$ , and  $W(Y_2, Y_3)$  and they do not depend on the choice of the matrices  $Z_j(t)$  with  $Y_j(t) = Z_j(t)E$  for  $j \in \{1, 2, 3\}$ . Indeed, by using the representation (compare with (2.20))

$$W(Y_k, Y_j) = -(I, 0)Z_k^{-1}Y_j, \quad Y_k = Z_kE,$$

and by (2.1), (2.2), and (2.3) we have (suppressing the argument  $t \in \{a, b\}$ )

$$\mu(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = \mu_1 + \mu_2, \quad \mu^*(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = \mu_1 + \mu_2^*,$$

where

$$\mu_1 = \mu_1(Z_1^{-1}Y_3, Z_1^{-1}Y_2) := \text{rank } \mathcal{M}, \quad \mathcal{M} = (I - [W(Y_1, Y_3)]^\dagger W(Y_1, Y_3))W(Y_3, Y_2),$$

while with  $V = I - \mathcal{M}^\dagger \mathcal{M}$  we have

$$\begin{aligned} \mu_2 &= \mu_2(Z_1^{-1}Y_3, Z_1^{-1}Y_2) := \text{ind } \mathcal{P}, \quad \mu_2^* = \mu_2^*(Z_1^{-1}Y_3, Z_1^{-1}Y_2) := \text{ind}(-\mathcal{P}), \\ \mathcal{P} &= V[W(Y_3, Y_2)]^T [W(Y_1, Y_3)]^\dagger W(Y_1, Y_2) V. \end{aligned}$$

In the above expressions we also simplified the Wronskian  $W(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = W(Y_3, Y_2)$ .

(ii) The comparative indices on the right-hand sides of (2.21) and (2.22) can be presented respectively in terms of  $\mu(Y_3, Y_2), \mu(Y_2, Y_1), \mu(Y_3, Y_1)$  and  $\mu^*(Y_3, Y_2), \mu^*(Y_2, Y_1), \mu^*(Y_3, Y_1)$  by the main theorem of the comparative index theory, see [9, Theorem 2.2, Eq. (2.14), (2.15)] or [7, Theorem 3.5, Corollary 3.12, Eq. (3.17), (3.26)]. More precisely, for an arbitrary  $2n \times 2n$  symplectic matrix  $S$  and for any Lagrangian planes  $Y$  and  $\hat{Y}$  we have

$$\mu(S^{-1}Y, S^{-1}\hat{Y}) = \mu(Y, \hat{Y}) + \mu(\hat{Y}, SE) - \mu(Y, SE), \tag{2.23}$$

$$\mu^*(S^{-1}Y, S^{-1}\hat{Y}) = \mu^*(Y, \hat{Y}) + \mu^*(\hat{Y}, SE) - \mu^*(Y, SE). \tag{2.24}$$

For example, by taking  $Y := Y_3, \hat{Y} := Y_2,$  and  $S := Z_1$  in (2.23) and (2.24) we deduce that

$$\mu(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = \mu(Y_3, Y_2) + \mu(Y_2, Y_1) - \mu(Y_3, Y_1), \tag{2.25}$$

$$\mu^*(Z_1^{-1}Y_3, Z_1^{-1}Y_2) = \mu^*(Y_3, Y_2) + \mu^*(Y_2, Y_1) - \mu^*(Y_3, Y_1). \tag{2.26}$$

The sums of the comparative indices on the right-hand sides of (2.25) and (2.26) are special cases of *cyclic sums* of comparative indices (of the second kind). Such cyclic sums are investigated in the recent paper [14], where they are denoted by  $\nu_c^-(Y_3, Y_2, Y_1)$  for (2.25), resp. by  $\nu_c^+(Y_3, Y_2, Y_1)$  for (2.26). We can see from (2.25) and (2.26) that these cyclic sums possess the symplectic invariance property  $\nu_c^\mp(SY_3, SY_2, SY_1) = \nu_c^\mp(Y_3, Y_2, Y_1)$ , compare with (2.18).

The results in Propositions 2.1 and 2.2 are fundamental for the connections of the comparative index with the Maslov index and with the Hörmander index presented below and in the next section. Observe that if  $Y_1 := E$  is the constant vertical Lagrangian path (i.e., the matrix  $Z_1(t) \equiv I$ ), then the result in Proposition 2.2 reduces exactly to that in Proposition 2.1.

**Remark 2.4.** Based on Proposition 2.2 we can provide useful estimates of the expressions on left-hand side of (2.21) and (2.22), namely

$$|\text{Mas}(Y_1, Y_2, [a, b]) + \text{Mas}(Y_2, Y_3, [a, b]) - \text{Mas}(Y_1, Y_3, [a, b])| \leq n, \tag{2.27}$$

$$|\text{Mas}^*(Y_1, Y_2, [a, b]) + \text{Mas}^*(Y_2, Y_3, [a, b]) - \text{Mas}^*(Y_1, Y_3, [a, b])| \leq n. \tag{2.28}$$

These general estimates are based on the simplest bounds for the comparative index and the dual comparative index shown in (2.1) and (2.2), and they are independent on the chosen Lagrangian paths  $Y_1, Y_2, Y_3$ . We note that more precise estimates for the lower and upper bounds for  $\mu(Y_1, Y_2)$  and  $\mu^*(Y_1, Y_2)$  are presented in [9, Property 7, pg. 449] and [7, Theorem 3.5(vii) and Remark 3.10] in terms of the quantities  $\text{rank } X_1, \text{rank } X_2,$  and  $\text{rank } W(Y_1, Y_2)$ . Therefore, the estimates in (2.27) and (2.28) can be improved in the same spirit.

### 3. Comparative index and Hörmander index in finite dimension

#### 3.1. Hörmander index

The following definition of the Hörmander index is motivated by [16, Section 3] and [30, Section 3]. Let us fix four Lagrangian planes  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$ . We consider two Lagrangian paths  $Y$  and  $\tilde{Y}$  on  $[a, b]$  connecting the Lagrangian plane  $Y_1$  with  $Y_2$ , and the Lagrangian plane  $\tilde{Y}_1$  with  $\tilde{Y}_2$ . That is, we have

$$Y(a) = Y_1, \quad Y(b) = Y_2, \quad \tilde{Y}(a) = \tilde{Y}_1, \quad \tilde{Y}(b) = \tilde{Y}_2. \tag{3.1}$$

Then, by using formula (2.16), we can easily derive the relation

$$\text{Mas}(Y, \tilde{Y}_1, [a, b]) + \text{Mas}(Y_2, \tilde{Y}, [a, b]) - \text{Mas}(Y, \tilde{Y}_2, [a, b]) - \text{Mas}(Y_1, \tilde{Y}, [a, b]) = 0, \tag{3.2}$$

where we used that  $\text{Mas}(E, C, [a, b]) = 0$  for every  $C \in \Lambda(n)$ , as the Maslov index of two constant Lagrangian paths (i.e., of two Lagrangian planes) is zero. Formula (3.2) can be interpreted by two combined Lagrangian paths in the arguments  $t \in [a, b]$  and  $\tilde{t} \in [a, b]$ , whose Maslov index is zero by the homotopy invariance, see e.g. the Maslov box in [16, Figure 2]. Equation (3.2) implies that

$$\text{Mas}(Y, \tilde{Y}_2, [a, b]) - \text{Mas}(Y, \tilde{Y}_1, [a, b]) = \text{Mas}(Y_2, \tilde{Y}, [a, b]) - \text{Mas}(Y_1, \tilde{Y}, [a, b]), \tag{3.3}$$

where the left-hand side depends on the endpoint values of  $\tilde{Y}$  (but not on  $\tilde{Y}$  itself) and the right-hand side depends on the endpoint values of  $Y$  (but not on  $Y$  itself). Equation (3.3) then shows that the difference  $\text{Mas}(Y, \tilde{Y}_2, [a, b]) - \text{Mas}(Y, \tilde{Y}_1, [a, b])$  does not depend on the choice of the Lagrangian path  $Y$  with  $Y(a) = Y_1$  and  $Y(b) = Y_2$ , and at the same time the difference  $\text{Mas}(Y_2, \tilde{Y}, [a, b]) - \text{Mas}(Y_1, \tilde{Y}, [a, b])$  does not depend on the choice of the Lagrangian path  $\tilde{Y}$  with  $\tilde{Y}(a) = \tilde{Y}_1$  and  $\tilde{Y}(b) = \tilde{Y}_2$ , and that these two differences are equal.

The Hörmander index of the Lagrangian planes  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$  is now defined as the integer

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) := \text{Mas}(Y, \tilde{Y}_2, [a, b]) - \text{Mas}(Y, \tilde{Y}_1, [a, b]) \tag{3.4}$$

$$= \text{Mas}(Y_2, \tilde{Y}, [a, b]) - \text{Mas}(Y_1, \tilde{Y}, [a, b]), \tag{3.5}$$

where the definitions in (3.4) and (3.5) do not depend on the choice of the Lagrangian paths  $Y$  and  $\tilde{Y}$  with (3.1), as we discussed above. Similarly, by using formula (2.17) for the dual Maslov index and the dual comparative index we easily obtain the equality

$$\text{Mas}^*(Y, \tilde{Y}_2, [a, b]) - \text{Mas}^*(Y, \tilde{Y}_1, [a, b]) = \text{Mas}^*(Y_2, \tilde{Y}, [a, b]) - \text{Mas}^*(Y_1, \tilde{Y}, [a, b]),$$

which leads to the definition of the dual Hörmander index as the integer

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) := \text{Mas}^*(Y, \tilde{Y}_2, [a, b]) - \text{Mas}^*(Y, \tilde{Y}_1, [a, b]) \tag{3.6}$$

$$= \text{Mas}^*(Y_2, \tilde{Y}, [a, b]) - \text{Mas}^*(Y_1, \tilde{Y}, [a, b]). \tag{3.7}$$

We note that it may seem artificial to introduce two Hörmander indices by the above equations. But in some situations we will take advantage of working with both Hörmander index  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$  and dual Hörmander index  $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$ , see e.g. the proof of Corollary 3.13.

**Remark 3.1.** Observe that the Hörmander index considered in [30, Definition 3.9], denote it by  $s_Z$ , is equal to the dual Hörmander index  $s^*$  defined in (3.6) and (3.7), since the ceiling function is used in the definition of the corresponding dual Maslov index in (2.14) as well as in [30, Definition 3.9] with the same angles  $\varphi_j(t)$ . On the other hand, the Hörmander index considered in [16, Section 3], denote it by  $s_H$ , is equal to  $-s^*$ , since it is defined by the Maslov index in (2.13) with roles of the pairs  $Y_1, Y_2$  and  $\tilde{Y}_1, \tilde{Y}_2$  interchanged. More precisely, according to [16, Eq. (3.2)–(3.3)] we have

$$s_H(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) := \text{Mas}(\tilde{Y}, Y_2, [a, b]) - \text{Mas}(\tilde{Y}, Y_1, [a, b]) \\ \stackrel{(2.15)}{=} -\text{Mas}^*(Y_2, \tilde{Y}, [a, b]) + \text{Mas}^*(Y_1, \tilde{Y}, [a, b]) \stackrel{(3.7)}{=} -s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2),$$

that is, the Hörmander index  $s_H$  from [16, Section 3] satisfies  $s_H = -s^* = -s_Z$ .

Formulas (3.4) and (3.6), resp. (3.5) and (3.7), suggest the interpretation the Hörmander index as the correction term in the target exchange on the second position in the Maslov index, resp. on the first position in the Maslov index. Moreover, the symplectic invariance of the Maslov index in (2.18) implies the same property for the Hörmander index, i.e., for any  $2n \times 2n$  symplectic matrix  $S$  we have

$$\left. \begin{aligned} s(SY_1, SY_2, S\tilde{Y}_1, S\tilde{Y}_2) &= s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2), \\ s^*(SY_1, SY_2, S\tilde{Y}_1, S\tilde{Y}_2) &= s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2). \end{aligned} \right\} \tag{3.8}$$

In the following theorem we show that the Hörmander index and the dual Hörmander index can be calculated from the data  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$  by means of linear algebra (matrix analysis) by evaluating the involved comparative indices and the dual comparative indices through (2.1) and (2.2) with the corresponding matrices in (2.3). It is the main result of this section.

**Theorem 3.2.** *Let  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$  be given Lagrangian planes with the associated symplectic matrices  $Z_1, Z_2$  such that  $Y_j = Z_j E$  for  $j \in \{1, 2\}$ . Then the Hörmander index defined in (3.4)–(3.5) is equal to*

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) - \mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2), \tag{3.9}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(Z_2^{-1}\tilde{Y}_2, Z_2^{-1}\tilde{Y}_1) - \mu(Z_1^{-1}\tilde{Y}_2, Z_1^{-1}\tilde{Y}_1), \tag{3.10}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(\tilde{Y}_2, Y_1) - \mu(\tilde{Y}_2, Y_2) - \mu(\tilde{Y}_1, Y_1) + \mu(\tilde{Y}_1, Y_2), \tag{3.11}$$

and the dual Hörmander index defined in (3.6)–(3.7) is equal to

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2) - \mu^*(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2), \tag{3.12}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(Z_1^{-1}\tilde{Y}_2, Z_1^{-1}\tilde{Y}_1) - \mu^*(Z_2^{-1}\tilde{Y}_2, Z_2^{-1}\tilde{Y}_1), \tag{3.13}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(\tilde{Y}_2, Y_2) - \mu^*(\tilde{Y}_2, Y_1) - \mu^*(\tilde{Y}_1, Y_2) + \mu^*(\tilde{Y}_1, Y_1). \tag{3.14}$$

Moreover, the Hörmander index and the dual Hörmander index are related by the formula

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = -s(\tilde{Y}_1, \tilde{Y}_2, Y_1, Y_2). \tag{3.15}$$

*Proof.* Let the matrices  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$  and  $Z_1, Z_2$  be as in the theorem. Let  $Z(t)$  be a continuous symplectic matrix on  $[a, b]$  connecting the matrices  $Z_1$  and  $Z_2$ , i.e.,  $Z(a) = Z_1$  and  $Z(b) = Z_2$ . Then we set  $Y(t) := Z(t)E$  on  $[a, b]$ , so that  $Y(a) = Y_1$  and  $Y(b) = Y_2$ . We now apply Proposition 2.2 with the Lagrangian paths  $Y_1(t) := Y(t)$ ,  $Y_2(t) \equiv \tilde{Y}_2$ , and  $Y_3(t) \equiv \tilde{Y}_1$ . Then by using that  $Y_2(t)$  and  $Y_3(t)$  are constant on  $[a, b]$  we obtain that

$$\text{Mas}(Y, \tilde{Y}_2, [a, b]) - \text{Mas}(Y, \tilde{Y}_1, [a, b]) \stackrel{(2.21)}{=} \mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) - \mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2),$$

where we also used that  $\text{Mas}(\tilde{Y}_2, \tilde{Y}_1, [a, b]) = 0$ . By the definition of  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$  in (3.4) we now obtain formula (3.9). For the proof of (3.10) we apply the first formula in (2.5) to the two comparative indices on the right-hand side of (3.9). Then by using that the Wronskians  $W(Z_j^{-1}\tilde{Y}_1, Z_j^{-1}\tilde{Y}_2) = W(\tilde{Y}_1, \tilde{Y}_2)$  for  $j \in \{1, 2\}$  for both terms are the same we derive from (3.9) the formula in (3.10). Equation (3.11) follows from the definition in (3.4) by expanding the two Maslov indices with the comparison formula (2.16). In more details, we have

$$\begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &\stackrel{(3.4)}{=} \text{Mas}(Y, \tilde{Y}_2, [a, b]) - \text{Mas}(Y, \tilde{Y}_1, [a, b]) \\ &\stackrel{(2.16)}{=} \left\{ \text{Mas}(E, \tilde{Y}_2, [a, b]) - \text{Mas}(E, Y, [a, b]) + \mu(\tilde{Y}_2, Y(a)) - \mu(\tilde{Y}_2, Y(b)) \right\} \\ &\quad - \left\{ \text{Mas}(E, \tilde{Y}_1, [a, b]) - \text{Mas}(E, Y, [a, b]) + \mu(\tilde{Y}_1, Y(a)) - \mu(\tilde{Y}_1, Y(b)) \right\} \\ &\stackrel{(3.1)}{=} \mu(\tilde{Y}_2, Y_1) - \mu(\tilde{Y}_2, Y_2) - \mu(\tilde{Y}_1, Y_1) + \mu(\tilde{Y}_1, Y_2), \end{aligned}$$



where we used that  $\text{Mas}(E, \tilde{Y}_j, [a, b]) = 0$ . The proof of (3.12) follows from equation (2.22) in Proposition 2.2 (with the Lagrangian paths  $Y_1 := Y$ ,  $Y_2 := \tilde{Y}_2$ , and  $Y_3 := \tilde{Y}_1$ ) and from the definition of  $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$  in (3.6). For the proof of (3.13) we apply the second formula in (2.5) to the two comparative indices on the right-hand side of (3.12). Equation (3.14) follows by expanding the two Maslov indices in (3.6) with the comparison formula (2.17) for the dual Maslov index. Finally, by using the relationship between the Maslov index and the dual Maslov index in (2.15) we get

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) \stackrel{(3.6)}{=} \text{Mas}^*(Y, \tilde{Y}_2, [a, b]) - \text{Mas}^*(Y, \tilde{Y}_1, [a, b]) \\ \stackrel{(2.15)}{=} -\text{Mas}(\tilde{Y}_2, Y, [a, b]) + \text{Mas}(\tilde{Y}_1, Y, [a, b]) \stackrel{(3.5)}{=} -s(\tilde{Y}_1, \tilde{Y}_2, Y_1, Y_2),$$

which proves formula (3.15). The proof is complete.  $\square$

**Remark 3.3.** (i) Recall that by Remark 2.3(i) the comparative indices on the right-hand sides of (3.9)–(3.10) and (3.12)–(3.13) are uniquely defined by the Wronskians  $W(Y_j, \tilde{Y}_1)$ ,  $W(Y_j, \tilde{Y}_2)$ , and  $W(\tilde{Y}_1, \tilde{Y}_2)$  for  $j \in \{1, 2\}$ , and then these comparative indices do not depend on the choice of the matrices  $Z_j$ . In particular, in view of (2.20) the transformed Lagrangian planes appearing in (3.9)–(3.10) and (3.12)–(3.13) can be taken in the form (for  $j, k \in \{1, 2\}$ )

$$Z_j^{-1}\tilde{Y}_k = -\mathcal{J}Z_j^T\mathcal{J}\tilde{Y}_k \stackrel{(2.20)}{=} \begin{pmatrix} -W(Y_j, \tilde{Y}_k) \\ K_j^2 Y_j^T \tilde{Y}_k \end{pmatrix}, \quad K_j := K_{Y_j} = (Y_j^T Y_j)^{-1/2}. \tag{3.16}$$

(ii) By using the duality relation in (3.15) we can derive from equations (3.12)–(3.14) alternative expressions for the Hörmander index  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$  in terms of the dual comparative index in the form

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(\tilde{Z}_1^{-1}Y_1, \tilde{Z}_1^{-1}Y_2) - \mu^*(\tilde{Z}_2^{-1}Y_1, \tilde{Z}_2^{-1}Y_2), \tag{3.17}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(\tilde{Z}_2^{-1}Y_2, \tilde{Z}_2^{-1}Y_1) - \mu^*(\tilde{Z}_1^{-1}Y_2, \tilde{Z}_1^{-1}Y_1), \tag{3.18}$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(Y_2, \tilde{Y}_1) - \mu^*(Y_2, \tilde{Y}_2) - \mu^*(Y_1, \tilde{Y}_1) + \mu^*(Y_1, \tilde{Y}_2), \tag{3.19}$$

where  $\tilde{Z}_1, \tilde{Z}_2$  are symplectic matrices such that  $\tilde{Y}_j = \tilde{Z}_j E$  for  $j \in \{1, 2\}$ . Similarly, from equations (3.9)–(3.11) we can derive alternative expressions for the dual Hörmander index  $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$  in terms of the comparative index in the form

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(\tilde{Z}_2^{-1}Y_1, \tilde{Z}_2^{-1}Y_2) - \mu(\tilde{Z}_1^{-1}Y_1, \tilde{Z}_1^{-1}Y_2), \tag{3.20}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(\tilde{Z}_1^{-1}Y_2, \tilde{Z}_1^{-1}Y_1) - \mu(\tilde{Z}_2^{-1}Y_2, \tilde{Z}_2^{-1}Y_1), \tag{3.21}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \mu(Y_2, \tilde{Y}_2) - \mu(Y_2, \tilde{Y}_1) - \mu(Y_1, \tilde{Y}_2) + \mu(Y_1, \tilde{Y}_1). \tag{3.22}$$

(iii) We note that equalities (3.11) and (3.14) also follow from Remark 2.3(ii) by applying formulas (2.25) and (2.26) to the comparative indices on the right-hand sides of (3.9)–(3.10) and (3.12)–(3.13).

As an application of Theorem 3.2 we derive a simple geometric interpretation of the comparative index of two Lagrangian planes  $Y_1$  and  $Y_2$ . Namely, it is the difference of the Maslov indices (i.e., the intersection numbers) of an arbitrarily chosen Lagrangian path  $Y$  connecting  $Y_1$  and  $Y_2$  with the Lagrangian planes  $Y_2$  and  $E$ . In other words, the comparative index is a special Hörmander index involving the Lagrangian planes  $Y_1, Y_2$ , and  $E$ .

**Theorem 3.4.** Let  $Y_1, Y_2 \in \Lambda(n)$  be given Lagrangian planes. Then we have the formulas

$$\mu(Y_1, Y_2) = s(E, Y_2, Y_1, Y_2) = \text{Mas}(Y_2, Y, [a, b]) - \text{Mas}(E, Y, [a, b]), \tag{3.23}$$

$$\mu^*(Y_1, Y_2) = -s^*(E, Y_2, Y_1, Y_2) = \text{Mas}^*(E, Y, [a, b]) - \text{Mas}^*(Y_2, Y, [a, b]), \tag{3.24}$$

where  $Y$  is an arbitrary Lagrangian path on  $[a, b]$  with  $Y(a) = Y_1$  and  $Y(b) = Y_2$ . Alternatively,

$$\mu(Y_1, Y_2) = -s^*(Y_1, Y_2, E, Y_2), \quad \mu^*(Y_1, Y_2) = s(Y_1, Y_2, E, Y_2). \tag{3.25}$$

*Proof.* By calculating the values  $s(E, Y_2, Y_1, Y_2)$  and  $s^*(E, Y_2, Y_1, Y_2)$  according to equations (3.11) and (3.14) we obtain

$$\begin{aligned} s(E, Y_2, Y_1, Y_2) &= \mu(Y_2, E) - \mu(Y_2, Y_2) - \mu(Y_1, E) + \mu(Y_1, Y_2) = \mu(Y_1, Y_2), \\ s^*(E, Y_2, Y_1, Y_2) &= \mu^*(Y_2, Y_2) - \mu^*(Y_2, E) - \mu^*(Y_1, Y_2) + \mu^*(Y_1, E) = -\mu^*(Y_1, Y_2), \end{aligned}$$

where we used the basic properties in (2.10) of the comparative index. This proves the first equations in (3.23) and (3.24). The second equations in (3.23) and (3.24) then follow from the definition of the Hörmander index and the dual Hörmander index in (3.5) and (3.7). Finally, the equalities in (3.25) follow from (3.23) and (3.24) by the relationship between the Hörmander index and the dual Hörmander index in (3.15).  $\square$

### 3.2. Several applications

Next we present several applications of Theorem 3.2. First we consider the special case when the upper blocks of the Lagrangian planes in (3.9)–(3.14) or in (3.17)–(3.22) are invertible matrices. In this case we can calculate the Hörmander index by using formula (2.8).

**Remark 3.5.** (i) Let  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$  be given Lagrangian planes with the associated symplectic matrices  $Z_1, Z_2$  in (3.16) such that  $Y_j = Z_j E$  for  $j \in \{1, 2\}$ . Assume that the Wronskians  $W(Y_j, \tilde{Y}_k)$  are invertible for  $j, k \in \{1, 2\}$ , i.e., the subspaces generated by  $Y_j$  and  $\tilde{Y}_k$  are transversal. Such an assumption is common in some references, such as in [30, Corollary 3.11]. Then we have the equality

$$\left. \begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \text{ind}(M_{11} - M_{12}) - \text{ind}(M_{21} - M_{22}), \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2), \\ M_{jk} &:= K_j^2 Y_j^T \tilde{Y}_k [W(Y_j, \tilde{Y}_k)]^{-1}, \end{aligned} \right\} \tag{3.26}$$

where we applied formula (2.8) to the comparative indices in (3.9) and to the the dual comparative indices in (3.12). Analogous results can be obtained by applying formula (2.8) to the other comparative indices and dual comparative indices appearing in Theorem 3.2 or in Remark 3.3(ii). Formulas of the type (3.26) were derived in [16, pp. 23–24]. Note also that the second formula in (3.26) is consistent with [30, Corollary 3.11].

(ii) Let  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$  be given Lagrangian planes, whose upper blocks  $X_j$  and  $\tilde{X}_k$  are invertible matrices for  $j, k \in \{1, 2\}$ , i.e., the Lagrangian planes  $Y_j$  and  $\tilde{Y}_k$  do not intersect the vertical plane  $E$ . Then we have the equalities

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \text{ind}(Q_1 - \tilde{Q}_2) - \text{ind}(Q_2 - \tilde{Q}_2) - \text{ind}(Q_1 - \tilde{Q}_1) + \text{ind}(Q_2 - \tilde{Q}_1), \tag{3.27}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \text{ind}(\tilde{Q}_2 - Q_2) - \text{ind}(\tilde{Q}_2 - Q_1) - \text{ind}(\tilde{Q}_1 - Q_2) + \text{ind}(\tilde{Q}_1 - Q_1), \tag{3.28}$$

where for  $j, k \in \{1, 2\}$  we have

$$Q_j := U_j X_j^{-1}, \quad \tilde{Q}_k := \tilde{U}_k \tilde{X}_k^{-1}, \quad Q_j - \tilde{Q}_k = -X_j^T W(Y_j, \tilde{Y}_k) \tilde{X}_k^{-1}. \tag{3.29}$$

For equality (3.27) we applied formula (2.8) to the comparative indices appearing in (3.11) or equivalently in (3.19), while for equality (3.28) we applied formula (2.8) to the dual comparative indices appearing in (3.14) or equivalently in (3.22).

The result in Remark 3.5(ii) can be generalized by using the symplectic invariance of the Hörmander index as follows. Note that with the choice of the symplectic matrix  $R = I$  the equalities in (3.30)–(3.32) below reduce to (3.27)–(3.29).

**Corollary 3.6.** Let  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$  be given Lagrangian planes. Let  $R$  be an arbitrary symplectic matrix, i.e.,  $R = (R_1, R_2)$  with  $R_1, R_2 \in \Lambda(n)$  and  $W(R_1, R_2) = I$ . Assume that the Lagrangian plane  $R_2 = RE$  is transversal

to  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2$ , i.e., the Wronskians  $W(R_2, Y_j)$  and  $W(R_2, \tilde{Y}_k)$  are invertible for  $j, k \in \{1, 2\}$ . Then we have the equalities

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \text{ind}(\tilde{N}_2 - N_1) - \text{ind}(\tilde{N}_2 - N_2) - \text{ind}(\tilde{N}_1 - N_1) + \text{ind}(\tilde{N}_1 - N_2), \tag{3.30}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = \text{ind}(N_2 - \tilde{N}_2) - \text{ind}(N_1 - \tilde{N}_2) - \text{ind}(N_2 - \tilde{N}_1) + \text{ind}(N_1 - \tilde{N}_1), \tag{3.31}$$

where for  $j, k \in \{1, 2\}$  the Riccati quotients  $N_j$  and  $\tilde{N}_k$  are defined by

$$N_j := W(R_1, Y_j)[W(R_2, Y_j)]^{-1}, \quad \tilde{N}_j := W(R_1, \tilde{Y}_j)[W(R_2, \tilde{Y}_j)]^{-1}. \tag{3.32}$$

*Proof.* By the symplectic invariance of the Hörmander index, i.e., equation (3.8) with the symplectic matrix  $S := R^{-1}$ , we know that

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = s(R^{-1}Y_1, R^{-1}Y_2, R^{-1}\tilde{Y}_1, R^{-1}\tilde{Y}_2), \tag{3.33}$$

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = s^*(R^{-1}Y_1, R^{-1}Y_2, R^{-1}\tilde{Y}_1, R^{-1}\tilde{Y}_2), \tag{3.34}$$

where the upper blocks of the transformed Lagrangian planes  $R^{-1}Y_j$  and  $R^{-1}\tilde{Y}_k$  are equal respectively to the Wronskians  $-W(R_2, Y_j)$  and  $-W(R_2, \tilde{Y}_k)$ , i.e., they are invertible by our assumptions. Therefore, equations (3.30) and (3.31) follow by the application of (3.27) and (3.28) to the right-hand side of (3.33) and (3.34), i.e., we take  $Q_j := -N_j$  and  $\tilde{Q}_k := -\tilde{N}_k$ .  $\square$

Next we obtain the following universal estimates for the values of the Hörmander index of four arbitrary Lagrangian planes. This result also implies that for a given Lagrangian path on  $[a, b]$  its Maslov indices (i.e., the intersection numbers) with respect to any two fixed Lagrangian planes can differ by at most  $n$ .

**Corollary 3.7.** *Let  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$  be given Lagrangian planes. Then we have*

$$|s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)| \leq n, \quad |s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)| \leq n. \tag{3.35}$$

*In addition, the Hörmander index attains its maximal value  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = n$  if and only if*

$$\mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) = n \quad \text{and} \quad \mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2) = 0, \tag{3.36}$$

*while the Hörmander index attains its minimal value  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = -n$  if and only if*

$$\mu(Z_1^{-1}\tilde{Y}_1, Z_1^{-1}\tilde{Y}_2) = 0 \quad \text{and} \quad \mu(Z_2^{-1}\tilde{Y}_1, Z_2^{-1}\tilde{Y}_2) = n, \tag{3.37}$$

where  $Z_1, Z_2$  are associated symplectic matrices such that  $Y_j = Z_j E$  for  $j \in \{1, 2\}$ .

*Proof.* The proof of the estimates in (3.35) follows from (3.9) and (3.12) by using the lower and upper bounds for the comparative index in (2.1) and (2.2). The statements in (3.36) and (3.37) about the maximal and minimal values of  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2)$  follow from equation (3.9).  $\square$

**Remark 3.8.** (i) Further equivalent conditions for the extreme value  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = n$  or for the extreme value  $s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = -n$  in the spirit of (3.36) and (3.37) can be obtained from equations (3.10), (3.17), and (3.18). Similarly, we may formulate equivalent conditions for the extreme values of the dual Hörmander index  $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = n$  or for  $s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = -n$  via equations (3.12), (3.13), (3.20), and (3.21).

(ii) The conditions on  $\mu(Y, \hat{Y}) = 0$  in (3.36) and (3.37) or in part (i) of this remark can be efficiently verified by checking the validity of the equivalent conditions presented in [9, Eq. (1.13), (1.14)] or [7, Theorem 3.14(iv)]. For this purpose we recall that the transformed Lagrangian planes appearing in (3.36) and (3.37) can have the form (3.16).

The third application of Theorem 3.2 is based on the additional assumptions on the Wronskians  $W(Y_j, \tilde{Y}_k) = 0$ , or equivalently on  $\dim(\text{Im } Y_j \cap \text{Im } \tilde{Y}_k) = n$  for the associated Lagrangian subspaces. In this case we can investigate the sign of the Hörmander index and the dual Hörmander index.

**Corollary 3.9.** Let  $Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$  be given Lagrangian planes with the associated symplectic matrices  $Z_1, Z_2$  such that  $Y_j = Z_j E$  for  $j \in \{1, 2\}$ . If  $W(Y_1, \tilde{Y}_1) = 0$ , then

$$\left. \begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \mu(Z_2^{-1} \tilde{Y}_2, Z_2^{-1} \tilde{Y}_1) \geq 0, \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= -\mu^*(Z_2^{-1} \tilde{Y}_2, Z_2^{-1} \tilde{Y}_1) \leq 0. \end{aligned} \right\} \quad (3.38)$$

Similarly, if  $W(Y_2, \tilde{Y}_2) = 0$ , then

$$\left. \begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \mu(Z_1^{-1} \tilde{Y}_1, Z_1^{-1} \tilde{Y}_2) \geq 0, \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= -\mu^*(Z_1^{-1} \tilde{Y}_1, Z_1^{-1} \tilde{Y}_2) \leq 0. \end{aligned} \right\} \quad (3.39)$$

Moreover, if  $W(Y_2, \tilde{Y}_1) = 0$ , then

$$\left. \begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= -\mu(Z_1^{-1} \tilde{Y}_2, Z_1^{-1} \tilde{Y}_1) \leq 0, \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \mu^*(Z_1^{-1} \tilde{Y}_2, Z_1^{-1} \tilde{Y}_1) \geq 0. \end{aligned} \right\} \quad (3.40)$$

Similarly, if  $W(Y_1, \tilde{Y}_2) = 0$ , then

$$\left. \begin{aligned} s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= -\mu(Z_2^{-1} \tilde{Y}_1, Z_2^{-1} \tilde{Y}_2) \leq 0, \\ s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) &= \mu^*(Z_2^{-1} \tilde{Y}_1, Z_2^{-1} \tilde{Y}_2) \geq 0. \end{aligned} \right\} \quad (3.41)$$

*Proof.* The proof is based on the fact that  $\mu(Y_1, Y_2) = 0 = \mu^*(Y_1, Y_2)$  under the assumption that the upper block of  $Y_2$  satisfies  $X_2 = 0$ , see (2.3) or (2.10). In our case the upper block of  $Z_j^{-1} \tilde{Y}_k$  is equal to  $-W(Y_j, \tilde{Y}_k)$  according to (3.16). Then (3.38) follows from (3.10) and (3.13) under the assumption  $W(Y_1, \tilde{Y}_1) = 0$ , while (3.39) follows from (3.9) and (3.12) under the assumption  $W(Y_2, \tilde{Y}_2) = 0$ . And similarly, (3.40) follows from (3.10) and (3.13) under the assumption  $W(Y_2, \tilde{Y}_1) = 0$ , while (3.41) follows from (3.9) and (3.12) under the assumption  $W(Y_1, \tilde{Y}_2) = 0$ . The proof is complete.  $\square$

As the fourth application of Theorem 3.2 we obtain the expression of the comparative index of two Lagrangian planes as a special Hörmander index, indicating that the Hörmander index plays a balancing role in exchanging the first Lagrangian plane in the comparative index by the vertical plane  $E$ .

**Corollary 3.10.** Let  $Y_1, Y_2 \in \Lambda(n)$  be given Lagrangian planes with partitions (2.4). Then

$$\mu(Y_1, Y_2) = \mu(E, Y_2) - s(E, Y_2, E, Y_1), \quad \mu(E, Y_2) = \text{rank } X_2, \quad (3.42)$$

$$\mu^*(Y_1, Y_2) = \mu^*(E, Y_2) + s^*(E, Y_2, E, Y_1), \quad \mu^*(E, Y_2) = \text{rank } X_2. \quad (3.43)$$

*Proof.* Formulas (3.42) and (3.43) follow from (3.11) and (3.14) by using property (2.10) of the comparative index.  $\square$

The result in Theorem 3.2 also allows to prove in a direct way the following well known exchange formulas, see [30, Eq. (18)–(20)] in combination with Remark 3.1.

**Proposition 3.11.** Let  $Y_1, Y_2, Y_3, \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3 \in \Lambda(n)$  be given Lagrangian planes. Then we have

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = s(Y_1, Y_3, \tilde{Y}_1, \tilde{Y}_2) + s(Y_3, Y_2, \tilde{Y}_1, \tilde{Y}_2), \quad (3.44)$$

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_3) + s(Y_1, Y_2, \tilde{Y}_3, \tilde{Y}_2), \quad (3.45)$$

$$s(Y_2, Y_1, \tilde{Y}_1, \tilde{Y}_2) = -s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2), \quad (3.46)$$

$$s(Y_1, Y_2, \tilde{Y}_2, \tilde{Y}_1) = -s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2), \quad (3.47)$$

$$s(\tilde{Y}_1, \tilde{Y}_2, Y_1, Y_2) = -s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) - \sum_{j,k \in \{1,2\}} (-1)^{j+k} \text{rank } W(Y_j, \tilde{Y}_k). \quad (3.48)$$

Formulas (3.44)–(3.47) hold also with the dual Hörmander index  $s^*$ , while (3.48) is replaced by

$$s^*(\tilde{Y}_1, \tilde{Y}_2, Y_1, Y_2) = -s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) + \sum_{j,k \in \{1,2\}} (-1)^{j+k} \text{rank } W(Y_j, \tilde{Y}_k). \quad (3.49)$$

Finally, we provide an example illustrating the applicability of Theorem 3.2 for special cases considered in [16, Section 3.3].

**Example 3.12.** Let  $Y_1, \tilde{Y}_1, \tilde{Y}_2 \in \Lambda(n)$  be given and assume that  $Y_2 = E$  is the vertical Lagrangian plane. Then by (3.11) and (3.14) together with (2.10) we derive

$$s(Y_1, E, \tilde{Y}_1, \tilde{Y}_2) = \mu(\tilde{Y}_2, Y_1) - \mu(\tilde{Y}_1, Y_1), \quad s^*(Y_1, E, \tilde{Y}_1, \tilde{Y}_2) = \mu^*(\tilde{Y}_1, Y_1) - \mu^*(\tilde{Y}_2, Y_1). \tag{3.50}$$

If in addition  $Y_1 = N$  is the horizontal Lagrangian plane, then by (2.11) and (3.50) we get

$$s(N, E, \tilde{Y}_1, \tilde{Y}_2) = \text{rank } \tilde{X}_1 - \text{rank } \tilde{X}_2 + \text{ind}(-\tilde{X}_2^T \tilde{U}_2) - \text{ind}(-\tilde{X}_1^T \tilde{U}_1), \tag{3.51}$$

$$s^*(N, E, \tilde{Y}_1, \tilde{Y}_2) = \text{rank } \tilde{X}_1 - \text{rank } \tilde{X}_2 + \text{ind}(\tilde{X}_2^T \tilde{U}_2) - \text{ind}(\tilde{X}_1^T \tilde{U}_1), \tag{3.52}$$

Alternatively, consider another special case of (3.50) when the upper blocks of  $Y_1$  and  $\tilde{Y}_1$  are invertible, then the comparative indices  $\mu(\tilde{Y}_1, Y_1)$  and  $\mu^*(\tilde{Y}_1, Y_1)$  in (3.50) reduce to the index of the difference of the associated Riccati quotients, as we show in (2.8). We also note that the case  $Y_2 = N = \mathcal{J}E$  can be reduced to  $Y_1 = E$  by the symplectic invariance property (3.8) with the matrix  $S := -\mathcal{J}$ .

### 3.3. Triple index

As the last result in this section we discuss the connection of the comparative index with the triple index  $i(Y_1, Y_2, Y_3)$ , which was first defined in [6, Eq. (2.16)], see also [30, Corollary 3.12]. In the present discussion we again identify the Lagrangian subspace  $\text{Im } Y_j \subseteq \mathbb{R}^{2n}$  with the matrix  $Y_j \in \Lambda(n)$  itself. For three Lagrangian planes  $Y_1, Y_2, Y_3 \in \Lambda(n)$  the *triple index*  $i(Y_1, Y_2, Y_3)$  is the integer defined as

$$i(Y_1, Y_2, Y_3) := \text{ind } Q(Y_1, Y_0, Y_2) + \text{ind } Q(Y_2, Y_0, Y_3) - \text{ind } Q(Y_1, Y_0, Y_3), \tag{3.53}$$

where  $Y_0 \in \Lambda(n)$  is a Lagrangian plane for which the Wronskians  $W(Y_j, Y_0)$  for  $j \in \{1, 2, 3\}$  are invertible. Here  $Q(\alpha, \beta, \gamma)$  with  $\alpha, \beta, \gamma \subseteq \mathbb{R}^{2n}$  being Lagrangian subspaces is a bilinear form defined on the subspace  $\alpha \cap (\beta + \gamma)$ . We refer to [30, Section 3.1] for more details regarding the form  $Q(\alpha, \beta, \gamma)$ . The definition in (3.53) does not depend on the choice of the Lagrangian plane  $Y_0 \in \Lambda(n)$ , as long as it has the required property regarding the invertibility of the Wronskians  $W(Y_j, Y_0)$ . Note that the number  $i(Y_1, Y_2, Y_3)$  is nonnegative, which follows e.g. from [30, Lemma 3.13].

A difference of two triple indices is used in [30, Theorem 1.1] for the expression of the (dual) Hörmander index, see Remark 3.1 and equation (1.2), which in our context reads as

$$s^*(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = i(Y_1, Y_2, \tilde{Y}_2) - i(Y_1, Y_2, \tilde{Y}_1) = i(Y_1, \tilde{Y}_1, \tilde{Y}_2) - i(Y_2, \tilde{Y}_1, \tilde{Y}_2). \tag{3.54}$$

Note that these results are proven in [30] on the basis of [6, Lemma 2.5]. In addition, by using the duality principle in (3.15) with the aid of (3.54) we get the alternative formulas

$$s(Y_1, Y_2, \tilde{Y}_1, \tilde{Y}_2) = i(\tilde{Y}_1, \tilde{Y}_2, Y_1) - i(\tilde{Y}_1, \tilde{Y}_2, Y_2) = i(\tilde{Y}_2, Y_1, Y_2) - i(\tilde{Y}_1, Y_1, Y_2). \tag{3.55}$$

By using Theorem 3.2 we shall connect the triple index with the comparative index. We recall that for the evaluation of the obtained comparative indices we may use the symplectic matrices  $Z_j$  considered in (3.16) in Remark 3.3(i).

**Corollary 3.13.** Let  $Y_1, Y_2, Y_3 \in \Lambda(n)$  be given Lagrangian planes with the associated symplectic matrices  $Z_1, Z_2, Z_3$  such that  $Y_j = Z_j E$  for  $j \in \{1, 2, 3\}$ . Then we have

$$i(Y_1, Y_2, Y_3) = \mu(Z_3^{-1} Y_1, Z_3^{-1} Y_2) = \mu(Y_1, Y_2) + \mu(Y_2, Y_3) - \mu(Y_1, Y_3), \tag{3.56}$$

$$i(Y_1, Y_2, Y_3) = \mu^*(Z_1^{-1} Y_3, Z_1^{-1} Y_2) = \mu^*(Y_3, Y_2) + \mu^*(Y_2, Y_1) - \mu^*(Y_3, Y_1). \tag{3.57}$$

In particular, we can express the comparative index and the dual comparative index as

$$\mu(Y_1, Y_2) = i(Y_1, Y_2, E), \quad \mu^*(Y_1, Y_2) = i(E, Y_2, Y_1). \tag{3.58}$$

*Proof.* For the proof we utilize [30, Corollary 3.16], where the authors formulate their result in terms of the endpoint values  $Y(a)$  and  $Y(b)$  of a Lagrangian path  $Y$  on  $[a, b]$ . By using the notation with fixed Lagrangian planes  $Y_1, Y_2, Y_3 \in \Lambda(n)$  and taking Remark 3.1 into account, we can reformulate the second equality in [30, Corollary 3.16] as

$$i(Y_1, Y_2, Y_3) = s^*(Y_1, Y_2, Y_2, Y_3). \quad (3.59)$$

Then we proceed by applying Theorems 3.2 and 3.4 and Proposition 3.11 together with the symplectic invariance of the Hörmander index (with the matrix  $S := Z_3^{-1}$ ). Namely, we obtain

$$\begin{aligned} i(Y_1, Y_2, Y_3) &\stackrel{(3.59)}{=} s^*(Y_1, Y_2, Y_2, Y_3) \stackrel{(3.15)}{=} -s(Y_2, Y_3, Y_1, Y_2) \stackrel{(3.46)}{=} s(Y_3, Y_2, Y_1, Y_2) \\ &\stackrel{(3.8)}{=} s(Z_3^{-1}Y_3, Z_3^{-1}Y_2, Z_3^{-1}Y_1, Z_3^{-1}Y_2) = s(E, Z_3^{-1}Y_2, Z_3^{-1}Y_1, Z_3^{-1}Y_2) \\ &\stackrel{(3.23)}{=} \mu(Z_3^{-1}Y_1, Z_3^{-1}Y_2), \end{aligned}$$

which proves the first equality in (3.56), and then by Remark 2.3(ii) we get the second equality in (3.56). The equations in (3.57) follow from (3.56) by using property (2.7) of the comparative index and from (2.26). Finally, the choice of  $Y_3 = E$  (i.e.,  $Z_3 = I$ ) in (3.56) proves the validity of the first equation in (3.58), while the choice of  $Y_1 = E$  (i.e.,  $Z_1 = I$ ) in (3.57) yields that  $\mu^*(Y_3, Y_2) = i(E, Y_2, Y_3)$ . By relabeling  $Y_3$  as  $Y_1$  we then obtain the second equation in (3.58). The proof is complete.  $\square$

#### 4. Conclusions

In this paper we investigated the relations between the comparative index and the Hörmander index (including the Maslov index and the triple index) in the finite dimensional case. As a main result we derived an algebraic expression for the Hörmander index of four given Lagrangian planes as a difference of two comparative indices involving certain transformed Lagrangian planes, or as a combination of four comparative indices (Theorem 3.2). This result is based on a generalization of the comparison theorem for the Maslov index from [15] involving three Lagrangian paths (Proposition 2.2), hence it is based entirely on the comparative index theory. Our approach allows to present a geometric interpretation of the comparative index as a special Hörmander index or as a special triple index involving the vertical Lagrangian plane (Theorem 3.4 and Corollary 3.13). We also derived estimates for the values of the Hörmander index and presented conditions allowing to determine its extreme values and its sign (Corollaries 3.7 and 3.9). In this way we contribute to the recent efforts in [16, 30] devoted to efficient calculation of the Hörmander index in the finite dimensional case.

#### References

- [1] A. Ben-Israel, T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Second Edition, Springer-Verlag, New York, NY, 2003.
- [2] B. Booss-Bavnbek, C. Zhu, The Maslov index in symplectic Banach spaces, *Memoirs of the American Mathematical Society* 252 (2018), no. 1201, x+118 pp.
- [3] S. L. Campbell, C. D. Meyer, *Generalized Inverses of Linear Transformations*, Reprint of the 1991 corrected reprint of the 1979 original, *Classics in Applied Mathematics*, Vol. 56, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009.
- [4] S. E. Cappell, R. Lee, E. Y. Miller, On the Maslov index, *Communications on Pure and Applied Mathematics* 47 (1994), no. 2, 121–186.
- [5] W. A. Coppel, *Disconjugacy*, *Lecture Notes in Mathematics*, Vol. 220, Springer-Verlag, Berlin, 1971.
- [6] J. J. Duistermaat, On the Morse index in variational calculus, *Advances in Mathematics* 21 (1976), no. 2, 173–195.
- [7] O. Došlý, J. V. Elyseeva, R. Šimon Hilscher, *Symplectic Difference Systems: Oscillation and Spectral Theory*, *Pathways in Mathematics*, Birkhäuser/Springer, Cham, 2019.
- [8] J. V. Elyseeva, The comparative index for conjoined bases of symplectic difference systems, in: “Difference Equations, Special Functions, and Orthogonal Polynomials”, *Proceedings of the International Conference (Munich, 2005)*, S. Elaydi, J. Cushing, R. Lasser, A. Ruffing, V. Papageorgiou, and W. Van Assche, editors, pp. 168–177, World Scientific, London, 2007.

- [9] J. V. Elyseeva, Comparative index for solutions of symplectic difference systems, *Differential Equations* 45 (2009), no. 3, 445–459; translated from *Differencial'nyje Uravnenija* 45 (2009), no. 3, 431–444.
- [10] J. V. Elyseeva, Comparison theorems for conjoined bases of linear Hamiltonian differential systems and the comparative index, *Journal of Mathematical Analysis and Applications* 444 (2016), no. 2, 1260–1273.
- [11] J. V. Elyseeva, Oscillation theorems for linear Hamiltonian systems with nonlinear dependence on the spectral parameter and the comparative index, *Applied Mathematics Letters* 90 (2019), 15–22.
- [12] J. V. Elyseeva, Comparison theorems for conjoined bases of linear Hamiltonian systems without monotonicity, *Monatshefte für Mathematik* 193 (2020), no. 2, 305–328.
- [13] J. V. Elyseeva, Relative oscillation theory for linear Hamiltonian systems with nonlinear dependence on the spectral parameter, *Mathematische Nachrichten*, 296 (2023), no. 1, 196–216.
- [14] J. V. Elyseeva, Cyclic sums of the comparative indices and their applications, preprint (2021). Available at Arxiv: <https://arxiv.org/abs/2202.01041>.
- [15] J. Elyseeva, P. Šepitka, R. Šimon Hilscher, Oscillation numbers for continuous Lagrangian paths and Maslov index, *Journal of Dynamics and Differential Equations*, in press (2022), doi: 10.1007/s10884-022-10140-7.
- [16] P. Howard, Hörmander's index and oscillation theory, *Journal of Mathematical Analysis and Applications* 500 (2021), no. 1, Art. 125076, 38 pp.
- [17] P. Howard, Maslov index and spectral counts for linear Hamiltonian systems on  $\mathbb{R}$ , *Journal of Dynamics and Differential Equations*, in press (2022), doi: 10.1007/s10884-021-10065-7.
- [18] P. Howard, S. Jung, B. Kwon, The Maslov index and spectral counts for linear Hamiltonian systems on  $[0, 1]$ , *Journal of Dynamics and Differential Equations* 30 (2018), no. 4, 1703–1729.
- [19] P. Howard, Y. Latushkin, A. Sukhtayev, The Maslov index for Lagrangian pairs on  $\mathbb{R}^{2n}$ , *Journal of Mathematical Analysis and Applications* 451 (2017), no. 2, 794–821.
- [20] P. Howard, Y. Latushkin, A. Sukhtayev, The Maslov and Morse indices for system Schrödinger operators on  $\mathbb{R}$ , *Indiana University Mathematics Journal* 67 (2018), no. 5, 1765–1815.
- [21] W. Kratz, Quadratic Functionals in Variational Analysis and Control Theory, *Mathematical Topics*, Vol. 6, Akademie Verlag, Berlin, 1995.
- [22] C.-G. Liu, Maslov-type index theory for symplectic paths with Lagrangian boundary conditions, *Advanced Nonlinear Studies* 7 (2007), no. 1, 131–161.
- [23] C.-G. Liu, *Index Theory in Nonlinear Analysis*, Springer, Singapore, 2019.
- [24] W. T. Reid, *Sturmian Theory for Ordinary Differential Equations*, Springer-Verlag, New York, 1980.
- [25] H. Schulz-Baldes, Sturm intersection theory for periodic Jacobi matrices and linear Hamiltonian systems, *Linear Algebra and its Applications* 436 (2012), no. 3, 498–515.
- [26] P. Šepitka, R. Šimon Hilscher, Comparative index and Sturmian theory for linear Hamiltonian systems, *Journal of Differential Equations* 262 (2017), no. 2, 914–944.
- [27] P. Šepitka, R. Šimon Hilscher, Lidskii angles and Sturmian theory for linear Hamiltonian systems on compact interval, *Journal of Differential Equations* 298 (2021), 1–29.
- [28] P. Šepitka, R. Šimon Hilscher, Comparative index and Lidskii angles for symplectic matrices, *Linear Algebra and its Applications* 624 (2021), 174–197.
- [29] V. A. Yakubovich, Oscillatory properties of solutions of canonical equations, in: “Fifteen Papers on Differential Equations”, American Mathematical Society Translations, Ser. 2, Vol. 42, pp. 247–288, American Mathematical Society, Providence, RI, 1964.
- [30] Y. Zhou, L. Wu, C. Zhou, Hörmander index in finite-dimensional case, *Frontiers of Mathematics in China* 13 (2018), no. 3, 725–761.