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Approximation by trigonometric polynomials and Faber-Laurent rational functions in grand Morrey spaces

Sadulla Z. Jafarov[a]

[a]*Department of Mathematics and Science Education, Faculty of Education, Muş Alparslan University, 49250, Muş, Turkey; Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan*

Abstract. Let G be finite Jordan domain bounded a Dini smoth curve Γ in the complex plane $\mathbb{C}$. We investigate the approximation properties of the partial sums of the Fourier series and prove direct theorem for approximation by polynomials in the subspace of Morrey spaces associated with grand Lebesgue spaces. Also, approximation properties of the Faber-Laurent rational series expansions in spaces $L^{p),\lambda}(\Gamma)$ are studied. Direct theorems of approximation theory in grand Morrey-Smirnov classes, defined in domains with a Dini- smooth boundary, are proved.

1. Introduction and main results

Let $\mathbb{T}$ denotes the interval $[0, 2\pi]$. Let $L^p(\mathbb{T})$, $1 \le p < \infty$ be the Lebesgue space of all measurable $2\pi-$periodic functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

The *Morrey spaces* $L_0^{p,\lambda}(\mathbb{T})$ for a given $0 \le \lambda \le 1$ and $p \ge 1$, we define as the set of functions $f \in L_{loc}^p(\mathbb{T})$ such that

$$\|f\|_{L_0^{p,\lambda}(\mathbb{T})} := \left\{ \sup_I \frac{1}{|I|^{1-\lambda}} \int_I |f(t)|^p \, dt \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $I \subset [0, 2\pi]$. Note that $L_0^{p,\lambda}(\mathbb{T})$ becomes a Banach spaces, $\lambda = 1$ coincides with $L^p(\mathbb{T})$ and for $\lambda = 0$ with $L^\infty(\mathbb{T})$. If $0 \le \lambda_1 \le \lambda_2 \le 1$, then $L_0^{p,\lambda_1}(\mathbb{T}) \subset L_0^{p,\lambda_2}(\mathbb{T})$. Also, if $f \in L_0^{p,\lambda}(\mathbb{T})$, then $f \in L^p(\mathbb{T})$ and hence $f \in L^1(\mathbb{T})$. The Morrey spaces, were introduced by C. B. Morrey in

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Email address: s.jafarov@alparslan.edu.tr (Sadulla Z. Jafarov)

1938. The properties of these spaces have been investigated intensively by various authors and together with Lebesgue spaces L^p play an important role in the theory of partial equations, especially in the study of local behavior of the solutions of elliptic differential equations and describe local reqularity more precisely than Lebesgue spaces L^p. The properties of the Morrey spaces have been investigated by several authors (see, for example, [13], [22], [24], [35], [44], [45], [46], [51] and [56]).

We denote by $L^{p),\theta}(\mathbb{T})$, $\theta \geq 0$, the Lebesgue space of all measurable functions f on $\mathbb{T}$, that is, the space of all such functions for which

$$\|f\|_{L^{p),\theta}(\mathbb{T})} := \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^{\theta}}{2\pi} \int_{\mathbb{T}} |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty.$$

The space $L^{p),\theta}(\mathbb{T})$, $\theta > 0$ is called the *generalized grand Lebesgue space*. $L^{p),\theta}(\mathbb{T})$ is Banach function space, nonreflexive and nonseparable. Note that if $\theta = 0$ then $L^{p),\theta}(\mathbb{T})$ turns into the grand Lebesgue space $L^{p)}(\mathbb{T})$. If $\theta = 1$ then we obtain grand Lebesgue space $L^{p)}(\mathbb{T}) : = L^{p),1}(\mathbb{T})$. The grand and generalized grand Lebesgue space were introduceed in the works [27] and [23], respectively. Note that the space $L^{p)}(\mathbb{T})$ is a rearrangement invariant Banach function space, but is not reflexive. We can write the following embeddings:

$$L^p(\mathbb{T}) \subset L^{p)_1}(\mathbb{T}) \subset L^{p-\varepsilon}(\mathbb{T}),\ 1 < p < \infty$$

If $\theta_1 < \theta_2$ then for $0 < \varepsilon < p - 1$ the embeddings

$$L^p(\mathbb{T}) \subset L^{p),\theta_1}(\mathbb{T}) \subset L^{p),\theta_2}(\mathbb{T}) \subset L^{p-\varepsilon}(\mathbb{T}),\ 1 < p < \infty$$

hold.Note that the space $L^p(\mathbb{T})$ is not dense in $L^{p)}(\mathbb{T})$ [18].The informations about properties and applications of the grand Lebesgue spaces can be found in [18], [23], [27], [39], [53] and [54].

We denote by $L^{p),\lambda}(\mathbb{T})$, $1 < p < \infty$, $0 \leq \lambda < 1$ the Lebesgue space of all measurable functions f on $\mathbb{T}$, that is, the space of all such functions for which

$$\|f\|_{L^{p),\lambda}(\mathbb{T})} := \sup_{0<\varepsilon<p-1} \left(\sup_I \frac{\varepsilon}{|I|^{\lambda}} \int_I |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where the supremum is taken over all intervals $I \subset [0, 2\pi]$. The space $L^{p),\lambda}(\mathbb{T})$ is called *grand Morrey space* .

It is easy to verify that the following embeddings are hold:

$$L^{p,\lambda}(\mathbb{T}) \subset L^{p),\lambda}(\mathbb{T}) \subset L^{p-\varepsilon}(\mathbb{T}),\ 0 < \varepsilon \leq p - 1.$$

Denote by $C^{\infty}(\mathbb{T})$ the set of all functions that are realized as the restriction to $\mathbb{T}$ of elements in $C^{\infty}(\mathbb{R})$. We define $\widetilde{L}^{p),\lambda}(\mathbb{T})$ as the closure of $C^{\infty}(\mathbb{T})$ in $L^{p),\lambda}(\mathbb{T})$.

Let $W_r^{p),\lambda}(\mathbb{T})$ $(r = 1, 2, ...)$ (resp. $\widetilde{W}_r^{p),\lambda}(\mathbb{T})$ $(r = 1, 2, ...)$ be the linear space of functions for which $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in \widetilde{L}^{p),\lambda}(\mathbb{T})$) be the linear space of functions for which $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L^{p),\lambda}(\mathbb{T})$

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by the rectifiable Jordan curve Γ. Without loss of generality we assume $0 \in Int\ \Gamma$. Let G^-: $=Ext\ \Gamma$. Let also $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$, $\mathbb{D} = Int\ \mathbb{T}$ and $\mathbb{D}^- = Ext\ \mathbb{T}$. We recall that if for a given analytic function f on G, there exists a sequence of rectifiable Jordan curves (Γ_n) in G tending to the boundary Γ in the sense that Γ_n eventually surrounds each compact subdomain of G such that

$$\int_{\Gamma_n} |f(z)|^p |dz| \leq M < \infty,$$

then we say that f belongs to the *Smirnov class* $E^p(G)$, $1 \le p < \infty$. Each function $f \in E^p(G)$ has non-tangential limit almost everywhere (*a.e.*) on Γ and the boundary function belongs to $L^p(\Gamma)$.

We denote by φ the conformal mapping of G^- onto $\mathbb{D}^-$ normalized by

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0.$$

Let ψ be the inverse of φ. The function φ and ψ have continuous extensions to Γ and $\mathbb{T}$, their derivatives φ' and ψ' have definite non-tangential limit values on Γ and $\mathbb{T}$ a.e., and they are integrable with respect to the Lebesgue measure on on Γ and $\mathbb{T}$, respectively. It is known that $\varphi' \in E^1(G^-)$ and $\psi' \in E^1(\mathbb{D}^-)$. Note that the general information about Smirnov classes can be found in [19, pp. 168-185] and [26 pp. 438-453].

Let $\Gamma \subset \mathbb{C}$ be a rectifiable Jordan curve in the complex plane . We denote $\Gamma(t,r) = \Gamma \cap B(t,r)$, $t \subset \Gamma, r > 0$, where $B(t,r) = \{z \in \mathbb{C} : |z-t| < r\}$. The *Morrey spaces* $L^{p,\lambda}(\Gamma)$ for a given $0 \le \lambda \le 1$ and $p \ge 1$, are defined as the set of functions $f \in L^p_{loc}(\Gamma)$ such that

$$\|f\|_{L^{p,\lambda}(\Gamma)} := \sup_{z\in\Gamma,\ 0<r<L} r^{\frac{-\lambda}{p}} \|f\|_{L^p(\Gamma(t,r))} < \infty,$$

where L is the length of the curve Γ.

Note that $L^{p,0}(\Gamma) = L^p(\Gamma)$, and if $\lambda < 0$ or $\lambda > 1$, then $L^{p,\lambda}(\Gamma) = \Theta$, where Θ is the set of all functions equivalent to 0 on Γ.

We define also the *Morrey– Smirnov classes* $E^{p,\lambda}(G)$, $0 \le \lambda \le 1$, and $p \ge 1$, of analytic functions in G as

$$E^{p,\lambda}(G) := \left\{ f \in E^1(G) : f \in L^{p,\lambda}(\Gamma) \right\}.$$

If we define the norm of $f \in E^{p,\lambda}(G)$, $0 \le \lambda \le 1$ and $p \ge 1$ by

$$\|f\|_{E^{p,\lambda}(G)} := \|f\|_{L^{p,\lambda}(\Gamma)},$$

then the class $E^{p,\lambda}(G)$, $0 \le \lambda \le 1$ and $p \ge 1$ will be a Banach space. Note that for $\lambda = 1$ the class $E^{p,\lambda}(G)$ coincides with the classical Smirnov class $E^p(G)$ and for $\lambda = 0$ with $E^\infty(G)$. If $G = D := \{z : |z| < 1\}$, then we have Morrey- Hardy space $H^{p,\lambda}(D) := E^{p,\lambda}(D)$ on the unit disk D.

Let $\Gamma \subset \mathbb{C}$ be a rectifiable curve and let ν be arc-length measure on Γ. Let $1 < p < \infty$ and let $0 \le \lambda < 1$. We say that a measurable locally integrable function f on Γ belongs to the class $L^{p),\lambda}(\Gamma)$ if the following condition holds:

$$\|f\|_{L^{p),\lambda}(\Gamma)} := \sup_{0<\varepsilon<p-1} \left(\sup_{\substack{z\in\Gamma \\ 0<r<L}} \frac{\varepsilon}{\nu(B(z,r)\cap\Gamma)^\lambda} \int_{B(z,r)\cap\Gamma} |f(t)|^{p-\varepsilon} d\nu(t) \right)^{\frac{1}{p-\varepsilon}} < \infty.$$

We define also the *grand Morrey– Smirnov classes* $E^{p),\lambda}(G)$, $0 \le \lambda < 1$, and $1 < p < \infty$, of analytic functions f in G as

$$E^{p),\lambda}(G) := \left\{ f \in E^1(G) : f \in L^{p),\lambda}(\Gamma) \right\}.$$

We define the norm of $f \in E^{p),\lambda}(G)$, $0 \le \lambda < 1$ and $1 < p < \infty$ by

$$\|f\|_{E^{p),\lambda}(G)} := \|f\|_{L^{p),\lambda}(\Gamma)}.$$

The closure of Smirnov class $E^p(G)$ in the space $E^{p),\lambda}(G)$, $0 \le \lambda < 1$ and $1 < p < \infty$ we denote by $\widetilde{E}^{p),\lambda}(G)$, $1 < p < \infty$ and $0 \le \lambda < 1$.

Let $1 < p < \infty$ and $0 \leq \lambda < 1$, and let

$$\Delta_t^r f(x) := \sum_{s=0}^{r} (-1)^{r+s+1} \binom{r}{s} f(x + st), \ t > 0,$$

for a given $r \in \mathbb{N}$. For $f \in \widetilde{L}^{p),\lambda}(\mathbb{T})$, we define the operator

$$\left(v_h^r f\right)(x) := \frac{1}{h} \int_0^h \left|\Delta_t^r f(x)\right| dt.$$

The function

$$\Omega_r(f, \delta)_{p),\lambda} := \sup_{0<h\leq\delta} \left\|\left(v_h^r\right) f(x)\right\|_{L^{p),\lambda}(\mathbb{T})}, \ \delta > 0,$$

is called the *rth mean modulus* of $f \in \widetilde{L}^{p),\lambda}(\mathbb{T})$, $1 < p < \infty$, $0 \leq \lambda < 1$.

It can easily be shown that $\Omega_r(f, \cdot)_{p),\lambda}$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega_r(f, \delta)_{p),\lambda} = 0, \ \Omega_r(f + g, \delta)_{p),\lambda} \leq \Omega_r(f, \delta)_{p),\lambda} + \Omega_r(g, \delta)_{p),\lambda}, \ \delta > 0$$

for $f, \ g \in L^{p),\lambda}(\mathbb{T})$.

Let $f \in L^1(\mathbb{T})$ and let $\widetilde{f}$ be its conjugate function , with Fourier series

$$f(x) \sim \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \ \ \widetilde{f}(x) \sim \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx).$$

Let $S_n(\cdot, f)$ $(n = 1, 2, ...)$ be the nth partial sum of the Fourier series of $f \in L^1(\mathbb{T})$, i.e.

$$S_n(x, f) = \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

Using the method of proof of [17,Theorem 2.1 and Proposition 2.2] one can shown that

$$\left\|S_n(\cdot, f)\right\|_{L^{p),\lambda}(\mathbb{T})} \leq c_1 \left\|f\right\|_{L^{p),\lambda}(\mathbb{T})}, \qquad \left\|\widetilde{f}\right\|_{L^{p),\lambda}(\mathbb{T})} \leq c_2 \left\|f\right\|_{L^{p),\lambda}(\mathbb{T})},$$

an as a corollary we obtain

$$\left\|f - S_n(\cdot, f)\right\|_{L^{p),\lambda}(\mathbb{T})} \leq c_3 E_n(f)_{p),\lambda}, \qquad E_n\left(\widetilde{f}\right)_{p),\lambda} \leq c_4 E_n(f)_{p),\lambda}. \tag{1}$$

Let $f \in \widetilde{L}^{p),\lambda}(\mathbb{T})$ and $r \in \dot{\mathbb{N}}$. We define $K-$functional as

$$K_r(f, \delta)_{p),\lambda} := \inf\left\{\left\|f - \psi\right\|_{L^{p),\lambda}} + \delta^r \left\|\psi^{(r)}\right\|_{L^{p),\lambda}(\mathbb{T})} : \psi \in \widetilde{W}_r^{p),\lambda}(\mathbb{T}), \ \delta > 0\right\}.$$

We denote by $\prod_n$ the class of trigonometric polynomials of degree not exceeding $n \in \mathbb{N}$. The best approximation of $f \in \widetilde{L}^{p),\lambda}$, $1 < p < \infty$, $0 \leq \lambda < 1$ in the class $\prod_n$ is defined by

$$E_n(f)_{p),\lambda} := \inf\left\{\left\|f - T_n\right\|_{L^{p),\lambda}(\mathbb{T})} : T_n \in \prod_n\right\}.$$

We denote by $w = \phi(z)$ the conformal mapping of G^- onto domain $\mathbb{D} := \{w \in \mathbb{C} : |w| > 1\}$ normalized by the conditions

$$\phi(\infty) = \infty, \lim_{z\to\infty} \frac{\phi(z)}{z} > 0$$

and let ψ be the inverse mapping of ϕ.

We denote by $w = \phi_1(z)$ the conformal mapping of G onto the domain $\mathbb{D}^- := \{w \in \mathbb{C} : |w| > 1\}$ normalized by the conditions

$$\phi_1(0) = \infty, \lim_{z\to 0}(z.\phi_1(z)) > 0,$$

and let ψ_1 be the inverse mapping of ϕ_1.

The functions ψ and $\psi_1$have in some deleted neighborhood of the point $w = \infty$ the following representations

$$\psi(w) = \gamma w + \gamma_0 + \frac{\gamma_1}{w} + \frac{\gamma_2}{w^2} + ..., \gamma > 0,$$

and

$$\psi_1(w) = \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + ... + \frac{\alpha_k}{w^k} + ..., \alpha_1 > 0.$$

The following expansions hold [14], [19] and [57] :

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{\Phi_k(z)}{w^{k+1}}, z \in G \text{ and } w \in \mathbb{D}^-, \tag{2}$$

and

$$\frac{\psi_1'(w)}{\psi_1(w) - z} = \sum_{k=0}^{\infty} -\frac{F\left(\frac{1}{z}\right)}{w^{k+1}}, z \in G^- \text{ and } w \in \mathbb{D}^-, \tag{3}$$

where $\Phi_k(z)$ and $F_k\left(\frac{1}{z}\right)$ are the Faber polynomials of degree k with respect to z and $\frac{1}{z}$ for the continuums $\overline{G}$ and $\overline{\mathbb{C}}\backslash G$, respectively. Also, for the Faber polynomials $\Phi_k(z)$ and rational functions $F_k\left(\frac{1}{z}\right)$ the integral representations

$$\Phi_k(z) = \left[\phi(z)\right]^k + \frac{1}{2\pi i} \int_{\Gamma} \frac{\left[\phi(\zeta)\right]^n}{\zeta - z} d\zeta, \ k = 0, 1, 2, ..., z \in G, \tag{4}$$

$$F_k\left(\frac{1}{z}\right) = \left[\phi_1(z)\right]^k - \frac{1}{2\pi i} \int_{\Gamma} \frac{\left[\phi_1(\zeta)\right]^n}{\zeta - z} d\zeta, \ k = 0, 1, 2, ..., z \in G \tag{5}$$

hold [14], [57].

Let also χ be a continuous function on 2π. Its modulus of continuity is defined by

$$\omega(t, \chi) := \sup_{t_1, t_2 \in [0, 2\pi], |t_1 - t_2| < t} |\chi(t_1) - \chi(t_2)|, \ t \geq 0.$$

The curve Γ is called Dini-smooth curve if it has the parametrization

$$\Gamma : \chi(t), 0 \leq t \leq 2\pi,$$

such that $\chi'(t)$ satisfied the Dini-continuous, i.e.

$$\int_0^\pi \frac{\omega(t,\chi')}{t}dt < \infty$$

and

$$\chi'(t) \neq 0$$

[50, p.48].

Let $f \in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\psi(w))\psi'(w)}{\psi(w) - z}dw, \quad z \in G \tag{6}$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\psi_1(w))\psi_1'(w)}{\psi_1(w) - z}dw, \quad z \in G^- \tag{7}$$

are analytic in G and G^-, respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon\to\infty}\frac{1}{2\pi i}\int_{\Gamma\cap\{\zeta:|\zeta - z|>\varepsilon\}} \frac{f(\zeta)}{\zeta - z}d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of f at $z \in \Gamma$.

According to the Privalov theorem [26, p.431] if one of the functions f^+ or f^- has the non-tangential limits a.e. on Γ, then $S_\Gamma(f)(z)$ exists a.e. on Γ and also the other one has the non-tangential limits a.e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a.e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have non-tangential limits a.e. on Γ. In both cases, the formulas

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z), \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z) \tag{8}$$

and hence

$$f(z) = f^+(z) - f^-(z) \tag{9}$$

holds a.e. on Γ. From the results in [47], it follows that if Γ is a Dini-smooth curve S_Γ is bounded on $L^{p),\lambda}(\Gamma)$. Note that some properties of the Cauchy singular integral in the different spaces were investigated in [12], [17], [20], [24], [38], [40], [41], [43] and [47] .

Let $f \in L^{p(\cdot),\lambda(\cdot)}(\Gamma)$. Using (3), (4), (7), (8) and (9) we can associate Faber- Laurent series

$$f(z) \backsim \sum_{k=0}^{\infty} a_k\Phi_k(z) + \sum_{k=1}^{\infty} b_kF_k\left(\frac{1}{z}\right),$$

where the coefficients a_k and b_k are defined by

$$a_k :== \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f[\psi(w)]}{w^{k+1}}d\omega,\ k = 0, 1, 2, ...$$

and

$$b_k := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f[\psi_1(w)]}{w^{k+1}} dw,\ k = 0, 1, 2, ...$$

The coefficients a_k and b_k are said to be the Faber- Laurent coefficients of f.

If Γ is a Dini-smooth curve, then from the results in [59], it follows that

$$\left. \begin{array}{ll} 0 < c_5 < \left|\phi'(w)\right| < c_6 < \infty, & 0 < c_7 < \left|\phi_1'(w)\right| < c_8 < \infty; \\ 0 < c_9 < \left|\psi'(w)\right| < c_{10} < \infty, & 0 < c_{11} < \left|\psi_1'(\omega)\right| < c_{12} < \infty \end{array} \right\} \tag{10}$$

where the constants c_5, c_6, c_7, c_8 and $c_9, c_{10}, c_{11}, c_{12}$ are independent of $z \in \bar{G}^-$ and $|w| \geq 1$, respectively.

Let Γ be a Dini-smooth curve and let $f_0(w) := f[\psi(w)]$ for $f \in L^{p),\lambda}(\Gamma)$ and let $f_1(w) := f[\psi_1(w)]$ for $f \in L^{p),\lambda}(\Gamma)$. Then using (10) and the method, applied for the proof of the similar result in [33, Lemma 1], we obtain $f_0 \in L^{p),\lambda}(\mathbb{T})$ and $f_1 \in L^{p),\lambda}(\mathbb{T})$.

Moreover, $f_0^-(\infty) = f_1^-(\infty) = 0$ and by (10)

$$f_0(w) = f_0^+(w) - f_0^-(w), \qquad f_1(w) = f_1^+(w) - f_1^-(w) \tag{11}$$

a.e. on $\mathbb{T}$.

We set

$$R_n(f, z) := \sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=0}^{n} b_k F_k(\frac{1}{z}).$$

The rational function $R_n(f, z)$ is called the Faber-Laurent rational function of degree n of f.

Note that, in this paper we study the problems of approximation theory in space $\widetilde{L}^{p),\lambda}(\mathbb{T})$, the closure of the set of the trigonometric polynomials in $L^{p),\lambda}(\mathbb{T})$, with $1 < p < \infty$ and $0 \leq \lambda < 1$.

The problems of approximation of the functions in classical Morrey spaces and variable exponent Morrey spaces were investigated in [10], [11], [12[, [14], [15], [21], [26], [27]. In this work we invwestigate the approximation properties of the partial sums of the Fourier series and some direct theorems for approximation by trigonometric polynomials in the subspace grand Morrey spaces. Similar results in different spaces have been investigated by several authors (see. for example, [1]-[4], [8] -[11], [15], [16], [15]-[17], [25], [30]-[35], [42]). Also, approximation of the functions by Faber-Laurent rational functions in the grand Morrey spaces defined on the Dini-smooth curve are investigated.. Direct theorems of approximation theory in grand Morrey- Smirnov classes, defined in domains with a Dini-smooth boundary, are proved. Similar problems of approximation of the functions by Faber polynomials and Faber- Laurent rational functions in different spaces were studied in [5]-[7], [28]-[33], [36], [37], [49], [58], [60] .

Our main results are as follows.

Theoem 1.1. *Let $1 < p < \infty$, $0 \leq \lambda < 1$ and $r \in \mathbb{N}$. Then for every $f \in \widetilde{W}_r^{p),\lambda}(\mathbb{T})$ the inequality*

$$E_n(f)_{p),\lambda} \leq \frac{c_{13}}{n^r} E_n\left(f^{(r)}\right)_{p),\lambda} \tag{12}$$

holds with a constant $c_{13} > 0$ independent of n.

From Theorem 1.1 we obtain the following Corollary.

Corollary 1.1. *Let $1 < p < \infty$, $0 \leq \lambda < 1$ and $r \in \mathbb{N}$. If $f \in \widetilde{W}_r^{p),\lambda}(\mathbb{T})$ then*

$$E_n(f)_{p),\lambda} \leq \frac{c_{14}}{n^r} \left\|f^{(r)}\right\|_{L^{p)\lambda}(\mathbb{T})}$$

with a constant $c_{14} > 0$, independent of n.

Theoem 1.2. *Let $1 < p < \infty$, $0 \leq \lambda < 1$ and $r \in \mathbb{N}$. Then for every $f \in \widetilde{L}^{p),\lambda}(\mathbb{T})$ the inequality*

$$E_n(f)_{p),\lambda} \leq c_{15} \Omega_r(f, \delta)_{p),\lambda}$$

holds with a constant $c_{15} > 0$, independent of n.

Theorem 1.3. *Let Γ be a Dini-smooth curve. If $f \in L^{p),\lambda}(\Gamma)$, $1 < p < \infty$, $0 \le \lambda < 1$, then for every natural number n there are a constant $c_{16} > 0$ and rational function*

$$R_n(z,f) := \sum_{k=-n}^{n} a_k^{(n)} z^k$$

such that

$$\left\| f - R_n(.,f) \right\|_{L^{p(.),\lambda(.)}(\Gamma)} \le c_{16} \left\{ \Omega_r(f_0, 1/n)_{p),\lambda} + \Omega_r(f_1, 1/n)_{p),\lambda} \right\},$$

where $R_n(.,f)$ is the n-th partial sum of the Faber-Laurent series of f.

Theorem 1.4. *Let Γ be a Dini-smooth curve. If $f \in \widetilde{E}^{p),\lambda}(G)$, $1 < p < \infty$, $0 \le \lambda < 1$, then for every natural number n the inequality*

$$\left\| f - \sum_{k=0}^{n} a_k \Phi_k(z \right\|_{L^{p),\lambda}(\Gamma)} \le c_{17} \Omega_r(f_0, 1/n)_{p),\lambda} \tag{13}$$

holds with a constant $c_{17} > 0$ independent of n.

Note that the order of polynomial approximation in $E^p(G)$, $p \ge 1$, has been investigated by several authors. In [58] Walsh an Rusel gave results when Γ is an analytic curve. When Γ is a Dini-smooth curve direct and inverse theorems were proved by S. Y. Alper [5], These results were later extended to domains with regular boundary for $p > 1$ by V.M. Kokilashvili [42] and for $p \ge 1$ by J. E. Andersson [6]. The approximation properties of the $p-$Faber series expansions in the $\omega-$weighted Smirnov class $E^p(G,\omega)$ of analytic functions in G whose boundary is a regular Jordan curve are investigated in [28].

Theorem 1.5. *Let Γ be a Dini-smooth curve. $f \in \widetilde{E}^{p),\lambda}(G^-)$, then for every natural number n the inequality*

$$\left\| f - f(\infty) - \sum_{k=0}^{n} -b_k F_k\left(\frac{1}{z}\right). \right\|_{L^{p),\lambda}(\Gamma)} \le c_{18} \Omega(f_1, 1/n)_{p)_1\lambda} \tag{14}$$

holds, with a constant $c_{18} > 0$, independent of n.

2. Auxiliary results

In the proof of Theorem 1.2 we use the following theorem.

Theorem 2.1. *Let $1 < p < \infty$, $0 \le \lambda < 1$ and $r \in \mathbb{N}$. Then for every $f \in \widetilde{L}^{p),\lambda}(\mathbb{T})$ the inequality*

$$K_r(f,\delta) \le c_{19} \Omega_r(f,\delta)_{p),\lambda}$$

holds, where the constant c_{19} independent of δ.

Proof of Theorem 2.1. Let $r \ge 1$ and $\delta > 0$. We define the function

$$\begin{aligned} f_{r,\delta}(x) \quad : \quad &= \frac{2}{\delta} \int_{\delta/2}^{\delta} \left\{ \frac{1}{h^r} \int_0^h \cdots \int_0^h \sum_{s=0}^{r-1} (-1)^{r+s+1} \binom{r}{s} \right. \\ &\quad \left. \times f\left(x + \frac{r-s}{r}(t_1 + \ldots + t_r)\right) dt_1 \ldots dt_r \right\} dh \end{aligned} \tag{15}$$

It is clear that

$$\begin{aligned} &\Delta^r_{(t_1+\ldots t_r)/r} f(x) \\ = \quad &\sum_{s=0}^{r-1} (-1)^{r+s+1} \binom{r}{s} f\left(x + \frac{r-s}{r}(t_1 + \ldots + t_r)\right) - f(x). \end{aligned} \tag{16}$$

Use of (15) and (16) we obtain

$$\begin{aligned}
&\left\| f_{r,\delta}(\cdot) - f(\cdot) \right\|_{L^{p),\lambda}(\mathbb{T})} \\
\leq\ & \frac{2}{\delta}\int\limits_{\delta/2}^{\delta} \left\| \frac{1}{h^r}\int\limits_0^h \ldots \int\limits_0^h \Delta^r_{(t_1+\ldots t_r)/r} f(\cdot)\, dt_1 \ldots dt_r \right\|_{L^{p),\lambda}(\mathbb{T})} dh \\
=\ & \frac{2}{\delta}\int\limits_{\delta/2}^{\delta} \sup_{\frac{\delta}{2}\leq h\leq\delta} \left\| \frac{1}{h^r}\int\limits_0^h \ldots \int\limits_0^h \Delta^r_{(t_1+\ldots t_r)/r} f(\cdot)\, dt_1 \ldots dt_r \right\|_{L^{p),\lambda}(\mathbb{T})} dh \\
=\ & \sup_{\frac{\delta}{2}\leq h\leq\delta} \left\| \frac{1}{h^r}\int\limits_0^h \ldots \int\limits_0^h \Delta^r_{(t_1+\ldots t_r)/r} f(\cdot)\, dt_1 \ldots dt_r \right\|_{L^{p),,\lambda}(\mathbb{T})} \\
=\ & \sup_{\frac{\delta}{2}\leq h\leq\delta} \left\| \frac{1}{h^r}\int\limits_0^h \ldots \int\limits_0^h \left(\int\limits_{t_2+\ldots+t_r}^{t_2+\ldots+t_r+h} \left| \Delta^r_{t/r} f(\cdot)\, dt \right| \right) dt_2 \ldots dt_r \right\|_{L^{p),\lambda}(\mathbb{T})} \\
\leq\ & c_{20} \sup_{\frac{\delta}{2}\leq h\leq\delta} \left\| \frac{1}{h^{r-1}}\int\limits_0^h \ldots \int\limits_0^h \left(\int\limits_{0_r}^{rh} \left| \Delta^r_{t/r} f(\cdot)\, dt \right| \right) dt_2 \ldots dt_r \right\|_{L^{p),\lambda}(\mathbb{T})} \\
\leq\ & \ldots \leq c_{21} \sup_{\frac{\delta}{2}\leq h\leq\delta} \left\| \frac{1}{h}\int\limits_{0_r}^{rh} \left| \Delta^r_{t/r} f(\cdot) \right| dt \right\|_{L^{p),,\lambda}(\mathbb{T})} \\
=\ & c_{22} r \sup_{0\leq h\leq\delta} \left\| \frac{1}{h}\int\limits_{0_r}^{h} \Delta^r_t f(\cdot)\, dt \right\|_{L^{p),\lambda}(\mathbb{T})} = c_{23}\Omega_r(f,\delta)_{p),\lambda} \qquad (17)
\end{aligned}$$

It is clear that for the function $f^{(r)}_{r,\delta}(\cdot)$ the eguality

$$f^{(r)}_{r,\delta}(x) = \frac{2}{\delta}\int\limits_{\delta/2}^{\delta} \left(\frac{1}{h^r}\sum_{s=0}^{r-1} (-1)^{r+s} \binom{r}{s} \left(\frac{r}{r-s} \right)^r \Delta_{(r-s)h/r} f(x) \right) dh \qquad (18)$$

holds. Then from (18), we have

$$\begin{aligned}
&\left| f^{(r)}_{r,\delta}(x) \right| \\
\leq\ & \frac{2}{\delta}\int\limits_{\delta/2}^{\delta} \frac{2^r}{\delta^r} \sum_{s=0}^{r-1} \binom{r}{s} \left(\frac{r}{r-s} \right)^r \left| \Delta_{(r-s)h/r} f(x) \right| dh \\
\leq\ & 2^{r+1}\delta^{-r} \sum_{s=0}^{r-1} \binom{r}{s} \left(\frac{r}{r-s} \right)^r \frac{1}{\delta}\int\limits_0^{\delta} \left| \Delta_{(r-s)h/r} f(x) \right| dh. \qquad (19)
\end{aligned}$$

From (19), we get:

$$\begin{aligned}
&\left\| f_{r,\delta}^{(r)} \right\|_{L^{p),\theta,\lambda}(\mathbb{T})} \\
\leq\ & 2^{r+1}\delta^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\left\| \frac{1}{\delta}\int_{0}^{\delta}\left|\Delta_{(r-s)h/r}f(\cdot)\right| dh \right\|_{L^{p),\theta,\lambda}(\mathbb{T})} \\
=\ & 2^{r+1}\delta^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\left\| \frac{1}{(r-s)\delta/r}\int_{0}^{(r-s)\delta/r}\left|\Delta_{t}^{r}f(x)\right| dt \right\|_{L^{p),\theta,\lambda}(\mathbb{T})} \\
\leq\ & 2^{r+1}\delta^{-r}\sum_{s=0}^{r-1}\binom{r}{s}\left(\frac{r}{r-s}\right)^{r}\Omega_r(f,\delta)_{p),\theta,\lambda} \\
=\ & 2^{2r}\delta^{-r}\Omega_r(f,\delta)_{p),\theta,\lambda}. \qquad (20)
\end{aligned}$$

Taking into account the relations (17) and (20) we have

$$\begin{aligned}
&K_r(f,\delta)_{p),\lambda} \\
\leq\ & \left\| f_{r,\delta}-f \right\|_{L^{p),,\lambda}(\mathbb{T})}+\delta^{r}\left\| f_{r,\delta}^{(r)} \right\|_{L^{p),\lambda}(\mathbb{T})} \\
\leq\ & 2^{r-1}r\Omega_r(f,\delta)_{p)\lambda}+2^{2r}\delta^{-r}\Omega_r(f,\delta)_{p),\lambda}\leq c_{24}\Omega_r(f,\delta)_{p),\lambda}.
\end{aligned}$$

The proof of Theorem 2.1 is completed.

In the proof of Theorem 1.3 we use the following Lemmas.

Using Theorem 1.2 and the method applied for the proof of a similar result in [14] we can prove the following Lemma:

Lemma 2.1. *Let* $g \in E^{p),\lambda}(\mathbb{D})$, $1 < p < \infty$, $0 \leq \lambda < 1$. *If* $\sum_{k=0}^{n} d_k(g)w^k$ *is the nth partial sum of the Taylor series of* g *at the origin, then*

$$\left\| g(w)-\sum_{k=0}^{n} d_k w^k \right\|_{L^{p),\lambda}(\mathbb{T})} \leq c_{25}\Omega\left(g,\frac{1}{n}\right)_{p),\lambda}, \forall n \in \mathbb{N},$$

holds *with some constant* $c_{25} > 0$, *independent of* n.

Lemma 2.2. *Let* $g \in L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})$, $1 < p < \infty$, $0 \leq \lambda < 1$ *Then the inequality*

$$\Omega(g^{+},\cdot)_{p(\cdot),\lambda(\cdot)} \leq c_{26}\Omega(g,\cdot)_{p(\cdot),\lambda(\cdot)} \qquad (21)$$

holds, with some constant $c_{26} > 0$, *independent of* n.

Proof of Lemma 2.2. It is clear that the equality

$$g^{+} = S_{\mathbb{T}}(g)+\frac{1}{2}g \qquad (22)$$

holds a.e. on $\mathbb{T}$. Using the method of proof of [14, Lemma 3.3] (see also, [33, Lemma 2]) and the boundedness of the singular operator $S_{\mathbb{T}}(g)$ in $L^{p),\lambda}(\mathbb{T})$ [47] we can prove that

$$\Omega_r(S_T(g)\cdot)_{p),\lambda} \leq c_{27}\Omega_r(g,\cdot)_{p),\lambda}. \qquad (23)$$

Then using the subadditivity of the modulus of smoothness $\Omega_r(g^{+},\cdot)_{p),\lambda}$, (22) and (23) we obtain inequality (21) of Lemma 2.1.

3. Proof of the main results

Proof of Theorem 1.1. Let $f \in \widetilde{W}^{p),\lambda}(\mathbb{T})$ and let $\sum_{k=0}^{\infty}(a_k \cos kx) + b_k \sin kx)$ be the Fourier series of f and

$$S_n(x, f) = \sum_{k=0}^{n}(a_k \cos kx) + b_k \sin kx)$$

be its nth partial sum. Note that for the conjugate fungtion $\widetilde{f}$ the Foruer expansion

$$S_n(x, f) = \sum_{k=1}^{\infty}(a_k \sin kx) - b_k \cos kx)$$

holds. We set

$$B_k(x, f) := a_k \cos kx + b_k \sin kx.$$

It is clear that

$$f(x)) = \sum_{k=0}^{\infty} B_k(x, f) \tag{24}$$

Note that, for $k = 1, 2, ...,$ we can write the following equalities:

$$\begin{aligned}
B_k(x, f) \quad : \quad &= a_k \cos kx + b_k \sin kx \\
&= a_k \cos k\left(x + \frac{r\pi}{2k} - \frac{r\pi}{2k}\right) + b_k \sin k\left(x + \frac{r\pi}{2k} - \frac{r\pi}{2k}\right) \\
&= a_k \cos\left(kx + \frac{r\pi}{2} - \frac{r\pi}{2}\right) + b_k \sin\left(kx + \frac{r\pi}{2} - \frac{r\pi}{2}\right) \\
&= a_k\left[\cos\left(kx + \frac{r\pi}{2}\right)\cos\frac{r\pi}{2} - \sin\left(kx + \frac{r\pi}{2}\right)\sin\frac{r\pi}{2}\right] \\
&\quad + b_k\left[\sin\left(kx + \frac{r\pi}{2}\right)\cos\frac{r\pi}{2} - \cos\left(kx + \frac{r\pi}{2}\right)\sin\frac{r\pi}{2}\right] \\
&= \cos\frac{r\pi}{2}\left[a_k \cos k\left(x + \frac{r\pi}{2}\right) + b_k \sin k\left(x + \frac{r\pi}{2}\right)\right] \\
&\quad + \sin\frac{r\pi}{2}\left[a_k \sin k\left(x + \frac{r\pi}{2}\right) - b_k \cos k\left(x + \frac{r\pi}{2}\right)\right] \\
&= B_k\left(x + \frac{r\pi}{2k}, f\right)\cos\frac{r\pi}{2} + B_k\left(x + \frac{r\pi}{2k}, \widetilde{f}\right)\sin\frac{r\pi}{2},
\end{aligned} \tag{25}$$

$$B_k\left(x, f^{(r)}\right) = k^r B_k\left(x + \frac{r\pi}{2k}, f\right). \tag{26}$$

From (25) and (26), we obtain:

$$\begin{aligned}\sum_{k=0}^{\infty} B_k(x,f) &= B_0(x,f) + \cos\frac{r\pi}{2}\sum_{k=1}^{\infty} B_k\left(x+\frac{r\pi}{2k},\ f\right)\\ &+\sin\frac{r\pi}{2}\sum_{k=1}^{\infty} B_k\left(x+\frac{r\pi}{2k},\ \widetilde{f}\right)\\ &= B_0(x,f) + \cos\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{r^k}r^k B_k\left(x+\frac{r\pi}{2k},\ f\right)\\ &+\sin\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{r^k}r^k B_k\left(x+\frac{r\pi}{2k},\ \widetilde{f}\right)\\ &= B_0(x,f) + \cos\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{r^k}B_k\left(x,\ f^{(r)}\right)\\ &+\sin\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{r^k}B_k\left(x,\ \widetilde{f^{(r)}}\right).\end{aligned} \tag{27}$$

Then, from equalities, (24) and (27), we have:

$$\begin{aligned}f(x)-S_n(x,f) &= \sum_{k=n+1}^{\infty} B_k(x,\ f)\\ &= \cos\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{r^k}B_k\left(x,\ f^{(r)}\right)\\ &+\sin\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{r^k}B_k\left(x,\ \widetilde{f^{(r)}}\right).\end{aligned} \tag{28}$$

On the other hand the following equalities hold:

$$\sum_{k=n+1}^{\infty}\frac{1}{k^r}B_k\left(\alpha,\ f^{(r)}\right) = \sum_{k=n+1}^{\infty}\frac{1}{k^r}\left[S_k\left(\alpha,\ f^{(r)}\right) - S_{k-1}\left(\alpha,\ f^{(r)}\right)\right]$$

$$\begin{aligned}&= \sum_{k=n+1}^{n}\frac{1}{k^r}\left\{\left[S_k\left(\alpha,\ f^{(r)}\right) - f^{(r)}(\alpha)\right] - \left[S_{k-1}\left(\alpha,\ f^{(r)}\right) - f^{(r)}(\alpha)\right]\right\}\\ &= \sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)\left[S_k\left(\alpha,\ f^{(r)}\right) - f^{(r)}(\alpha)\right]\\ &-\frac{1}{(n+1)^r}\left[S_n\left(\alpha,\ f^{(r)}\right) - f^{(r)}(\alpha)\right],\\ &-\end{aligned} \tag{29}$$

$$\begin{aligned}\sum_{k=n+1}^{\infty}\frac{1}{k^r}B_k\left(\alpha,\ \widetilde{f^{(r)}}\right) &= \sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)\left[S_k\left(\alpha,\ \widetilde{f^{(r)}}\right) - \widetilde{f^{(r)}}\right]\\ &-\frac{1}{(n+1)^r}\left[S_n\left(\alpha,\ \widetilde{f^{(r)}}\right) - \widetilde{f^{(r)}}(\alpha)\right].\end{aligned} \tag{30}$$

Use of (28), (29) , (30) and (1), we get:

$$\begin{aligned}
&\left\|f-S_n(\cdot,f)\right\|_{L^{p),\lambda}(\mathbb{T})}\\
\leq\;& \sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)\left\|S_k\left(\cdot,\ f^{(r)}\right)-f^{(r)}\right\|_{L^{p),\lambda}(\mathbb{T})}\\
&+-\frac{1}{(n+1)^r}\left\|S_n\left(\cdot,\ f^{(r)}\right)-f^{(r)}\right\|_{L^{p),\lambda}(\mathbb{T})}\\
&+\sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)\left\|S_k\left(\cdot,\ \widetilde{f^{(r)}}\right)-\widetilde{f^{(r)}}\right\|_{L^{p),\lambda}(\mathbb{T})}\\
&+\frac{1}{(n+1)^r}\left\|S_n\left(\cdot,\ \widetilde{f^{(r)}}\right)-\widetilde{f^{(r)}}\right\|_{L^{p),\lambda}(\mathbb{T})}\\
\leq\;& c_{28}\left\{\sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)E_n\left(f^{(r)}\right)_{p),\lambda}+\frac{1}{(n+1)^r}{}_{L^{p),\lambda}(\mathbb{T})}E_n\left(f^{(r)}\right)_{p),\lambda}\right\}\\
&+c_{29}\sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)E_n\left(\widetilde{f^{(r)}}\right)_{p),\lambda}.\\
&+c_{29}\frac{1}{(n+1)^r}{}_{L^{p),\lambda}(\mathbb{T})}E_n\left(\widetilde{f^{(r)}}\right)_{p),\lambda}
\end{aligned}\tag{31}$$

Note that the sequence $\left\{E_n\left(f^{(r)}\right)_{p),\lambda}\right\}$ is desreasing. Then using (31) and (1) we have

$$\begin{aligned}
&\left\|f-S_n(\cdot,f)\right\|_{L^{p),\lambda}(\mathbb{T})}\\
\leq\;& c_{30}E_n\left(f^{(r)}\right)_{p),\lambda}\left\{\sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)+\frac{1}{(n+1)^r}\right\}\\
&+c_{31}E_n\left(\widetilde{f^{(r)}}\right)_{p),\lambda}\left\{\sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)+\frac{1}{(n+1)^r}\right\}\\
\leq\;& c_{32}E_n\left(f^{(r)}\right)_{p),\lambda}\left\{\sum_{k=n+1}^{n}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)+\frac{1}{(n+1)^r}\right\}\\
=\;& \frac{2c_{33}}{(n+1)^r}E_n\left(f^{(r)}\right)_{p),\lambda}.
\end{aligned}\tag{32}$$

On the other hand the inequality

$$E_n(f)_{p),\lambda}\leq\left\|f-S_n(\cdot,f)\right\|_{L^{p),\lambda}(\mathbb{T})}\tag{33}$$

holds. According to (32) and (33), we have inequality (12) . The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2. Let $f\in\mathcal{L}^{p),\lambda}(\mathbb{T})$ and $h\in\widetilde{W}_r^{p),\lambda}$. According to Theorem 1.1 and Corollary 1.1, we obtain:

$$\begin{aligned}
E_n(f)_{p),\lambda}\;&\leq\; E_n(f-h+h)_{p),\lambda}\leq E_n(f-h)_{p),\lambda}+E_n(h)_{p),\lambda}\\
&\leq\; c_{34}\left\{\left\|f-h\right\|_{L^{p),\lambda}(\mathbb{T})}+\frac{1}{n^r}\left\|h^{(r)}\right\|_{L^{p),\lambda}(\mathbb{T})}\right\}
\end{aligned}\tag{34}$$

Using (34) and definition of the K−functional we have

$$E_n(f)_{p),\lambda}\leq c_{35}K_r\left(f,\frac{1}{n}\right)_{p),\lambda}.$$

The last inequality and Theorem 2.1 imply that:

$$E_n(f)_{p),\lambda} \le c_{36}\Omega_r(f,\delta)_{p),\lambda}.$$

The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. Let $f \in L^{p),\lambda}(\Gamma)$. Then, from (11) we have $f_0 \in L^{p),\lambda}(\mathbb{T})$, $f_1 \in L^{p)\lambda}(\mathbb{T})$. According to (12), we obtain

$$f(\zeta) = f_0^+(\phi(\zeta)) - f_0^-(\phi(\zeta)),\ f(\xi) = f_1^+(\phi_1(\xi)) - f_1^-(\phi_1(\xi)). \tag{35}$$

a.e. on Γ.

We prove that the rational function

$$R_n(f,z) = \sum_{k=0}^{\infty} a_k \Phi_k(z) + \sum_{k=1}^{\infty} b_k F_k\left(\frac{1}{z}\right)$$

satisfies the condition of Theorem 1.3 .

Let $z^* \in G^-$. Using the method of proof in [32], we can prove that $f_0^-\left(\phi(\zeta)\right) \in E^{p),\lambda}(G^-) \in E^1(G^-)$. Then it is clear that

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f_0^-\left(\phi(\zeta)\right)}{\zeta - z^*}d\zeta = -f_0^-\left(\phi(z^*)\right).$$

Then last equality, (5) and (15) imply that

$$\sum_{k=0}^{n} a_k\Phi_k(z^*) = \sum_{k=0}^{n} a_k\left[\phi(z^*)\right]^k + \frac{1}{2\pi i}\int_{\Gamma}\frac{\sum_{k=0}^{n} a_k\left[\phi(\zeta)\right]^k}{\zeta - z^*}d\zeta$$

$$= \sum_{k=0}^{n} a_k\left[\phi(z^*)\right]^k + \frac{1}{2\pi i}\int_{\Gamma}\frac{\sum_{k=0}^{n} a_k\left[\phi(\zeta)\right]^k - f_0^+\left[\phi(\zeta)\right]}{\zeta - z^*}d\zeta$$

$$+\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - z^*}d\zeta - f_0^-\left[\phi(z^*)\right]. \tag{36}$$

Using (7) and (36), we find:

$$\sum_{k=0}^{n} a_k\Phi_k(z^*) = \sum_{k=0}^{n} a_k\left[\phi(z^*)\right]^k + \frac{1}{2\pi i}\int_{\Gamma}\frac{\sum_{k=0}^{n} a_k\left[\phi(\zeta)\right]^k}{\zeta - z^*}d\zeta$$

$$= \sum_{k=0}^{n} a_k\left[\phi(z^*)\right]^k + \frac{1}{2\pi i}\int_{\Gamma}\frac{\sum_{k=0}^{n} a_k\left[\phi(\zeta)\right]^k - f_0^+\left[\phi(\zeta)\right]}{\zeta - z^*}d\zeta$$

$$+f^-(z^*) - f_0^-\left[\phi(z^*)\right]. \tag{37}$$

Taking the limit as $z^* \to z \in \Gamma$ along all non-tangential paths outside Γ and considering (8),(9), (35) and (37), we have

$$f^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z^*) = \frac{1}{2}\left[f_0^+\left[\phi(z^*)\right] - \sum_{k=0}^{n} a_k\left[\phi(z^*)\right]^k\right]$$

$$+S_\Gamma\left(\left[f_0^+\left[\phi(z^*)\right] - \sum_{k=0}^{n} a_k\left[\phi(z^*)\right]^k\right]\right). \tag{38}$$

By [47] the singular operator $S_\Gamma : L^{p),\lambda}(\Gamma) \to L^{p),\lambda}(\Gamma)$ is bounded. Then using (35), Minkowski's inequality, Lemma 2.1 and 2.2, we have:

$$\begin{aligned}
&\left\| f^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z^*)\right\|_{L^{p),\lambda}(\Gamma)} \\
\leq\ & \frac{1}{2}\left\|f_0^+\left[\phi(z^*)\right] - \sum_{k=0}^{n} a_k\left[\phi(z^*)\right]^k\right\|_{L^{p),\lambda}(\Gamma)} + \left\|S_\Gamma\left(\left[f_0^+\left[\phi(z^*)\right] - \sum_{k=0}^{n} a_k\left[\phi(z^*)\right]^k\right]\right).\right\|_{L^{p),\lambda}(\Gamma)} \\
\leq\ & \frac{1}{2}\left\|f_0^+(w) - \sum_{k=0}^{n} a_k w^k\right\|_{L^{p),\lambda}(\mathbb{T})} + c_{37}\left\|f_0^+(w) - \sum_{k=0}^{n} a_k w^k\right\|_{L^{p),\lambda}(\mathbb{T})} \\
\leq\ & c_{38}\left\|f_0^+(w) - \sum_{k=0}^{n} a_k w^k\right\|_{L^{p),\lambda}(T)} \leq c_{39}\left\|f_0^+(w) - \sum_{k=0}^{n} \alpha_k\left(f_0^+\right) w^k\right\|_{L^{p),\lambda}(T)} \\
\leq\ & c_{40}\Omega_r\left(f_0^+, 1/n\right)_{p),\lambda} \leq c_{41}\Omega_r\left(f_0, 1/n\right)_{p),\lambda}.
\end{aligned} \tag{39}$$

Let $z^* \in G$. Using the method of proof in [32] we can prove that $f^-\left(\phi_1(\zeta)\right) \in E^{p),\lambda}(G^-) \in E^1(G^-)$. Therefore,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f_1^-\left(\phi_1(\zeta)\right)}{\zeta - z^*}d\zeta = f_1^{--}\left(\phi_1(z^*)\right),$$

Then the last equality, (5) and (35) imply that

$$\begin{aligned}
\sum_{k=1}^{n} b_k F_k\left(\tfrac{1}{z^*}\right) &= \sum_{k=1}^{n} b_k\left[\phi_1(z^*)\right]^k - \frac{1}{2\pi i}\int_\Gamma \frac{\sum_{k=1}^{n} b_k\left[\phi_1(\xi)\right]^k}{\xi - z^*}d\xi \\
&= \sum_{k=1}^{n} b_k\left[\phi_1(z^*)\right]^k - \frac{1}{2\pi i}\int_\Gamma \frac{\left(\sum_{k=1}^{n} b_k\left[\phi_1(\xi)\right]^k - f_1^+\left[\phi_1(\xi)\right]\right)}{\xi - z^*}d\xi \\
&- \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z^*}d\xi - \frac{1}{2\pi i}\int_\Gamma \frac{f_1^-\left(\phi_1(\zeta)\right)}{\zeta - z^*}d\zeta \\
&= \sum_{k=1}^{n} b_k\left[\phi_1(z^*)\right]^k - \frac{1}{2\pi i}\int_\Gamma \frac{\left(\sum_{k=1}^{n} b_k\left[\phi_1(\xi)\right]^k - f_1^+\left[\phi_1(\xi)\right]\right)}{\xi - z^*}d\xi \\
&- f^+(z^*) - f_1^-\left[\phi_1(z^*)\right].
\end{aligned}$$

Taking the limit as $z^* \to z$ along all nontangential paths inside of Γ we have

$$\sum_{k=1}^{n} b_k F_k\left(\tfrac{1}{z}\right) = \sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k - \frac{1}{2}\left(\sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k - f_1^+\left[\phi_1(z)\right]\right)$$

$$-S_\Gamma\left(\sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k - f_1^+\left[\phi_1(z)\right]\right) - -f^+(z) - f_1^-\left[\phi_1(z)\right]$$

a.e.on Γ. Consideration of (9) and (35) gives us

$$f^-(z) + \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right) = \frac{1}{2}\left(\sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k - f_1^+\left[\phi_1(z)\right]\right) - S_\Gamma\left(\sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k - f_1^+\left[\phi_1(z)\right]\right) \tag{40}$$

Using (40), Minkowski's inequality and the boundedness of S_Γ in $L^{p),\lambda}(\Gamma)$ [47], Lemma 2.1 and 2.2, we obtain:

$$\begin{aligned}
&\left\| f^-(z) + \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right)\right\|_{L^{p),,\lambda}(\Gamma)} \\
&\le \left\|\frac{1}{2}\left(\sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k - f_1^+\left[\phi_1(z)\right]\right)\right\|_{L^{p),,\lambda}(\Gamma)} + \left\| S_\Gamma\left(\sum_{k=1}^{n} b_k\left[\phi_1(z)\right]^k - f_1^+\left[\phi_1(z)\right]\right)\right\|_{L^{p),\lambda}(\Gamma)} \\
&\le \frac{1}{2}\left\|\sum_{k=1}^{n} b_k w^k - f_1^+(w)\right\|_{L^{p),\lambda}(\mathbb{T})} + c_{42}\left\|\sum_{k=1}^{n} b_k w^k - f_1^+(w)\right\|_{L^{p),\lambda}(\mathbb{T})} \\
&\le c_{43}\left\|\sum_{k=1}^{n} b_k w^k - f_1^+(w)\right\|_{L^{p),\lambda}(\mathbb{T})} \\
&= c_{44}\left\|\sum_{k=1}^{n} \beta_k\left(f_1^+\right) w^k - f_1^+(w)\right\|_{L^{p),\lambda}(\mathbb{T})} \\
&\le c_{45}\Omega_r\left(f_1^+, 1/n\right)_{p),\lambda} \le c_{46}\Omega_r\left(f_1, 1/n\right)_{p),\lambda}.
\end{aligned} \tag{41}$$

Now combining (8), (39) and (41) we obtain

$$\left\| f - R_n(., f)\right\|_{L^{p),\lambda}(\Gamma)} \le c_{47}(p)\left\{\Omega_r\left(f_0, 1/n\right)_{p),\lambda} + \Omega_r\left(f_1, 1/n\right)_{p),\lambda}\right\}$$

The proof of Theorem 1.3 is completed.

Proof of Theorem 1.4. Let $z^* \in G^-$.If $f \in E^{p),\lambda}(G)$, then $f \in E^p(G)$ and $\frac{f(\zeta)}{\zeta - z^*} \in E^p(G)$. Therefore, $\int\limits_\Gamma \frac{f(\zeta)}{\zeta - z^*} = 0$. That is $f^-(z) = 0$ a.e. on Γ. Then taking into account (11), we have:

$$\left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k\right\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{T})} \le c_{48}\Omega_r\left(f_0, \frac{1}{n}\right)_{p),\lambda}, \forall n \in \mathbb{N},$$

$$\left\| f_0^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z)\right\|_{L^{p),\lambda}(\Gamma)} \le c_{49}\left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k\right\|_{L^{p),\lambda}(\mathbb{T})}$$

we have the inequality (13) of Theorem 1.4.

Proof of Theorem 1.5. Let $z^* \in G$ and $f \in E^{p),\lambda}(G^-)$. It is clear that $\int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z^*} = f(\infty)$. Then we obtain $f^+(z) = f(\infty)$ a.e. on Γ. Now combining (11), we get:

$$\left\| f_1^+(w) - \sum_{k=0}^{n} b_k w^k \right\|_{L^{p),\lambda}(\mathbb{T})} \leq c_{50}(p)\Omega_r\left(f_1, \frac{1}{n}\right)_{p),\lambda}, \forall n \in \mathbb{N},$$

$$\left\| f^-(z) - \sum_{k=0}^{n} b_k F_k\left(\frac{1}{z}\right) \right\|_{L^{p),\lambda}(\Gamma)} \leq c_{51} \left\| f_1^+(w) - \sum_{k=0}^{n} b_k w^k \right\|_{L^{p),\lambda}(\mathbb{T})}$$

we obtain the inequality (14) of Theorem 1.5.

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