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Domination and Korenblum constants for some function spaces

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Abstract. The paper shows the domination in M_λ which is associated with the analytic Morrey space $\mathcal{L}^{2,\lambda}$. We get the explicit expressions for upper bounds of Korenblum constants for the analytic Morrey space $\mathcal{L}^{2,\lambda}$, the weighted Dirichlet space $\mathcal{D}^p_\alpha$, and the general function space $F(p,q,s)$, etc. Furthermore, the domination in the Dirichlet space $\mathcal{D}_\alpha$ is investigated.

1. Preliminaries and basic facts

We denote by $\mathbb{D}$ the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T}$ the unit circle in the complex plane $\mathbb{C}$. Let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For $0 < p < +\infty$, $-1 < \alpha$, the weighted Bergman space $\mathcal{A}^p_\alpha$ and the weighted Dirichlet space $\mathcal{D}^p_\alpha$ are consisted of functions $f \in \mathcal{H}ol(\mathbb{D})$ satisfying, respectively

$$\|f\|_{\mathcal{A}^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z)\right)^{\frac{1}{p}} < \infty,$$

and

$$\|f\|^p_{\mathcal{D}^p_\alpha} = |f(0)|^p + \|f'\|^p_{\mathcal{A}^p_\alpha} < \infty.$$

The normalized Lebesgue measure on $\mathbb{D}$ will be denoted by $dA(z)$. Obviously,

$$dA(z) = \frac{1}{\pi} dxdy = \frac{r}{\pi} drd\theta$$

for $z = x + iy = re^{i\theta}$. More introduction about the weighted Bergman spaces $\mathcal{A}^p_\alpha$ and the weighted Dirichlet spaces $\mathcal{D}^p_\alpha$ are mentioned in references [2][4][32][33]. If $p < \alpha + 1$, then $\mathcal{D}^p_\alpha = \mathcal{A}^p_{\alpha-p}$ [8]. If $p > \alpha + 2$, then $\mathcal{D}^p_\alpha \subset \mathcal{H}^\infty$. Trivially, $\mathcal{D}^2_1 = \mathcal{H}^2$, for $2 \leqslant p < \infty$, $\mathcal{H}^p \subset \mathcal{D}^p_{p-1}$ [21], and $\mathcal{D}^p_{p-1} \subset \mathcal{H}^p$ when $0 < p \leqslant 2$ [8][23]. In particular, $\mathcal{D}^2_0$ is the classical Dirichlet spaces $\mathcal{D}$. For $0 < p < \infty$, the space of Hardy $\mathcal{H}^p$ consists of all analytic functions f defined on $\mathbb{D}$ such that

$$\|f\|^p_{\mathcal{H}^p} = \left(\sup_{0\leqslant r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \frac{|d\zeta|}{2\pi}\right)^{\frac{1}{p}} < \infty.$$

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For $p=\infty$, it is said that $h\in\mathcal{H}^\infty$ if h is a bounded analytic function on $\mathbb{D}$ with

$$\|h\|_\infty=\sup\{|h(z)|:z\in\mathbb{D}\}.$$

For $p=2$, it is well known that $f\in\mathcal{H}^2$ has the boundary value $f(\zeta)=\lim_{r\to 1^-}f(r\zeta)$ for all $\zeta\in\mathbb{T}$. For $\lambda\in\mathbb{R}$, define the Morrey space $\mathcal{L}^{2,\lambda}$ to be the sets of all $f\in\mathcal{H}^2$ such that

$$\sup_{I\subset\mathbb{T}}\frac{1}{|I|^\lambda}\int_I|f(\zeta)-f_I|^2\frac{|d\zeta|}{2\pi}<\infty.$$

Clearly, for $\lambda\leqslant 0$, $\mathcal{L}^{2,\lambda}=\mathcal{H}^2$, $\mathcal{L}^{2,1}=BMOA$, and $\mathcal{L}^{2,\lambda}$ is Bloch type space if $\lambda>1$. Hence, $0<\lambda<1$ is the most interesting range for Morrey space $\mathcal{L}^{2,\lambda}$ and this paper will study Morrey space $\mathcal{L}^{2,\lambda}$ for $0<\lambda<1$ and $BMOA$ when $\lambda=1$. See [5][6][7][10][20][24] and the references therein for further details.

Let $f_I=\frac{1}{|I|}\int_I f(\zeta)\frac{|d\zeta|}{2\pi}$ and $S(I)$ be the Carleson square with

$$S(I)=\{z\in\mathbb{D}:|z|>1-|I|,\ \textit{as well as}\ z/|z|\in I\}.$$

Here $|I|=\int_I|d\zeta|/(2\pi)$ is the normalized length of arc I, which is an interval of $\mathbb{T}$. If $I=\mathbb{T}$, then $S(I)=\mathbb{D}$.

Wulan, Zhou [24] and Xiao [31] give some equivalent conditions for the norm of $\mathcal{L}^{2,\lambda}$. Thanks to their work, a analytic function $f\in\mathcal{L}^{2,\lambda}$ can be well defined with the following norms:

$$\begin{aligned}\|f\|_{\mathcal{L}^{2,\lambda}}&\approx|f(0)|+\sup_{I\subset\mathbb{T}}\left(\frac{1}{|I|^\lambda}\int_{S(I)}|f'(z)|^2(1-|z|^2)dA(z)\right)^{\frac12}\\&\approx|f(0)|+\sup_{a\in\mathbb{D}}\left((1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}|f'(z)|^2(1-|\phi_a(z)|^2)dA(z)\right)^{\frac12}\\&\approx|f(0)|+\sup_{a\in\mathbb{D}}\left((1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}|f'(z)|^2\log\frac{1}{|\phi_a(z)|}dA(z)\right)^{\frac12}\end{aligned}$$

where we denote $\phi_a(z)$ as the conformal automorphisms with

$$\phi_a(z)=\frac{a-z}{1-\bar{a}z},\quad z\in\mathbb{D}.$$

For $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$, the general function space $F(p,q,s)$ is the class of each $f\in\mathcal{H}ol(\mathbb{D})$ satisfying

$$\|f\|_{F(p,q,s)}=\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^s dA(z)\right)^{\frac1p}<\infty.$$

In addition, functions $f\in F_0(p,q,s)$ if $f\in\mathcal{H}ol(\mathbb{D})$ and satisfies

$$\lim_{|a|\to 1^-}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\phi_a(z)|^2)^s dA(z)=0.$$

Zhao [34] systematically introduced the family of spaces $F(p,q,s)$, he showed that $F(p,q,s)$ is a Banach space with the norm $\|f\|_*=|f(0)|+\|f\|_{F(p,q,s)}$ and $F_0(p,q,s)$ is a closed subspace of $F(p,q,s)$ when $1\leqslant p<\infty$. Also, $F(p,q,s)$ are well known spaces for certain parameters p,q,s. Namely,

$F(2,0,s)$ is the space Q_s for $0<s<\infty$;

$F(2,1-\lambda,\lambda)$ is the analytic Morrey space $\mathcal{L}^{2,\lambda}$, for $0<\lambda\leqslant 1$;

$F(p,\alpha p+p-2,q-(\alpha p+p-2))$ is the Dirichlet-Morrey space $\mathcal{D}^{p,q}_\alpha$ when $0<p<\infty$, $0\leqslant\alpha<\infty$, $\alpha p+p-2<q$;

$F(p,p-\lambda,\lambda)$ is the Bergman-Morrey space $\mathcal{A}^{p,\lambda}$ for $0<p<\infty$, $0<\lambda<2$, etc.

For $0<p<\infty$, $-2<q<\infty$, $0<s<\infty$, $q+s\leqslant -1$, $F(p,q,s)$ is trivial, that is , $F(p,q,s)$ only contains constants [34].

Similarly, for $0 < p < +\infty,\ -1 < \alpha,\ 0 < \lambda \leqslant 1$, a analytic function f is said to belong to the spaces $M_{p,\alpha}$ or M_λ if

$$\|f\|_{M_{p,\alpha}} = \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z)\right)^{\frac{1}{p}} < \infty$$

or

$$\|f\|_{M_\lambda} = \sup_{a\in\mathbb{D}} \left((1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f(z)|^2 (1-|\phi_a(z)|^2) dA(z)\right)^{\frac{1}{2}} < \infty.$$

Clearly, $M_{p,\alpha}(\ 1 \leqslant p < +\infty, -1 < \alpha)$ and $M_\lambda(\ 0 < \lambda \leqslant 1)$ are Banach spaces with the above norms and $M_{p,\alpha}$ is exactly the weighted Bergman space $\mathcal{A}^p_\alpha$.

The Gamma function $\Gamma(z)$ and the Beta function $B(u, v)$ are essential tools in the process of our main results. Recall that, the Gamma function $\Gamma(z)$ is defined by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt. \tag{1}$$

whenever the complex variable z has a positive real part, i.e., $\Re(z) > 0$. The Beta function $B(u, v)$ is defined by the formula

$$B(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1} dx, \quad (\Re(u) > 0, \Re(v) > 0) \tag{2}$$

and satisfies the following property

$$B(u+1, v+1) = \frac{u}{u+v+1} B(u, v+1). \tag{3}$$

The formula relating between the beta function and the gamma function is the following:

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{4}$$

Furthermore, the gamma function has the following properties [22]:

(i) $\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}$;

(ii) $\Gamma(n+1) = n!, \quad n \in \mathbb{N}^+$.

Duren [3] characterized the theory of Hardy space $\mathcal{H}^p$. Indeed, he proved the following lemma :

Lemma 1.1. *Suppose that $f(z) = \sum_{n=0}^\infty a_n z^n$ is analytic in $\mathbb{D}$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^\infty |a_n|^2 r^{2n} \tag{5}$$

for any $r, 0 \leqslant r < 1$.

By the definition of the upper bound, we can easily have the result.

Lemma 1.2.

$$\sup_{z\in\mathbb{D}} f(z) - \sup_{z\in\mathbb{D}} g(z) \leqslant \sup_{z\in\mathbb{D}} (f(z) - g(z)) \tag{6}$$

for any f, g are two real-valued functions in $\mathbb{D}$.

2. Introduction

Korenblum [16] introduced the idea of domination in the Bergman space $\mathcal{A}^2$. Recent years, Wang [29] proved that

$$z^n + a \prec z(1 + \bar{a}z)$$

for $n \geqslant 2$ in $\mathcal{A}^2$ under certain conditions, and applies it to obtain a new upper bound for the Korenblum constant. The theory of domination has been a subject of high interest during the past decades. The domination has been studied quite extensively on $\mathcal{A}^2$. Most of their efforts aimed at the Korenblum constants. There are more and more researchers focusing on the domination of spaces of analytic function to obtain a upper bound on Korenblum constant. Korenblum [16] conjectured that if $f, g \in \mathcal{A}^2$ with $|f(z)| \leqslant g(z)$ in the annulus $c < |z| < 1$, then there is an absolute constant c, $0 < c < 1$ such that $\|f\|_{\mathcal{A}^2} \leqslant \|g\|_{\mathcal{A}^2}$. Hayman [14] proved that Korenblum's conjecture for $c = 0.04$. Hinkkanen [11] improved Hayman's result that $c = 0.157\cdots$. Wang [25] showed that for any functions $f(z)$ and $g(z)$ (which depend on n and $a > 0$) defined in Theorem of [25], $c = 0.6947116\cdots$ is the best one. More about the theory of domination, see references [15][16][17][18][29] for the study of domination in Bergman space. Based on Wang's work [29], the paper studies the domination in the space $\mathcal{D}_\alpha$. Later, Gao and Wang [9] prove that $z^2 + a \prec z(1 + \bar{a}z^2)$ in M_p (associated with the Q_p spaces) under the condition

$$|a| \leqslant \frac{3(p+6)}{p^2 + 14p + 60}.$$

Motivated by [9], this paper characterizes the domination relationships in the spaces M_λ and $M_{p,\alpha}$.

Božin and Karapetrović [1] proved that the maximum principle does not hold in Bergman space $\mathcal{A}^p$ $(0 < p < 1)$. However, Korenblum maximum principle holds in Bergman spaces $\mathcal{A}^p$ if and only if $p \geqslant 1$. This is improved by Karapetrović [19], where it is shown that Korenblum maximum principle holds in weighted Bergman spaces $\mathcal{A}^p_\alpha$ if and only if $p \geqslant 1$. In addition, results related to Korenblum maximum principle in mixed norm spaces $H(p, q, \alpha)$ were also obtained in [19]. Hu and Lou [13] showed that the maximum principle in Fock space $\mathcal{F}^p_\alpha$ $(0 < p < 1)$ is invalid. Subsequently, Wee and Le [30] gave the explicit expressions for the upper bounds of Korenblum constants for the weighted Fock spaces and the weighted Bergman spaces with exponential weights, and showed that there is a failure of the Korenblum Maximum Principle for weighted Bergman spaces $\mathcal{A}^p_\alpha$ $(0 < \alpha, 0 < p < 1)$. Inspired by the above work [30], we study the Korenblum Maximum Principle on these analytic function spaces $\mathcal{L}^{2,\lambda}$, BMOA, M_λ, $\mathcal{D}^p_\alpha$, $M_{p,\alpha}$, $F(p, q, s)$.

The paper is organized as follows:

In Section 3, we study the domination in the spaces M_λ and $M_{p,\alpha}$; In section 4, the Korenblum constants are investigated for the analytic Morrey space $\mathcal{L}^{2,\lambda}$, the weighted Dirichlet space $\mathcal{D}^p_\alpha$, the general function space $F(p, q, s)$ and some function spaces associated with the above spaces; In section 5, the domination in the Dirichlet space $\mathcal{D}_\alpha$ is given.

3. Domination in certain spaces M_λ and $M_{p,\alpha}$

In this section, we investigate domination in certain spaces M_λ and $M_{p,\alpha}$, which are associated with $\mathcal{L}^{2,\lambda}$ and $\mathcal{D}^p_\alpha$, respectively. And we get the following results:

(i) For the analytic function space M_λ:

- The domination between $f(z) = z^2 + a$ and $g(z) = z(1 + \bar{a}z^2)$ in M_λ, $0 < \lambda \leqslant 1$.
- The domination between $f(z) = z + a$ and $g(z) = z(1 + \bar{a}z)$ in M_λ, $0 < \lambda \leqslant 1$.
- The domination between $f(z) = z^m + a$ and $g(z) = z(1 + \bar{a}z^m)$ in M_λ, $m \geqslant 3$, $0 < \lambda \leqslant 1$.

(ii) For the analytic function space $M_{p,\alpha}$:

- The domination between $f(z) = z^2 + a$ and $g(z) = z(1 + \bar{a}z^2)$ in $M_{p,\alpha}$, $-1 < \alpha$, $0 < p < \infty$.
- The domination between $f(z) = z + a$ and $g(z) = z(1 + \bar{a}z)$ in $M_{p,\alpha}$, $-1 < \alpha$, $0 < p < \infty$.
- The domination between $f(z) = z_m + a$ and $g(z) = z(1 + \bar{a}z_m)$ in $M_{p,\alpha}$, $m \geqslant 3$, $-1 < \alpha$, $0 < p < \infty$.

Generally, let X is a Banach space, if $f, g \in X$, g is said to dominate f in X if $\|fh\|_X \leqslant \|gh\|_X$ for any arbitrary $h \in \mathcal{H}^\infty$, and we denote it by $f \prec g$.

3.1 Domination in M_λ

Theorem 3.1. *Suppose* $a \in \mathbb{C},\ 0 < \lambda \leqslant 1$, *let*

$$f(z) = z^2 + a, \quad g(z) = z(1 + \bar{a}z^2),$$

where

$$|a| \leqslant \sqrt{\frac{7}{43}}. \tag{7}$$

Then $f \prec g$ *in* M_λ.

Proof. Suppose $h(z)$ is analytic in the open disk $\mathbb{D}$ and

$$H(z) = \frac{h(z)}{1 - \bar{b}z}$$

is also analytic in $\mathbb{D}$. Thus we can rewrite $H(z)$ as

$$H(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where a_n is a function of a.

By the definition of domination and Lemma 1.2, we obtain

$$\begin{aligned}
\|fh\|_{M_\lambda}^2 - \|gh\|_{M_\lambda}^2 &\leqslant \sup_{b \in \mathbb{D}} \left((1 - |b|^2)^{1-\lambda} \int_{\mathbb{D}} (|f(z)|^2 - |g(z)|^2)|h(z)|^2 (1 - |\phi_b(z)|^2) dA(z) \right) \\
&= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{2-\lambda} \int_{\mathbb{D}} \left(|f(z)|^2 - |g(z)|^2)|H(z)|^2 (1 - |z|^2) dA(z) \right) \\
&= \sup_{b \in \mathbb{D}} (1 - |b|^2)^{2-\lambda} \left(I_1(b) - I_2(b) \right),
\end{aligned}$$

where

$$I_1(b) = \int_{\mathbb{D}} |f(z)|^2 |H(z)|^2 (1 - |z|^2) dA(z).$$

$$I_2(b) = \int_{\mathbb{D}} |g(z)|^2 |H(z)|^2 (1 - |z|^2) dA(z).$$

It is easy to see that $I_2(b) < \infty$. Now, we proceed to show that $I_1(b) - I_2(b) \leqslant 0$ holds for any $b \in \mathbb{D}$.

Notice that

$$f(z)H(z) = a \sum_{n=0}^{1} a_n z^n + \sum_{n=0}^{\infty} (a_n + a a_{n+2}) z^{n+2},$$

$$g(z)H(z) = \sum_{n=0}^{1} a_n z^{n+1} + \sum_{n=0}^{\infty} (a_{n+2} + \bar{a} a_n) z^{n+3}.$$

Employing Lemma 1.1, we get

$$\begin{aligned}
I_1(b) &= 2 \int_0^1 \left(|a|^2 \sum_{n=0}^{1} |a_n|^2 |z|^{2n} + \sum_{n=0}^{\infty} |a_n + a a_{n+2}|^2 |z|^{2(n+2)} \right) (1 - r^2) r dr \\
&= \int_0^1 \left(|a|^2 \sum_{n=0}^{1} |a_n|^2 r^n + \sum_{n=0}^{\infty} |a_n + a a_{n+2}|^2 r^{n+2} \right) (1 - r) dr \\
&= |a|^2 \sum_{n=0}^{1} |a_n|^2 B(n+1, 2) + \sum_{n=0}^{\infty} |a_n + a a_{n+2}|^2 B(n+3, 2).
\end{aligned}$$

and

$$I_2(b) = \sum_{n=0}^{1} |a_n|^2 B(n+2,2) + \sum_{n=0}^{\infty} |a_{n+2} + \bar{a}a_n|^2 B(n+4,2)$$
$$= \sum_{n=0}^{1} \frac{n+1}{n+3} |a_n|^2 B(n+1,2) + \sum_{n=0}^{\infty} \frac{n+3}{n+5} |a_{n+2} + \bar{a}a_n|^2 B(n+3,2).$$

Then

$$\begin{aligned} I_1(b) - I_2(b) &= \sum_{n=0}^{1} |a_n|^2 \left(|a|^2 - \frac{n+1}{n+3}\right) B(n+1,2) \\ &\quad + \sum_{n=0}^{\infty} \left(|a_n + aa_{n+2}|^2 - \frac{n+3}{n+5} \times |a_{n+2} + \bar{a}a_n|^2\right) B(n+3,2) \\ &= \sum_{n=0}^{1} |a_n|^2 \left(|a|^2 - \frac{n+1}{n+3}\right) B(n+1,2) \\ &\quad + \sum_{n=0}^{\infty} \Big[|a_n|^2 + |a|^2 |a_{n+2}|^2 + 2\Re(a_n \overline{aa_{n+2}}) \\ &\quad - \frac{n+3}{n+5} \cdot \left(|a_{n+2}|^2 + |a|^2 |a_n|^2 + 2\Re(a_n \overline{aa_{n+2}}) \right) \Big] B(n+3,2). \end{aligned}$$

Using the following inequality

$$2\Re(a_n \overline{aa_{n+2}}) \leqslant |a|^2 \left(|a_n|^2 + |a_{n+2}|^2 \right), \tag{8}$$

we obtain

$$\begin{aligned} I_1(b) - I_2(b) &\leqslant \sum_{n=0}^{1} \left(|a|^2 - \frac{n+1}{n+3}\right) |a_n|^2 B(n+1,2) + \sum_{n=0}^{\infty} \Big[|a_n|^2 + 2|a|^2 |a_{n+2}|^2 + |a|^2 |a_n|^2 \\ &\quad - \frac{n+3}{n+5} \cdot \left(|a_{n+2}|^2 + 2|a|^2 |a_n|^2 + |a|^2 |a_{n+2}|^2 \right) \Big] B(n+3,2) \\ &= \sum_{n=0}^{1} \left(|a|^2 - \frac{n+1}{n+3}\right) |a_n|^2 B(n+1,2) + \sum_{n=0}^{\infty} \left(1 - \frac{n+1}{n+5}|a|^2\right) |a_n|^2 B(n+3,2) \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{n+7}{n+5}|a|^2 - \frac{n+3}{n+5} \right) |a_{n+2}|^2 B(n+3,2) \\ &= \sum_{n=0}^{1} \left(|a|^2 - \frac{n+1}{n+3}\right) |a_n|^2 B(n+1,2) + \sum_{n=0}^{1} \left(1 - \frac{n+1}{n+5}|a|^2\right) |a_n|^2 B(n+3,2) \\ &\quad + \sum_{n=2}^{\infty} \left(1 - \frac{n+1}{n+5}|a|^2\right) |a_n|^2 B(n+3,2) \\ &\quad + \sum_{n=2}^{\infty} \left(\frac{n+5}{n+3}|a|^2 - \frac{n+1}{n+3} \right) |a_n|^2 B(n+1,2) \\ &= \sum_{n=0}^{1} \left(|a|^2 - \frac{n+1}{n+3} + \tau_1 (1 - \frac{n+1}{n+5}|a|^2) \right) |a_n|^2 B(n+1,2) \\ &\quad + \sum_{n=2}^{\infty} \left(\frac{n+5}{n+3}|a|^2 - \frac{n+1}{n+3} + \tau_1 \Big(1 - \frac{n+1}{n+5}|a|^2\Big) \right) |a_n|^2 B(n+1,2) \end{aligned}$$

where $\tau_1 = \frac{(n+1)(n+2)}{(n+3)(n+4)}$.

Then for

$$|a| \leqslant \left(\frac{\frac{n+1}{n+3} - \tau_1}{1 - \tau_1 \frac{n+1}{n+5}} \right)^{\frac{1}{2}}, \quad n = 0, 1,$$

and

$$|a| \leqslant \left(\frac{\frac{n+1}{n+3} - \tau_1}{\frac{n+5}{n+3} - \tau_1 \frac{n+1}{n+5}} \right)^{\frac{1}{2}}, \quad n = 2, 3, \cdots.$$

we have

$$I_1(b) - I_2(b) \leqslant 0.$$

Thus

$$|a| \leqslant \min \left\{ \left\{ \left(\frac{\frac{n+1}{n+3} - \tau_1}{\frac{n+5}{n+3} - \tau_1 \frac{n+1}{n+5}} \right)^{\frac{1}{2}} : n = 2, 3, \cdots. \right\}, \left\{ \left(\frac{\frac{n+1}{n+3} - \tau_1}{1 - \tau_1 \frac{n+1}{n+5}} \right)^{\frac{1}{2}} : n = 0, 1 \right\} \right\}. \tag{9}$$

An easy calculation shows that the right side of the above formula takes the minimum value when n takes 2, then

$$|a| \leqslant \sqrt{\frac{7}{43}},$$

which completes the theorem. □

Corollary 3.2. *It seems that , for*

$$f(z) = z + a, \quad g(z) = z(1 + \bar{a}z)$$

$f \prec g$ does not hold in M_λ unless $a = 0$.

Corollary 3.3. *Suppose $a \in \mathbb{C}$, $0 < \lambda \leqslant 1$, let*

$$f(z) = z^m + a, \quad g(z) = z(1 + \bar{a}z^m), \quad m \geqslant 3$$

where

$$|a| \leqslant \frac{1}{\sqrt{3}}. \tag{10}$$

Then $f \prec g$ in M_λ.

Proof. An application of Theorem 3.1 yields the following:

$$\begin{aligned} I_1(b) - I_2(b) \leqslant & \sum_{n=0}^{m-1} \left(|a|^2 - \frac{n+1}{n+3} + \tau_2 \Big(1 - \frac{n+m-1}{n+m+3} |a|^2 \Big) \right) |a_n|^2 B(n+1, 2) \\ & + \sum_{n=m}^{\infty} \left(\frac{n+5}{n+3} |a|^2 - \frac{n+1}{n+3} + \tau_2 \Big(1 - \frac{n+m-1}{n+m+3} |a|^2 \Big) \right) |a_n|^2 B(n+1, 2) \end{aligned}$$

where

$$\tau_2 = \frac{(n+m)(n+m-1)\cdots(n+1)}{(n+m+2)(n+m+1)\cdots(n+3)} = \frac{(n+2)(n+1)}{(n+m+2)(n+m+1)}.$$

Also, for

$$|a| \leqslant \left(\frac{\frac{n+1}{n+3} - \tau_2}{1 - \tau_2 \frac{n+m-1}{n+m+3}} \right)^{\frac{1}{2}}, \quad n = 0, 1, \cdots, m-1$$

and

$$|a| \leqslant \left(\frac{\frac{n+1}{n+3} - \tau_2}{\frac{n+5}{n+3} - \tau_2 \frac{n+m-1}{n+m+3}} \right)^{\frac{1}{2}}, \quad n = m, m+1, \cdots,$$

we have

$$I_1(b) - I_2(b) \leqslant 0.$$

Let

$$L(x) = \frac{\frac{x+1}{x+3} - \tau_2}{1 - \tau_2 \frac{x+m-1}{x+m+3}}, \quad x \in [0, m-1].$$

Straightforward calculations show that

$$L(0) \leqslant L(1) \leqslant \cdots \leqslant L(m-1),$$

so

$$\begin{aligned} |a|^2 &\leqslant L(0) \\ &= \frac{(m+3)(m^2+5m)}{3(m+1)(m+2)(m+3) - 6(m-1)} \\ &\leqslant \lim_{m\to\infty} \frac{(m+3)(m^2+5m)}{3(m+1)(m+2)(m+3) - 6(m-1)} \\ &= \frac{1}{3}. \end{aligned} \tag{11}$$

On the other hand, consider

$$N(x) = \frac{\frac{x+1}{x+3} - \tau_2}{\frac{x+5}{x+3} - \tau_2 \frac{x+m-1}{x+m+3}}, \quad x \in [m, +\infty).$$

Put $x = m, m+1, \ldots$ into $N(x)$ for calculations and we get

$$N(m) \leqslant N(m+1) \leq \cdots,$$

then

$$\begin{aligned} |a|^2 &\leqslant N(m) \\ &\leqslant \lim_{m\to\infty} \frac{6m^3 + 11m^2 - 5m - 12}{6m^3 + 47m^2 + 79m + 36} = 1. \end{aligned} \tag{12}$$

Hence, we obtain the desired result. □

3.2 Domination in $M_{p,\alpha}$

Now, we study the domination in certain spaces $M_{p,\alpha}$ associated with weighted Dirichlet spaces $\mathcal{D}^p_\alpha$.

Theorem 3.4. *Suppose* $0 < p < \infty$, $-1 < \alpha$, *let*

$$f(z) = z^2 + a, \quad g(z) = z(1 - \bar{a}z^2), \quad z \in \mathbb{D}$$

where

$$|a| \leqslant \begin{cases} \left(\frac{3(\alpha+6)}{2\alpha^2+25\alpha+102}\right)^{\frac{1}{2}}, & -1 < \alpha < 2, \\ \left(\frac{\alpha+4}{\alpha^2+8\alpha+20}\right)^{\frac{1}{2}}, & 2 \leqslant \alpha. \end{cases} \tag{13}$$

Then $f \prec g$ *in* $M_{p,\alpha}$.

Proof. Suppose $h(z)$ is analytic in the open disk $\mathbb{D}$ hence we can write $h(z)$ as

$$h(z) = \sum_{n=0}^{\infty} a_n z^n.$$

where a_n is a function of a.

By the definition of domination, consider

$$\begin{aligned}\|fh\|^p_{M_{p,\alpha}} - \|gh\|^p_{M_{p,\alpha}} &= \int_{\mathbb{D}} (|f(z)|^p - |g(z)|^p)|h(z)|^p(1-|z|^2)^\alpha dA(z)\\ &= I_1(b) - I_2(b).\end{aligned}$$

where

$$I_1(b) = \int_{\mathbb{D}} |f(z)|^p|h(z)|^p(1-|z|^2)^\alpha dA(z).$$

$$I_2(b) = \int_{\mathbb{D}} |g(z)|^p|h(z)|^p(1-|z|^2)^\alpha dA(z).$$

Be the same as the proof of Theorem 3.1, we need only to consider the case $I_1(b) - I_2(b) \leqslant 0$ holds for any $b \in \mathbb{D}$.

With the proof technique of Theorem 3.1, one has

$$\begin{aligned}I_1(b) - I_2(b) \leqslant &\sum_{n=0}^{1}\Big(|a|^2 - \frac{n+1}{n+\alpha+2} + \tau_3\Big(1 - \frac{n-\alpha+2}{n+\alpha+4}|a|^2\Big)\Big)|a_n|^2 B(n+1,\alpha+1)\\ &+ \sum_{n=2}^{\infty}\Big(\frac{n+2\alpha+3}{n+\alpha+2}|a|^2 - \frac{n+1}{n+\alpha+2} + \tau_3\Big(1 - \frac{n-\alpha+2}{n+\alpha+4}|a|^2\Big)\Big)\\ &\cdot |a_n|^2 B(n+1,\alpha+1).\end{aligned}$$

where

$$\tau_3 = \frac{(n+1)(n+2)}{(n+\alpha+2)(n+\alpha+3)}.$$

For $n = 0, 1$, it follows that

$$\begin{aligned}|a| &\leqslant \min\left\{\left(\frac{\alpha+4}{\alpha^2+8\alpha+20}\right)^{\frac{1}{2}}, \left(\frac{2(\alpha+5)}{\alpha^2+11\alpha+42}\right)^{\frac{1}{2}}\right\}\\ &= \left(\frac{\alpha+4}{\alpha^2+8\alpha+20}\right)^{\frac{1}{2}}.\end{aligned}$$

And for $n = 2, 3, \cdots$

$$\begin{aligned}|a| &\leqslant \min\left\{\left(\frac{\frac{n+1}{n+\alpha+2} - \tau_3}{\frac{n+2\alpha+3}{n+\alpha+2} - \tau_3\frac{n-\alpha+2}{n+\alpha+4}}\right)^{\frac{1}{2}}\right\}\\ &= \left(\frac{3(\alpha+6)}{2\alpha^2+25\alpha+102}\right)^{\frac{1}{2}}.\end{aligned}$$

It is proper to verify that when

$$|a| \leqslant \begin{cases}\left(\frac{3(\alpha+6)}{2\alpha^2+25\alpha+102}\right)^{\frac{1}{2}}, & -1 < \alpha < 2,\\ \left(\frac{\alpha+4}{\alpha^2+8\alpha+20}\right)^{\frac{1}{2}}, & 2 \leqslant \alpha.\end{cases} \tag{14}$$

we obtain

$$I_1(b) - I_2(b) \leqslant 0.$$

This theorem is completed. □

Corollary 3.5. *It seems that , for*

$$f(z) = z + a, \quad g(z) = z(1 + \bar{a}z)$$

$f \prec g$ does not hold in $M_{p,\alpha}$ unless $a = 0$.

Corollary 3.6. *Suppose $0 < p < \infty$, $-1 < \alpha$, let*

$$f(z) = z^m + a, \quad g(z) = z(1 - \bar{a}z^m), \quad m \geqslant 3$$

where

$$|a| \leqslant \min\left\{\left(\frac{(\alpha+3)\cdots(m+\alpha+2) - m!(m+\alpha+2)}{(\alpha+2)\cdots(m+\alpha+2) - m!(m-\alpha)}\right)^{\frac{1}{2}}, \left(\frac{\frac{m+1}{m+\alpha+2} - \tau_5}{\frac{m+2\alpha+3}{m+\alpha+2} - \tau_5\frac{2m-\alpha}{2m+\alpha+2}}\right)^{\frac{1}{2}}\right\}, \tag{15}$$

with

$$\tau_5 = \frac{\Gamma(2m+1)/\Gamma(m+1)}{\Gamma(2m+\alpha+2)/\Gamma(m+\alpha+2)}.$$

Then $f \prec g$ in $M_{p,\alpha}$.

Proof. By the techniques of Theorems 3.1,3.4 and Corollary 3.3, we have

$$\begin{aligned} I_1(b) - I_2(b) \leqslant & \sum_{n=0}^{m-1}\left(|a|^2 - \frac{n+1}{n+\alpha+2} + \tau_4\left(1 - \frac{n+m-\alpha}{n+m+\alpha+2}|a|^2\right)\right)|a_n|^2 B(n+1,\alpha+1) \\ & + \sum_{n=m}^{\infty}\left(\frac{n+2\alpha+3}{n+\alpha+2}|a|^2 - \frac{n+1}{n+\alpha+2} + \tau_4\left(1 - \frac{n+m-\alpha}{n+m+\alpha+2}|a|^2\right)\right) \\ & \cdot |a_n|^2 B(n+1,\alpha+1). \end{aligned}$$

where

$$\tau_4 = \frac{\Gamma(n+m+1)/\Gamma(n+1)}{\Gamma(n+m+\alpha+2)/\Gamma(n+\alpha+2)}.$$

It is simple to show that for $n = 0$,

$$|a| \leqslant \left(\frac{(\alpha+3)\cdots(m+\alpha+2) - m!(m+\alpha+2)}{(\alpha+2)\cdots(m+\alpha+2) - m!(m-\alpha)}\right)^{\frac{1}{2}}$$

and for $n = m$,

$$|a| \leqslant \left(\frac{\frac{m+1}{m+\alpha+2} - \tau_5}{\frac{m+2\alpha+3}{m+\alpha+2} - \tau_5\frac{2m-\alpha}{2m+\alpha+2}}\right)^{\frac{1}{2}}, \quad \tau_5 = \frac{\Gamma(2m+1)/\Gamma(m+1)}{\Gamma(2m+\alpha+2)/\Gamma(m+\alpha+2)},$$

we get

$$I_1(b) - I_2(b) \leqslant 0.$$

Thus

$$|a| \leqslant \min\left\{\left(\frac{(\alpha+3)\cdots(m+\alpha+2) - m!(m+\alpha+2)}{(\alpha+2)\cdots(m+\alpha+2) - m!(m-\alpha)}\right)^{\frac{1}{2}}, \left(\frac{\frac{m+1}{m+\alpha+2} - \tau_5}{\frac{m+2\alpha+3}{m+\alpha+2} - \tau_5\frac{2m-\alpha}{2m+\alpha+2}}\right)^{\frac{1}{2}}\right\}, \tag{16}$$

we get the desired result. □

4. Korenblum constants for analytic spaces

In this section, we obtain the following results:
(i) For the analytic Morrey space $\mathcal{L}^{2,\lambda}$:
–The explicit expression for upper bounds of $\kappa_{\mathcal{L}^{2,\lambda}}$ in $\mathcal{L}^{2,\lambda}$, $0 < \lambda < 1$.
–The explicit expression for upper bounds of κ_{BMOA} in $BMOA$.
–The explicit expression for upper bounds of κ_{BMOA} in M_λ, $0 < \lambda \leqslant 1$.
(ii) For the weighted Dirichlet space $\mathcal{D}^p_\alpha$:
–The explicit expression for upper bounds of $\kappa_{\mathcal{D}^p_\alpha}$ in $\mathcal{D}^p_\alpha$, $0 < \alpha, p < \infty$.
–The explicit expression for upper bounds of $\kappa_{M_{p,\alpha}}$ in $M_{p,\alpha}$, $-1 < \alpha$, $0 < p < \infty$.
(iii) For the general function space $F(p,q,s)$:
–The explicit expression for upper bounds of $\kappa_{F(p,q,s)}$ in $F(p,q,s)$, $0 < p, s < \infty$, $-2 < q < \infty$, $-1 < q+s$.
(iv) Open questions.

Throughout the paper, c is called a Korenblum constant and denote by κ as the largest value of c. To avoid ambiguity, for the remainder of this paper we denote the largest Korenblum constant, unless otherwise stated, by $\kappa_{\mathcal{L}^{2,\lambda}}$ for $\mathcal{L}^{2,\lambda}$, κ_{BMOA} for $BMOA$, etc.

4.1 For the analytic Morrey space $\mathcal{L}^{2,\lambda}$

Theorem 4.1. *For any $0 < \lambda \leqslant 1$, we consider the spaces $\mathcal{L}^{2,\lambda}$.*
(i) For $\lambda = 1$, suppose

$$\sup_{a\in\mathbb{D}} \left((1-|a|^2) \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}} < c < 1. \tag{17}$$

(ii) For $0 < \lambda < 1$, suppose

$$\sup_{a\in\mathbb{D}} \left((1-|a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}} < c < 1. \tag{18}$$

Then there exist analytic functions f and g defined on $\mathbb{D}$ such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{BMOA} > \|g\|_{BMOA}$ (or $\|f\|_{\mathcal{L}^{2,\lambda}} > \|g\|_{\mathcal{L}^{2,\lambda}}$).
Therefore,

$$\kappa_{BMOA} \leqslant \sup_{a\in\mathbb{D}} \left((1-|a|^2) \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}} \tag{19}$$

and

$$\kappa_{\mathcal{L}^{2,\lambda}} \leqslant \sup_{a\in\mathbb{D}} \left((1-|a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} \right)^{\frac{1}{2}}. \tag{20}$$

Proof. Choose functions $f(z) = c$, $g(z) = z \in \mathcal{L}^{2,\lambda}$, one obtains

$$|f(z)| = |c| < |z| = |g(z)|$$

for any $|z| > c$.

We aim to proof

$$\|f\|_{\mathcal{L}^{2,\lambda}} > \|g\|_{\mathcal{L}^{2,\lambda}}.$$

or equivalently,

$$c > \sup_{a\in\mathbb{D}} \left((1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} (1-|\phi_a(z)|^2) dA(z) \right)^{\frac{1}{2}}.$$

or,

$$c > \sup_{a\in\mathbb{D}}\left((1-|a|^2)^{2-\lambda}\int_{\mathbb{D}}\frac{1-|z|^2}{|1-\bar{a}z|^2}dA(z)\right)^{\frac{1}{2}}.$$

Also, by [33, Lemma 3.10] , we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\left((1-|a|^2)^{2-\lambda}\int_{\mathbb{D}}\frac{1-|z|^2}{|1-\bar{a}z|^2}dA(z)\right)^{\frac{1}{2}}\\
&=\sup_{a\in\mathbb{D}}\left((1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{n!}{(n+2)!}|a|^{2n}\right)^{\frac{1}{2}}\\
&=\sup_{a\in\mathbb{D}}\left((1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|a|^{2n}\right)^{\frac{1}{2}}.
\end{aligned}$$

Then,

$$\|f\|_{\mathcal{L}^{2,\lambda}} > \|g\|_{\mathcal{L}^{2,\lambda}} \Longleftrightarrow c > \sup_{a\in\mathbb{D}}\left((1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|a|^{2n}\right)^{\frac{1}{2}}. \tag{21}$$

Therefore, in order to get the Korenblum Maximum Principle, we must have

$$\kappa_{\mathcal{L}^{2,\lambda}} \leqslant \sup_{a\in\mathbb{D}}\left((1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|a|^{2n}\right)^{\frac{1}{2}}. \tag{22}$$

□

Theorem 4.2. *For any $0 < \lambda \leqslant 1$, we consider the space M_λ. Suppose*

$$\sup_{a\in\mathbb{D}_\delta}(1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}(n+1)|a|^nB(\frac{n}{2}+2,2) < \sup_{a\in\mathbb{D}_\delta}(1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|a|^{2n}c \tag{23}$$

with $c < 1$. Then there exist analytic functions f and g defined on $\mathbb{D}$ such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{M_\lambda} > \|g\|_{M_\lambda}$.

Therefore,

$$\kappa_{M_\lambda} \leqslant \frac{\sup_{a\in\mathbb{D}_\delta}(1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}(n+1)|a|^nB(\frac{n}{2}+2,2)}{\sup_{a\in\mathbb{D}_\delta}(1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|a|^{2n}}. \tag{24}$$

Proof. Choose functions $f(z) = c,\ g(z) = z \in M_\lambda$, one obtains

$$|f(z)| = |c| < |z| = |g(z)|$$

for any $|z| > c$.

Since

$$\begin{aligned}
\|f\|^2_{M_\lambda} &= c\cdot\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}1-|\phi_a(z)|^2dA(z)\\
&= c\cdot\sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\int_{\mathbb{D}}\frac{1-|z|^2}{|1-\bar{a}z|^2}dA(z)\\
&= c\cdot\sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{\Gamma^2(n+1)}{n!\Gamma(n+3)}|a|^{2n}\\
&= c\cdot\sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|a|^{2n}.
\end{aligned}$$

$$
\begin{aligned}
\|g\|_{M_\lambda}^2 &= \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\cdot\int_{\mathbb{D}}|z|^2(1-|\phi_a(z)|^2)dA(z)\\
&= \sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\cdot\int_{\mathbb{D}}|z|^2(1-|z|^2)\frac{1}{|1-\bar{a}z|^2}dA(z)\\
&= \sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\cdot\sum_{n=0}^{\infty}\frac{\Gamma(n+2)}{n!\Gamma(2)}|a|^n\int_{\mathbb{D}}|z|^{n+2}(1-|z|^2)dA(z)\\
&= \sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\cdot\sum_{n=0}^{\infty}(n+1)|a|^n\int_{\mathbb{D}}|z|^{n+2}(1-|z|^2)dA(z)\\
&= \sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\cdot\sum_{n=0}^{\infty}(n+1)|a|^n\int_0^1 r^{\frac{n}{2}+1}(1-r)dr\\
&= \sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\lambda}\cdot\sum_{n=0}^{\infty}(n+1)|a|^nB(\frac{n}{2}+2,2).
\end{aligned}
$$

Using Mathematica, we know that there exists $\delta > 0$ such that

$$(1-|a|^2)^{2-\lambda}\cdot\sum_{n=0}^{\infty}(n+1)|a|^nB(\frac{n}{2}+2,2) < (1-|a|^2)^{2-\lambda}\cdot\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|a|^{2n}$$

where $a \in \mathbb{D}_\delta = \{z; |z| < \delta\}$. Then

$$\|f\|_{M_\lambda} > \|g\|_{M_\lambda}$$

if and only if (23) holds. Also, we get the Korenblum Maximum Principle (24). □

4.2 For the weighted Dirichlet space $\mathcal{D}_\alpha^p$

Theorem 4.3. *For any $0 < \alpha, p < \infty$, let*

$$c > \left(\frac{1}{\alpha+1}\right)^{\frac{1}{p}}. \tag{25}$$

Then there exist analytic functions f and g on the open unit disk $\mathbb{D}$ such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{\mathcal{D}_\alpha^p} > \|g\|_{\mathcal{D}_\alpha^p}$.

Therefore,

$$\kappa_{\mathcal{D}_\alpha^p} \leqslant \left(\frac{1}{\alpha+1}\right)^{\frac{1}{p}}. \tag{26}$$

Proof. Similarly, we choose $f(z) = c, g(z) = z \in \mathcal{D}_\alpha^p$, and

$$|f(z)| = |c| < |z| = |g(z)|$$

for any $|z| > c$.

Next, we are going to proof

$$\|f\|_{\mathcal{D}_\alpha^p}^p > \|g\|_{\mathcal{D}_\alpha^p}^p.$$

that is,

$$c^p > \int_{\mathbb{D}}(1-|z|^2)^\alpha dA(z) = \int_0^1(1-r)^\alpha dr = \frac{1}{\alpha+1}.$$

Hence c satisfies (25).

Clearly, in order for the Korenblum Maximum Principle to satisfy, one must get (26). □

Theorem 4.3 yields the corollary, Korenblum constant for Dirichlet space $\mathcal{D}$.

Corollary 4.4. *Let $c > 1$. There exist functions f and g in $\mathcal{D}$ such that $|f(z)| < |g(z)|$ for all $|z| > c$, but $\|f\|_{\mathcal{D}} > \|g\|_{\mathcal{D}}$. Therefore,*

$$\kappa_{\mathcal{D}} \leqslant 1. \tag{27}$$

Wee and Le [30] studied the Korenblum Maximum Principle on the weighted Bergman space with exponential weight $e^{-\frac{p\alpha}{2}|z|^2}$, they get the explicit expression for the upper bounds of the largest Korenblum constant for the weighted Bergman spaces, $0 < \alpha$, $1 \leqslant p < \infty$. and prove that a failure of the Korenblum Maximum Principle for the weighted Bergman spaces, $0 < \alpha$, $0 < p < 1$. Now, we focus on the weighted Bergman spaces $M_{p,\alpha}$ with the weight $\mathbf{v}_\alpha(z) = (1 - |z|^2)^\alpha$.

Theorem 4.5. *Let $0 < p < \infty$, $-1 < \alpha$. Let*

$$c > \left((\alpha + 1)B(\frac{p}{2} + 1, \alpha + 1)\right)^{\frac{1}{p}}. \tag{28}$$

There exist functions f and g on $\mathbb{D}$ such that $|f(z)| < |g(z)|$ for all $|z| > c$, but $\|f\|_{M_{p,\alpha}} > \|g\|_{M_{p,\alpha}}$. Therefore,

$$\kappa_{M_{p,\alpha}} \leqslant \left((\alpha + 1)B(\frac{p}{2} + 1, \alpha + 1)\right)^{\frac{1}{p}}. \tag{29}$$

Proof. Choose

$$f(z) = c, \ g(z) = z \in M_{p,\alpha}$$

such that $|f(z)| < |g(z)|$. Since

$$\|f\|^p_{M_{p,\alpha}} = c^p \cdot \int_{\mathbb{D}} (1 - |z|^2)^\alpha dA(z) = \frac{c^p}{\alpha + 1}.$$

$$\|g\|^p_{M_{p,\alpha}} = \int_{\mathbb{D}} |z|^p (1 - |z|^2)^\alpha dA(z) = B(\frac{p}{2} + 1, \alpha + 1).$$

Then

$$\|f\|_{M_{p,\alpha}} > \|g\|_{M_{p,\alpha}} \Leftrightarrow c > \left((\alpha + 1)B(\frac{p}{2} + 1, \alpha + 1)\right)^{\frac{1}{p}}.$$

This completes the theorem. □

4.3 For the general function space $F(p, q, s)$

Based on the analytic Morrey space $\mathcal{L}^{2,\lambda}$, we have the general case.

Theorem 4.6. *For any $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, let*

$$c > \left(\sup_{a \in \mathbb{D}} h_a(q, s)\right)^{\frac{1}{p}}. \tag{30}$$

Then there exist analytic functions f and g on the open unit disk $\mathbb{D}$ such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{F(p,q,s)} > \|g\|_{F(p,q,s)}$.

Therefore,

$$\kappa_{F(p,q,s)} \leqslant \left(\sup_{a \in \mathbb{D}} h_a(q, s)\right)^{\frac{1}{p}}. \tag{31}$$

where

$$h_a(q, s) = \int_{\mathbb{D}} (1 - |z|^2)^q (1 - |\phi_a(z)|^2)^s dA(z) \tag{32}$$

$$= (1 - |a|^2)^s \cdot \frac{\Gamma(q + s + 1)}{\Gamma^2(s)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma^2(n + s)}{n!\Gamma(n + q + s + 2)} |a|^{2n}.$$

Proof. Let $f(z) = c, g(z) = z \in F(p, q, s)$ with

$$|f(z)| = |c| < |z| = |g(z)|$$

for each $|z| > c$.

Since

$$\|f\|^p_{F(p,q,s)} = c^p,$$

and

$$\|g\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1-|z|^2)^q (1-|\phi_a(z)|^2)^s dA(z) \tag{33}$$

$$= \sup_{a\in\mathbb{D}} (1-|a|^2)^s \int_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{|1-\bar{a}z|^{2s}} dA(z)$$

$$= \sup_{a\in\mathbb{D}} h_a(q, s). \tag{34}$$

where $h_a(q, s)$ is defined as (32).

So,

$$\|f\|_{F(p,q,s)} > \|g\|_{F(p,q,s)}$$

if and only if (30) holds, as desired. □

Corollary 4.7. *Let $0 < s < \infty$. Suppose*

$$\left(\sup_{a\in\mathbb{D}} h_a(0, s)\right)^{\frac{1}{2}} < c < 1. \tag{35}$$

Then there exist functions f and g in Q_s such that $|f(z)| < |g(z)|$ for all $|z| > c$, but $\|f\|^2_{Q_s} > \|g\|^2_{Q_s}$. Therefore,

$$\kappa_{Q_s} \leqslant \left(\sup_{a\in\mathbb{D}} h_a(0, s)\right)^{\frac{1}{2}} \tag{36}$$

$$= \left(\sup_{a\in\mathbb{D}} (1-|a|^2)^s \cdot \frac{s}{\Gamma(s)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{n!(n+s)(n+s+1)} |a|^{2n}\right)^{\frac{1}{2}}.$$

4.4 Open questions

Theorems 4.1-4.6 lead us to have the following questions to consider.

Question 1. For $0 < \lambda \leqslant 1$, suppose

$$c = \sup_{a\in\mathbb{D}} \left((1-|a|^2)^{2-\lambda} \cdot \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n}\right)^{\frac{1}{2}}. \tag{37}$$

Do there exist analytic functions f and g defined on $\mathbb{D}$ such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{\mathcal{L}^{2,\lambda}} > \|g\|_{\mathcal{L}^{2,\lambda}}$?

Question 2. For any $0 < \lambda \leqslant 1$, suppose

$$\sup_{a\in\mathbb{D}_\delta} (1-|a|^2)^{2-\lambda} \sum_{n=0}^{\infty} (n+1)|a|^n B(\frac{n}{2}+2, 2) = \sup_{a\in\mathbb{D}_\delta} (1-|a|^2)^{2-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} |a|^{2n} c. \tag{38}$$

Do there exist analytic functions f and g defined on $\mathbb{D}$ such that $|f(z)| < |g(z)|$ whenever $c < |z| < 1$, but $\|f\|_{M_\lambda} > \|g\|_{M_\lambda}$?

Question 3. For any $0 < \alpha, p < \infty$, let

$$c = \left(\frac{1}{\alpha+1}\right)^{\frac{1}{p}}. \tag{39}$$

Do there exist analytic functions f and g in $\mathcal{D}_\alpha^p$ for which $|f(z)| < |g(z)|$ with $|z| > c$ and $\|f\|_{\mathcal{D}_\alpha^p} > \|g\|_{\mathcal{D}_\alpha^p}$?

Question 4. Let $c = 1$. Do there exist functions $f(z)$ and $g(z)$ in $\mathcal{D}$ for which $|f(z)| < |g(z)|$ with $|z| > c$ and $\|f\|_{\mathcal{D}} > \|g\|_{\mathcal{D}}$?

Question 5. Let $-1 < \alpha, 0 < p < \infty$. Let

$$c = \left((\alpha+1)B(\frac{p}{2}+1, \alpha+1)\right)^{\frac{1}{p}}. \tag{40}$$

Do there exist analytic functions f and g in $M_{p,\alpha}$ for which $|f(z)| < |g(z)|$ with $|z| > c$ and $\|f\|_{M_{p,\alpha}} > \|g\|_{M_{p,\alpha}}$?

Question 6. For any $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, let

$$c = \left(\sup_{a\in\mathbb{D}} h_a(q,s)\right)^{\frac{1}{p}}. \tag{41}$$

Do there exist analytic functions f and g in $F(p,q,s)$ for which $|f(z)| < |g(z)|$ with $|z| > c$ and $\|f\|_{F(p,q,s)} > \|g\|_{F(p,q,s)}$?

5. Domination in Dirichlet space $\mathcal{D}_u$

For $\alpha \in \mathbb{R}$, the Dirichlet space $\mathcal{D}_\alpha$ is the set of functions $f(z) = \sum_{n=0}^{\infty} a_n^{z^n} \in \mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_\alpha} = \left(\sum_{n=0}^{\infty}(n+1)^{1-\alpha}|a_n|^2\right)^{\frac{1}{2}}. \tag{42}$$

Remark that $\mathcal{D}_0$ is the classical Dirichlet space, and $\mathcal{D}_1$ is the Hardy space $\mathcal{H}^2$ [2][3][32].

Motivated by the paper [29], in which the special case of domination on the space $\mathcal{D}_\alpha$ when $\alpha = 2$ was investigated, the following theorem generalizes the results to the general cases for $2 < \alpha < \infty$.

Theorem 5.1. *Let $2 < \alpha < \infty$, $a \in \mathbb{C}$, and m be a positive integer. Suppose that one of the following conditions is satisfied :*

(i) $m \geqslant 3$, and

$$|a| \leqslant \sqrt{\frac{m-2}{2(m-1)}} \cdot \frac{1}{(k+m+1)^{\alpha-2}},\ k = 0, 1, \cdots; \tag{43}$$

(ii) $m = 2$, and $|a| \leqslant |a^*|$, where $|a^*|$ is the solution of (49).

Then

$$\|(z^m + a)h(z)\|_{\mathcal{D}_\alpha} \leqslant \|z(1 + \bar{a}z^m)h(z)\|_{\mathcal{D}_\alpha}.$$

for any $h(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{H}^\infty$.

Proof. Define

$$f(z) = (z_m + a)h(z),\ g(z) = z(1 + \bar{a}z^m)h(z)$$

where $h(z) = \sum_{k=0}^{\infty} c_k z^k$ is the power series expansion of $h(z)$ and $a \in \mathbb{C}$, $m \in \mathbb{N}$. Then

$$f(z) = a\sum_{k=0}^{m-1} c_k z^k + \sum_{k=0}^{\infty}(c_k + ac_{k+m})z^{k+m},$$

$$g(z)=\sum_{k=0}^{m-1}c_kz^{k+1}+\sum_{k=0}^{\infty}(\overline{a}c_k+c_{k+m})z^{k+m+1}.$$

Clearly, $\|g\|_{\mathcal{D}_\alpha}<\infty$.

Followed by (42), we have

$$\begin{aligned}\|f\|^2_{\mathcal{D}_\alpha}-\|g\|^2_{\mathcal{D}_\alpha}&=\sum_{k=0}^{m}|c_k|^2\left(\frac{|a|^2}{(k+1)^{\alpha-1}}-\frac{1}{(k+2)^{\alpha-1}}\right)\\&\quad+\sum_{k=0}^{\infty}\left(\frac{|c_k+ac_{k+m}|^2}{(k+m+1)^{\alpha-1}}-\frac{|\overline{a}c_k+c_{k+m}|^2}{(k+m+2)^{\alpha-1}}\right)\\&=\sum_{k=0}^{\infty}|c_k|^2|a|^2\left(\frac{1}{(k+1)^{\alpha-1}}-\frac{1}{(k+m+2)^{\alpha-1}}\right)\\&\quad-\sum_{k=0}^{\infty}|c_k|^2\left(\frac{1}{(k+2)^{\alpha-1}}-\frac{1}{(k+m+1)^{\alpha-1}}\right)\\&\quad+\sum_{k=0}^{\infty}2\Re(a\overline{c_k}c_{k+m})\left(\frac{1}{(k+m+1)^{\alpha-1}}-\frac{1}{(k+m+2)^{\alpha-1}}\right)\qquad(44)\\&\leqslant\sum_{k=0}^{\infty}(\alpha-1)|c_k|^2|a|^2\left(\frac{1}{(k+1)^{\alpha-2}}\cdot\frac{m+1}{(k+1)(k+m+2)}\right)\qquad(45)\\&\quad-\sum_{k=0}^{\infty}(\alpha-1)|c_k|^2\left(\frac{1}{(k+2)^{\alpha-2}}\cdot\frac{m-1}{(k+2)(k+m+1)}\right)\\&\quad+\sum_{k=0}^{\infty}2(\alpha-1)\Re(a\overline{c_k}c_{k+m})\left(\frac{1}{(k+m+1)^{\alpha-2}}\cdot\frac{1}{(k+m+1)(k+m+2)}\right),\qquad(46)\end{aligned}$$

where a well-known inequality of [12, p39] is used in the inequality (45), which states that

$$py^{p-1}(x-y)\leqslant x^p-y^p\leqslant px^{p-1}(x-y)\qquad(47)$$

whenever $x\geqslant 0, y\geqslant 0$ for $p\geqslant 1$.

Notice that

$$2\Re(a\overline{c_kc_{k+m}})\leqslant|a|b\frac{k+2m}{k+m}|c_k|^2+\frac{|a|}{b}\frac{k+m}{k+2m}|c_{k+m}|^2.\qquad(48)$$

for all nonnegative integer k and all positive number b. So

$$\begin{aligned}&\sum_{k=0}^{\infty}2\Re(a\overline{c_k}c_{k+m})\left(\frac{1}{(k+m+1)^{\alpha-1}(k+m+2)}\right)\\&\leqslant\sum_{k=0}^{\infty}|a|b\frac{k+2m}{k+m}\cdot\frac{|c_k|^2}{(k+m+1)^{\alpha-1}(k+m+2)}+\sum_{k=m}^{\infty}\frac{|a|}{b}\frac{k}{k+m}\cdot\frac{|c_k|^2}{(k+1)^{\alpha-1}(k+2)}\\&\leqslant\sum_{k=0}^{\infty}|c_k|^2\left(|a|b\frac{k+2m}{(k+m)(k+m+1)^{\alpha-1}(k+m+2)}+\frac{|a|k}{b(k+m)(k+1)^{\alpha-1}(k+2)}\right).\end{aligned}$$

Hence, we get

$$
\begin{aligned}
\|f\|^2_{\mathcal{D}_\alpha} - \|g\|^2_{\mathcal{D}_\alpha} \leqslant &(\alpha-1)\cdot\sum_{k=0}^{\infty}\frac{|c_k|^2}{(k+1)^{\alpha-1}(k+m+2)}\cdot\Bigg[(m+1)|a|^2 \\
&-\frac{(m-1)(k+1)^{\alpha-1}(k+m+2)}{(k+2)(k+m+1)^{\alpha-1}}+\frac{|a|b(k+1)^{\alpha-1}(k+2m)}{(k+m)(k+m+1)^{\alpha-1}} \\
&+\frac{|a|k(k+m+2)}{b(k+2)(k+m)}\Bigg].
\end{aligned}
$$

We denote the expression in the bracket above by d_k. Then, we are ready to show that $d_k \leqslant 0$.

(i) For $m \geqslant 3$.

Suppose $|a| \leqslant \sqrt{\frac{m-2}{2(m-1)}}\cdot\frac{1}{(k+m+1)^{u-2}}, k=0,1,\cdots$, and $b=\sqrt{\frac{2}{(m-1)(m-2)}}$.

Then

$$
|a|b \leqslant \frac{1}{(m-1)(k+m+1)^{\alpha-2}},\quad \frac{|a|}{b}\leqslant\frac{m-2}{2}\frac{1}{(k+m+1)^{\alpha-2}}.
$$

Thus,

$$
\begin{aligned}
d_k \leqslant&(m+1)\frac{m-2}{2(m-1)}\frac{1}{(k+m+1)^{\alpha-2}}-\frac{(m-1)}{(k+m+1)^{\alpha-2}}\left[1-\frac{m}{(k+2)(k+m+1)}\right] \\
&+\frac{(k+1)^{\alpha-2}}{(m-1)(k+m+1)^{\alpha-2}}\left[1-\frac{m(m-1)}{(k+m)(k+m+1)}\right] \\
&+\frac{m-2}{2}\frac{1}{(k+m+1)^{\alpha-2}}\left[1-\frac{2m}{(k+m)(k+2)}\right] \\
\leqslant&\frac{(k+1)^{\alpha-2}}{(k+m+1)^{\alpha-2}}\Bigg\{(m+1)\frac{m-2}{2(m-1)}-(m-1)\left[1-\frac{m}{(k+2)(k+m+1)}\right] \\
&+\frac{1}{m-1}\left[1-\frac{m(m-1)}{(k+m)(k+m+1)}\right]+\frac{m-2}{2}\left[1-\frac{2m}{(k+m)(k+2)}\right]\Bigg\} \\
=&0.
\end{aligned}
$$

(ii) For $m = 2$.

Choose $b = 1$. Consider

$$
\begin{aligned}
d_k \leqslant&3|a|^2-\frac{1}{(k+3)^{\alpha-2}}\left[1-\frac{2}{(k+2)(k+3)}\right] \\
&+|a|b(k+1)^{\alpha-2}\left[1-\frac{2}{(k+2)(k+3)}\right]+\frac{|a|}{b}\left[1-\frac{4}{(k+2)^2}\right] \\
=&3|a|^2+\frac{(k+4)\left[(k+1)^{\alpha-1}(k+2)+k(k+3)\right]}{(k+2)^2(k+3)}|a|-\frac{(k+1)(k+4)}{(k+2)(k+3)^{\alpha-1}}.
\end{aligned}
$$

Let

$$
F(|a|)=3|a|^2+\frac{(k+4)\left[(k+1)^{u-1}(k+2)+k(k+3)\right]}{(k+2)^2(k+3)}|a|-\frac{(k+1)(k+4)}{(k+2)(k+3)^{\alpha-1}}, \tag{49}
$$

with $|a| \in [0,1)$. Then $d_k \leqslant 0$ if and only if the above binary linear equation of $|a|$ (that is, $F(|a|)$) have solutions. Of course, it does have. Suppose $F(|a^*|) = 0$, hence $|a| \leqslant |a^*|$. □

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