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Best proximity point for generalized proximal contraction in a complete metric space

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Abstract. In this article, we have proved some best proximity point theorems for a non-self mapping by using generalized proximal contraction in a complete metric space. An example is also given in the support of our result.

1. Introduction and Preliminaries

In 1922, Banach[3] has given fixed point theorem in complete metric space by using a contraction condition defined on a self map $P : U \to U$ and obtain a unique fixed point. But in nonlinear functional analysis there is no restriction to be a map always a self map to get fixed point theorems. During last decades a question arises by the researchers that what happend when the mapping is non-self map, then how can we find fixed point. Answer of this question is the origin of best proximity point.
A non-self mapping $P : E \to G$ does not necessarily have a fixed point. If the fixed point equation $Pu = u$ has no exact solution, then we have to find an approximate solution u such that the $d(u, Pu)$ is minimum.
Now what is best proximty point? Let E and G be non-empty subsets of a metric space (U, d). Let $P : E \to G$ is a non-self mapping there exists a point $u \in E$ is called best proximity point if $d(u, Pu) = d(E, G)$, where $d(E, G) = \inf\{d(e, g) : e \in E, g \in G\}$.
In 2010, Basha[4] by considering the concept of best proximity point has given extensions of Banach contraction principle and researchers can see more generalized fixed point theorems on best proximity point by refering [1], [2], [5], [8], [9], [10], [12].

In this article, $U, \mathbb{R}^+, \mathbb{N}, \mathbb{N}_0$ denote the non-empty set, set of positive real number, set of positive integer, and set of non-negative integer respectively.

Now we recall some fundamental concepts of best proximity points.

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If E and G are non-empty subsets of U, then we denote by.

$$\begin{aligned} d(e,G) &= \inf\{d(e,g) : g \in G\}, e \in E, \\ E_0 &= \{e \in E : d(e,g) = d(E,G) \text{ for some } g \in G\}, \\ G_0 &= \{g \in G : d(e,g) = d(E,G) \text{ for some } e \in E\}. \end{aligned}$$

Wardowski [13] introduced a new class of functions to defined the notion of F-contractions and proved the following fixed point theorem, where $\mathfrak{I}$ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$.

Definition 1.1. *[13] Let (U, d) is a metric space and P be a self mapping on U. Then P is called an F-contraction, if there exists $F \in \mathfrak{I}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(d(Pu, Pv)) \leq F(d(u, v)),$$

for all $u, v \in U$ with $d(Pu, Pv) > 0$.

Definition 1.2. *[11] Let (U, d) be a metric space and (E, G) be a pair of non-empty subsets of (U, d) with $E_0 \neq \phi$. If for every $u_1, u_2 \in E$ and every $v_1, v_2 \in G$, $d(u_1, u_2) = d(v_1, v_2)$ whenever $d(u_1, v_1) = d(E, G)$ and $d(u_2, v_2) = d(E, G)$, then the pair (E, G) is said to have the p-property.*

Definition 1.3. *[6] A set G is called approximately compact with respect to E if every sequence $\{g_n\}$ of G with $d(e, g_n) \to d(e, G)$ for some $e \in E$ has a convergent subsequence.*

2. Main Results

In this section, Ismat et al. [7] introduced a new generalization of F-proximal contractions of the first and second kind and proved some best proximity point theorems for generalized F-proximal contractions of first and second kind on complete metric space. They used family of the functions $\mathfrak{I}$ having such kind of property.
Let $\mathfrak{I}$ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:
($F1$) F is strictly increasing;
($F2$) for each sequence $\{\alpha_n\}$ of positive number, we have

$$\lim_{n\to+\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to+\infty} F(\alpha_n) = -\infty;$$

($F3$) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
When we study the Ismat et. al.[7] paper and found that without using ($F2$) and ($F3$) properties by only consideration of ($F1$) we can prove the Theorem 2.4, 2.7 and 2.10. Now we are going to rewrite the Defintion 2.1 and 2.2 and also prove Theorem 2.4, 2.7 and 2.10 according to our assumptions.

Definition 2.1. *A mapping $P : E \to G$ is said to be a generalized proximal contraction of the first kind if F is strictly increasing function defined of $F : \mathbb{R}^+ \to \mathbb{R}$ and $a, b, c, h, \tau > 0$ with $a + b + c + 2h = 1, c \neq 1$ and $0 < a + 2h \leq 1$ such that the conditions*

$$\left.\begin{aligned} d(u_1, Pv_1) = d(E, G) \\ d(u_2, Pv_2) = d(E, G) \end{aligned}\right\} \textit{implies } \tau + F(d(u_1, u_2)) \leq F(ad(v_1, v_2) + bd(u_1, v_1) + cd(u_2, v_2) + h(d(v_1, u_2) + d(v_2, u_1)))$$

for all $u_1, u_2, v_1, v_2 \in E$ and $u_1 \neq u_2$.

Definition 2.2. *A mapping $P : E \to G$ is said to be a generalized proximal contraction of the second kind if F is strictly increasing function defined of $F : \mathbb{R}^+ \to \mathbb{R}$ and $a, b, c, h, \tau > 0$ with $a+b+c+2h = 1, c \neq 1$ and $0 < a+2h \leq 1$ such that the coditions*

$$\left.\begin{array}{l} d(u_1, Pv_1) = d(E, G) \\ d(u_2, Pv_2) = d(E, G) \end{array}\right\} \text{ implies } \tau + F(d(Pu_1, Pu_2)) \leq F(ad(Pv_1, Pv_2) + bd(Pu_1, Pv_1) + cd(Pu_2, Pv_2) + h(d(Pv_1, Pu_2) + d(Pv_2, Pu_1)))$$

for all $u_1, u_2, v_1, v_2 \in E$ and $Pu_1 \neq Pu_2$.

Definition 2.3. *Let $P : E \to G$ be a mapping and $u_0 \in E$ be any arbitrary point. Then P has q-property if for a sequence $\{u_n\}$ defined by*

$$d(u_{n+1}, Pu_n) = d(E, G).$$

There exists subsequences $\{u_{p(n)}\}_{n\in\mathbb{N}}$ and $\{u_{q(n)}\}_{n\in\mathbb{N}}$ of $\{u_n\}$ such that

$$\lim_{n\to+\infty} d(u_{p(n)}, u_{q(n)}) = 0,$$

where $p(n) > q(n) > n,\ n \in \mathbb{N}$, then

$$d(u_{p(n)}, Pu_{p(n)-1}) = d(E, G) \text{ and } d(u_{q(n)}, Pu_{q(n)-1}) = d(E, G).$$

Now, we are ready to state and prove our main results.

Theorem 2.4. *Let (U, d) be a complete metric space and (E, G) be a pair of non-empty closed subsets of (U, d). If G is approximately compact with respect to E and $P : E \to G$ satisfy the following conditions:*
(i) $P(E_0) \subseteq G_0$ and (E, G) satisfies the p -property;
(ii) P is a generalized proximal contraction of the first kind;
(iii) P has q-property.
Then, there exists a unique $u \in E$ such that $d(u, Pu) = d(E, G)$. Moreover, for any fixed element $u_0 \in E_0$, sequence $\{u_n\}$ defined by

$$d(u_{n+1}, Pu_n) = d(E, G),$$

converges to the best proximity point u.

Proof. Let $u_0 \in E_0$. Since, $P(E_0) \subseteq G_0$, By the definition of G_0, there is an element $u_1 \in E_0$ satisfying

$$d(u_1, Pu_0) = d(E, G).$$

Again, in veiw of the fact that $Pu_1 \in P(E_0) \subseteq G_0$, it is guranteed that there exists an element $u_2 \in E_0$ such that

$$d(u_2, Pu_1) = d(E, G).$$

Continuing in this way, we can construct a sequence $\{u_n\} \in E_0$ such that

$$d(u_{n+1}, Pu_n) = d(E, G), \tag{1}$$

for all non-negative integer n.
From the p-property and (1) we get

$$d(u_n, u_{n+1}) = d(Pu_{n-1}, Pu_n), \text{ for all } n \in \mathbb{N}.$$

If for some n_0, $d(u_{n_0}, u_{n_0+1}) = 0$, consequently

$$d(Pu_{n_0-1}, Pu_{n_0}) = 0 \text{ implies } Pu_{n_0-1} = Pu_{n_0} \text{ implies } d(u_{n_0}, Pu_{n_0}) = d(E, G).$$

Thus the enclusion is immediate.
So let for any $n \geq 0$, $d(u_n, u_{n+1}) > 0$. By our hypothesis P is a generalized proximal contraction of first kind, we have

$$\begin{aligned}\tau + F(d(u_n, u_{n+1})) \leq & F(ad(u_{n-1}, u_n) + bd(u_{n-1}, u_n) + cd(u_n, u_{n+1}) \\ & + h(d(u_{n-1}, u_{n+1}) + d(u_n, u_n))) \\ \leq & F(ad(u_{n-1}, u_n) + bd(u_{n-1}, u_n) + cd(u_n, u_{n+1}) \\ & + h(d(u_{n-1}, u_n) + d(u_n, u_{n+1}))) \\ = & F((a + b + h)d(u_{n-1}, u_n) + (c + h)d(u_n, u_{n+1}).\end{aligned}$$

Since F is strictly increasing, we obtain

$$\begin{aligned} d(u_n, u_{n+1}) &\leq (a + b + h)d(u_{n-1}, u_n) + (c + h)d(u_n, u_{n+1}) \\ (1 - c - h)d(u_n, u_{n+1}) &\leq (a + b + h)d(u_{n-1}, u_n) \\ d(u_n, u_{n+1}) &\leq \left(\frac{a + b + h}{1 - c - h}\right) d(u_{n-1}, u_n), \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Now, since $a + b + c + 2h = 1$ and $c \neq 1$, we obtain $a + b + h = 1 - c - h$ and $1 - c - h > 0$ and so

$$d(u_n, u_{n+1}) \leq d(u_n, u_{n-1}), \text{ for all } n \in \mathbb{N}. \tag{2}$$

This implies that $d(u_n, u_{n+1})$ is monotonic decreasing sequence in U.
Consequently,

$$\begin{aligned} \tau + F(d(u_n, u_{n+1})) &\leq F(d(u_n, u_{n-1})), \text{ for all } n \in \mathbb{N}, \tau > 0 \\ F(d(u_n, u_{n+1})) &\leq F(d(u_n, u_{n-1})) - \tau \end{aligned} \tag{3}$$

Now again,

$$\begin{aligned} F(d(u_n, u_{n-1})) &\leq F(d(u_{n-1}, u_{n-2})) - \tau \leq ... \\ &\leq F(d(u_0, u_1)) - n\tau, \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{4}$$

Since $\{d(u_n, u_{n+1})\}$ is monotonic decreasing sequence, so we claim that $\lim_{n\to+\infty} d(u_n, u_{n+1}) = 0$. Put $t_n = d(u_n, u_{n+1})$. Let $\lim_{n\to+\infty} d(u_n, u_{n+1}) = r > 0$, by (3) taking $n \to +\infty$

$$\begin{aligned} \lim_{n\to+\infty} F(d(u_n, u_{n+1})) &\leq \lim_{n\to+\infty} F(d(u_n, u_{n-1})) - \tau \\ F(r) &\leq F(r) - \tau < F(r), \ \tau > 0 \end{aligned}$$

we get contradiction. This implies

$$\lim_{n\to+\infty} d(u_n, u_{n+1}) = 0 \text{ implies } t_n \to 0 \text{ as } n \to +\infty. \tag{5}$$

Now, we have to show that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. For this we shall use contrapositive method. By assuming that there exists $\epsilon > 0$ and sequences $\{p(n)\}_{n\in\mathbb{N}}$ and $\{q(n)\}_{n\in\mathbb{N}}$ of positive integers such that $p(n) > q(n) > n$,

$$d(u_{p(n)}, u_{q(n)}) > \epsilon, \ d(u_{p(n)-1}, u_{q(n)}) \leq \epsilon, \text{ for all } n \in \mathbb{N}.$$

Then, we have

$$\begin{aligned}\epsilon < d(u_{p(n)}, u_{q(n)}) &\leq d(u_{p(n)}, u_{p(n)-1}) + d(u_{p(n)-1}, u_{q(n)})\\ &\leq d(u_{p(n)}, u_{p(n)-1}) + \epsilon.\end{aligned}$$

It follows from (5) and the above inequality that

$$\lim_{n\to+\infty} d(u_{p(n)}, u_{q(n)}) = \epsilon.$$

By the q-property, we have

$$\left.\begin{aligned}d(u_{p(n)}, Pu_{p(n)-1}) = d(E, G)\\ d(u_{q(n)}, Pu_{q(n)-1}) = d(E, G)\end{aligned}\right\}\text{implies}$$

$$\begin{aligned}\tau + F(d(u_{p(n)}, u_{q(n)})) \leq& F(ad(u_{p(n)-1}, u_{q(n)-1}) + bd(u_{p(n)}, u_{p(n)-1}) + cd(u_{q(n)}, u_{q(n)-1})\\ &+ h(d(u_{p(n)-1}, u_{q(n)}) + d(u_{q(n)-1}, u_{p(n)})))\\ \leq& F(a(d(u_{p(n)-1}, u_{p(n)}) + d(u_{p(n)}, u_{q(n)}) + d(u_{q(n)}, u_{q(n)-1}))\\ &+ bd(u_{p(n)}, u_{p(n)-1}) + cd(u_{q(n)}, u_{q(n)-1})\\ &+ h(d(u_{p(n)-1}, u_{p(n)}) + d(u_{p(n)}, u_{q(n)}) + d(u_{q(n)-1}, u_{q(n)})\\ &+ d(u_{q(n)}, u_{p(n)})))\\ =& F((a + 2h)d(u_{p(n)}, u_{q(n)}) + (a + b + h)d(u_{p(n)-1}, u_{p(n)})\\ &+ (a + c + h)d(u_{q(n)-1}, u_{q(n)})).\end{aligned}$$

Since $0 < a + 2h \leq 1$, letting $n \to +\infty$ in the above inequality, we get

$$\tau + F(\epsilon + 0) \leq F(\epsilon + 0)$$

which is a contradiction. This shows that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since the space (U, d) is complete, the sequnece $\{u_n\}$ converges to some element u in E.
Furthermore,

$$\begin{aligned}d(u, G) \leq d(u, Pu_n) &\leq d(u, u_{n+1}) + d(u_{n+1}, Pu_n)\\ &= d(u, u_{n+1}) + d(E, G)\\ &\leq d(u, u_{n+1}) + d(u, G).\end{aligned}$$

So, $d(u, Pu_n) \to d(u, G)$.
Therefore, G is approximately compact with respect to E, the sequence $\{Pu_n\}$ has a subsequence $\{Pu_{n_k}\}$ converging to some element v in G. So that

$$d(u, v) = \lim_{n\to+\infty} d(u_{n_{k+1}}, Pu_{n_k}) = d(E, G). \tag{6}$$

Thus u must be an element of E_0. Since $P(E_0) \subseteq G_0$,

$$d(t, Pu) = d(E, G)$$

for some element t in E. Using the p-property and (6), we have

$$d(u_{n_{k+1}}, t) = d(Pu_{n_k}, Pu), \text{ for all } n_k \in \mathbb{N}.$$

If for some n_0, $d(t, u_{n_0+1}) = 0$, consequently $d(Pu_{n_0}, Pu) = 0$ implies $Pu_{n_0} = Pu$ implies $d(u, Pu) = d(E, G)$. Thus the enclusion is immediate.

So let for any $n \geq 0$, $d(t, u_{n+1}) > 0$. By our hypothesis P is a generalized proximal contraction of first kind, we have

$$\tau + F(d(t, u_{n+1})) \leq F(ad(u, u_n) + bd(t, u) + cd(u_n, u_{n+1}) + h(d(u, u_{n+1}) + d(u_n, t))).$$

Since F is strictly increasing, we obtain

$$d(t, u_{n+1}) \leq ad(u, u_n) + bd(t, u) + cd(u_n, u_{n+1}) + h(d(u, u_{n+1}) + d(u_n, t)).$$

Letting $n \to +\infty$,

$$\begin{aligned} d(t, u) &\leq bd(t, u) + hd(t, u) \\ d(t, u) &\leq (b + h)d(t, u), \end{aligned}$$

this shows that u and t must be an identical. It follows, that

$$d(u, Pu) = d(t, Pu) = d(E, G).$$

To prove the uniqueness of the best proximity point. Let u^* is an another best proximity point of the mapping P such that

$$d(u^*, Pu^*) = d(E, G).$$

Since P is a generalized proximal contraction of the first kind, therefore

$$\tau + F(d(u, u^*)) \leq F((a + 2h)d(u, u^*)).$$

Since F is strictly increasing,

$$d(u, u^*) \leq (a + 2h)d(u, u^*).$$

Therefore, u and u^* must be identical. Hence, P has a unique best proximity point. □

We can obtain the following corollaries from the Theorem 2.4.

Corollary 2.5. *Let (U, d) be a complete metric space and (E, G) be a pair of non-empty closed subsets of (U, d). If G is approximately compact with respect to E and $P : E \to G$ satisfy the following proximal contraction condition:*

$$\left.\begin{aligned} d(u_1, Pv_1) = d(E, G) \\ d(u_2, Pv_2) = d(E, G) \end{aligned}\right\} \text{ implies } \tau + F(d(u_1, u_2)) \leq F(ad(v_1, v_2) + bd(u_1, v_1) + cd(u_2, v_2)),$$

where F is strictly increasing function and $a, b, c, \tau > 0$ with $a + b + c = 1$, $c \neq 1$, for all $u_1, u_2, v_1, v_2 \in E$ and $u_1 \neq u_2$. Also,
(i) $P(E_0) \subseteq G_0$ and (E, G) satisfies the p -property;
(ii) P has q-property.
Then, there exists a unique $u \in E$ such that $d(u, Pu) = d(E, G)$. Moreover, for any fixed element $u_0 \in E_0$, sequence $\{u_n\}$ defined by

$$d(u_{n+1}, Pu_n) = d(E, G),$$

converges to the best proximity point u.

Corollary 2.6. *Let (U, d) be a complete metric space and (E, G) be a pair of non-empty closed subsets of (U, d). If G is approximately compact with respect to E and $P : E \to G$ satisfy the following proximal contraction condition:*

$$\left.\begin{aligned} d(u_1, Pv_1) = d(E, G) \\ d(u_2, Pv_2) = d(E, G) \end{aligned}\right\} \text{ implies } \tau + F(d(u_1, u_2)) \leq F(d(v_1, v_2)),$$

where F is strictly increasing function and $\tau > 0$, *for all* $u_1, u_2, v_1, v_2 \in E$ *and* $u_1 \neq u_2$.
Also,
(i) $P(E_0) \subseteq G_0$ *and* (E, G) *satisfies the p -property;*
(ii) P has q-property.
Then, there exists a unique $u \in E$ *such that* $d(u, Pu) = d(E, G)$. *Moreover, for any fixed element* $u_0 \in E_0$, *sequence* $\{u_n\}$ *defined by*

$$d(u_{n+1}, Pu_n) = d(E, G),$$

converges to the best proximity point u.

Now, we will state and prove the result for non-self generalized proximal contraction of the second kind.

Theorem 2.7. *Let* (U, d) *be a complete metric space and* (E, G) *be a pair of non-empty closed subsets of* (U, d). *If E is approximately compact with respect to G and* $P : E \to G$ *satisfy the following conditions:*
(i) $P(E_0) \subseteq G_0$ *and* (E, G) *satisfies the p -property;*
(ii) P is a continuous generalized proximal contraction of the second kind;
(iii) P has q-property.
Then, there exists a unique $u \in E$ *such that* $d(u, Pu) = d(E, G)$. *Moreover, for any fixed element* $u_0 \in E_0$, *sequence* $\{u_n\}$ *defined by*

$$d(u_{n+1}, Pu_n) = d(E, G),$$

converges to the best proximity point u. Further, if u^* *is another best proximity point of P, then* $Pu = Pu^*$.

Proof. Similar to Theorem 2.4, we can construct a sequence $\{u_n\}$ in E_0 such that

$$d(u_{n+1}, Pu_n) = d(E, G), \tag{7}$$

for all non-negative integer n.
From the p-property and (7) we get

$$d(u_n, u_{n+1}) = d(Pu_{n-1}, Pu_n), \text{ for all } n \in \mathbb{N}.$$

If for some n_0, $d(u_{n_0}, u_{n_0+1}) = 0$, consequently

$$d(Pu_{n_0-1}, Pu_{n_0}) = 0 \text{ implies } Pu_{n_0-1} = Pu_{n_0} \text{ implies } d(u_{n_0}, Pu_{n_0}) = d(E, G).$$

Thus the enclusion is immediate.
So let for any $n \geq 0$, $d(Pu_n, Pu_{n+1}) > 0$. By our hypothesis P is a generalized proximal contraction of second kind, we have

$$\begin{aligned}\tau + F(d(Pu_n, Pu_{n+1})) \leq & F(ad(Pu_{n-1}, Pu_n) + bd(Pu_{n-1}, Pu_n) + cd(Pu_n, Pu_{n+1}) \\ & + h(d(Pu_{n-1}, Pu_{n+1}) + d(Pu_n, Pu_n))) \\ \leq & F(ad(Pu_{n-1}, Pu_n) + bd(Pu_{n-1}, Pu_n) + cd(Pu_n, Pu_{n+1}) \\ & + h(d(Pu_{n-1}, Pu_n) + d(Pu_n, Pu_{n+1}))) \\ = & F((a + b + h)d(Pu_{n-1}, Pu_n) + (c + h)d(Pu_n, Pu_{n+1})).\end{aligned}$$

Since F is strictly increasing, we obtain

$$\begin{aligned}d(Pu_n, Pu_{n+1}) &\leq (a + b + h)d(Pu_{n-1}, Pu_n) + (c + h)d(Pu_n, Pu_{n+1}) \\ (1 - c - h)d(Pu_n, Pu_{n+1}) &\leq (a + b + h)d(Pu_{n-1}, Pu_n) \\ d(Pu_n, Pu_{n+1}) &\leq \left(\frac{a + b + h}{1 - c - h}\right)d(Pu_{n-1}, Pu_n), \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Now, since $a+b+c+2h=1$ and $c \neq 1$, we obtain
$a+b+h=1-c-h$ and $1-c-h>0$ and so

$$d(Pu_n, Pu_{n+1}) \leq d(Pu_n, Pu_{n-1}), \text{ for all } n \in \mathbb{N}. \tag{8}$$

This implies that $d(Pu_n, Pu_{n+1})$ is monotonic decreasing sequence in U.
Consequently,

$$\begin{aligned} \tau + F(d(Pu_n, Pu_{n+1})) &\leq F(d(Pu_n, Pu_{n-1})), \text{ for all } n \in \mathbb{N}, \tau > 0 \\ F(d(Pu_n, Pu_{n+1})) &\leq F(d(Pu_n, Pu_{n-1})) - \tau \end{aligned} \tag{9}$$

Now again,

$$\begin{aligned} F(d(Pu_n, Pu_{n-1})) &\leq F(d(Pu_{n-1}, Pu_{n-2})) - \tau \leq ... \\ &\leq F(d(Pu_0, Pu_1)) - n\tau, \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{10}$$

Since $\{d(Pu_n, Pu_{n+1})\}$ is monotonic decreasing sequence, so we claim that $\lim_{n\to+\infty} d(Pu_n, Pu_{n+1}) = 0$. Put $s_n = d(Pu_n, Pu_{n+1})$. Let $\lim_{n\to+\infty} d(Pu_n, Pu_{n+1}) = s > 0$, by (9) taking $n \to +\infty$

$$\begin{aligned} \lim_{n\to+\infty} F(d(Pu_n, Pu_{n+1})) &\leq \lim_{n\to+\infty} F(d(Pu_n, Pu_{n-1})) - \tau \\ F(s) &\leq F(s) - \tau < F(s), \ \tau > 0 \end{aligned}$$

we get contradiction. This implies

$$\lim_{n\to+\infty} d(Pu_n, Pu_{n+1}) = 0 \text{ implies } s_n \to 0 \ as \ n \to +\infty. \tag{11}$$

Now, we have to show that $\{Pu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. For this we shall use contrapositive method. By assuming that there exists $\epsilon > 0$ and sequences $\{p(n)\}_{n\in\mathbb{N}}$ and $\{q(n)\}_{n\in\mathbb{N}}$ of positive integers such that $p(n) > q(n) > n$,

$$d(Pu_{p(n)}, Pu_{q(n)}) > \epsilon, \ d(Pu_{p(n)-1}, Pu_{q(n)}) \leq \epsilon, \text{ for all } n \in \mathbb{N}.$$

Then, we have

$$\begin{aligned} \epsilon < d(Pu_{p(n)}, Pu_{q(n)}) &\leq d(Pu_{p(n)}, Pu_{p(n)-1}) + d(Pu_{p(n)-1}, Pu_{q(n)}) \\ &\leq d(Pu_{p(n)}, Pu_{p(n)-1}) + \epsilon. \end{aligned}$$

It follows from (11) and the above inequality that

$$\lim_{n\to+\infty} d(Pu_{p(n)}, Pu_{q(n)}) = \epsilon.$$

By the q-property, we have

$$\left.\begin{aligned} d(u_{p(n)}, Pu_{p(n)-1}) = d(E, G) \\ d(u_{q(n)}, Pu_{q(n)-1}) = d(E, G) \end{aligned}\right\} \text{ implies}$$

$$\begin{aligned}
\tau + F(d(Pu_{p(n)}, Pu_{q(n)})) \leq & F(ad(Pu_{p(n)-1}, Pu_{q(n)-1}) + bd(Pu_{p(n)}, Pu_{p(n)-1}) \\
& + cd(Pu_{q(n)}, Pu_{q(n)-1}) + h(d(Pu_{p(n)-1}, Pu_{q(n)}) \\
& + d(Pu_{q(n)-1}, Pu_{p(n)}))) \\
\leq & F(a(d(Pu_{p(n)-1}, Pu_{p(n)}) + d(Pu_{p(n)}, Pu_{q(n)}) \\
& + d(Pu_{q(n)}, Pu_{q(n)-1})) + bd(Pu_{p(n)}, Pu_{p(n)-1}) \\
& + cd(Pu_{q(n)}, Pu_{q(n)-1}) + h(d(Pu_{p(n)-1}, Pu_{p(n)}) \\
& + d(Pu_{p(n)}, Pu_{q(n)}) + d(Pu_{q(n)-1}, Pu_{q(n)}) \\
& + d(Pu_{q(n)}, Pu_{p(n)}))) \\
= & F((a + 2h)d(Pu_{p(n)}, Pu_{q(n)}) + (a + b + h)d(Pu_{p(n)-1}, Pu_{p(n)}) \\
& + (a + c + h)d(Pu_{q(n)-1}, Pu_{q(n)})).
\end{aligned}$$

Since $0 < a + 2h \leq 1$, letting $n \to +\infty$ in the above inequality, we get

$$\tau + F(\epsilon + 0) \leq F(\epsilon + 0)$$

which is a contradiction. This shows that $\{Pu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since the space (U, d) is complete, the sequnece $\{Pu_n\}$ converges to some element v in G.
Furthermore,

$$\begin{aligned}
d(v, E) \leq d(v, u_{n+1}) & \leq d(v, Pu_n) + d(Pu_n, u_{n+1}) \\
& = d(v, Pu_n) + d(E, G) \\
& \leq d(v, Pu_n) + d(v, E).
\end{aligned}$$

So, $d(v, u_n) \to d(v, E)$.
Therefore, E is approximately compact with respect to G, the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ converging to some element u in E. So that

$$d(u, Pu) = \lim_{n\to+\infty} d(u_{n+1}, Pu_n) = d(E, G). \tag{12}$$

To prove the uniqueness of the best proximity point. Let u^* is an another best proximity point of the mapping P such that

$$d(u^*, Pu^*) = d(E, G).$$

Since P is a generalized proximal contraction of the second kind, therefore

$$\tau + F(d(Pu, Pu^*)) \leq F((a + 2h)d(Pu, Pu^*)).$$

Since F is strictly increasing,

$$d(Pu, Pu^*) \leq (a + 2h)d(Pu, Pu^*).$$

Therefore, u and u^* must be identical. Hence, P has a unique best proximity point. □

We can obtain the following corollaries from the Theorem 2.7.

Corollary 2.8. *Let (U, d) be a complete metric space and (E, G) be a pair of non-empty closed subsets of (U, d). If E is approximately compact with respect to G and $P : E \to G$ satisfies the following proximal contraction condition:*

$$\left.\begin{aligned} d(u_1, Pv_1) = d(E, G) \\ d(u_2, Pv_2) = d(E, G) \end{aligned}\right\} \textit{ implies } \tau + F(d(Pu_1, Pu_2)) \leq F(ad(Pv_1, Pv_2) + bd(Pu_1, Pv_1) + cd(Pu_2, Pv_2)),$$

where F is strictly increasing function and $a, b, c, \tau > 0$ *with* $a + b + c = 1$, $c \neq 1$, *for all* $u_1, u_2, v_1, v_2 \in E$ *and* $Pu_1 \neq Pu_2$.
Also,
(i) $P(E_0) \subseteq G_0$ *and* (E, G) *satisfies the p -property;*
(ii) P has q-property.
Then, there exists a unique $u \in E$ *such that* $d(u, Pu) = d(E, G)$. *Moreover, for any fixed element* $u_0 \in E_0$, *sequence* $\{u_n\}$ *defined by*

$$d(u_{n+1}, Pu_n) = d(E, G),$$

converges to the best proximity point u. Further, if u^* *is another best proximity point of P, then* $Pu = Pu^*$.

Corollary 2.9. *Let* (U, d) *be a complete metric space and* (E, G) *be a pair of non-empty closed subsets of* (U, d). *If E is approximately compact with respect to G and* $P : E \to G$ *satisfies the following proximal contraction condition:*

$$\left.\begin{array}{l} d(u_1, Pv_1) = d(E, G) \\ d(u_2, Pv_2) = d(E, G) \end{array}\right\} \text{ implies } \tau + F(d(Pu_1, Pu_2)) \leq F(d(Pv_1, Pv_2)),$$

where F is strictly increasing function and $\tau > 0$, *for all* $u_1, u_2, v_1, v_2 \in E$ *and* $Pu_1 \neq Pu_2$.
Also,
(i) $P(E_0) \subseteq G_0$ *and* (E, G) *satisfies the p -property;*
(ii) P has q- property.
Then, there exists a unique $u \in E$ *such that* $d(u, Pu) = d(E, G)$. *Moreover, for any fixed element* $u_0 \in E_0$, *sequence* $\{u_n\}$ *defined by*

$$d(u_{n+1}, Pu_n) = d(E, G),$$

converges to the best proximity point u. Further, if u^* *is another best proximity point of P, then* $Pu = Pu^*$.

Our next result is for non-self generalized proximal contractions of the first kind as well as second kind without the assumption of approximately compactness of the domains or the co-domain of the mappings.

Theorem 2.10. *Let* (U, d) *be a complete metric space and* (E, G) *be a pair of non-empty closed subsets of* (U, d). *Let* $P : E \to G$ *satisfy the following conditions:*
(i) $P(E_0) \subseteq G_0$ *and* (E, G) *satisfies the p -property;*
(ii) P is a generalized proximal contraction of the first kind as well as a second kind;
(iii) P has q-property.
Then, there exists a unique element $u \in E$ *such that* $d(u, Pu) = d(E, G)$. *Moreover, for any fixed element* $u_0 \in E_0$, *sequence* $\{u_n\}$ *defined by*

$$d(u_{n+1}, Pu_n) = d(E, G),$$

converges to the best proximity point u. Further, if u^* *is another best proximity point of P, then* $Pu = Pu^*$.

Proof. Similar to Theorem 2.4, we can construct a sequence $\{u_n\}$ in E_0 such that

$$d(u_{n+1}, Pu_n) = d(E, G), \tag{13}$$

for all non-negative integer n, $P(E_0) \subseteq G_0$. Similar to Theorem 2.4, we can show that sequence $\{u_n\}$ is a Cauchy sequence. Thus converges to some element u in E. As in Theorem 2.7, it can be shown that the sequence $\{Pu_n\}$ is a Cauchy sequence and converges to some element v in G. Therefore,

$$d(u, v) = \lim_{n \to +\infty} d(u_{n+1}, Pu_n) = d(E, G). \tag{14}$$

Thus, u becomes an element of E_0. Since $P(E_0) \subseteq G_0$,

$$d(t, Pu) = d(E, G)$$

for some element t in E. Using the p-property and (14), we have

$$d(u_{n+1},t) = d(Pu_n, Pu), \text{ for all } n_k \in \mathbb{N}.$$

If for some n_0, $d(t,u_{n_0+1}) = 0$, consequently $d(Pu_{n_0}, Pu) = 0$ implies $Pu_{n_0} = Pu$, hence $d(E,G) = d(u,Pu)$. Thus the enclusion is immediate.
So let for any $n \geq 0$, $d(t,u_{n+1}) > 0$. Since P is a generalized proximal contraction of first kind, it follows from this that

$$\tau + F(d(t,u_{n+1})) \leq F(ad(u,u_n) + bd(t,u) + cd(u_n,u_{n+1}) + h(d(u,u_{n+1}) + d(u_n,t))).$$

Since F is strictly increasing, we have

$$d(t,u_{n+1}) \leq ad(u,u_n) + bd(t,u) + cd(u_n,u_{n+1}) + h(d(u,u_{n+1}) + d(u_n,t)).$$

As $n \to +\infty$,

$$\begin{aligned} d(t,u) &\leq bd(t,u) + hd(t,u) \\ d(t,u) &\leq (b+h)d(t,u), \end{aligned}$$

which implies that u and t must be identical. It follows, that

$$d(u,Pu) = d(t,Pu) = d(E,G).$$

Also, as in the Theorem 2.4 , the uniqueness of the best proximity point of mapping P follows. □

Example 2.11. *Let $U = \mathbb{R}^2$ and d be a metric defined on U by*

$$d((u_1,u_2),(v_1,v_2)) = |u_1 - v_1| + |u_2 - v_2|, \text{ for all } (u_1,u_2),(v_1,v_2) \in \mathbb{R}^2$$

and (U,d) be a complete metric space. Let $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(t) = 2t$, which is increasing function.
Let $E = \{(e,0) : e \geq 0\}$ and $G = \{(g,1) : g \geq 0\}$. Here we have $E = E_0$ and $G = G_0$. Let $P : E \to G$ be a mapping defind by, for each $(e,0) \in E$,

$$P((e,0)) = (T(e),1),$$

where

$$T(e) = \frac{e}{1+e}.$$

Consider,

$$u_n = \left(\frac{1}{n},0\right), n \in \mathbb{N}.$$

$$\begin{aligned} P(u_n) &= P\left(\left(\frac{1}{n},0\right)\right) \\ &= \left(T\left(\frac{1}{n}\right),1\right) \\ \text{implies } P(u_n) &= \left(\frac{1}{1+n},1\right) \end{aligned}$$

Consider $u_{n_k} = \left(\frac{1}{2k},0\right)$ and $u_{m_k} = \left(\frac{1}{3k},0\right)$ where $3k > 2k > n$.

$$\begin{aligned} d\left(u_{n_k},u_{m_k}\right) &= d\left(\left(\frac{1}{2k},0\right),\left(\frac{1}{3k},0\right)\right) \\ &= \left|\frac{1}{2k} - \frac{1}{3k}\right| \\ &\to 0 \text{ as } k \to +\infty. \end{aligned}$$

Now,

$$\begin{aligned} d\left(\left(\frac{1}{2k},0\right),P\left(\frac{1}{2k-1},0\right)\right) &= d\left(\left(\frac{1}{2k},0\right),\left(T\left(\frac{1}{2k-1}\right),1\right)\right) \\ &= d\left(\left(\frac{1}{2k},0\right),\left(\frac{1}{2k},1\right)\right) \\ &= \left|\frac{1}{2k}-\frac{1}{2k}\right| + |0-1| \\ &= 1 = d(E,G). \end{aligned}$$

Similarly,

$$d\left(\left(\frac{1}{3k},0\right),P\left(\frac{1}{3k-1},0\right)\right) = 1 = d(E,G).$$

Hence, P satisfies q-property. Now, it is clear that, for each $x, y \geq 0$,

$$\begin{aligned} |T(x)-T(y)| &= \left|\frac{x}{1+x}-\frac{y}{1+y}\right| \\ &= \left|\frac{x-y}{(1+x)(1+y)}\right| \\ &\leq \left|x-y\right|. \end{aligned}$$

Hence, E is approximately compact with respect to G, (E, G) *satisfies the p-property, P is continuous and* $P(E_0) \subseteq G_0$. *Now* $r, s, i, j \in E$ *such that* $d(r, Pi) = d(E, G)$ *and* $d(s, Pj) = d(E, G)$. *Let* $i = (e_1, 0)$ *and* $j = (e_2, 0)$ *for some* $e_1, e_2 \geq 0$. *So*

$$r = (T(e_1), 1) = \left(\frac{e_1}{1+e_1}, 1\right),\ \ s = (T(e_2), 1) = \left(\frac{e_2}{1+e_2}, 1\right).$$

We obtain that

$$\begin{aligned} d(Pr,Ps) &= d\left(P\left(\frac{e_1}{1+e_1},1\right),P\left(\frac{e_2}{1+e_2},1\right)\right) \\ &= d\left(\left(T\left(\frac{e_1}{1+e_1}\right),1\right),\left(T\left(\frac{e_2}{1+e_2}\right),1\right)\right) \\ &= \left|T\left(\frac{e_1}{1+e_1}\right)-T\left(\frac{e_2}{1+e_2}\right)\right| \\ &= \left|\frac{e_1}{1+2e_1}-\frac{e_2}{1+2e_2}\right|. \end{aligned}$$

and

$$\begin{aligned} d(Pi,Pj) &= d(P(e_1,0),P(e_2,0)) \\ &= d((T(e_1),1),(T(e_2),1)) \\ &= d\left(\left(\frac{e_1}{1+e_1},1\right)-\left(\frac{e_2}{1+e_2},1\right)\right) \\ &= \left|\frac{e_1}{1+e_1}-\frac{e_2}{1+e_2}\right| \end{aligned}$$

Now,

$$\begin{aligned} \tau + F(d(Pr,Ps)) &\leq F(ad(Pi,Pj)) \ \ \textit{if}\ c,b,h=0 \\ \tau + 2(d(Pr,Ps)) &\leq 2ad(Pi,Pj) \\ \tau + 2\left|\frac{e_1}{1+2e_1}-\frac{e_2}{1+2e_2}\right| &\leq 2a\left|\frac{e_1}{1+e_1}-\frac{e_2}{1+e_2}\right| \\ \tau + 2\left|\frac{e_1}{1+2e_1}-\frac{e_2}{1+2e_2}\right| &\leq 2\left|\frac{e_1}{1+e_1}-\frac{e_2}{1+e_2}\right|, \quad \textit{if}\ a=1 \end{aligned}$$

When we put $e_1 = 0$ *and* $e_2 = 1$ *we get* $\tau \in (0, \frac{1}{3})$. *Hence P is a generalized proximal contraction of second kind. Thus, all the condition of Theorem 2.7 are satisfied. Hence P has a unique best proximity point* $(0, 0)$.

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