Some results on the maximal safety distance in a graph

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Abstract. The work in this paper is motivated by I. Banič and A. Taranenko's recent paper, where they introduced a new notion, the span of a graph. Their goal was to solve the problem of keeping the safety distance while two players are moving through a graph and they presented three different types of graph spans, depending on the movement rules. We observe the same goal, but give a different approach to that problem by directly defining the maximal safety distance for different movement rules two players can take. This allowed us to solve several problems, prove some relations between different graph spans, and calculate the span values for some classes of graphs.

1. Introduction

The inspiration for this work is found in Banič and Taranenko's recent paper, [1], in which they present a graph theoretical equivalent to the Lelek's topological notion of a span, first defined in [3]. It was motivated by the problem of keeping a safety distance between two players moving in public spaces, the relevance of which has become quite apparent during the recent pandemic years. Many maps can be modeled as a graph, be it city streets, classrooms in schools or rooms in the museum. The authors presented three different concepts of a span of a graph, depending on the movement rules the players were bound by, and for each span they defined the vertex and the edge variant. They analyzed some families of graphs with regard to different spans and gave characterizations for some span values. In this paper we observe the original problem of keeping the safety distance and we define this notion of the safety distance a bit differently than it was defined in [1].

We will now introduce the problem in more detail. The authors of the original paper used Alice and Bob as well-known designations for two players moving through a graph, so we will continue that approach. Let Alice and Bob be two players moving through graph vertices via graph edges.

Most importantly, Alice and Bob want to keep the maximal possible safety distance from each other in all the steps. For example, one possible movement through the graph G in Figure 1 where Alice and Bob visit all the vertices while keeping the distance 2 from each other, is presented in Table 1.

Table 1: Alice and Bob's order of movement

	1	2	3	4	5	6
Alice	u_4	u_5	u_6	u_1	u_2	u_3
Bob	u_1	u_2	u_4	u_3	u_6	u_5

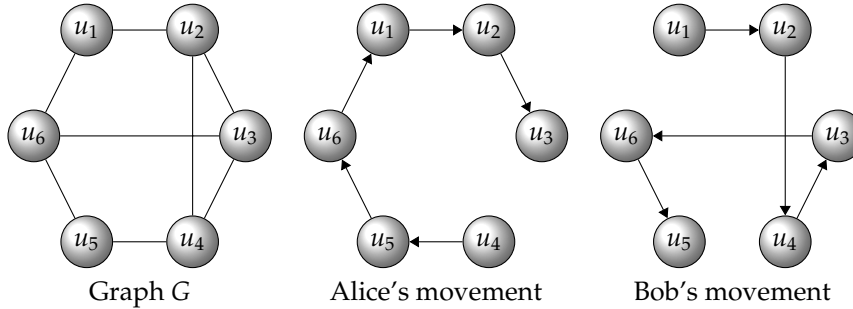Graph G Alice's movement   Bob's movement

Figure 1: Example of a graph with Alice and Bob's movement keeping the distance 2

We will describe their movements as functions, from some set $\mathbb{N}_l$, representing the order of their movement, to the set of vertices of a given graph. Of course, they can only move from vertex to vertex if they are adjacent. Generally, in each step, we will allow both players to move to an adjacent vertex, or to stay in place, and our definitions of movement functions will reflect this.

However, there are three possible ways in which Alice and Bob may move through a graph simultaneously:

- Traditional movement rules: Alice and Bob move independently, and at each step each of them may move to the adjacent vertex or stay in place.

- Active movement rules: Both Alice and Bob must move to the adjacent vertex in each step.

- Lazy movement rules: In each step, exactly one of the players moves to an adjacent vertex, and the other player stays in place.

These different rules, together with the requirement to visit all the vertices or all the edges, led the authors of [1] to define six different spans of a graph. Mathematical definitions of these spans are quite complicated and we present our approach to the problem in Section 3. Intuitively, one can imagine the value of a particular span as the maximal possible safety distance that two players can keep on a given graph with given movement rules. In this paper, we will observe vertex spans only, and we will use the same notation as used in [1], i.e.:

- $\sigma_V^{\boxtimes}(G)$ - strong vertex span, corresponding to the traditional movement rules;

- $\sigma_V^{\times}(G)$ - direct vertex span, corresponding to the active movement rules;

- $\sigma_V^{\square}(G)$ - Cartesian vertex span, corresponding to the lazy movement rules.

Our main result, the relation between direct and Cartesian vertex span for each connected simple graph is presented in Section 4. Further, we observe values of spans for some classes of graphs, based on [1] and our own results, we prove and summarize this in Section 5.

2. Preliminaries

We now give an overview of definitions and notation. Well-known terminology of graph theory can be found in [4], however, to facilitate reading, we will define some concepts important for our research.

Let G be a connected graph. The **distance** $d_G(u, v)$, between the vertices $u, v \in V(G)$ is defined as the length of the shortest path between u and v. Let u be a vertex in a simple connected graph G. The **eccentricity** of a vertex u in G is defined as

$$\text{ecc}_G(u) = \max\{d_G(u, v) : v \in V(G)\}.$$

If it is clear which graph we are observing, we use $\text{ecc}(u)$ and $d(u, v)$. The **radius** of a simple connected graph G is

$$\text{rad}(G) = \min\{\text{ecc}(u) : u \in V(G)\},$$

and the **diameter** of a simple connected graph G is

$$\text{diam}(G) = \max\{\text{ecc}(u) : u \in V(G)\}.$$

Based on the motivation, it is clear that it makes sense to observe only simple connected graphs, so the term graph will refer to a simple connected graph from now on.

Let G and H be graphs. By $H \subseteq G$ we denote that H is a subgraph of G, and by $H \subseteq_C G$ we denote that H is a connected subgraph of G.

In paper [1] the authors make a connection between the maximal safety distance in a graph, as described in the Introduction, and particular subgraphs of graph products. They observe strong, Cartesian and direct graph products, corresponding to their definitions of strong, Cartesian and direct span. We will not repeat all their definitions here, but refer the reader to their paper as well as a book about graph products, for instance [2]. Instead, we proceed to our own definitions of the maximal safety distance and later show how it is connected to the spans in paper [1].

By $\mathbb{N}_l$ we denote the set of first l natural numbers, i.e. the set $\{1, \ldots, l\}$.

Definition 2.1. *Let $G = (V, E)$ be a graph and $l \in \mathbb{N}$. We say that a surjective function $f_l : \mathbb{N}_l \longrightarrow V(G)$ is an l-**track** on G if $f(i)f(i + 1) \in E(G)$ holds, for each $i \in \{1, \ldots, l-1\}$.*

Definition 2.2. *Let $G = (V, E)$ be a graph and $l \in \mathbb{N}$. We say that a surjective function $f_l : \mathbb{N}_l \longrightarrow V(G)$ is a **lazy** l-**track** on G if $f(i)f(i + 1) \in E(G)$ or $f(i) = f(i + 1)$ holds, for each $i \in \{1, \ldots, l-1\}$.*

It is easy to see that each l-track is also a lazy l-track. These functions represent a walk through all the vertices one player visits in the graph, as described in the Introduction.

One can immediately observe that if f_l is a lazy l-track on some graph G, then $l \geq |V(G)|$ must hold.

Proposition 2.3. *Let G be a graph. There exists $l \in \mathbb{N}$ such that there exists a function $f_l : \mathbb{N}_l \longrightarrow V(G)$ which is a lazy l-track on G, and for each $l' > l$ there exists a lazy l'-track on G.*

Proof. The claim that there exists $l \in \mathbb{N}$ such that there exists a function $f_l : \mathbb{N}_l \longrightarrow V(G)$ that is a lazy l-track on G, follows directly from the fact that G is connected so there exists a walk through all the vertices. Let G be a graph and $l \in \mathbb{N}$ such that $f_l : \mathbb{N}_l \longrightarrow V(G)$ is a lazy l-track on G. Let $l' > l$. Function $f_{l'} : \mathbb{N}_{l'} \longrightarrow V(G)$ for which $f_{l'}(i) = f_l(i)$, for $i \in \{1, \ldots, l\}$, and $f_{l'}(i) = f_l(l)$, for $l < i \leq l'$ is a lazy l'-track on G. $\square$

Now let us define the distance between two lazy l-tracks on a graph.

Definition 2.4. *Let G be a graph and f, g lazy l-tracks on G. The **distance between f and g** is defined as*

$$m_G(f, g) = \min\{d_G(f(i), g(i)) : i \in \mathbb{N}_l\}.$$

Lemma 2.5. *Let G be a graph and f, g lazy l-tracks on G, for some l. It holds*

$$m_G(f, g) \leq \text{rad}(G).$$

Proof. Let G be a graph and f, g lazy l-tracks on G. Let $v \in V(G)$ be a vertex for which $\text{ecc}(v) = \text{rad}(G)$ holds. From surjectivity of f it follows that there exists $i \in \mathbb{N}_l$ such that $f(i) = v$. But then we have $d_G(f(i), g(i)) \leq \text{rad}(G)$, so $m_G(f, g) \leq \text{rad}(G)$ follows. $\square$

Remark 2.6. *Lemma 2.5 is stated and proved in paper [1] in different terms. Since we will use it on occasion, we restate it here in the terms of lazy tracks.*

3. Maximal safety distances and graph spans

In this section we aim to define the maximal safety distance formally and to show how these definitions correspond to the concept of graph spans.

Strong vertex span

Let G be a graph and $l \in \mathbb{N}$ such that at least one lazy l-track on G exists. We define

$$M_l^{\boxtimes} = \max\{m_G(f, g) : f \text{ and } g \text{ lazy } l\text{-tracks on } G\}.$$

The existence of $M_l^{\boxtimes}$ follows from Lemma 2.5.

Let G be a graph and let $S \subseteq \mathbb{N}$ be the set of all integers l for which at least one lazy l-track on G exists. Obviously, S is non-empty. We define

$$M^{\boxtimes}(G) = \max\{M_l^{\boxtimes} : l \in S\}.$$

This number is the maximal safety distance that can be kept while two players visit all the vertices of a graph with respect to traditional movement rules.

To show how $M^{\boxtimes}(G)$ equals $\sigma_V^{\boxtimes}(G)$ we need one more definition, the same as used in paper [1].

Definition 3.1. *Let G and H be graphs such that $V(H) \subseteq V(G) \times V(G)$. We define*

$$\varepsilon_G(H) = \min\{d_G(u, v) : (u, v) \in V(H)\}.$$

Theorem 3.2. *Let G be a graph. It holds $M^{\boxtimes}(G) = \sigma_V^{\boxtimes}(G)$.*

Proof. Let G be a graph and let $S \subseteq \mathbb{N}$ be the set of all integers l for which at least one lazy l-track on G exists. Let

$$A = \{M_l^{\boxtimes} : l \in S\}$$

and

$$B = \{\varepsilon_G(H) : H \subseteq_C G \boxtimes G \text{ with } p_1(V(H)) = p_2(V(H)) = V(G)\},$$

where $\boxtimes$ is the strong product of the two graphs, and p_1, p_2 are projections to the corresponding coordinates, [1]. By Theorem 3.3. in [1] we need to prove that $\max A = \max B$. We will prove that $A = B$. Let $r \in A$. Let f and g be lazy l-tracks such that $m_G(f, g) = r$. Denote

$$V(H) = \{(f(i), g(i)) : i \in \mathbb{N}_l\}$$

and

$$E(H) = \{f(i)g(i) : i \in \mathbb{N}_l\}.$$

From the definition of f and g it follows that H is a connected subgraph of $G \boxtimes G$ and that $p_1(V(H)) = p_2(V(H)) = V(G)$. Moreover, $\varepsilon_G(H) = r$, therefore $r \in B$. Now, let $r \in B$. Let $H \subseteq_C G \boxtimes G$ with $p_1(V(H)) = p_2(V(H)) = V(G)$ such that $\varepsilon_G(H) = r$.

We will now construct a walk through all the vertices of H which will allow us to define lazy tracks on G and conclude that $r \in A$. Choose any $(u_1, v_1) \in V(H)$. We know that H is connected and therefore we can obtain its spanning tree with root (u_1, v_1). Now we construct a walk that visits all the vertices of H, starting from (u_1, v_1) and we denote the steps of this walk by natural numbers $1, ..., l$, so that we are in the vertex (u_i, v_i) in step i. Note that, for some $i \neq j$, (u_i, v_i) and (u_j, v_j) may be the same. Also note that l is not necessarily the smallest possible number of steps needed to visit all the vertices of H, but it holds $V(H) = \{(u_i, v_i) : i \in \mathbb{N}_l\}$. Now we define functions $f : \mathbb{N}_l \to V(G)$ and $g : \mathbb{N}_l \to V(G)$ with $f(i) = u_i$ and

$g(i) = v_i$. From the definition of the strong product $\boxtimes$ and from $p_1(V(H)) = p_2(V(H)) = V(G)$ it follows that f and g are lazy l-tracks. Furthermore, we have

$$
\begin{aligned}
m_G(f, g) &= \min\{d_G(f(i), g(i)) : i \in \mathbb{N}_l\} \\
&= \min\{d_G(u_i, v_i) : (u_i, v_i) \in V(H)\} \\
&= \varepsilon_G(H).
\end{aligned}
$$

Moreover, $m_G(f, g) = r$, therefore $r \in A$.

We have proved that $A = B$ so it follows that $\max A = \max B$. $\quad\square$

Direct vertex span

Let G be a graph and $l \in \mathbb{N}$ such that at least one l-track on G exists. We define

$$
M_l^\times = \max\{m_G(f, g) : f \text{ and } g \text{ } l\text{-tracks on } G\}.
$$

The existence of this number follows from Lemma 2.5.

Let G be a graph and let $D \subseteq \mathbb{N}$ be the set of all integers l for which at least one l-track on G exists. We define

$$
M^\times(G) = \max\{M_l^\times : l \in D\}.
$$

This number is the maximal safety distance that can be kept while two players visit all the vertices of a graph with respect to the active movement rules.

Theorem 3.3. *Let G be a graph. It holds $M^\times(G) = \sigma_V^\times(G)$.*

Proof. Proof is analogous to the proof of the Theorem 3.2 so we omit it. $\quad\square$

Cartesian vertex span

In order to formally describe lazy movement rules, we introduce the notion of opposite lazy l-tracks.

Definition 3.4. *Let G be a graph, $f, g : \mathbb{N}_l \longrightarrow V(G)$ lazy l-tracks on G. We say that f and g are **opposite lazy l-tracks** on G if*
$$
f(i)f(i+1) \in E(G) \iff g(i) = g(i+1),
$$
for all $i \in \mathbb{N}_{l-1}$.

Let G be a graph and $l \in \mathbb{N}$ such that at least one pair of opposite lazy l-tracks on G exists. We define

$$
M_l^\square = \max\{m_G(f, g) : f \text{ and } g \text{ opposite lazy } l\text{-tracks on } G\}.
$$

The existence of this number follows from Lemma 2.5.

Let G be a graph and let $C \subseteq \mathbb{N}$ be the set of all integers l for which opposite lazy l-tracks on G exist. We define

$$
M^\square(G) = \max\{M_l^\square : l \in C\}.
$$

This number is the maximal safety distance that can be kept while two players visit all the vertices of a graph with respect to the lazy movement rules.

Theorem 3.5. *Let G be a graph. It holds $M^\square(G) = \sigma_V^\square(G)$.*

Proof. Proof is analogous to the proof of the Theorem 3.2 so we omit it. $\square$

We have shown that the definitions of maximal safety distance presented here indeed define vertex spans presented in [1] so from now on we only use the span notation, $\sigma_V^{\boxtimes}(G)$, $\sigma_V^{\times}(G)$ and $\sigma_V^{\square}(G)$.

Remark 3.6. *As we said in the Introduction, some claims will be proven by describing Alice and Bob's movement in the graph, instead of formally defining corresponding l-tracks. In those cases we will use expressions such as Alice and Bob's lazy walks, walks, or opposite lazy walks in a graph, which correspond to traditional, active and lazy movement rules, respectively. This approach is used to make proofs more fluid and more understandable to the reader.*

4. Relation between different vertex spans

Here we take a look at the relations between strong, direct and Cartesian vertex span.

Proposition 4.1. *Let G be a graph. It holds $\sigma_V^{\boxtimes}(G) \geq \max\{\sigma_V^{\times}(G), \sigma_V^{\square}(G)\}$.*

Proof. By the definition of $M^{\times}(G)$ and Theorem 3.3 there exist l-tracks, f and g on G, such that $m_G(f,g) = \sigma_V^{\times}(G)$. But f and g are then also lazy l-tracks and by the definition of $M^{\boxtimes}(G)$ and Theorem 3.2 it follows $\sigma_V^{\boxtimes}(G) \geq \sigma_V^{\times}(G)$. The proof for $\sigma_V^{\boxtimes}(G) \geq \sigma_V^{\square}(G)$ is analogous. $\square$

Theorem 4.2. *Let G be a graph. Then*
$$|\sigma_V^{\times}(G) - \sigma_V^{\square}(G)| \leq 1.$$

Proof. If G is trivial then $\sigma_V^{\times}(G) = \sigma_V^{\square}(G) = 0$ so the claim holds. For any non-trivial graph G, we will prove the following:

(1) $\sigma_V^{\square}(G) \geq \sigma_V^{\times}(G) - 1$;

(2) $\sigma_V^{\times}(G) \geq \sigma_V^{\square}(G) - 1$.

Proof of (1):
Let $\sigma_V^{\times}(G) = r$. Then there exist l-tracks f and g such that for each $i \in \mathbb{N}_l$, $d(f(i), g(i)) \geq r$. Define functions $f', g' : \mathbb{N}_{2l-1} \to V(G)$ with:
$$f'(i) = f\left(\left\lceil \frac{i+1}{2} \right\rceil\right);$$
$$g'(i) = g\left(\left\lceil \frac{i}{2} \right\rceil\right).$$

We claim that f' and g' are opposite lazy tracks and that for each $i \in \mathbb{N}_{2l-1}$, $d(f'(i), g'(i)) \geq r - 1$. First, let us prove that f' and g' are lazy tracks.
For each odd i, i.e., $i = 2k - 1$, $k \in \mathbb{N}$, $f'(i)f'(i+1)$ is in $E(G)$ because
$$f'(i)f'(i+1) = f\left(\left\lceil \frac{2k}{2} \right\rceil\right) f\left(\left\lceil \frac{2k+1}{2} \right\rceil\right) = f(k)f(k+1) \in E(G);$$

and it holds
$$g'(i) = g\left(\left\lceil \frac{2k-1}{2} \right\rceil\right) = g(k) = g\left(\left\lceil \frac{2k}{2} \right\rceil\right) = g'(i+1).$$

Similarly, for each even i, i.e., $i = 2k$, $k \in \mathbb{N}$, it holds
$$f'(i) = f\left(\left\lceil \frac{2k+1}{2} \right\rceil\right) = f(k+1) = f\left(\left\lceil \frac{2k+2}{2} \right\rceil\right) = f'(i+1);$$

and $g'(i)g'(i+1)$ is in $E(G)$:

$$g'(i)g'(i+1) = g\left(\left\lceil\frac{2k}{2}\right\rceil\right)g\left(\left\lceil\frac{2k+1}{2}\right\rceil\right) = g(k)g(k+1) \in E(G).$$

By Definition 2.2 f' and g' are lazy $(2l-1)$-tracks. Moreover, we can see that for each $i \in \mathbb{N}_{2l-1}$ we have $f'(i)f'(i+1) \in E(G) \Rightarrow g'(i) = g'(i+1)$ and $g'(i)g'(i+1) \in E(G) \Rightarrow f'(i) = f'(i+1)$, so it follows that f' and g' are indeed opposite lazy $(2l-1)$-tracks. Now, let us prove that for each $i \in \mathbb{N}_{2l-1}$, $d(f'(i), g'(i)) \geq r - 1$. Again, for each odd i, i.e., $i = 2k - 1$, $k \in \mathbb{N}$, it holds:

$$d(f'(i), g'(i)) = d\left(f\left(\left\lceil\frac{2k}{2}\right\rceil\right), g\left(\left\lceil\frac{2k-1}{2}\right\rceil\right)\right) = d(f(k), g(k)) \geq r.$$

Similarly, for each even i, i.e., $i = 2k$, $k \in \mathbb{N}$, it holds:

$$d(f'(i), g'(i)) = d\left(f\left(\left\lceil\frac{2k+1}{2}\right\rceil\right), g\left(\left\lceil\frac{2k}{2}\right\rceil\right)\right) = d(f(k+1), g(k)).$$

Since $f(k)f(k+1) \in E(G)$, for each $k \in \mathbb{N}_{l-1}$, then

$$d(f(k+1), g(k)) \geq d(f(k), g(k)) - 1 \geq r - 1.$$

Proof of (2):
Let $\sigma_V^{\square}(G) = r$. Then there exist opposite lazy l-tracks f and g such that for each $i \in N_l$, $d(f(i), g(i)) \geq r$. Because f and g are opposite, we may define the function $X : \mathbb{N}_{l-1} \rightarrow \{1, 2\}$ in the following manner:

$$X(i) := \begin{cases} 1 & \text{if } f(i)f(i+1) \in E(G); \\ 2 & \text{if } g(i)g(i+1) \in E(G). \end{cases}$$

Now, we observe the pairs $(X(2k-1), X(2k))$, for $k \in \mathbb{N}_{\lfloor\frac{l-1}{2}\rfloor}$. If $l-1$ is odd, we leave the element $X(l-1)$ unpaired. All the pairs must be elements of the set $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Let a be the number of $(1, 2)$ and $(2, 1)$ pairs combined. We now define functions $f', g' : \mathbb{N}_{l-a} \rightarrow V(G)$, which will be $(l-a)$-tracks, using the following algorithm:

We first define $f'(1) = f(1)$ and $g'(1) = g(1)$, and set the value of b to 0 and the value of i to 1. Now, we go through pairs $(X(2k-1), X(2k))$, for $k \in \mathbb{N}_{\lfloor\frac{l-1}{2}\rfloor}$, one by one, and do the following:

If the pair is $(1, 1)$ we increase the value of i by two and define

$$f'(i-1) = f(i-1+b); f'(i) = f(i+b),$$

and also

$$g'(i-1) = x; g'(i) = g(i+b),$$

where x is any neighbour of $g(i+b)$. Note that in this case $f'(i-1)f'(i) \in E(G)$ and $g'(i-1)g'(i) \in E(G)$.

Similarly, if the pair is $(2, 2)$ we increase the value of i by two and define

$$f'(i-1) = x; f'(i) = f(i+b),$$

and also

$$g'(i-1) = g(i-1+b); g'(i) = g(i+b),$$

where x is any neighbour of $f(i+b)$. Again, note that $f'(i-1)f'(i) \in E(G)$ and $g'(i-1)g'(i) \in E(G)$.

If the pair is $(1, 2)$ or $(2, 1)$ we increase the value of b and i by one and define

$$f'(i) = f(i+b); g'(i) = g(i+b).$$

Lastly, if we have any unpaired $X(l-1)$, we increase the value of i by one and define

$$f'(i) = f(i+b); g'(i) = x,$$

if $X(l-1) = 1$, or

$$f'(i) = y; g'(i) = g(i+b),$$

if $X(l-1) = 2$, where x and y are any neighbours of $g(i+b)$ and $f(i+b)$ respectively.

Observe that at the end of the algorithm we have $b = a$, where $0 \leq a \leq \left\lfloor \frac{l-1}{2} \right\rfloor$ and in the last step $i = l - a$, so the domain of f' and g' is indeed $\mathbb{N}_{l-a}$. Let us show that f' is an $(l-a)$-track, i.e. that $f'(i)f'(i+1) \in E(G)$, for each $i \in \mathbb{N}_{l-a-1}$. Let $j \in \mathbb{N}_{l-a-1}$ and let b_0 be the value of the parameter b in the step in which $f'(j)$ is defined. If $f'(j+1)$ is defined in the same pair as $f'(j)$ (when the pairs are $(1,1)$ or $(2,2)$), then $f'(j)f'(j+1) \in E(G)$, as previously noted. Let us analyze the cases when $f'(j)$ is defined in one pair and $f'(j+1)$ is defined in the next one, or with the last unpaired value, $X(l-1)$. Let $f'(j)$ be defined in any pair $(x,y) \in \{(1,1),(1,2),(2,1),(2,2)\}$. In all these cases, $f'(j)$ is defined as $f'(j) = f(j+b_0)$, where $b_0 \in \{0,...,a\}$.

1. If $f'(j+1)$ is defined in pair $(1,1)$, then $f'(j+1) = f(j+1+b_0)$, and since in this case it holds $f(j+b_0)f(j+1+b_0) \in E(G)$, we have $f'(j)f'(j+1) \in E(G)$.
2. If $f'(j+1)$ is defined in pair $(1,2)$ then $f'(j+1) = f(j+1+b_0+1) = f(j+1+b_0)$, so again $f'(j)$ and $f'(j+1)$ are neighbors.
3. If $f'(j+1)$ is defined in pair $(2,1)$, then $f'(j+1) = f(j+1+b_0+1)$. In this case $f(j+1+b_0) = f(j+b_0)$ and $f(j+1+b_0)f(j+1+b_0+1) \in E(G)$, so again it follows $f'(j)f'(j+1) \in E(G)$.
4. If $f'(j+1)$ is defined in pair $(2,2)$, then $f'(j+1)$ is defined as the neighbor of $f'(j+2) = f(j+1+b_0+1)$. In this case $f(j+1+b_0+1) = f(j+1+b_0) = f(j+b_0) = f'(j)$ so the claim follows.

The proof that g' is also an $(l-a)$-track is analogous.

From the construction it follows that for each $i \in \mathbb{N}_{l-a}$, $d(f'(i), g'(i)) \geq r-1$.

Now from (1) and (2) we have

$$\sigma_V^\times(G) - 1 \leq \sigma_V^\square(G) \leq \sigma_V^\times(G) + 1,$$

which ends our proof. $\square$

Remark 4.3. *To fully clarify the second, more algorithmic, part of the proof of Theorem 4.2, we will additionally explain it in terms of Alice and Bob. When Alice and Bob are walking in opposite lazy walks, while one of them is walking, the other is standing still, and vice versa. That means, if we have their lazy walks that keep the safety distance r, we can first derive the order in which the two of them walked through all the vertices. When constructing their walks in active movement rules, we are starting from the same position they started in lazy movement rules, and then we go through the order in which they moved. Two different situations may happen. Either one of them moved, and then the other one, or one of them moved twice in a row, while the other stood still for two steps. In case one of them moved and then the other one, those two movements will just happen simultaneously in active movement rules, and Alice and Bob will end up in a position they were in with lazy movement rules, still at safety distance r. In case of one of them moving twice in a row, while the other stands still, we can transfer those movements to active movement rules just by moving the one who stood still to any neighbor vertex and back, thus possibly lowering the safety distance by 1, but still keeping it at least $r-1$.*

Notice that the bound in Theorem 4.2 is tight. For example, observe the two graphs in Figure 2.
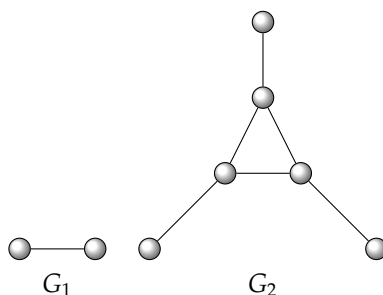

G_1 G_2

Figure 2: Graphs for which the bounds of Theorem 4.2 are obtained

It can be easily seen that $\sigma_V^\times(G_1) = 1$ and $\sigma_V^\square(G_1) = 0$. Similarly, we can easily see that $\sigma_V^\times(G_2) = 1$ and $\sigma_V^\square(G_2) = 2$.

5. Spans of some graph classes

First we take a look at the way a cut-edge in a graph might help us reduce the problem of calculating the span values of a graph to its subgraphs determined by the cut-edge. This result might lead to an algorithm for calculating upper bonds for spans, for some graphs better then the radius. We used the reasoning from proof of Proposition 5.1 to calculate some results for this paper, for instance in Proposition 5.5.

Proposition 5.1. *Let G be a graph such that $|V(G)| \geq 3$. Let xy be a cut-edge in G and G_1 and G_2 subgraphs of G induced by components of $G - xy$ such that $x \in V(G_1)$ and $y \in V(G_2)$. Then*

$$\sigma_V^\boxtimes(G) \leq \max\{ecc_{G_1}(x), ecc_{G_2}(y)\}.$$

Proof. We will provide this proof in terms of Alice and Bob. Depending on where they start their walks, we have three different situations.
Situation 1: Both Alice and Bob start their walks in some vertex of G_1. In that case, at some point, both of them will have to visit vertices of G_2, and the only way to get there is through the cut-edge xy. Without any loss of generality, we can assume that Alice leaves G_1 first. When she gets to vertex x, Bob is still in G_1, so at that point their distance is for sure less than or equal to $ecc_{G_1}(x)$. So our claim holds.
Situation 2: Both Alice and Bob start their walks in some vertex of G_2. Analogously as in situation 1, we see that, at some point, when leaving G_2, the distance between Alice and Bob will have to be less than or equal to $ecc_{G_2}(y)$.
Situation 3: Alice and Bob start their walks in vertices of different subgraphs G_1 and G_2. Let us say, without any loss of generality, that Alice starts in G_1, and Bob in G_2. Either one of them will walk through cut-edge xy to the other subgraph, at which point we will end up in the situation 1 or 2, or they will both switch to the other subgraph at the same time, in which case, they will have to be in vertices x and y at the same step, and at that point their distance will be 1, which is less than or equal to $\max\{ecc_{G_1}(x), ecc_{G_2}(y)\}$, since $|V(G)| \geq 3$ implies that at least one of the graphs G_1 and G_2 is non-trivial. □

Remark 5.2. *Since we already had that $\sigma_V^\boxtimes(G) \leq rad(G)$, one might wonder if this is an improvement of that result, and how does $\max\{ecc_{G_1}(x), ecc_{G_2}(y)\}$ compare to $rad(G)$. The two measures are incomparable as there are examples of graphs where either one is greater than the other, as is shown in Figure 3. So this is an additional result that one might use when calculating upper bonds for spans of certain types of graphs.*

$rad(G) = 2$ $\qquad\qquad\qquad$ $rad(G) = 3$
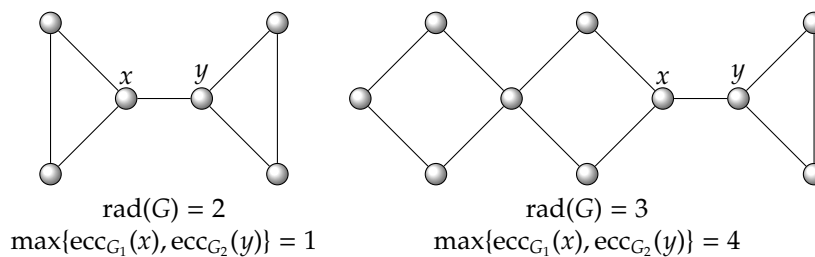$\max\{ecc_{G_1}(x), ecc_{G_2}(y)\} = 1$ $\qquad$ $\max\{ecc_{G_1}(x), ecc_{G_2}(y)\} = 4$

Figure 3: Graphs that illustrate the difference between the radius of a graph and the maximum of the eccentricities of the ends of a cut-edge

In further observations of spans of different graph families, we mostly encountered graphs for which the direct vertex span was greater than or equal to their Cartesian vertex span. So it prompted us to find a family of graphs for which a greater safety distance can be obtained following lazy movement rules rather than active ones. One such family is known as sunlet graphs.

Definition 5.3. *Let C_n be a cycle with n vertices $\{v_1, ..., v_n\}$, so that $v_i v_{i+1} \in E(C_n)$, $i \in \{1, ..., n-1\}$ and $v_n v_1 \in E(C_n)$. A graph obtained by adding n leaves $\{u_1, ..., u_n\}$ to C_n in a way that u_i and v_i are adjacent for each $i \in \mathbb{N}_n$ is called an n-**sunlet graph**. We will denote it by SC_n.*

It is easily seen that $|V(SC_n)| = 2n$ and $\mathrm{rad}(SC_n) = \mathrm{rad}(C_n) + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Theorem 5.4. *It holds*

$$\sigma_V^\times(SC_n) = \left\lfloor \frac{n}{2} \right\rfloor$$

and

$$\sigma_V^\square(SC_n) = \sigma_V^\boxtimes(SC_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Let SC_n be an n-sunlet graph with vertex labels as in Definition 5.3. We will refer to the vertices $\{u_1, ..., u_n\}$ as the pendant vertices. As we can see from the claim, the value of different spans will depend on the parity of number n, so our proof will have two cases, that is, we will prove:

For $k \geq 1$ it holds $\sigma_V^\square(SC_{2k+1}) = \sigma_V^\boxtimes(SC_{2k+1}) = k + 1$ and $\sigma_V^\times(SC_{2k+1}) = k$;

For $k \geq 2$ it holds $\sigma_V^\times(SC_{2k}) = \sigma_V^\square(SC_{2k}) = \sigma_V^\boxtimes(SC_{2k}) = k$.

We now prove the claim for SC_{2k+1}. We have $\sigma_V^\square(SC_{2k+1}) \leq \mathrm{rad}(SC_{2k+1}) = \lfloor \frac{2k+1}{2} \rfloor + 1 = k + 1$. Now, let us show that $\sigma_V^\square(SC_{2k+1}) = k + 1$. We will prove this by describing Alice and Bob's movements in SC_{2k+1} that follow lazy movement rules, which keep the distance at least $k + 1$. First, observe that for each pendant vertex u_i, there are two cycle vertices that are on the distance $k + 1$ from it. For $i \neq k + 1$ those will be the vertices v_m and v_{m+1}, where $m = (i + k) \mod (2k + 1)$. For $i = k + 1$ those will be the vertices v_{2k+1} and v_1. We now describe the first few Alice and Bob's steps with Table 2.

	1	2	3	4	5	6	7
Alice's position	u_1	u_1	u_1	u_1	v_1	v_2	u_2
Bob's position	u_{k+1}	v_{k+1}	v_{k+2}	u_{k+2}	u_{k+2}	u_{k+2}	u_{k+2}

Table 2: The first few Alice and Bob's steps in SC_{2k+1} keeping the distance at least $k + 1$ with lazy movement rules

In Figure 4 we can see the illustration of these steps for SC_5, for which $k = 2$. It is easy to see that, by continuing this way, Alice and Bob will visit all the vertices, while always keeping the distance greater or equal to $k + 1$.
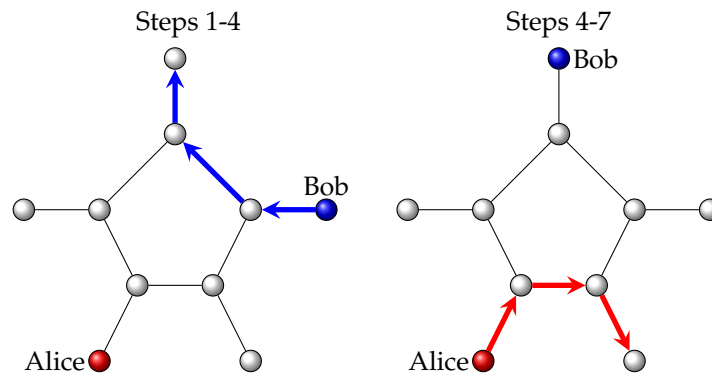

Figure 4: The first 7 steps of Alice and Bob in SC_5, following lazy movement rules and keeping the distance at least 3

We have shown that $\sigma_V^{\square}(SC_{2k+1}) = k + 1$ and from Proposition 4.1 it follows $\sigma_V^{\boxtimes}(SC_{2k+1}) = k + 1$.
We now prove that $\sigma_V^{\times}(SC_{2k+1}) = k$. First, we will show that $\sigma_V^{\times}(SC_{2k+1}) < k + 1$. We know that $\mathrm{rad}(C_{2k+1}) = k$,
so if at any point Alice and Bob are both in the cycle vertices, their distance is less then $k + 1$. Let us assume
they can keep the distance at least $k + 1$ at all times. If Alice and Bob both start in the pendant vertices, in the
next step, they are both in cycle vertices, so the only possible start is for one of them to be in a cycle vertex,
and for the other to be in a pendant vertex. Moreover, since we are trying to keep the maximal distance, they
have to start at the diametrically opposite sides of the cycle. Formally, and without the loss of generality,
let Alice start in the pendant vertex u_1 and let Bob start in the cycle vertex v_{k+1}, since $d(u_1, v_{k+1}) = k + 1$. In
the next step, Alice has only one option, to move to v_1, and Bob has 3 options. If he moves to v_k or v_{k+2}, they
are both in the cycle vertices, and the third option takes him to u_{k+1}, which leaves them in an analogous
situation as in the starting positions. So $\sigma_V^{\times}(SC_{2k+1}) < k + 1$. Now the fact that $\sigma_V^{\times}(SC_{2k+1}) = k$ follows from
$\sigma_V^{\square}(SC_{2k+1}) = k + 1$, $\sigma_V^{\times}(SC_{2k+1}) < k + 1$ and Theorem 4.2.
We now prove the claim for SC_{2k}. Its radius is $k + 1$, so that is the maximal possible span value for each of
the spans. Since $\mathrm{rad}(C_{2k}) = k$ we immediately see that for all three movement rules, if at any point Alice
and Bob are both in the vertices of the cycle, their distance will be at most k.
For active movement rules, the proof that $\sigma_V^{\times}(SC_{2k}) < k + 1$ is the same as for SC_{2k+1}. To show that
$\sigma_V^{\times}(SC_{2k}) = k$ let us describe Alice and Bob's movements in SC_{2k}, following active movement rules and
keeping the distance k. We will describe the first few steps with Table 3.

	1	2	3	4	5	6
Alice's position	u_1	v_1	v_2	u_2	v_2	v_3
Bob's position	u_{k+1}	v_{k+1}	v_{k+2}	u_{k+2}	v_{k+2}	v_{k+3}

Table 3: The first few Alice and Bob's steps in SC_{2k} keeping the distance k with active movement rules

By continuing in this way, Alice and Bob will visit all the vertices, keeping their distance, so we have proven
$\sigma_V^{\times}(SC_{2k}) = k$. The illustration of the first 3 steps in SC_6 is given in Figure 5.
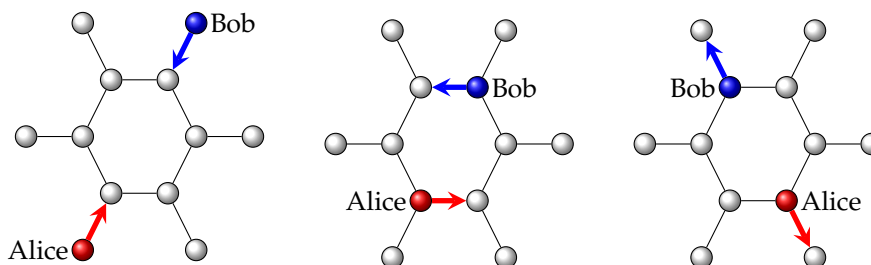

Figure 5: The first 3 steps of Alice and Bob in SC_6, following active movement rules and keeping the distance
at least 3

We now observe traditional movement rules. The situation is a little bit different, since it doesn't
necessarily have to occur that both of them are in the cycle vertices at the same time. We now show that
$\sigma_V^{\boxtimes}(SC_{2k}) < k + 1$. Let us assume the opposite, that there are lazy walks that Alice and Bob can take in order
to keep the distance $k + 1$. Without the loss of generality, let Alice start in the pendant vertex u_1. That leaves
Bob starting in either the cycle vertex v_{k+1} or the pendant vertex u_{k+1}. It can be easily seen that neither
of them can ever move to any cycle vertex other than v_1 and v_{k+1} without getting closer than $k + 1$ to one
another. It immediately follows that $\sigma_V^{\boxtimes}(SC_{2k}) < k + 1$. To show that $\sigma_V^{\square}(SC_{2k}) = k$, we can easily construct
Alice and Bob's walks, similar as with the active movement rules, just following lazy movement rules, so
each of the players doesn't move in every other step. So $\sigma_V^{\square}(SC_{2k}) = k$, which also gives us $\sigma_V^{\boxtimes}(SC_{2k}) = k$. $\quad\square$

Also, worth noticing is that graph SC_3 is the graph of the smallest order for which its Cartesian vertex
span is greater than its direct vertex span. We now prove that claim.

Proposition 5.5. *Graph SC_3 is the graph of the smallest order such that its Cartesian vertex span is greater than its direct vertex span.*

Proof. Let G be a graph such that its Cartesian vertex span is greater than its direct vertex span. Since any graph, for which its direct vertex span equals 0, has only one vertex, [1], it follows that the Cartesian vertex span of that graph is also 0. So $\sigma_V^\times(G)$ must be at least 1 and $\sigma_V^\square(G)$ at least 2. All connected graphs up to 3 vertices have radii at most 1. Only connected graphs with 4 vertices of radius 2 are P_4 ($\sigma_V^\times(P_4) = 1$, $\sigma_V^\square(P_4) = 0$) and C_4 ($\sigma_V^\times(C_4) = 2$, $\sigma_V^\square(C_4) = 1$).

Lastly, we have 10, up to isomorphism, connected graphs with 5 vertices of radius 2: P_5 ($\sigma_V^\times(P_5) = 1$, $\sigma_V^\square(P_5) = 0$), C_5 ($\sigma_V^\times(C_5) = 2$, $\sigma_V^\square(C_5) = 2$), and the remaining 8 are presented in Figure 6, alongside their respective direct and Cartesian vertex spans:
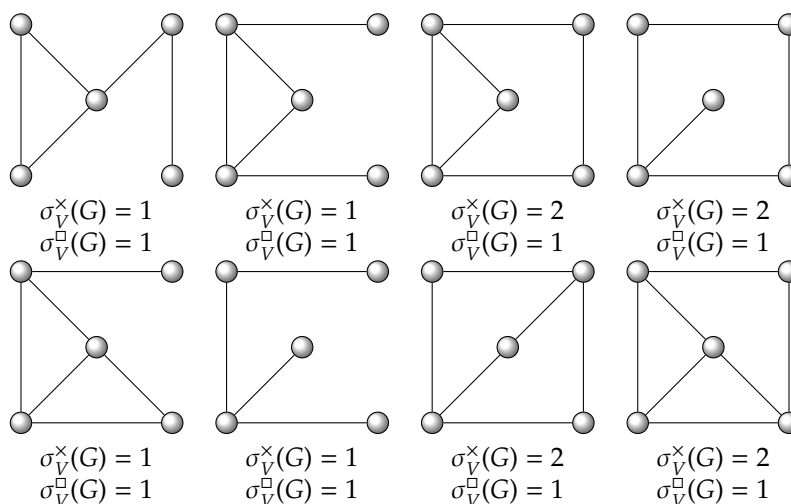

$$\sigma_V^\times(G) = 1 \qquad \sigma_V^\times(G) = 1 \qquad \sigma_V^\times(G) = 2 \qquad \sigma_V^\times(G) = 2$$
$$\sigma_V^\square(G) = 1 \qquad \sigma_V^\square(G) = 1 \qquad \sigma_V^\square(G) = 1 \qquad \sigma_V^\square(G) = 1$$

$$\sigma_V^\times(G) = 1 \qquad \sigma_V^\times(G) = 1 \qquad \sigma_V^\times(G) = 2 \qquad \sigma_V^\times(G) = 2$$
$$\sigma_V^\square(G) = 1 \qquad \sigma_V^\square(G) = 1 \qquad \sigma_V^\square(G) = 1 \qquad \sigma_V^\square(G) = 1$$

Figure 6: Connected graphs of order 5 with radius 2, other than P_5 and C_5

So, from these observations it is obvious that such graph needs to have at least 6 vertices. And that proves our claim. □

Another class of graphs that we observed are binary trees. They are a useful class of graphs in observing hierarchical structures and are sometimes embedded in networks to help solve some network problems. They are also important in data science, because of the efficient way of searching through the data organized in this way. As searching through graph-like structures is somewhat similar to walking around that graph, we thought it might be useful to observe the spans of binary trees, for the possibility of 2 simultaneous searches may arise. The structure for which we solved the span values is a perfect binary tree. **A perfect binary tree** is a rooted binary tree in which all interior vertices have exactly 2 children and all the leaves are on the same distance from the root. The height of a binary tree is the number of non-root levels within the tree, and it is usually denoted by h. The height of the binary tree is also its radius. We will denote the perfect binary tree of height h by BT_h.

Theorem 5.6. *For $h > 1$ it holds*

$$\sigma_V^\boxtimes(BT_h) = \sigma_V^\times(BT_h) = \sigma_V^\square(BT_h) = h - 1.$$

Proof. First, we will prove that $\sigma_V^\square(BT_h) < h$, which will prove that all vertex spans are less than h, and then we will describe Alice and Bob's walks through such graphs in a way that leaves them on a safety distance $h - 1$ using different movement rules.

We now denote the root of BT_h by x and we refer to the two subgraphs induced by components of $BT_h - x$ as the left branch and the right branch. Alice and Bob can start their walks in the same branch, in different

branches, or one of them can start their walk in the vertex x. If they start in the same branch, then at some point one of them will have to move to the other branch, and they will have to go there through the root of the branch they are in. At that point the distance between the two players will be at most $h-1$. If they start in different branches, one of them will have to move to the other branch, while the other one is still in that branch or in x, as no movement rules allow them to simultaneously change branches, because they have to pass through x first. As they have to enter the branch through its root at that point the distance between the two players will be at most $h-1$. Which brings us to the third option, one of them being in one branch and the other in x. While one is in x, they can only be in distance h if the other is in some leaf of the tree. But then, the only movement that will leave them in the safety distance of at least h will be to move from x to the different branch than the other player. Which brings us back to the second situation. Therefore, it is impossible for Alice and Bob to walk through all the vertices of a perfect binary tree of height h keeping the safety distance at least h at all times.

Now, let us describe Alice and Bob's walks that keep them at the safety distance $h-1$.

Opposite lazy walks described: Alice starts her walk in any leaf of the left branch, and Bob in any leaf of the right branch. Alice now walks through all of the left branch while Bob stands still. Since he is in a leaf of the other branch, Alice has never gotten closer than $h+1$ to him. Now she can go back to any of the leaves of the left branch so Bob can visit all vertices of the right branch. Now it is time to switch branches. While Alice is in a leaf of the left branch, Bob can go through the root x and go down the left branch, always choosing a vertex that is more distant from Alice, until he reaches a leaf. That way he will be closest to her at the root of the left branch, where his distance from Alice is exactly $h-1$. Now Alice can go up the left branch and switch to the right branch and go down to any leaf. Exactly as in the beginning, now both of them can in turns visit all the vertices of their respective branches, never getting closer to one another than $h-1$. So $\sigma_V^\square(BT_h) = h-1$. Also, since $h-1 = \sigma_V^\square(BT_h) \leq \sigma_V^\boxtimes(BT_h) < h$, then $\sigma_V^\boxtimes(BT_h) = h-1$.

Active walks described: Alice and Bob can visit all the vertices as they did in the opposite lazy walks, with few adjustments, as there is no more option for someone to stand still while the other one is walking. So when Alice is visiting all the vertices of the left branch, Bob can just switch from his leaf to the neighbor vertex and back. And the same goes for Alice when Bob is visiting his right branch vertices. And the other thing we need to keep in mind is that, when switching sides, when Bob is in the root of the left branch, Alice needs to be in a leaf, or they would come closer to each other than $h-1$, and for that to be possible, we just have to make sure to place Alice in the beginning of her walk in a leaf of the left branch if h is odd and in a leaf neighbor vertex if h is even. The illustration for $h=3$ is given in Figure 7.
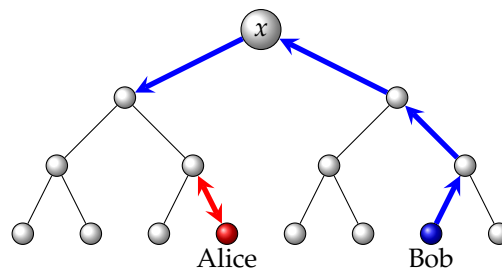

Figure 7: Bob needs 4 steps to get from the leaf of the right branch to the root of the left branch. At the same time Alice goes from the leaf to the neighbour and back, finishing on the leaf, keeping the distance 2.

That way such switch of the positions is possible so the rest can take place as in lazy movement rules. So $\sigma_V^\times(BT_h) = h-1$. $\square$

Based on the definition of an n-friendly graph and Observation 6.3. from [1] we observed different vertex span values for some classes of graphs. Theorem 4.2 allowed us to expand these results. First note that if there is a vertex v in a graph G of order n, such that $d(v) = n-1$, then $\text{rad}(G) = 1$ and $\sigma_V^\boxtimes(G) \leq 1$. Graphs with radius equal to 1 are observed in [1], in Theorems 4.2, 4.5 and 4.8. From those results we conclude that for complete graphs K_n, $n \geq 3$, wheel graphs W_n, $n \geq 4$, and star graphs S_n, $n \geq 4$, the value

of all spans will be equal to 1. We present an overview of the other classes in Table 4. P_n is a path with n vertices, Q_n is an n-dimensional cube with 2^n vertices and $K_{r,s}$ is a complete bipartite graph $G = (X, Y)$, where $|X| = r$ and $|Y| = s$.

Table 4: Span values for some classes of graphs

Graph class	Radius	$\sigma_V^{\boxtimes}$	$\sigma_V^{\times}$	$\sigma_V^{\square}$	Comment
$P_n, n \geq 2$	$\lfloor \frac{n}{2} \rfloor$	1	1	0	[1]
$C_n, n \geq 3$	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	$\begin{cases} \lfloor \frac{n}{2} \rfloor, & n \text{ odd} \\ \frac{n}{2} - 1, & n \text{ even} \end{cases}$	[1], Thm 4.2
$Q_n, n \geq 2$	n	n	n	$n - 1$	[1], Thm 4.2
$K_{r,s}, r,s \geq 2$	2	2	2	1	[1], Thm 4.2
SC_n	$\lfloor \frac{n}{2} \rfloor + 1$	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	Thm 5.4
BT_h	h	$h - 1$	$h - 1$	$h - 1$	Thm 5.6

6. Conclusion

Inspired by the work in [1], we observed the same problem in a somewhat different way. With that, we obtained a result for the relation between direct and Cartesian vertex spans, and using this result we calculated spans for some graph classes. We saw that in the majority of the observed cases, the direct vertex span is greater or equal to the Cartesian vertex span; but we also found a family for which the opposite holds. It would be interesting to find the characterization for the graphs for which Cartesian vertex span is greater than their direct vertex span and it would be interesting to discuss what is necessary in the graph structure for greater safety distance to be achieved by lazy movement rules instead of active movement rules, and vice versa. Further work may also include generalizing the notion of spans to more than two players or analyzing the minimal number of steps in which the maximal safety distance can be achieved.

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