Filomat 37:15 (2023), 5053–5063
https://doi.org/10.2298/FIL2315053C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterizations of weaker forms of the Rothberger and Menger properties in hyperspaces

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Abstract. In this paper, we introduce the notions of almost $\pi_\Delta(\Lambda)$-network and weakly $\pi_\Delta(\Lambda)$-network to characterize the properties of almost Rothberger (Menger) and weakly Rothberger (Menger), respectively, in the hyperspaces $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$, endowed with the hit-and-miss topology. Also, we introduce the concepts of groupable $c_\Delta(\Lambda)$-cover and weakly (Δ, Λ)-groupable cover of X to give equivalences of the selection principles $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$, $\mathbf{S}_{\mathrm{fin}}(\mathscr{D}, \mathscr{D}^{gp})$, $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$ and $\mathbf{S}_{\mathrm{fin}}(\mathscr{D}, \mathscr{D}^{wgp})$ in the same hyperspaces.

1. Introduction and preliminaries

The hyperspace theory started in the first half of the 20th century with the works [11, 19, 21, 28]. Given a topological space X, we denote by $\mathrm{CL}(X)$ the family of all nonempty closed subsets of X. The set $\mathrm{CL}(X)$, endowed with some topology, is known as hyperspace of X. Numerous relations between properties of the space X and their hyperspaces have been widely studied. On the other hand, the study of selection principles started in [2, 12, 18, 22, 23]. Some lines of research generated are the study of selection principles concerning groupability properties [7, 8, 16] and weaker versions of Rothberger and Menger properties [13, 20, 26].

The relationships between selection principles and hyperspaces have been developed by several authors. Namely, in [8] the authors used π-networks to characterize topological spaces whose hyperspace, endowed with the upper Fell topology, satisfies the Rothberger property. Then, in [17] are defined the concepts of π_F-network, π_V-network, k_F-cover and c_V-cover and they are used to study the $\mathbf{S}_1(\mathscr{A}, \mathscr{B})$ and $\mathbf{S}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$ principles in $\mathrm{CL}(X)$ endowed with the Fell and Vietoris topologies, for different families $\mathscr{A}$ and $\mathscr{B}$. Later, in [3] the authors introduce the generic notions of $\pi_\Delta(\Lambda)$-networks (and $c_\Delta(\Lambda)$-covers), which are a generalization of π_F-networks and π_V-networks (and of k_F-cover and c_V-cover, respectively). These concepts are used to characterize Menger-type star selection principles [3], star and strong star-type versions

2020 *Mathematics Subject Classification*. Primary 54B20; Secondary 54A05, 54A25, 54D20.
Keywords. Almost Rothberger (Menger); Groupable cover; Hit-and-miss topologies; Hyperspaces; Selection principles; Weakly groupable cover; Weakly Rothberger (Menger).
Received: 25 August 2022; Revised: 01 October 2022; Accepted: 07 October 2022
Communicated by Ljubiša D. R. Kočinac
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of Rothberger and Menger principles [4] and Hurewicz like properties [5] in hyperspaces endowed with the hit-and-miss topology.

Next, we recall two known notions both defined in 1996 by M. Scheepers [23]. Given an infinite set X, let $\mathscr{A}$ and $\mathscr{B}$ be collections of families of subsets of X.

- $\mathbf{S}_1(\mathscr{A}, \mathscr{B})$ denotes the principle: For any sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $B_n \in \mathcal{A}_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

- $\mathbf{S}_{\mathsf{fin}}(\mathscr{A}, \mathscr{B})$ denotes the principle: for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n \in \mathscr{B}$.

Let $\mathscr{O}$ be the collection of open covers of a topological space X. When we take $\mathbf{S}_1(\mathscr{O}, \mathscr{O})$ and $\mathbf{S}_{\mathsf{fin}}(\mathscr{O}, \mathscr{O})$, we get the well known *Rothberger property* [22] and the *Menger property* [12, 18], respectively.

On the other hand, a topological space X is *almost Rothberger* [24] (resp., *weakly Rothberger* [6]) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(U_n : n \in \mathbb{N})$ such that for any $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\bigcup\{cl_X(U_n) : n \in \mathbb{N}\} = X$ (resp., $cl_X(\bigcup_{n\in\mathbb{N}} U_n) = X$).

In turn, a topological space X is *almost Menger* [24] (resp., *weakly Menger* [6]) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{cl_X(U) : U \in \mathcal{V}_n\}$ is a cover of X (resp., $cl_X\left(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n\right) = X$).

Diagram 1 provides relationships between the properties defined previously. These follow immediately from the definitions and are not reversible.

$$\begin{tikzcd}
\text{Menger} \arrow[r] & \text{almost Menger} \arrow[r] & \text{weakly Menger} \\
\text{Rothberger} \arrow[r] \arrow[u] & \text{almost Rothberger} \arrow[r] \arrow[u] & \text{weakly Rothberger} \arrow[u]
\end{tikzcd}$$

Diagram 1: Relationships between selection principles.

Many authors have made investigations on selection principles and interesting results have been obtained, see [1, 14, 25, 27], among other works.

For the purposes of this work, we present some basic concepts about the theory of hyperspaces. All spaces are assumed to be Hausdorff noncompact and, even, nonparacompact.

For a space (X, τ), we denote by $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$ the family of all nonempty closed subsets, the family of all nonempty compact subsets, the family of all nonempty finite subsets of X and the family of all convergent sequences of X, respectively.

For every subset $U \subseteq X$ and any family $\mathcal{U}$ of subsets of X, we write:

$$\begin{array}{rcl}
U^- & = & \{A \in \mathrm{CL}(X) : A \cap U \neq \emptyset\}; \\
U^+ & = & \{A \in \mathrm{CL}(X) : A \subseteq U\}; \\
U^c & = & X \setminus U; \\
\mathcal{U}^c & = & \{U^c : U \in \mathcal{U}\}.
\end{array}$$

Let $\Delta \subseteq \mathrm{CL}(X)$ a subfamily of $\mathrm{CL}(X)$ closed under finite unions and containing all singletons. Then, the *hit-and-miss topology on* $\mathrm{CL}(X)$ *respect to* Δ, denoted by τ^+_Δ, has as a base, the family

$$\left\{\left(\bigcap_{i=1}^{m} V_i^-\right) \cap (B^c)^+ : B \in \Delta \text{ and } V_i \in \tau \text{ for } i \in \{1, \dots, m\}\right\}.$$

Following [29], the basic element $\left(\bigcap_{i=1}^{m} V_i^-\right) \cap (B^c)^+$ will be denoted by $(V_1, \dots, V_m)^+_B$.

It is known that two important particular cases of the hit-and-miss topology are the *Vietoris topology*, τ_V, when $\Delta = \mathrm{CL}(X)$ (see [19, 28]), and the *Fell topology*, τ_F, when $\Delta = \mathbb{K}(X)$ (see [10]). Although in the literature there are several topologies that can be defined on $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$, throughout this work we will consider them as subspaces of the $(\mathrm{CL}(X), \tau^+_\Delta)$.

Along this paper, unless we say the opposite, we will consider a family $\Lambda \subseteq \mathrm{CL}(X)$ such that it is closed under finite unions. Recall that $[A]^{<\omega}$ denotes the collection of all finite subsets of any set A.

Now, we recall the definitions of $\pi_\Delta(\Lambda)$-network and $c_\Delta(\Lambda)$-cover of a space X, a remark and a couple of lemmas which will be used along this work (see [3]).

Given a family $\Delta \subseteq \mathrm{CL}(X)$, we denote

$$\zeta_\Delta = \{(B; V_1, \dots, V_n) : B \in \Delta \text{ and } V_1, \dots, V_n \text{ are open subsets of } X \text{ with } V_i \cap B^c \neq \emptyset \ (1 \leq i \leq n),\ \ n \in \mathbb{N}\}.$$

Definition 1.1. A family $\mathcal{J} \subseteq \zeta_\Delta$ is called a $\pi_\Delta(\Lambda)$-*network of* X, if for each $U \in \Lambda^c$, there exist $(B; V_1, \dots, V_n) \in \mathcal{J}$ with $B \subseteq U$ and $F \in [X]^{<\omega}$ such that $F \cap U = \emptyset$ and for each $i \in \{1, \dots, n\}$, $F \cap V_i \neq \emptyset$. The family of all $\pi_\Delta(\Lambda)$-networks is denoted by $\Pi_\Delta(\Lambda)$.

Remark 1.2. We have the following statements.

(1) If $\Delta = \mathbb{K}(X)$ and $\Lambda = \mathrm{CL}(X)$, then the notion of $\pi_\Delta(\Lambda)$-network of X coincides with the definition of π_F-network of X (see [17, Definition 3.7]).
(2) If $\Delta = \Lambda = \mathrm{CL}(X)$, then every $\pi_\Delta(\Lambda)$-network of X induces a π_V-network of X (see [17, Definition 3.11]), and vice versa.

Lemma 1.3. *Let (X, τ) be a topological space. Suppose that $\mathcal{J} = \{(B_s; V_{1,s}, \dots, V_{m_s,s}) : s \in S\}$ and $\mathscr{U} = \left\{(V_{1,s}, \dots, V_{m_s,s})^+_{B_s} : (B_s; V_{1,s}, \dots, V_{m_s,s}) \in \mathcal{J}\right\}$. Then, $\mathcal{J}$ is a $\pi_\Delta(\Lambda)$-network of X if and only if $\mathscr{U}$ is an open cover of (Λ, τ^+_Δ).*

Definition 1.4. Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$-*cover of* X, if for any $B \in \Delta$ and open subsets $V_1, \dots, V_m$ of X, with $B^c \cap V_i \neq \emptyset$ for any $i \in \{1, \dots, m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \emptyset$ and for each $i \in \{1, \dots, m\}$, $F \cap V_i \neq \emptyset$. We denote by $\mathbb{C}_\Delta(\Lambda)$ the family of all $c_\Delta(\Lambda)$-covers of X.

Lemma 1.5. *Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is a $c_\Delta(\Lambda)$-cover of X if and only if the family $\mathcal{U}^c$ is a dense subset of (Λ, τ^+_Δ).*

Continuing the work done in [3–5, 9], in this paper, we introduce the generic notions of almost and weakly $\pi_\Delta(\Lambda)$-networks, groupable $c_\Delta(\Lambda)$-cover, weakly dense (Δ, Λ)-groupable family and weakly (Δ, Λ)-groupable cover. Those notions are used to characterize the properties: selection principles of almost Rothberger (Theorem 2.3), almost Menger (Theorem 2.7), weakly Rothberger (Theorem 2.13), weakly Menger (Theorem 2.17), $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$ (Theorem 3.4), $\mathbf{S}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D}^{gp})$ (Theorem 3.8), $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$ (Theorem 3.15) and $\mathbf{S}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D}^{wgp})$ (Theorem 3.19) in the hyperspace (Λ, τ^+_Δ). As particular cases, we get some results obtained by Li in [17] (Corollaries 3.6, 3.7, 3.21, 3.22).

2. Almost and weakly Rothberger and Menger properties

In order to characterize the almost Rothberger and almost Menger properties in hyperspaces endowed with the hit-and-miss topology, we introduce the notion of almost $\pi_\Delta(\Lambda)$-network.

Definition 2.1. A family $\zeta \subseteq \zeta_\Delta$ is called an *almost $\pi_\Delta(\Lambda)$-network of X*, if for every $U \in \Lambda^c$, there is $(B; V_1, \dots, V_n) \in \zeta$ such that for any $K \in \Delta$ and $U_1, \dots, U_m$ open sets in X, with $U^c \cap K = \emptyset$ and $U^c \cap U_i \neq \emptyset$, for $1 \leq i \leq m$, there exist $W \in \Lambda^c$ and $F \in [X]^{<\omega}$ which satisfy $F \cap W = \emptyset$, $B \cup K \subseteq W$, $F \cap V_i \neq \emptyset$, for each $1 \leq i \leq n$ and $F \cap U_j \neq \emptyset$, for any $1 \leq j \leq m$. The family of all the almost $\pi_\Delta(\Lambda)$-networks is denoted by $a\Pi_\Delta(\Lambda)$.

Remark 2.2. It can be shown that every $\pi_\Delta(\Lambda)$-network is an almost $\pi_\Delta(\Lambda)$-network.

Theorem 2.3. *Given a topological space (X, τ), the following conditions are equivalent:*

(1) (Λ, τ_Δ^+) *is almost Rothberger;*
(2) (X, τ) *satisfies* $\mathbf{S}_1(\Pi_\Delta(\Lambda), a\Pi_\Delta(\Lambda))$.

Proof. (1) ⇒ (2) Let $(\mathcal{J}_n : n \in \mathbb{N})$ be a sequence in $\Pi_\Delta(\Lambda)$. Denote, for any $n \in \mathbb{N}$,

$$\mathcal{U}_n = \left\{(V_1, \ldots, V_m)_B^+ : (B; V_1, \ldots, V_m) \in \mathcal{J}_n\right\}.$$

By Lemma 1.3, we have that for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is an open cover of (Λ, τ_Δ^+). Applying (1) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, there exists $(V_1^n, \ldots, V_{m_n}^n)_{B^n}^+ \in \mathcal{U}_n$, for each $n \in \mathbb{N}$ such that the collection $\mathcal{U} = \left\{cl_\Lambda\left(\left(V_1^n, \ldots, V_{m_n}^n\right)_{B^n}^+\right) : n \in \mathbb{N}\right\}$ is a cover of (Λ, τ_Δ^+).

We claim that the collection $\mathcal{J} = \left\{\left(B^n; V_1^n, \ldots, V_{m_n}^n\right) : n \in \mathbb{N}\right\}$ is an almost $\pi_\Delta(\Lambda)$-network of X. Indeed, let $U \in \Lambda^c$, then there is $(B^n; V_1, \ldots, V_{m_n}) \in \mathcal{J}$ such that $U^c \in cl_\Lambda((V_1^n, \ldots, V_{m_n}^n)_{B^n}^+)$. Let $K \in \Delta$ and $U_1, \ldots, U_l$ open sets in X such that $U^c \cap K = \emptyset$ and $U^c \cap U_i \neq \emptyset$, for $1 \leq i \leq l$. Then $U^c \in (U_1, \ldots, U_l)_K^+$. Let $D \in (V_1^n, \ldots, V_{m_n}^n)_{B^n}^+ \cap (U_1, \ldots, U_l)_K^+$ and $F = \{x_1, \ldots, x_{m_n}, y_1, \ldots, y_l\}$, where $x_i \in D \cap V_i^n$ and $y_j \in D \cap U_j$ for $1 \leq i \leq m_n$ and $1 \leq j \leq l$. It can be shown that $W = D^c$ and F satisfy the conditions required in Definition 2.1. So, (X, τ) satisfies $\mathbf{S}_1(\Pi_\Delta(\Lambda), a\Pi_\Delta(\Lambda))$.

(2) ⇒ (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of (Λ, τ_Δ^+). Suppose that for any $n \in \mathbb{N}$, the open cover $\mathcal{U}_n$ consists in basic open subsets. For each $n \in \mathbb{N}$, let $\mathcal{J}_n = \{(B; V_1, \ldots, V_m) : (V_1, \ldots, V_m)_B^+ \in \mathcal{U}_n\}$. Then, by Lemma 1.3, $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network of X, for any $n \in \mathbb{N}$. Applying (2) to the sequence $(\mathcal{J}_n : n \in \mathbb{N})$, there is $(B^n; V_1^n, \ldots, V_{m_n}^n) \in \mathcal{J}_n$, for every $n \in \mathbb{N}$ such that $\mathcal{J} = \{(B^n; V_1^n, \ldots, V_{m_n}^n) : n \in \mathbb{N}\}$ is an almost $\pi_\Delta(\Lambda)$-network of X.

We see that the collection $\mathcal{U} = \{cl_\Lambda((V_1^n, \ldots, V_{m_n}^n)_{B^n}^+) : n \in \mathbb{N}\}$ is a cover of (Λ, τ_Δ^+). Indeed, let $D \in \Lambda$ and $U = D^c$, then there is $(B^n; V_1^n, \ldots, V_{m_n}^n) \in \mathcal{J}$ which satisfies Definition 2.1. So, if $(U_1, \ldots, U_l)_K^+$ is an open basic neighbourhood of U^c, then $U^c \cap K = \emptyset$ and $U^c \cap U_i \neq \emptyset$, for $1 \leq i \leq l$. Hence, there exist $W \in \Lambda^c$ and $F \in [X]^{<\omega}$ which satisfy $F \cap W = \emptyset$, $B^n \cup K \subseteq W$, $F \cap V_i^n \neq \emptyset$, for each $1 \leq i \leq m_n$ and $F \cap U_j \neq \emptyset$, for any $1 \leq j \leq l$. It means that $W \in (U_1, \ldots, U_l)_K^+ \cap (V_1^n, \ldots, V_{m_n}^n)_{B^n}^+$. So, $D \in cl_\Lambda((V_1^n, \ldots, V_{m_n}^n)_{B^n}^+)$. We conclude that (Λ, τ_Δ^+) is almost Rothberger. □

From Theorem 2.3, we obtain some interesting particular cases.

Corollary 2.4. *Let* (X, τ) *be a topological space. If* Λ *is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then* (Λ, τ_Δ^+) *is almost Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_\Delta(\Lambda), a\Pi_\Delta(\Lambda))$.

Corollary 2.5. *Let* (X, τ) *be a topological space. If* Λ *is any of the hyperspaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then:*

(a) (Λ, τ_F) *is almost Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_{\mathbb{K}(X)}(\Lambda), a\Pi_{\mathbb{K}(X)}(\Lambda))$.
(b) (Λ, τ_V) *is almost Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_{\mathrm{CL}(X)}(\Lambda), a\Pi_{\mathrm{CL}(X)}(\Lambda))$.

From Remark 1.2, we have:

Corollary 2.6. *Let* (X, τ) *be a topological space, then:*

(a) $(\mathrm{CL}(X), \tau_F)$ *is almost Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_F, a\Pi_{\mathbb{K}(X)}(\mathrm{CL}(X)))$.
(b) $(\mathrm{CL}(X), \tau_V)$ *is almost Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_V, a\Pi_{\mathrm{CL}(X)}(\mathrm{CL}(X)))$.

Similarly as in Theorem 2.3, we obtain the next result.

Theorem 2.7. *Given a topological space* (X, τ), *the following conditions are equivalent:*

(1) (Λ, τ_Δ^+) *is almost Menger;*
(2) (X, τ) *satisfies the property* $\mathbf{S}_{fin}(\Pi_\Delta(\Lambda), a\Pi_\Delta(\Lambda))$.

From Theorem 2.7, we get some interesting particular cases.

Corollary 2.8. *Let (X,τ) be a topological space. If Λ is any of the hyperspaces $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then (Λ,τ^+_Δ) is almost Menger if and only if X satisfies $\mathbf{S}_{fin}(\Pi_\Delta(\Lambda), a\Pi_\Delta(\Lambda))$.*

Corollary 2.9. *Let (X,τ) be a topological space. If Λ is any of the hyperspaces $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then:*

(a) (Λ,τ_F) is almost Menger if and only if X satisfies $\mathbf{S}_{fin}(\Pi_{\mathbb{K}(X)}(\Lambda), a\Pi_{\mathbb{K}(X)}(\Lambda))$.
(b) (Λ,τ_V) is almost Menger if and only if X satisfies $\mathbf{S}_{fin}(\Pi_{\mathrm{CL}(X)}(\Lambda), a\Pi_{\mathrm{CL}(X)}(\Lambda))$.

From Remark 1.2, we obtain:

Corollary 2.10. *Let (X,τ) be a topological space, then;*

(a) $(\mathrm{CL}(X),\tau_F)$ is almost Menger if and only if X satisfies $\mathbf{S}_{fin}(\Pi_F, a\Pi_{\mathbb{K}(X)}(\mathrm{CL}(X)))$.
(b) $(\mathrm{CL}(X),\tau_V)$ is almost Menger if and only if X satisfies $\mathbf{S}_{fin}(\Pi_V, a\Pi_{\mathrm{CL}(X)}(\mathrm{CL}(X)))$.

Now, we introduce the notion of weakly $\pi_\Delta(\Lambda)$-network to characterize the weakly Rothberger and weakly Menger properties in hyperspaces endowed with the hit-and-miss topology.

Definition 2.11. A family $\zeta \subseteq \zeta_\Delta$ is called a *weakly $\pi_\Delta(\Lambda)$-network of X*, if for every $U \in \Lambda^c$, and any $K \in \Delta$ and $U_1,\dots,U_m$ open sets in X, with $U^c \cap K = \emptyset$ and $U^c \cap U_i \neq \emptyset$, for $1 \le i \le m$, there exist $(B; V_1,\dots,V_N) \in \zeta$, $W \in \Lambda^c$ and $F \in [X]^{<\omega}$ which satisfy $F \cap W = \emptyset$, $B \cup K \subseteq W$, $F \cap V_i \neq \emptyset$, for each $1 \le i \le N$ and $F \cap U_j \neq \emptyset$, for any $1 \le j \le m$. The family of all the weakly $\pi_\Delta(\Lambda)$-networks is denoted by $w\Pi_\Delta(\Lambda)$.

Remark 2.12. It can be shown that an almost $\pi_\Delta(\Lambda)$-cover is a weakly $\pi_\Delta(\Lambda)$-cover. So, every $\pi_\Delta(\Lambda)$-cover is a weakly $\pi_\Delta(\Lambda)$-cover.

Theorem 2.13. *Given a topological space (X,τ), the following conditions are equivalent:*

(1) (Λ,τ^+_Δ) is weakly Rothberger;
(2) (X,τ) satisfies the property $\mathbf{S}_1(\Pi_\Delta(\Lambda), w\Pi_\Delta(\Lambda))$.

Proof. (1) $\Rightarrow$ (2) Let $(\mathcal{J}_n : n \in \mathbb{N})$ be a sequence in $\Pi_\Delta(\Lambda)$. Denote, for any $n \in \mathbb{N}$,

$$\mathcal{U}_n = \left\{(V_1,\dots,V_m)^+_B : (B; V_1,\dots,V_m) \in \mathcal{J}_n\right\}.$$

By Lemma 1.3, we have that for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is an open cover of (Λ,τ^+_Δ). Applying (1) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, there exists $(V^n_1,\dots,V^n_{m_n})^+_{B^n} \in \mathcal{U}_n$, for any $n \in \mathbb{N}$, such that $cl_\Lambda\left(\bigcup\left\{\left(V^n_1,\dots,V^n_{m_n}\right)^+_{B^n} : n \in \mathbb{N}\right\}\right) = \Lambda$.

We see that the collection $\mathcal{J} = \left\{\left(B^n; V^n_1,\dots,V^n_{m_n}\right) : n \in \mathbb{N}\right\}$ is a weakly $\pi_\Delta(\Lambda)$-network of X. Indeed, let $U \in \Lambda^c$, $K \in \Delta$ and $U_1,\dots,U_l$ open sets in X such that $U^c \cap K = \emptyset$ and $U^c \cap U_i \neq \emptyset$, for $1 \le i \le l$. Then $U^c \in (U_1,\dots,U_l)^+_K$. Hence, given that $cl_\Lambda\left(\bigcup\left\{\left(V^n_1,\dots,V^n_{m_n}\right)^+_{B^n} : n \in \mathbb{N}\right\}\right) = \Lambda$, we have that

$$\bigcup\left\{\left(V^n_1,\dots,V^n_{m_n}\right)^+_{B^n} : n \in \mathbb{N}\right\} \bigcap (U_1,\dots,U_l)^+_K \neq \emptyset.$$

Thus, there exist $N \in \mathbb{N}$ and $D \in \Delta$ with $D \in (V^N_1,\dots,V^N_{m_N})^+_{B^N} \cap (U_1,\dots,U_l)^+_K$ and $F = \{x_1,\dots,x_{m_N}, y_1,\dots,y_l\}$, where $x_i \in D \cap V^N_i$ and $y_j \in D \cap U_j$ for $1 \le i \le m_N$ and $1 \le j \le l$. It can be shown that $W = D^c$ and F satisfy the conditions required in Definition 2.11. So, (X,τ) satisfies $\mathbf{S}_1(\Pi_\Delta(\Lambda), w\Pi_\Delta(\Lambda))$.

(2) $\Rightarrow$ (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of (Λ,τ^+_Δ). Suppose that for any $n \in \mathbb{N}$, the open cover $\mathcal{U}_n$ consists in basic open subsets. For each $n \in \mathbb{N}$, let $\mathcal{J}_n = \{(B; V_1,\dots,V_m) : (V_1,\dots,V_m)^+_B \in \mathcal{U}_n\}$. Then, by Lemma 1.3, $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network of X, for any $n \in \mathbb{N}$. Applying (2) to the sequence $(\mathcal{J}_n : n \in \mathbb{N})$, there is $(B^n; V^n_1,\dots,V^n_{m_n}) \in \mathcal{J}_n$, for every $n \in \mathbb{N}$, such that $\mathcal{J} = \{(B^n; V^n_1,\dots,V^n_{m_n}) : n \in \mathbb{N}\}$ is a weakly $\pi_\Delta(\Lambda)$-network of X.

We claim that $cl_\Lambda\left(\bigcup\{(V_1^n,\ldots,V_{m_n}^n)^+_{B^n}) : n\in\mathbb{N}\}\right)=\Lambda$. Indeed, let $D\in\Lambda$ and $(U_1,\ldots,U_l)^+_K$ an open basic neighbourhood of D. Hence, applying Definition 2.11 to $U=D^c$, K, $U_1,\ldots,U_l$, there exist $N\in\mathbb{N}$, $W\in\Lambda^c$ and $F\in[X]^{<\omega}$ which satisfy $F\cap W=\emptyset$, $B^N\cup K\subseteq W$, $F\cap V_i^N\neq\emptyset$, for each $1\le i\le m_N$ and $F\cap U_j\neq\emptyset$, for any $1\le j\le l$. It means that $W\in(U_1,\ldots,U_l)^+_K\cap(V_1^N,\ldots,V_{m_N}^N)^+_{B^N}$. So, $D\in cl_\Lambda\left(\bigcup\{(V_1^n,\ldots,V_{m_n}^n)^+_{B^n}) : n\in\mathbb{N}\}\right)$. We conclude that (Λ,τ^+_Δ) is weakly Rothberger. □

We get, from Theorem 2.13, some interesting particular cases.

Corollary 2.14. *Let (X,τ) be a topological space. If Λ is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then (Λ,τ^+_Δ) is weakly Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_\Delta(\Lambda), w\Pi_\Delta(\Lambda))$.

Corollary 2.15. *Let (X,τ) be a topological space. If Λ is any of the hyperspaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then:*

(a) (Λ,τ_F) *is weakly Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_{\mathbb{K}(X)}(\Lambda), w\Pi_{\mathbb{K}(X)}(\Lambda))$.
(b) (Λ,τ_V) *is weakly Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_{\mathrm{CL}(X)}(\Lambda), w\Pi_{\mathrm{CL}(X)}(\Lambda))$.

From Remark 1.2, we have:

Corollary 2.16. *Let (X,τ) be a topological space, then:*

(a) $(\mathrm{CL}(X),\tau_F)$ *is weakly Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_F, w\Pi_{\mathbb{K}(X)}(\mathrm{CL}(X)))$.
(b) $(\mathrm{CL}(X),\tau_V)$ *is weakly Rothberger if and only if X satisfies* $\mathbf{S}_1(\Pi_V, w\Pi_{\mathrm{CL}(X)}(\mathrm{CL}(X)))$.

Similarly as in Theorem 2.13, we obtain the next result.

Theorem 2.17. *Given a topological space (X,τ), the following conditions are equivalent:*

(1) (Λ,τ^+_Δ) *is weakly Menger;*
(2) (X,τ) *satisfies the property* $\mathbf{S}_{fin}(\Pi_\Delta(\Lambda), w\Pi_\Delta(\Lambda))$.

We obtain, from Theorem 2.17, some interesting particular cases.

Corollary 2.18. *Let (X,τ) be a topological space. If Λ is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then (Λ,τ^+_Δ) is weakly Menger if and only if X satisfies* $\mathbf{S}_{fin}(\Pi_\Delta(\Lambda), w\Pi_\Delta(\Lambda))$.

Corollary 2.19. *Let (X,τ) be a topological space. If Λ is any of the hyperspaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then:*

(a) (Λ,τ_F) *is weakly Menger if and only if X satisfies* $\mathbf{S}_{fin}(\Pi_{\mathbb{K}(X)}(\Lambda), w\Pi_{\mathbb{K}(X)}(\Lambda))$.
(b) (Λ,τ_V) *is weakly Menger if and only if X satisfies* $\mathbf{S}_{fin}(\Pi_{\mathrm{CL}(X)}(\Lambda), w\Pi_{\mathrm{CL}(X)}(\Lambda))$.

From Remark 1.2, we have:

Corollary 2.20. *Let (X,τ) be a topological space, then:*

(a) $(\mathrm{CL}(X),\tau_F)$ *is weakly Menger if and only if X satisfies* $\mathbf{S}_{fin}(\Pi_F, w\Pi_{\mathbb{K}(X)}(\mathrm{CL}(X)))$.
(b) $(\mathrm{CL}(X),\tau_V)$ *is weakly Menger if and only if X satisfies* $\mathbf{S}_{fin}(\Pi_V, w\Pi_{\mathrm{CL}(X)}(\mathrm{CL}(X)))$.

3. Groupable and weakly groupable covers

We denote by $\mathscr{D}$ the family of dense subset of (Λ,τ^+_Δ). In [15, 16] it is defined $\mathscr{D}^{gp}$, the family of every groupable element of $\mathscr{D}$, where $\mathcal{D}\in\mathscr{D}$ is *groupable* if there is a partition $\mathcal{D}=\bigcup_{n\in\mathbb{N}}\mathcal{D}_n$ into finite sets such that each open set of the space intersects $\mathcal{D}_n$, for all but finitely many n. Now, we introduce the notion of groupable $c_\Delta(\Lambda)$-cover to characterize the selection principles $\mathbf{S}_1(\mathscr{D},\mathscr{D}^{gp})$ and $\mathbf{S}_{fin}(\mathscr{D},\mathscr{D}^{gp})$ in hyperspaces.

Definition 3.1. Let (X, τ) be a topological space. A $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of X is called *groupable*, if it can be represented as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n \subset \mathcal{U}$ such that for any $B \in \Delta$ and open sets $V_1, \ldots, V_m$ in X, with $V_i \cap B^c \neq \emptyset$, for each $i \in \{1, \ldots, m\}$, there is $n_0 \in \mathbb{N}$, such that for each $n \geq n_0$, there exist $U_n \in \mathcal{U}_n$ and $F_n \in [X]^{<\omega}$ with $F_n \cap V_i \neq \emptyset$ such that $B \subseteq U_n$ and $F_n \cap U_n = \emptyset$. We denote by $\mathbb{C}_\Delta(\Lambda)^{gp}$ the family of all groupable $c_\Delta(\Lambda)$-covers of a space.

In a similar way as [3, Remark 2.21], we obtain.

Remark 3.2. We have the following statements:

(1) If $\Delta = \mathbb{K}(X)$ and $\Lambda = \mathrm{CL}(X)$, then the notion of groupable $c_\Delta(\Lambda)$-cover in X coincides with the definition of F-groupable cover of X, as defined in [17, Definition 5.1].

(2) If $\Delta = \Lambda = \mathrm{CL}(X)$, then $\mathcal{U}$ is a groupable $c_\Delta(\Lambda)$-cover of X if and only if $\mathcal{U}$ is a V-groupable cover of X, as defined in [17, Definition 5.1].

Lemma 3.3. *Let (X, τ) be a topological space. Then $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ is a groupable dense subset of (Λ, τ_Δ^+) if and only if $\mathcal{V} = \{D_n^c : n \in \mathbb{N}\}$ is a groupable $c_\Delta(\Lambda)$-cover of X.*

Proof. Suppose that $\mathcal{D}$ is a groupable dense subset of (Λ, τ_Δ^+) and let $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$ a partition into finite sets such that each open set of (Λ, τ_Δ^+) intersects $\mathcal{D}_n$, for all but finitely many n. By Lemma 1.5, $\mathcal{V} = \{D_n^c : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of X. Furthermore, let $\mathcal{V}_n = \mathcal{D}_n^c$. It can be proved that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a partition into finite sets which witness that $\mathcal{V}$ is a groupable $c_\Delta(\Lambda)$-cover of X.

Reciprocally, suppose that $\mathcal{V} = \{D_n^c : n \in \mathbb{N}\}$ is a groupable $c_\Delta(\Lambda)$-cover of X and let $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ a partition into finite sets which witness that $\mathcal{V}$ is a groupable $c_\Delta(\Lambda)$-cover of X. By Lemma 1.5, $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ is a dense subset of (Λ, τ_Δ^+). Now, let $\mathcal{D}_n = \mathcal{V}_n^c$. It is not difficult to show that $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$ is a partition into finite sets such that each open set of (Λ, τ_Δ^+) intersects $\mathcal{D}_n$, for all but finitely many n. □

Theorem 3.4. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) (Λ, τ_Δ^+) satisfies $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$;

(2) (X, τ) satisfies $\mathbf{S}_1(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda)^{gp})$.

Proof. (1) ⇒ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $c_\Delta(\Lambda)$-covers of X. For any $n \in \mathbb{N}$, we put $\mathcal{D}_n = \mathcal{U}_n^c$. By Lemma 1.5, we obtain that, for any $n \in \mathbb{N}$, $\mathcal{D}_n$ is a dense subset of (Λ, τ_Δ^+). Hence, applying (1) to the sequence $(\mathcal{D}_n : n \in \mathbb{N})$, we obtain, for each $n \in \mathbb{N}$, $D_n \in \mathcal{D}_n$ such that $\{D_n : n \in \mathbb{N}\}$ is a groupable dense subset of (Λ, τ_Δ^+). For any $n \in \mathbb{N}$, let $U_n = D_n^c$. So, by Lemma 3.3, we have that $\{U_n : n \in \mathbb{N}\}$ is a groupable $c_\Delta(\Lambda)$-cover of X.

(2) ⇒ (1) Let $(\mathcal{D}_n : n \in \mathbb{N})$ be a sequence of dense subsets of (Λ, τ_Δ^+). For each $n \in \mathbb{N}$, take $\mathcal{V}_n = \mathcal{D}_n^c$. By Lemma 1.5, we have that, for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a $c_\Delta(\Lambda)$-cover of X. Hence, applying (2) to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, exists $V_n \in \mathcal{V}_n$, for each $n \in \mathbb{N}$, such that $\{V_n : n \in \mathbb{N}\}$ is a groupable $c_\Delta(\Lambda)$-cover of X. For any $n \in \mathbb{N}$, let $D_n = V_n^c$. Thus, from Lemma 3.3, we have that $\{D_n : n \in \mathbb{N}\}$ is a groupable dense subset of (Λ, τ_Δ^+). □

As a consequence of Theorem 3.4, in Corollaries 3.6 and 3.7, we generalize some results obtained by Li (see [17, Theorems 5.2, 5.4]) and provide characterizations to the property $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$ for another hyperspaces.

Corollary 3.5. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then (Λ, τ_Δ^+) satisfies $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$ if and only if X satisfies $\mathbf{S}_1(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda)^{gp})$.*

From Remark 3.2, we obtain:

Corollary 3.6. *Let (X, τ) be a topological space and let $\Delta = \mathbb{K}(X)$. Then:*

(a) $(\mathrm{CL}(X), \tau_F)$ satisfies $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$ if and only if X satisfies $\mathbf{S}_1(\mathbb{K}_F, \mathbb{K}_F^{gp})$ (see [17, Theorem 5.2]).

(b) *Let* Λ *any of the hyperspaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$. *Then* (Λ, τ_F) *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$ *if and only if X satisfies* $\mathbf{S}_1(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)^{gp})$.

Corollary 3.7. *Let* (X, τ) *be a topological space and let* $\Delta = \mathrm{CL}(X)$. *Then:*

(a) $(\mathrm{CL}(X), \tau_V)$ *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$ *if and only if X satisfies* $\mathbf{S}_1(\mathbb{C}_V, \mathbb{C}_V^{gp})$ *(see [17, Theorem 5.4]).*

(b) *Let* Λ *any of the hyperspaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$. *Then* (Λ, τ_V) *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{gp})$ *if and only if X satisfies* $\mathbf{S}_1(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}_{\mathrm{CL}(X)}(\Lambda)^{gp})$.

Similarly as in Theorem 3.4, we get the next result.

Theorem 3.8. *Let* (X, τ) *be a topological space. The following conditions are equivalent:*

(1) (Λ, τ_Δ^+) *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{gp})$;

(2) (X, τ) *satisfies* $\mathbf{S}_{fin}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda)^{gp})$.

As a consequence of Theorem 3.8, we get the following results.

Corollary 3.9. *Let* (X, τ) *be a topological space. If* Λ *is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then* (Λ, τ_Δ^+) *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{gp})$ *if and only if X satisfies* $\mathbf{S}_{fin}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda)^{gp})$.

When $\Delta = \mathbb{K}(X)$, from Remark 3.2, we obtain the characterization of $\mathbf{S}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D}^{gp})$ for Λ endowed with Fell topology.

Corollary 3.10. *Let* (X, τ) *be a topological space,* $\Delta = \mathbb{K}(X)$ *and* Λ *in* $\{\mathrm{CL}(X), \mathbb{K}(X), \mathbb{F}(X), \mathbb{CS}(X)\}$. *Then* (Λ, τ_F) *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{gp})$ *if and only if X satisfies* $\mathbf{S}_{fin}(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)^{gp})$.

If $\Delta = \mathrm{CL}(X)$, from Remark 3.2, we get the characterization of $\mathbf{S}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D}^{gp})$, for Λ endowed with Vietoris topology.

Corollary 3.11. *Let* (X, τ) *be a topological space,* $\Delta = \mathrm{CL}(X)$ *and* Λ *in* $\{\mathrm{CL}(X), \mathbb{K}(X), \mathbb{F}(X), \mathbb{CS}(X)\}$. *Then* (Λ, τ_V) *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{gp})$ *if and only if X satisfies* $\mathbf{S}_{fin}(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}_{\mathrm{CL}(X)}(\Lambda)^{gp})$.

Now, following the ideas of Li (see [17, Theorems 5.6, 5.8 and Definitions 5.5, 5.7]), we introduce the notions of a family weakly dense (Δ, Λ)-groupable and a weakly (Δ, Λ)-groupable cover to characterize the selection principles $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$ and $\mathbf{S}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D}^{wgp})$ in hyperspaces.

Definition 3.12. Let (X, τ) be a topological space and consider the hyperspace (Λ, τ_Δ^+). A family $\mathcal{A} \subseteq \Lambda$ is called *weakly dense* (Δ, Λ)*-groupable* if it can be partitioned into a countable union of finite sets C_n, $n \in \mathbb{N}$, so that $\{\bigcap C_n : n \in \mathbb{N}\}$ is dense in (Λ, τ_Δ^+). We denote by $\mathscr{D}^{wgp}$ the family of every weakly dense (Δ, Λ)-groupable family of a space.

Definition 3.13. Let (X, τ) be a topological space. An open cover $\mathcal{U} \subseteq \Lambda^c$ is called a *weakly* (Δ, Λ)*-groupable cover of* X, if it can be represented as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n \subset \mathcal{U}$ such that $\{\bigcup \mathcal{U}_n : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of X. We denote by $\mathbb{C}_\Delta(\Lambda)^{wgp}$ the family of all weakly (Δ, Λ)-groupable covers of a space.

Proposition 3.14. *We have the following statements.*

(1) *If* $\Delta = \mathbb{K}(X)$ *and* $\Lambda = \mathrm{CL}(X)$, *then the notion of weakly* (Δ, Λ)*-groupable cover of X coincides with the definition of weakly F-groupable cover of X (see [17, Definition 5.5]).*

(2) *If* $\Delta = \Lambda = \mathrm{CL}(X)$, *then* $\mathcal{U}$ *is a weakly* (Δ, Λ)*-groupable cover of X if and only if* $\mathcal{U}$ *is a weakly V-groupable cover of X (see [17, Definition 5.7]).*

Proof. We will show (2). Let $\mathcal{U}$ be a weakly (Δ, Λ)-groupable cover of X and $\mathcal{U}_n$ the finite subfamilies of $\mathcal{U}$ which witness it. Now, we show that the same families $\mathcal{U}_n$ work. Let $V_1, \ldots, V_n$ be open subsets of X. We apply the hypothesis to the sets $B = \bigcap_{i=1}^{n} V_i^c \in \mathrm{CL}(X)$ and $V_1, \ldots, V_n$, to obtain $N \in \mathbb{N}$ and $F \in [X]^{<\omega}$ which satisfy Definition 3.13. The same $\mathcal{U}_N$ and F satisfy [17, Definition 5.7].

On the other hand, let $\mathcal{U}$ be a weakly V-groupable cover of X and $\mathcal{U}_n$ the finite subfamilies of $\mathcal{U}$ which witness it. Lets see that the same families $\mathcal{U}_n$ work. Let $B \in \Delta$ and $V_1, \ldots, V_n$ be open sets such that $V_i \cap B^c \neq \emptyset$, for every $i \in \{1, \ldots, n\}$. We consider the non empty open sets $V_1 \cap B^c, \ldots, V_n \cap B^c$. By hypothesis, there exist $N \in \mathbb{N}$ and $F \in [X]^{<\omega}$ which satisfy [17, Definition 5.7]. The same N and F satisfy Definition 3.13. $\square$

Theorem 3.15. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) (Λ, τ_Δ^+) *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$;
(2) (X, τ) *satisfies* $\mathbf{S}_1(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda)^{wgp})$.

Proof. (1) $\Rightarrow$ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $c_\Delta(\Lambda)$-covers of X. For any $n \in \mathbb{N}$, we put $\mathcal{D}_n = \mathcal{U}_n^c$. By Lemma 1.5, we obtain that, for any $n \in \mathbb{N}$, $\mathcal{D}_n$ is a dense subset of (Λ, τ_Δ^+). Hence, applying (1) to the sequence $(\mathcal{D}_n : n \in \mathbb{N})$, we obtain, for each $n \in \mathbb{N}$, $B_n \in \mathcal{D}_n$ such that $\{B_n : n \in \mathbb{N}\}$ can be partitioned into a union of finite sets C_n, $n \in \mathbb{N}$, so that $\{\bigcap C_n : n \in \mathbb{N}\}$ is dense in (Λ, τ_Δ^+).

Let $\mathcal{V}_n = C_n^c$, we claim that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a weakly (Δ, Λ)-groupable cover of X. Indeed, consider $B \in \Delta$ and open sets $V_1, \ldots, V_m$ in X, with $V_i \cap B^c \neq \emptyset$, for each $i \in \{1, \ldots, m\}$. There is some $N \in \mathbb{N}$ such that $\bigcap C_N \in (V_1, \ldots, V_m)_B^+$. For any $i \in \{1, \ldots, m\}$, choose some $x_i \in V_i \cap (\bigcap C_N)$ and let $F = \{x_i : i \in \{1, \ldots, m\}\}$. It can be shown that $B \subseteq (\bigcap C_N)^c = \bigcup \mathcal{V}_N$ and $F \cap \bigcup \mathcal{V}_N = \emptyset$. So, the same partition $\{\mathcal{V}_n : n \in \mathbb{N}\}$ works.

(2) $\Rightarrow$ (1) Let $(\mathcal{D}_n : n \in \mathbb{N})$ be a sequence of dense subsets of (Λ, τ_Δ^+). For each $n \in \mathbb{N}$, we put $\mathcal{U}_n = \mathcal{D}_n^c$. By Lemma 1.5, we have that, for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is a $c_\Delta(\Lambda)$-cover of X. Hence, applying (2) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, for every $n \in \mathbb{N}$ there exists $V_n \in \mathcal{U}_n$ such that $\{V_n : n \in \mathbb{N}\}$ is a weakly (Δ, Λ)-groupable cover of X. So, $\mathcal{W} = \{V_n : n \in \mathbb{N}\}$ can be written as a countable union of finite, pairwise disjoint subfamilies $\mathcal{W}_n \subset \mathcal{W}$ such that for any $B \in \Delta$ and open sets $V_1, \ldots, V_m$ in X, with $V_i \cap B^c \neq \emptyset$, for each $i \in \{1, \ldots, m\}$, there exists $N \in \mathbb{N}$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \emptyset$ for each $i \in \{1, \ldots, m\}$, such that $B \subseteq \bigcup \mathcal{W}_N$ and $F \cap \bigcup \mathcal{W}_N = \emptyset$.

For each $n \in \mathbb{N}$, let $C_n = \mathcal{W}_n^c$. So, $\{V_n^c : n \in \mathbb{N}\} = \bigcup\{C_n : n \in \mathbb{N}\}$, where every C_n is a finite subset, which satisfy, by Lemma 1.5, that $\{\bigcap C_n : n \in \mathbb{N}\}$ is dense in (Λ, τ_Δ^+). It is, $\{V_n^c : n \in \mathbb{N}\}$ is weakly dense (Δ, Λ)-groupable. $\square$

As a consequence of Theorem 3.15, we obtain the following particular cases.

Corollary 3.16. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then* (Λ, τ_Δ^+) *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if X satisfies* $\mathbf{S}_1(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda)^{wgp})$.

From Proposition 3.14, we obtain:

Corollary 3.17. *Let (X, τ) be a topological space and let $\Delta = \mathbb{K}(X)$. Then:*

(a) $(\mathrm{CL}(X), \tau_F)$ *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if X satisfies* $\mathbf{S}_1(\mathbb{K}_F, \mathbb{K}_F^{wgp})$.
(b) Let Λ be any of the hyperspaces $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$. *Then* (Λ, τ_F) *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if X satisfies* $\mathbf{S}_1(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)^{wgp})$.

Corollary 3.18. *Let (X, τ) be a topological space and let $\Delta = \mathrm{CL}(X)$. Then:*

(a) $(\mathrm{CL}(X), \tau_V)$ *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if X satisfies* $\mathbf{S}_1(\mathbb{C}_V, \mathbb{C}_V^{wgp})$.
(b) et Λ be any of the hyperspaces $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$. *Then* (Λ, τ_V) *satisfies* $\mathbf{S}_1(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if X satisfies* $\mathbf{S}_1(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}_{\mathrm{CL}(X)}(\Lambda)^{wgp})$.

Similarly as in Theorem 3.15, it can be proved the following result.

Theorem 3.19. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) $(\Lambda, \tau_{\Delta}^{+})$ *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{wgp})$*;*
(2) (X, τ) *satisfies* $\mathbf{S}_{fin}(\mathbb{C}_{\Delta}(\Lambda), \mathbb{C}_{\Delta}(\Lambda)^{wgp})$*.*

As a consequence of Theorem 3.19, in Corollaries 3.21 and 3.22, we generalize some results obtained by Li in [17, Theorems 5.6, 5.8] and provide characterizations to the $\mathbf{S}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D}^{wgp})$ property for another hyperspaces.

Corollary 3.20. *Let* (X, τ) *be a topological space. If* Λ *is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$*, then* $(\Lambda, \tau_{\Delta}^{+})$ *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if* X *satisfies* $\mathbf{S}_{fin}(\mathbb{C}_{\Delta}(\Lambda), \mathbb{C}_{\Delta}(\Lambda)^{wgp})$*.*

From Proposition 3.14, we obtain:

Corollary 3.21. *Let* (X, τ) *be a topological space and let* $\Delta = \mathbb{K}(X)$*. Then:*

(a) $(\mathrm{CL}(X), \tau_F)$ *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if* X *satisfies* $\mathbf{S}_{fin}(\mathbb{K}_F, \mathbb{K}_F^{wgp})$ *(see [17, Theorem 5.6]).*
(b) Let Λ *be any of the hyperspaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$*. Then* (Λ, τ_F) *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if* X *satisfies* $\mathbf{S}_{fin}(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)^{wgp})$*.*

Corollary 3.22. *Let* (X, τ) *be a topological space and let* $\Delta = \mathrm{CL}(X)$*. Then:*

(a) $(\mathrm{CL}(X), \tau_V)$ *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if* X *satisfies* $\mathbf{S}_{fin}(\mathbb{C}_V, \mathbb{C}_V^{wgp})$ *(see [17, Theorem 5.8]).*
(b) Let Λ *be any of the hyperspaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$*. Then* (Λ, τ_V) *satisfies* $\mathbf{S}_{fin}(\mathscr{D}, \mathscr{D}^{wgp})$ *if and only if* X *satisfies* $\mathbf{S}_{fin}(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}_{\mathrm{CL}(X)}(\Lambda)^{wgp})$*.*

Acknowledgement

The authors are very grateful to the referee for careful reading and remarks that helped to improve the paper.

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