Filomat 37:15 (2023), 5027–5038
https://doi.org/10.2298/FIL2315027A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On strongly r-ideals on commutative rings and some related graphs

Khaled Alhazmy[a], Fuad Ali Ahmed Almahdi[a], El Mehdi Bouba[b], Mohammed Tamekkante[c]

[a]*Department of Mathematics, Faculty of Sciences, King Khalid University, P. O. Box. 9004, Abha, Saudi Arabia*
[b]*Mathematics Department, Pluridisciplinary faculty, Mohammed First University, B.P. 300, Selouane, Nador 62700, Morocco*
[c]*Department of Mathematics, Faculty of Science, University Moulay Ismail Meknes, Box 11201, Zitoune, Morocco*

Abstract. A proper ideal I of a ring R is called an r-ideal if, whenever $x, y \in R$ with $xy \in I$, we have $x \in I$ or $y \in \mathrm{Z}(R)$ [R. Mohamadian, r-ideals in commutative rings, Turkish J. Math. 39(5) (2015),733-749]. In this article, we are interested in a subclass of the class of r-ideals which we call the class of strongly r-ideals. A proper ideal I of a ring R is called a strongly r-ideal if, whenever $x, y \in R$ with $xy \in I$, we have $x \in I$ or $y \in \mathrm{Z}(I)$. First, we give a basic study of this new concept which includes, among others, characterizations, properties and examples. After that, we use the introduced concept to characterize rings for which the diameter of the zero-divisor graph is less than or equal to two, rings for which the annihilator graph is complete, and rings for which the zero-annihilator graph is empty.

1. Introduction

Throughout, all rings considered are commutative with nonzero unity. Let R be a ring, I be an ideal of R, and S be a subset of R. Set $S^* := S\backslash\{0\}$ and $(I : S) := \{x \in R \mid xS \subseteq I\}$. The annihilator of S over R is defined by $\mathrm{ann}_R(S) := (0 : S) = \{x \in R \mid xS = (0)\}$, and the annihilator of S in I is defined by $\mathrm{ann}_I(S) = \mathrm{ann}_R(S) \cap I = \{x \in I \mid xS = (0)\}$. An element $a \in R$ is said to be a zero-divisor following I if $\mathrm{ann}_I(a) \neq (0)$. The set of zero-divisors following I is denoted by $\mathrm{Z}(I)$; that is $\mathrm{Z}(I) = \{x \in R \mid \text{there exists } i \in I^* \text{ such that } xi = 0\}$. In particular, $\mathrm{Z}(R) = \{x \in R \mid \text{there exists } y \in R^* \text{ scuh that } xy = 0\}$ is the set of zero-divisors of R. The set of regular elements following I is $\mathrm{Reg}(I) = R\backslash\mathrm{Z}(I) = \{x \in R \mid \mathrm{ann}_I(x) = (0)\}$, and the set of regular elements of R is $\mathrm{Reg}(R) = R\backslash\mathrm{Z}(R)$. The ideal I is said to be proper if $I \neq R$. The radical of I is denoted by $\sqrt{I} := \{x \in R \mid x^n \in I \text{ for some integer } n \geq 1\}$, and the nil-radical of R is denoted by $\mathrm{nil}(R) := \sqrt{(0)}$. The total ring of fractions of R is denoted $Q(R) := \left\{\frac{a}{b} \mid a \in R \text{ and } b \in \mathrm{Reg}(R)\right\}$. The ring R is said to be a total fractions ring if $R = Q(R)$, equivalently, every element in R is either a zero-divisor or a unit.

In [15], Mohamadian introduced the concept of r-ideals. A proper ideal I of a ring R is called an r-ideal if, whenever $x, y \in R$ with $xy \in I$, we have $x \in I$ or $y \in \mathrm{Z}(R)$. Let $a \in R$ and set P_a the intersection of all minimal prime ideals containing a. Following [7], a proper ideal I is said a z^0-ideal if, for each $a \in I$, we have $P_a \subseteq I$. It is proved that every z^0-ideal is an r-ideal, however the two concepts are different ([15]). Let $C(X)$ be the ring of real valued continuous functions on a Tychonoff space X. In [15, Proposition 5.4], it is proved that,

---

2020 *Mathematics Subject Classification*. Primary 13A15, 13E05; Secondary 13F20.
Keywords. (Strongly) r-ideals; Total ring of fractions; The zero-divisor graph; McCoy rings.
Received: 24 August 2022; Accepted: 23 November 2022
Communicated by Dijana Mosić
Email addresses: khazmy@kku.edu.sa (Khaled Alhazmy), fuadalialmahdy@hotmail.com (Fuad Ali Ahmed Almahdi), mehdi8bouba@hotmail.fr (El Mehdi Bouba), tamekkante@yahoo.fr (Mohammed Tamekkante)

over $C(X)$, every r-ideal is a z^0-ideal if and only if X is a ∂-space (a space in which the boundary of any zeroset is contained in a zeroset with empty interior). Several recently introduced notions are related to the notion of r-ideals (see, for example, [17, 19])

In this paper, we are interested in a subclass of the class of r-ideals which we call the class of strongly r-ideals. A proper ideal of a ring R is called a strongly r-ideal if, whenever $x, y \in R$ with $xy \in I$, we have $x \in I$ or $y \in Z(I)$.
Section 2 gives a preliminary study of strongly r-ideals. Among other results, it is proved that an ideal I is a strongly r-ideal if and only if I is an r-ideal and $\mathrm{Z}(R) = Z(I)$ (Theorem 2.2). Other characterizations of strongly r-ideals are given in Theorem 2.6. Theorem 2.7 gives some characterizations of strongly r-ideal of $Q(R)$. Theorem 2.13 states that if a minimal prime ideal of a reduced ring R is a strongly r-ideal then R admits an infinity of minimal prime ideals. Let I be a proper ideal of a reduced ring R. Theorem 2.17 characterizes when $I[X]$ is a strongly r-ideal of $R[X]$.

For a graph G, we set $V(G)$ and $E(G)$ to be the sets of vertices and edges of G, respectively. Two elements $x, y \in V(G)$ are defined to be adjacent, denoted by $x - y$, if there exists an edge between them. A path between two elements $a_1, a_n \in V(G)$ is an ordered sequence of distinct vertices of G, $\{a_1, a_2, ..., a_n\}$, such that $a_i - a_{i+1}$. We say that G is connected if there is a path between any two distinct vertices of G. The length of a path between x and y is the number of edges crossed to get from x to y in the path. The distance between $x, y \in V(G)$, denoted $d(x, y)$, is the length of a shortest path between x and y, if such a path exists; otherwise, $d(x, y) = \infty$. The diameter of a graph is $\mathrm{diam}(G) = \max\{d(x, y) \mid x, y \in V(G)\}$. The graph G is said to be complete if each pair of distinct vertices forms an edge. For a general background on graph theory, we refer the reader to [18].
In section 3, we use the concept of strongly r-ideal to study some graphic properties of some well-know graphs of rings. The zero-divisor graph of a ring R, introduced by Anderson and Livingston in [5] and denoted by $\Gamma(R)$, is the simple graph associated to R such that its vertex set is $\mathrm{Z}(R)^*$ and that two distinct vertices x and y are adjacent if and only if $xy = 0$. This graph turns out to best exhibit the properties of the set of zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us to explore some interesting results in the algebraic structures of rings. The zero-divisor graphs of commutative rings have attracted the attention of several researchers. It was proved, among other things, that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ ([5, Theorem 2.3]). For a survey and recent results concerning zero-divisor graphs, we refer the reader to [3].
Theorem 3.1 proves that $\mathrm{ann}_R(x)$ is a strongly r-ideal for each $x \in \mathrm{Z}(R)^*$ if and only if $\mathrm{diam}(\Gamma(R)) \leq 2$ and $\mathrm{Z}(R)$ is an ideal of R. As a consequence, Theorem 3.5 shows that $\mathrm{ann}_{R[X]}(f)$ is a strongly r-ideal for each $f \in Z(R[X])^*$ if and only if R is McCoy and $\mathrm{Z}(R)$ is an ideal of R (recall that a ring R is said to be a McCoy ring if each finitely generated ideal contained in $\mathrm{Z}(R)$ has a nonzero annihilator [10] (referred to as Property A in [11–13])).
In [8], Badawi defined the annihilator graph of a commutative ring R, denoted by $AG(R)$, as a simple graph whose vertex set is the set of all nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$. Badawi studied some graph theoretical parameters of this graph such as diameter and girth. In addition, he studied some conditions under which the annihilator graph is identical to it's sub-graph, zero-divisor graph. He also determined when $AG(R)$ is a complete graph provided R is reduced. Theorem 3.11 and Corollary 3.12 study when $AG(R)$ is a complete graph in the case when R is non-reduced.

2. Strongly r-ideals in commutative rings

We begin by recalling the key definition of this paper.

Definition 2.1. *Let R be a ring. A proper ideal I is called a strongly r-ideal if, whenever $a, b \in R$ with $ab \in I$, then $\mathrm{ann}_I(a) \neq (0)$ or $b \in I$.*

The following characterization of strongly r-ideals will be used often throughout this article.

Theorem 2.2. *Let R be a ring and I be a proper ideal of R. The following are equivalent.*

1. *I is a strongly r-ideal.*
2. *I is an r-ideal and $\mathrm{Z}(I) = \mathrm{Z}(R)$.*

Proof. ($\Rightarrow$) It is clear that every strongly r-ideal is an r-ideal. Moreover, the inclusion $\mathrm{Z}(I) \subseteq \mathrm{Z}(R)$ is always satisfied for each ideal I of R. Let I be a strongly r-ideal. We need to show now that $\mathrm{Z}(R) \subseteq \mathrm{Z}(I)$. Let $a \in \mathrm{Z}(R)$ and consider $b \in R^*$ such that $ab = 0$. If $b \in I$ then $a \in \mathrm{Z}(I)$. So, suppose that $b \notin I$. Since $ab = 0 \in I$ and $b \notin I$, we obtain that $\mathrm{ann}_I(a) \neq (0)$. Thus, $a \in \mathrm{Z}(I)$. Accordingly, $\mathrm{Z}(R) \subseteq \mathrm{Z}(I)$, as desired.
($\Leftarrow$) Let $a, b \in R$ such that $ab \in I$ and $b \notin I$. Since I is an r-ideal , we have $\mathrm{ann}_R(a) \neq (0)$. Thus, $a \in \mathrm{Z}(R) = \mathrm{Z}(I)$. So, $\mathrm{ann}_I(a) \neq (0)$. Hence, I is a strongly r-ideal of R. □

Note that the zero ideal is an r-ideal while it is not a strongly r-ideal.

Proposition 2.3. *Let R be a ring and I be a proper ideal of R. If I is a strongly r-ideal then $(0) \neq I \subseteq \mathrm{Z}(I)$. The equivalence holds if I is prime.*

Proof. Let I be a strongly r-ideal of R. Note that (0) is never a strongly r-ideal since $Z((0)) = \emptyset$. Hence, $I \neq (0)$. On the other hand, since I is an r-ideal, I is consisting entirely of zero-divisors. Thus, $I \subseteq \mathrm{Z}(R) = \mathrm{Z}(I)$.
Let P be a nonzero prime ideal with $P \subseteq \mathrm{Z}(P)$. Let $a, b \in R$ with $ab \in P$ and $b \notin P$. Then, $a \in P \subseteq \mathrm{Z}(P)$. Hence, $\mathrm{ann}_P(a) \neq (0)$. So, P is a strongly r-ideal. □

Remarks 2.4.

1. *For a ring R, if $\mathrm{Z}(R)$ is a nonzero ideal of R, then it is a strongly r-ideal.*
2. *For a nonzero example of an r-ideal that is not a strongly r-ideal, consider the ring $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ and the ideal $I = 2\mathbb{Z}_4 \times (\bar{0})$. It is well known that R is zero-dimensional, and then nonunit elements of R are the zero-divisors of R. Hence, every ideal of R is an r-ideal. On the other hand, $2\mathbb{Z}_4 \times \mathbb{Z}_4 = \mathrm{Z}(I) \neq \mathrm{Z}(R) = (2\mathbb{Z}_4 \times \mathbb{Z}_4) \cup (\mathbb{Z}_4 \times 2\mathbb{Z}_4)$. Thus, I is not a strongly r-ideal.*

Let S be a multiplicatively closed subset of R and consider $f : R \to S^{-1}R$; the natural homomorphism defined by $f(x) = \frac{x}{1}$. Let J^c denote the contraction of J in R, i.e, $J^c := f^{-1}(J) = \left\{x \in R \mid \frac{x}{1} \in J\right\}$.

Lemma 2.5. *Let S be a multiplicatively closed subset of R. If J is a strongly r-ideal of $S^{-1}R$, then J^c is a strongly r-ideal of R. The equivalence holds when $S = \mathrm{Reg}(R)$.*

Proof. Let $a, b \in R$ with $ab \in J^c$ and $a \notin J^c$. Thus, $\frac{ab}{1} \in J$ and $\frac{a}{1} \notin J$. Then, $\mathrm{ann}_J\left(\frac{b}{1}\right) \neq \left(\frac{0}{1}\right)$. So, there exists $\frac{0}{1} \neq \frac{x}{s} \in J$ such that $\frac{xb}{s} = \frac{0}{1}$. Hence, $0 \neq x \in J^c$ (since $\frac{x}{1} = \frac{x}{s}.\frac{s}{1} \in J$) and $xs' \neq 0$ for each $s' \in S$. Moreover, there exists $s'' \in S$ such that $xbs'' = 0$. Then, $0 \neq xs'' \in \mathrm{ann}_{J^c}(b)$. Therefore, J^c is a strongly r-ideal of R.
Now, suppose that $S = \mathrm{Reg}(R)$ and that J^c is a strongly r-ideal of R. Let $\frac{a}{r}, \frac{b}{r'} \in Q(R) = (\mathrm{Reg}(R))^{-1}(R)$ with $\frac{a}{r}.\frac{b}{r'} \in J$ and $\frac{a}{r} \notin J$. Then, $ab \in J^c$ and $a \notin J^c$. Thus, there exists a nonzero element $x \in J^c$ such that $xb = 0$. Hence, $\frac{0}{1} \neq \frac{x}{1} \in J$ and $\frac{x}{1}.\frac{b}{r'} = \frac{0}{1}$. So, J is a strongly r-ideal of $Q(R)$. □

Let R be a ring and I be a nonzero proper ideal of R. It is easy to see that $\mathrm{Reg}(I)$ is a multiplicative closed subset of R (with $0 \notin S$). So, set $Q_I(R) := (\mathrm{Reg}(I))^{-1} R = \left\{\frac{a}{b} \mid a \in R \text{ and } b \in \mathrm{Reg}(I)\right\}$.
Next, we give some other characterizations of strongly r-ideals.

Theorem 2.6. *Let R be a ring and I be a nonzero proper ideal of R. The following statements are equivalent.*

1. *I is a strongly r-ideal of R.*
2. *$I = (I : r)$ for each $r \in \mathrm{Reg}(I)$.*
3. *$I = J^c$ for some ideal J of $Q_I(R)$.*

Proof. (1) ⇒ (2) Let $r \in \mathrm{Reg}(I)$. By Proposition 2.3, $r \notin I$ since $I \subseteq \mathrm{Z}(I)$. Hence, $(I : r)$ is a proper ideal of R. Let $a \in (I : r)$. We have $ar \in I$ and $\mathrm{ann}_I(r) = (0)$. Thus, $a \in I$. Hence, $(I : r) \subseteq I$. Consequently, $I = (I : r)$ since the inclusion $I \subseteq (I : r)$ is always satisfied.
(2) ⇒ (3) Set $J = (\mathrm{Reg}(I))^{-1} I$. We have $I \subseteq J^c$. Now, let $x \in J^c$. Then, $\frac{x}{1} \in J$. Therefore, $xr \in I$ for some $r \in \mathrm{Reg}(I)$. Hence, $x \in (I : r) = I$. Accordingly, $I = J^c$.
(3) ⇒ (1) Let $a, b \in R$ such that $ab \in I$ and $\mathrm{ann}_I(a) = (0)$. Then, $\frac{ab}{1} \in J$ and $a \in \mathrm{Reg}(I)$. Thus, $\frac{b}{1} = \frac{ab}{1}.\frac{1}{a} \in J$. So, $b \in J^c = I$. □

Let R be a ring. It is clear that every ideal of $Q(R)$ is an r-ideal, and so strongly r-ideals of $Q(R)$ are the ideals J such that $Z(J) = Z(Q(R))$.
Next, we characterize strongly r-ideals of $Q(R)$ in terms of strongly r-ideals of R.

Theorem 2.7. *Let R be a ring and J be a proper ideal of $Q(R)$. Then, the following are equivalent.*

1. *J is a strongly r-ideal of $Q(R)$.*
2. *J^c is a strongly r-ideal of R.*
3. *$Z(J^c) = \mathrm{Z}(R)$.*
4. *$Z(J) = Z(Q(R))$.*

Proof. The equivalence (1) ⇔ (4) follows from Theorem 2.2 since every ideal of $Q(R)$ is an r-ideal.
(1) ⇔ (2) Follows from Lemma 2.5.
(2) ⇒ (3) Follows from Theorem 2.2.
(3) ⇒ (4) Let $\frac{x}{s} \in Z(Q(R))$. There exists $\frac{0}{1} \neq \frac{y}{s'} \in Q(R)$ such that $\frac{xy}{ss'} = \frac{0}{1}$. Hence, $y \neq 0$ and $xy = 0$. So, $x \in \mathrm{Z}(R) = Z(J^c)$. Then, there exists $0 \neq z \in J^c$ such that $xz = 0$. So, $\frac{x}{s}.\frac{z}{1} = \frac{0}{1}$ and $\frac{0}{1} \neq \frac{z}{1} \in J$. Then, $\frac{x}{s} \in Z(J)$. Thus, $Z(Q(R)) \subseteq Z(J)$ and we are done. □

Proposition 2.8. *Let R be a ring and $e \neq 1$ be an idempotent element of R. Then, Re is an r-ideal which is not a strongly r-ideal.*
Consequently, every decomposable ring (in particular, every non-local ring) contains a nonzero r-ideal which is not a strongly r-ideal.

Proof. Let $x, y \in R$ with $xy \in Re$ and $y \notin Re$. Then, $xy(1-e) = 0$ and $y(1-e) = y - ye \neq 0$ since $y \notin Re$. Thus, $\mathrm{ann}_R(x) \neq (0)$. Hence, Re is an r-ideal. On the other hand, $e(1-e) = 0 \in Re$, $1 - e \notin Re$, and $\mathrm{ann}_{Re}(e) = (0)$. Thus, Re is not a strongly r-ideal. □

Proposition 2.9. *Let R be a reduced ring and $0 \neq a \in R$. Then, Ra is never a strongly r-ideal of R.*

Proof. Suppose that Ra is a strongly r-ideal. Then, by Proposition 2.3, $0 \neq a \in Ra \subseteq Z(Ra)$. Thus, there exists $r \in R$ such that $ra \neq 0$ and $ra^2 = 0$. Since R is reduced and $(ra)^2 = 0$, we get $ra = 0$, which leads to a contradiction. □

Proposition 2.10. *Let R be a ring which is not a domain and $x \in \mathrm{Z}(R)^*$. Then, the following are equivalent.*

1. *$\mathrm{ann}_R(x)$ is a strongly r-ideal of R.*
2. *$\mathrm{Z}(\mathrm{ann}_R(x)) = \mathrm{Z}(R)$.*
3. *For each $y \in \mathrm{Z}(R)$, $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(y) \neq (0)$.*

Proof. The equivalence (2) ⇔ (3) is clear. Moreover, it is easy to see that $\mathrm{ann}_R(x)$ is an r-ideal. Hence, the equivalence (1) ⇔ (2) follows immediately from Theorem 2.2. □

Corollary 2.11. *Let R be a non-reduced ring and $a \in \mathrm{nil}(R)^*$. If k is the smallest positive integer such that $a^k = 0$ then $\mathrm{ann}_R(a^{k-1})$ is a strongly r-ideal.*

Proof. Necessarily, $k \geq 2$ and $\mathrm{ann}_R(a^{k-1})$ is proper. Let $x \in Z(R)$. There exists $y \in R^*$ such that $xy = 0$. If $y \in \mathrm{ann}_R(a^{k-1})$ then $x \in Z\left(\mathrm{ann}_R(a^{k-1})\right)$. So, assume that $y \notin \mathrm{ann}_R(a^{k-1})$. Then, $ya^{k-1} \neq 0$ and $ya^{2k-2} = 0$ since $2k-2 \geq k$. Therefore, $ya^{k-1} \in \mathrm{ann}_R(a^{k-1})$. Thus, $x \in Z\left(\mathrm{ann}_R(a^{k-1})\right)$ since $xya^{k-1} = 0$. Consequently, $Z(R) = Z\left(\mathrm{ann}_R(a^{k-1})\right)$. Thus, $\mathrm{ann}_R(a^{k-1})$ is a strongly r-ideal (by Proposition 2.10). □

Proposition 2.12. *Let R be a ring such that Q(R) is von Neumann regular. Then,* $\mathrm{ann}_R(a)$ *is never a strongly r-ideal for each* $a \in R$*.*

Proof. Note first that R must be reduced since $Q(R)$ is so. If $a = 0$ then $\mathrm{ann}_R(a) = R$, and if $a \in \mathrm{Reg}(R)$ then $\mathrm{ann}_R(a) = (0)$. In the both cases, $\mathrm{ann}_R(a)$ is not a strongly r-ideal. So, we may assume that $a \in Z(R)^*$. Since $Q(R)$ is von Neumann regular, we can write $\frac{a}{1} = \frac{r_1}{r_2}\frac{b}{r_3}$ with $\frac{r_1}{r_2}$ is a unit element of $Q(R)$ and $\frac{b}{r_3}$ is an idempotent element of $Q(R)$. Since $\frac{r_1}{r_2}$ is unit, we have also $r_1 \in \mathrm{Reg}(R)$. Moreover, $\frac{b(r_3-b)}{r_3^2} = \frac{b}{r_3}\left(\frac{1}{1} - \frac{b}{r_3}\right) = \frac{0}{1}$. Thus, $ar_2r_3 = r_1b$ and $b(r_3 - b) = 0$. Hence, $ar_2r_3(r_3 - b) = r_1b(r_3 - b) = 0$. So, $a(r_3 - b) = 0$. If $r_3 = b$ then $ar_2 = r_1$, and so a is a regular element of R, a contradiction. Thus, $x = r_3 - b \in Z(R)^*$. Now, we claim that $\mathrm{ann}_{\mathrm{ann}_R(a)}(x) = (0)$. Let $\alpha \in R$ such that $\alpha a = \alpha x = 0$. Hence, $\alpha b = 0$ since $ar_2r_3 = r_1b$. Then, $\alpha r_3 = \alpha(x + b) = 0$, and so $\alpha = 0$. Thus, $\mathrm{ann}_{\mathrm{ann}_R(a)}(x) = (0)$. Finally, we get $ax = 0 \in \mathrm{ann}_R(a)$, $a \notin \mathrm{ann}_R(a)$ and $\mathrm{ann}_{\mathrm{ann}_R(a)}(x) = (0)$. Consequently, $\mathrm{ann}_R(a)$ is not a strongly r-ideal. □

It is well-known that a reduced ring R with only a finite number of minimal prime ideals (in particular, a reduced Noetherian ring) has von Neumann regular total ring of fractions $Q(R)$. Propositions 2.9 and 2.12 say that in such rings neither Ra nor $\mathrm{ann}_R(a)$ is a strongly r-ideal for each $a \in R$. The next result shows that minimal prime ideals are also not strongly r-ideals in such rings.
Let $\mathrm{Min}(R)$ denote the set of minimal prime ideals of the ring R.

Theorem 2.13. *Let R be a reduced ring. If* $\mathrm{Min}(R)$ *contains a strongly r-ideal then* $\mathrm{Min}(R)$ *is infinite.*

Proof. Suppose that there exists a minimal prime ideal P_1 which is a strongly r-ideal. Assume that $\mathrm{Min}(R)$ is finite and set $\mathrm{Min}(R) = \{P_1, \cdots, P_n\}$. If $n = 1$ then R is a domain and $P_1 = (0)$, a contradiction since (0) cannot be a strongly r-ideal. Hence, $n \geq 2$. Let $x \in P_1$ and suppose that $x \notin \bigcup_{i=2}^{n} P_i$. Since $P_1 \subseteq Z(P_1)$, there exists $y \in P_1^*$ such that $xy = 0$. Thus, $y \in \bigcap_{i=1}^{n} P_i = (0)$, a contradiction. Thus, $x \in \bigcup_{i=2}^{n} P_i$, and so $P_1 \subseteq \bigcup_{i=2}^{n} P_i$, a contradiction (by [11, Theorem 2.5]). Consequently, $\mathrm{Min}(R)$ is infinite. □

Here are some properties of strongly r-ideals.

Theorem 2.14. *Let R be a ring. Then, the following hold.*

1. *If I is a strongly r-ideal of R then,* $(I : S)$ *is a strongly r-ideal of R for each nonempty subset S of R such that* $S \nsubseteq I$*.*
2. *Every maximal strongly r-ideal of R is a prime ideal.*
3. *If I is a strongly r-ideal of R and P is a minimal prime ideal over I, then P is a strongly r-ideal.*
4. *If I is a strongly r-ideal of R then so is* $\sqrt{I}$*.*

Proof. (1) It is easy to see that $(I : S) \neq R$. Let $a, b \in R$ with $ab \in (I : S)$ and $b \notin (I : S)$. There exists $x \in S$ such that $abx \in I$ and $bx \notin I$. Hence, $\mathrm{ann}_I(a) \neq (0)$. Since $I \subseteq (I : S)$, we get $\mathrm{ann}_{(I:S)}(a) \neq (0)$. Thus, $(I : S)$ is a strongly r-ideal.
(2) Suppose that P is a maximal strongly r-ideal of R, and let $a, b \in R$ with $ab \in P$ and $a \notin P$. We have to show that $b \in P$. By (1), $(P : a)$ is a strongly r-ideal which contains P. By the maximality of P, we obtain that $P = (P : a)$. Hence, $b \in (P : a) = P$.
(3) Firstly, $(0) \neq I \subseteq P$. Let $a, b \in R$ such that $ab \in P$ and $\mathrm{ann}_P(a) = (0)$ (so $a \neq 0$). By [11, Theorem 2.1], there exists $x \notin P$ and a positive integer $n \geq 1$ such that $xa^nb^n \in I$. Let $\alpha \in \mathrm{ann}_P(a^n)$. Then, $\alpha a^{n-1}a = 0$.

Since $\mathrm{ann}_P(a) = (0)$, we get $\alpha a^{n-1} = 0$. By induction, we obtain that $\alpha = 0$. Hence, $\mathrm{ann}_P(a^n) = (0)$. Thus, $\mathrm{ann}_I(a^n) = (0)$. Thus, $xb^n \in I \subseteq P$. Then, $b \in P$.
(4) Note that $(0) \neq I \subseteq \sqrt{I}$. Let $a, b \in R$ with $ab \in \sqrt{I}$ and $b \notin \sqrt{I}$. We have to show that $\mathrm{ann}_{\sqrt{I}}(a) \neq (0)$. We may assume that $a \neq 0$. We have $a^n b^n \in I$ for some integer $n \geq 1$ and $b^n \notin I$. Thus, $\mathrm{ann}_I(a^n) \neq (0)$. Let $x \in I^*$ such that $xa^n = 0$, and k be the smallest integer such that $xa^k = 0$. Note that $k \geq 1$ since $x \neq 0$. Hence, $0 \neq xa^{k-1} \in \mathrm{ann}_{\sqrt{I}}(a)$. Thus, $\sqrt{I}$ is a strongly r-ideal of R □

Remark 2.15.
1. *Let R be a ring. If $\mathrm{Z}(R)$ is a nonzero ideal of R then it is the unique maximal strongly r-ideal of R.*
2. *The previous results about reduced rings do not mean that reduced rings, in general, do not contain strongly r-ideals. Indeed, seeing Remark 3.2 and Corollary 3.4 of section 3, [14, Example 5.1] is a witness of a reduced ring with strongly r-ideals.*
3. *Let I be a proper ideal of a ring R. If $\sqrt{I}$ is a strongly r-ideal of R then I need not be a strongly r-ideal of R. For example, for $R = \mathbb{Z}/4\mathbb{Z}$, the ideal $\mathrm{Z}(\mathbb{Z}/4\mathbb{Z}) = 2\mathbb{Z}/4\mathbb{Z} = \sqrt{(0)}$ is a strongly r-ideal of $\mathbb{Z}/4\mathbb{Z}$ but (0) is not.*

Proposition 2.16. *Let R be a ring.*
1. *R admits a strongly r-ideal if and only if R admits a (nonzero) prime ideal P such that $P \subseteq \mathrm{Z}(P)$.*
2. *If (R, M) is local then, M is a strongly r-ideal if and only if R is a total ring of fractions.*

Proof. (1) If R admits a strongly r-ideal of R, then R admits a maximal strongly r-ideal P which is nonzero prime ideal (by Theorem 2.14), and so, by Proposition 2.3, $(0) \neq P \subseteq \mathrm{Z}(P)$. Conversely, if P is a prime ideal of R such that $P \subseteq \mathrm{Z}(P)$ then, $P \neq (0)$. Indeed, $Z(0) = \emptyset$. Now, using Proposition 2.3, P must be a strongly r-ideal.
(2) If M is a strongly r-ideal, then $(0) \neq M \subseteq Z(M) = \mathrm{Z}(R) \subseteq M$. Thus, $M = \mathrm{Z}(R)$. So, R is a total ring of fractions. Conversely, if R is a local total ring of fractions, then $\mathrm{Z}(R)$ is a maximal ideal, and then $\mathrm{Z}(R)$ is a strongly r-ideal. □

Let R be a ring and let $f \in R[x]$ be a polynomial in one variable over R. The content of f, denoted by $c(f)$, is the ideal of R generated by the coefficients of f. The content of a polynomial, $c(f)$, satisfies a number of multiplicative properties. For example, the Dedekind-Mertens Lemma (see, for example, [6, Theorem 1]) asserts that for every two polynomials f and g in $R[x]$:

$$\mathrm{c}(f)\mathrm{c}(g)^{k+1} = \mathrm{c}(g)^k \mathrm{c}(fg), \text{ where } k = \deg(f).$$

Let R be a reduced ring and I be a proper ideal of R. The next result characterizes when $I[X]$ is a strongly r-ideal of $R[X]$.

Theorem 2.17. *Let R be a reduced ring and I be a proper ideal of R. Then, the following are equivalent.*
1. *$I[X]$ is a strongly r-ideal of $R[X]$.*
2. *For each ideal A of R and each finitely generated ideal B of R, $AB \subseteq I$ implies that $A \subseteq I$ or $\mathrm{ann}_I(B) \neq (0)$.*
3. *For each finitely generated ideals A and B of R, $AB \subseteq I$ implies that $A \subseteq I$ or $\mathrm{ann}_I(B) \neq (0)$.*

Proof. Let $g \in R[X]$. Using [2, Theorem 3.3], we have

$$\mathrm{ann}_{I[X]}(g) = \mathrm{ann}_{R[X]}(g) \cap I[X] = \mathrm{ann}_R(c(g))[X] \cap I[X] = \mathrm{ann}_I(c(g))[X].$$

(1) ⇒ (2) Let A be an ideal of R and B a finitely generated ideal of R with $AB \subseteq I$ and $A \nsubseteq I$. Set $B = \sum_{i=1}^n Rb_i$ and $g = \sum_{i=1}^n b_i X^i$. Then, $B = c(g)$. Let $a \in A \backslash I$. We have $aB \subseteq I$. Then, $ag \in I[X]$. Thus, since $a \notin I[X]$, we get $\mathrm{ann}_I(B)[X] = \mathrm{ann}_I(c(g))[X] = \mathrm{ann}_{I[X]}(g) \neq (0)$. So, $\mathrm{ann}_I(B) \neq (0)$, as desired.
(2) ⇒ (3) Clear.
(3) ⇒ (1) Let $f, g \in R[X]$ with $fg \in I[X]$ and $\mathrm{ann}_{I[X]}(g) = (0)$. We have to show that $f \in I[X]$. We may assume that $f \neq 0$. We have $\mathrm{c}(f)\mathrm{c}(g)^{k+1} = \mathrm{c}(g)^k \mathrm{c}(fg)$ where $k = \deg(f)$. Hence, $\mathrm{c}(f)\mathrm{c}(g)^{k+1} \subseteq I$ since $\mathrm{c}(fg) \subseteq I$. Moreover, $\mathrm{ann}_I(c(g))[X] = \mathrm{ann}_{I[X]}(g) = (0)$. Thus, $\mathrm{ann}_I(c(g)) = (0)$. Set $g = \sum_{i=1}^n b_i X^i$ and let $x \in \mathrm{ann}_I(c(g)^{k+1})$. Then, for each $i \in \{1, \cdots, n\}$, $xab_i^{k+1} = 0$. Hence, $(xb_i)^{k+1} = 0$. Since R is reduced, we get $xb_i = 0$. Hence, $x \in \mathrm{ann}_I(c(g)) = (0)$. Thus, $\mathrm{ann}_I(c(g)^{k+1}) = (0)$. Since $c(f)$ and $c(g)^{k+1}$ are finitely generated, we get $c(f) \subseteq I$. Hence, $f \in I[X]$. □

Corollary 2.18. *Let I be a proper ideal of a reduced ring R. If $I[X]$ is a strongly r-ideal of $R[X]$ then*

1. *I is a strongly r-ideal of R, and*
2. *for each finitely generated sub-ideal B of I, $\mathrm{ann}_I(B) \neq (0)$.*

Proof. (1) Let $a, b \in R$ with $ab \in I$. Set $A = Ra$ and $B = Rb$. Since $AB \subseteq I$, we have $A \subseteq I$ or $\mathrm{ann}_I(B) \neq (0)$ (by Theorem 2.17). Thus, $a \in I$ or $\mathrm{ann}_I(b) \neq (0)$. Accordingly, I is a strongly r-ideal of R.
(2) We have $RB = B \subseteq I$ and $R \nsubseteq I$. Then, $\mathrm{ann}_I(B) \neq (0)$. □

A ring R is said to have the strong annihilator condition or briefly R satisfies (s.a.c.) if for each finitely generated ideal I of R there exists an element $a \in I$ with $\mathrm{ann}_R(I) = \mathrm{ann}_R(a)$ ([2]).

Proposition 2.19. *Let R be a ring satisfying the (s.a.c) and I be a proper ideal of R. Then, $\sqrt{I}$ is a strongly r-ideal of R if and only if $\sqrt{I}[X]$ is a strongly r-ideal of $R[X]$.*

Proof. ($\Rightarrow$) Suppose that $\sqrt{I}$ is a strongly r-ideal of R. Let $f, g \in R[X]$ with $fg \in \sqrt{I}[X]$ and $\mathrm{ann}_{\sqrt{I}[X]}(g) = (0)$. We have to show that $f \in \sqrt{I}[X]$. We may assume that $f \neq 0$. We have $\mathrm{ann}_{\sqrt{I}}(c(g)) \subseteq \mathrm{ann}_{\sqrt{I}[X]}(g) = (0)$. Hence, $\mathrm{ann}_{\sqrt{I}}(c(g)) = (0)$. There exists so $a \in c(g)$ such that $\mathrm{ann}_{\sqrt{I}}(a) = \mathrm{ann}_R(a) \cap \sqrt{I} = \mathrm{ann}_R(c(g)) \cap \sqrt{I} = \mathrm{ann}_{\sqrt{I}}(c(g)) = (0)$. As in the proof of Theorem 2.17, we can show that $\mathrm{c}(f)^{k+1}\mathrm{c}(g) \subseteq \sqrt{I}$ where $k = \deg(g)$. Thus, $\mathrm{c}(f)^{k+1}a \subseteq \sqrt{I}$ and $\mathrm{ann}_{\sqrt{I}}(a) = (0)$. Hence, $\mathrm{c}(f)^{k+1} \subseteq \sqrt{I}$, and so $\mathrm{c}(f) \subseteq \sqrt{I}$. Thus, $f \in \sqrt{I}[X]$. Consequently, $\sqrt{I}[X]$ is a strongly r-ideal of $R[X]$.
($\Leftarrow$) Clear. □

Proposition 2.20. *Let $I_1 \times I_2$ be a proper ideal of a direct product of rings $R_1 \times R_2$. Then $I_1 \times I_2$ is a strongly r-ideal of $R_1 \times R_2$ if and only if I_1 and I_1 are strongly r-ideals of R_1 and R_2, respectively.*

Proof. ($\Rightarrow$) Let $x, y \in R_1$ with $xy \in I_1$ and $x \notin I_1$. Then, $(x, 0)(y, 1) = (xy, 0) \in I_1 \times I_2$ and $(x, 0) \notin I_1 \times I_2$. Thus, there exists a nonzero element $(r_1, r_2) \in I_1 \times I_2$ such that $(r_1, r_2)(y, 1) = (0, 0)$. Thus, $r_2 = 0$, and so $r_1 \in I_1^*$ with $r_1 y = 0$. So, I_1 is a strongly r-ideal of R_1. Similarly, I_2 is a strongly r-ideal of R_2.
($\Leftarrow$) Let $(x_1, x_2), (y_1, y_2) \in R_1 \times R_2$ with $(x_1, x_2)(y_1, y_2) \in I_1 \times I_2$ and $(x_1, x_2) \notin I_1 \times I_2$. Thus, $x_1 \notin I_1$ or $x_2 \notin I_2$. Suppose, for example, that $x_1 \notin I_1$. Since $x_1 y_1 \in I_1$, there exists $r \in I_1^*$ such that $r y_1 = 0$. Thus, $(r, 0)(y_1, y_2) = (0, 0)$ and $(r, 0) \in (I_1 \times I_2)^*$. Hence, $I_1 \times I_2$ is a strongly r-ideal of $R_1 \times R_2$. □

Proposition 2.21. *Let R be a ring, $S \subseteq \mathrm{Reg}(R)$ be a multiplicatively closed subset of R, and I be a proper ideal of R. If I is a strongly r-ideal of R then $S^{-1}I$ is a strongly r-ideal of $S^{-1}R$.*

Proof. Since $I \cap S \subseteq \mathrm{Z}(R) \cap \mathrm{Reg}(R) = \emptyset$, $S^{-1}I$ is a proper ideal of $S^{-1}R$. Let $\frac{a}{s_1}, \frac{b}{s_2} \in S^{-1}R$ with $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I$ and $\frac{a}{s_1} \notin S^{-1}I$. Then, $as \notin I$ for each $s \in S$ and $abs_3 \in I$ for some $s_3 \in S$. Hence, there exists $c \in I^*$ such that $bc = 0$. Since $S \subseteq \mathrm{Reg}(R)$, $\frac{c}{1} \in \left(S^{-1}I\right)^*$ and $\frac{b}{s_2}.\frac{c}{1} = \frac{0}{1}$. Hence, $S^{-1}I$ is a strongly r-ideal of $S^{-1}R$. □

3. Strong *r*-ideals and some related graphs

This section is devoted to the use of the notion of strongly r-ideals to characterize some graphical properties of some well-known graph of rings.
The first result of this section characterizes rings R such that $\mathrm{diam}(\Gamma(R)) \leq 2$ provided $\mathrm{Z}(R)$ is an ideal of R.

Theorem 3.1. *Let R be a ring which is not a domain. Then, the following are equivalent:*

1. *For each $x \in \mathrm{Z}(R)^*$, $\mathrm{ann}_R(x)$ is a strongly r-ideal.*
2. *For each $x \in \mathrm{Z}(R)^*$, $\mathrm{Z}(\mathrm{ann}_R(x)) = \mathrm{Z}(R)$.*
3. *For each $x, y \in \mathrm{Z}(R)^*$, $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(y) \neq (0)$.*
4. *$\mathrm{diam}(\Gamma(R)) \leq 2$ and $\mathrm{Z}(R)$ is an ideal of R.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Proposition 2.10.
(3) $\Rightarrow$ (4) Let $x, y \in Z(R)^*$. There exists $r \in R^*$ such that $rx = ry = 0$. Thus, $r(x+y) = 0$. So, $x + y \in Z(R)$. Then, $Z(R)$ is an ideal of R. Moreover, if $x \neq y$ and $xy \neq 0$ then $x - r - y$ is a path. Thus, $d(x, y) = 2$. Hence, $\operatorname{diam}(\Gamma(R)) \leq 2$.
(4) $\Rightarrow$ (3) Let $x, y \in Z(R)^*$. If $x = y$ then, $\operatorname{ann}_R(x) \cap \operatorname{ann}_R(y) = \operatorname{ann}_R(x) \neq (0)$. So, suppose that $x \neq y$. Since $\operatorname{diam}(\Gamma(R)) \leq 2$, we get either $xy = 0$ or there exists $r \in Z(R)^*$ such that $rx = ry = 0$. In the last case, $0 \neq r \in \operatorname{ann}_R(x) \cap \operatorname{ann}_R(y)$, as desired. Suppose now, that $x \neq y$ and that $xy = 0$. Since $Z(R)$ is an ideal, $x + y \in Z(R)$, and then there exists $\alpha \in Z(R)^*$ such that $\alpha(x+y) = 0$. Set $r = \alpha x = -\alpha y$. If $r = 0$ then $0 \neq \alpha \in \operatorname{ann}_R(x) \cap \operatorname{ann}_R(y)$. Now, if $r \neq 0$ then $rx = ry = 0$ and so $0 \neq r \in \operatorname{ann}_R(x) \cap \operatorname{ann}_R(y)$. In all cases, we get $\operatorname{ann}_R(x) \cap \operatorname{ann}_R(y) \neq (0)$. □

Remark 3.2. *An example of a reduced (resp. non-reduced) ring R with* $\operatorname{diam}(\Gamma(R)) \leq 2$ *such that* $Z(R)$ *is an ideal of R is given in [14, Example 5.1] (resp. [14, Example 5.5]).*

Let R be a non-reduced ring. Using [5, Theorem 2.2] and [14, Corollary 2.5], we conclude that if $\operatorname{diam}(\Gamma(R)) \leq 2$ then $Z(R)$ is an ideal. Hence, we have the following corollary.

Corollary 3.3. *Let R be a non-reduced ring. Then,* $\operatorname{diam}(\Gamma(R)) \leq 2$ *if and only if* $\operatorname{ann}_R(x)$ *is a strongly for each* $x \in Z(R)^*$.

Let R be a reduced ring with $\operatorname{diam}(\Gamma(R)) \leq 2$ such that $Z(R)$ is an ideal of R. Following [14, Theorem 2.6], $\operatorname{diam}(R) \geq 1$. Suppose now that $\operatorname{diam}(\Gamma(R)) = 1$. Then, $xy = 0$ for each distinct pair of zero-divisors and R has at least two nonzero zero-divisors. Let $x \in Z(R)^*$. Then, $x^2 \neq 0$ since R is reduced. If $x \neq x^2$ then $x^3 = xx^2 = 0$, and so $x = 0$, a contradiction. Thus, $x = x^2$, and so x is a non trivial idempotent. Moreover, $1 - x$ is a non trivial idempotent, and so is a zero-divisor. Hence, $1 = x + (1 - x) \in Z(R)$ since $Z(R)$ is an ideal, a contradiction. Thus, $\operatorname{diam}(\Gamma(R)) = 2$. We conclude then the following corollary.

Corollary 3.4. *Let R be a reduced ring. Then,* $\operatorname{ann}_R(x)$ *is a strongly r-ideal for each* $x \in Z(R)^*$ *if and only if* $\operatorname{diam}(\Gamma(R)) = 2$ *and* $Z(R)$ *is an ideal of R.*

Theorem 3.5. *Let R be a ring which is not a domain. Then, the following are equivalent:*

1. *For each* $f \in Z(R[X])^*$, $\operatorname{ann}_{R[X]}(f)$ *is a strongly r-ideal.*
2. *For each* $f, g \in Z(R[X])^*$, $\operatorname{ann}_{R[X]}(f) \cap \operatorname{ann}_{R[X]}(g) \neq (0)$.
3. $\operatorname{diam}(\Gamma(R[X])) \leq 2$ *and* $Z(R[X])$ *is an ideal of R.*
4. $Z(R[X])$ *is an ideal of R.*
5. *R is McCoy and* $Z(R)$ *is an ideal of R.*
6. *For each finitely generated ideals* $A, B \subseteq Z(R)$, $A + B$ *has a nonzero annihilator.*

Proof. As R is not a domain, $R[X]$ is not either. Then, the equivalences between (1), (2), and (3) follow from Theorem 3.1. Moreover, the equivalence (4) $\Leftrightarrow$ (5) is just [14, Theorem 3.3].
(3) $\Rightarrow$ (4) Clear.
(5) $\Rightarrow$ (3) Note first that $\operatorname{diam}(\Gamma(R[X])) \geq 1$ (By [14, Theorem 3.4]) and that $Z(R[X])$ is an ideal. If $Z(R)^2 \neq (0)$, then $\operatorname{diam}(\Gamma(R[X])) = 2$ ((By [14, Theorem 3.4]). So, assume that $Z(R)^2 = (0)$. If R is reduced then for each $x \in Z(R)$, we have $x^2 = 0$ and so $x = 0$. Hence, R is a domain, a contradiction. Hence, R is not reduced. Thus, $\operatorname{diam}(\Gamma(R[X])) = 1$ (By [14, Theorem 3.4]).
(5) $\Rightarrow$ (6) Let $A, B \subseteq Z(R)$ be two finitely generated ideals of R. Since $Z(R)$ is an ideal, $A + B \subseteq Z(R)$. Hence, since $A + B$ is finitely generated and R is a McCoy ring, $A + B$ has a nonzero annihilator.
(6) $\Rightarrow$ (5) Let $A \subseteq Z(R)$ be a finitely generated ideal. By hypothesis, $A = A + (0)$ has a nonzero annihilator. So, R is a McCoy ring. Let $x, y \in Z(R)$. Then, $Rx + Ry$ has a nonzero annihilator. Thus, $Rx + Ry \subseteq Z(R)$, and so $Z(R)$ is an ideal of R. □

Remark 3.6. *An example of a reduced (resp. non-reduced) McCoy ring R such that* $Z(R)$ *is an ideal of R is given in [14, Example 5.3] (resp. [14, Example 5.5]).*

As is the case of corollaries 3.3 and 3.4, we deduce the following ones

Corollary 3.7. *Let R be a non-reduced ring. Then, the following are equivalent.*

1. *For each $f \in Z(R[X])^*$, $\mathrm{ann}_{R[X]}(f)$ is a strongly r-ideal.*
2. $\mathrm{diam}(\Gamma(R[X])) \leq 2$.
3. *R is McCoy and $Z(R)$ is an ideal of R.*

Corollary 3.8. *Let R be a reduced ring. Then, the following are equivalent.*

1. *For each $f \in Z(R[X])^*$, $\mathrm{ann}_{R[X]}(f)$ is a strongly r-ideal.*
2. $\mathrm{diam}(\Gamma(R[X])) = 2$ *and $Z(R)$ is an ideal of R.*
3. *R is McCoy and $Z(R)$ is an ideal of R.*

The zero-annihilator graph of R, introduced by Mostafanasab in [16] and denoted by $ZA(R)$, is the graph whose vertex set is the set of all nonzero nonunit elements of R and two distinct vertices x and y are adjacent whenever $\mathrm{ann}_R(Rx + Ry) = \mathrm{ann}_R(x) \cap \mathrm{ann}_R(y) = (0)$.
Recall that an empty graph is a graph in which there are no edges between its vertices.

Corollary 3.9. *Let R be a ring which is not a domain. Then, the following are equivalent:*

1. *For each nonzero nonunit element $x \in R$, $\mathrm{ann}_R(x)$ is a strongly r-ideal.*
2. *For each nonzero nonunit elements $x, y \in R$, $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(y) \neq (0)$.*
3. *R is local with maximal ideal $Z(R)$ and $\mathrm{diam}(\Gamma(R)) \leq 2$.*
4. *$ZA(R)$ is empty.*

Proof. If one of the assertions (1), (2), or (3) holds then $\mathrm{ann}_R(x) \neq (0)$ for each nonzero nonunit element $x \in R$ (recall that (0) cannot be a strongly r-ideal). Thus, R is a total ring of fractions. So, the equivalences between (1), (2), and (3) follow from Theorem 3.1.
(2) ⇒ (4) follows directly from the definition of the zero-annihilator graph.
(4) ⇒ (2) Let $x, y \in R$ be nonzero nonunit elements. If $x \neq y$ then, since $ZA(R)$ is empty, $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(y) \neq (0)$, as desired. Hence, it suffices to prove that $\mathrm{ann}_R(x) = \mathrm{ann}_R(x) \cap \mathrm{ann}_R(x) \neq (0)$, that is x is a zero-divisor. Suppose that x is regular. Hence, $x \neq x^2$, and x and x^2 are regular. Thus, $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(x^2) = (0)$. However, $ZA(R)$ is empty, and so $\mathrm{ann}_R(x) \cap \mathrm{ann}_R(x^2) \neq (0)$, a contradiction. □

Remark 3.10. *An example of a local reduced (resp. non-reduced) ring R with maximal ideal $Z(R)$ and $\mathrm{diam}(\Gamma(R)) \leq 2$ is given in [14, Example 5.2] (resp. [14, Example 5.6]).*

Next, we are interested to rings R such that every nonzero proper ideal of $Q(R)$ is a strongly r-ideal of $Q(R)$. Suppose that R is reduced. Let $x \in Z(R)^*$. Then, $Q(R)\frac{x}{1}$ is a strongly r-ideal of $Q(R)$. Thus, $Z\left(Q(R)\frac{x}{1}\right) = Z(Q(R))$. Since $\frac{x}{1} \in Z(Q(R))$, there exists $\frac{a}{s} \in Q(R)$ such that $\frac{a}{s}.\frac{x}{1} \neq \frac{0}{1}$ and $\frac{a}{s}.\frac{x}{1}.\frac{x}{1} = \frac{0}{1}$. That is $ax \neq 0$ and $ax^2 = 0$. Hence, $(ax)^2 = 0$. Since R is reduced, we get $ax = 0$, a contradiction. Thus, $Z(R) = \{0\}$, and so R is a domain. So, we turn our attention to the case when R is non-reduced.

Theorem 3.11. *Let R be a non-reduced ring and set $Q = Q(R)$. Then, the following are equivalent.*

1. *Every nonzero proper ideal of Q is a strongly r-ideal of Q.*
2. *For every nonzero proper ideal J of Q, J^c is a strongly r-ideal of R.*
3. *For each $x \in Z(R)^*$, $Q\frac{x}{1}$ is a strongly r-ideal of Q.*
4. *For each $x \in Z(R)^*$, $Z(Rx) = Z(R)$.*
5. *For each $x, y \in Z(R)^*$, $\mathrm{ann}_R(y) \neq \mathrm{ann}_R(xy)$.*
6. *For each $x, y \in Z(R)^*$, $\mathrm{ann}_R(x) \cup \mathrm{ann}_R(y) \neq \mathrm{ann}_R(xy)$.*
7. *For each $x, y \in Z(R)^*$, $Rx \cap \mathrm{ann}_R(y) \neq (0)$.*
8. *For each nonzero ideal I of R and $y \in Z(R)^*$, $I \cap \mathrm{ann}_R(y) \neq (0)$.*

9. *For each ideal* $(0) \neq I \subseteq Z(R)$ *and* $y \in Z(R)^*$, $I \cap \mathrm{ann}_R(y) \neq (0)$.
10. *For each ideal* $(0) \neq I \subseteq Z(R)$, $Z(I) = Z(R)$.
11. *R is indecomposable and $AG(R)$ is complete.*
12. $Z(R)$ *is an ideal and $AG(R)$ is complete.*
13. *Q is local and $AG(Q)$ is complete.*

Proof. (1) ⇔ (2) Follows from Theorem 2.7.
(1) ⇒ (3) Clear.
(3) ⇒ (4) Let $x \in Z(R)^*$ and $r \in Z(R)$. Then, $\frac{r}{1} \in Z(Q)$. Following Theorem 2.2, we have $Z\left(Q\frac{x}{1}\right) = Z(Q)$. Then, there exists $\frac{a}{s} \in Q$ such that $\frac{ax}{s} \neq \frac{0}{1}$ and $\frac{axr}{s} = \frac{0}{1}$. Then, $ax \neq 0$ and $axr = 0$. Thus, $r \in Z(Rx)$. Consequently, $Z(Rx) = Z(R)$.
(4) ⇒ (5) Let $x, y \in Z(R)^*$. We have $x \in Z(R) = Z(Ry)$. So, there exists $r \in R$ such that $ry \neq 0$ and $rxy = 0$. Then, $r \in \mathrm{ann}_R(xy) \backslash \mathrm{ann}_R(y)$.
(5) ⇒ (6) Let $x, y \in Z(R)^*$. If $\mathrm{ann}_R(x) \cup \mathrm{ann}_R(y) = \mathrm{ann}_R(xy)$ then $\mathrm{ann}_R(x)$ and $\mathrm{ann}_R(y)$ are comparable and $\mathrm{ann}_R(x) = \mathrm{ann}_R(xy)$ or $\mathrm{ann}_R(y) = \mathrm{ann}_R(xy)$, a contradiction.
(6) ⇒ (7) Let $x, y \in Z(R)^*$. By hypothesis, we have $\mathrm{ann}_R(x) \cup \mathrm{ann}_R(y) \subsetneq \mathrm{ann}_R(xy)$. Let $\alpha \in \mathrm{ann}_R(xy) \backslash \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$. Then, $\alpha xy = 0$ and $\alpha x \neq 0$. Thus, $0 \neq \alpha x \in Rx \cap \mathrm{ann}_R(y)$, as desired.
(7) ⇒ (8) Let I be a nonzero ideal of R and let $y \in Z(R)^*$. Let $r \in I$. Then, either $r \in Z(R)$ or r is regular and $0 \neq ry \in I \cap Z(R)$. Thus, we have always $I \cap Z(R) \neq (0)$. Consider $0 \neq x \in I \cap Z(R)$. We have $(0) \neq Rx \cap \mathrm{ann}_R(y) \subseteq I \cap \mathrm{ann}_R(y)$.
(8) ⇒ (9) Clear.
(9) ⇒ (10) Let $(0) \neq I \subseteq Z(R)$ be an ideal of R. Let $y \in Z(R)^*$ and consider $0 \neq x \in I \cap \mathrm{ann}_R(y)$. Then, $x \in I^*$ and $xy = 0$. Thus, $\mathrm{ann}_I(y) \neq (0)$. Hence, $y \in Z(I)$. So, $Z(R) \subseteq Z(I)$. Consequently, $Z(I) = Z(R)$.
(10) ⇒ (1) Let J be a nonzero proper ideal of Q. Thus, $(0) \neq J^c \subseteq Z(R)$. Then, $Z(J^c) = Z(R)$. So, by Theorem 2.7, J is a strongly r-ideal of Q.
(6) ⇒ (11) Clearly $AG(R)$ is complete. Let $e \in R$ be an idempotent. If e is not trivial (that is $e \notin \{0, 1\}$) then $e \in Z(R)^*$. Hence, $\mathrm{ann}_R(e) = \mathrm{ann}_R(e) \cup \mathrm{ann}_R(e) \neq \mathrm{ann}_R(e^2) = \mathrm{ann}_R(e)$, a contradiction. Thus R is indecomposable.
(11) ⇒ (6) Since $AG(R)$ is complete, it suffices to show that $\mathrm{ann}_R(a) \neq \mathrm{ann}(a^2)$ for each $a \in Z(R)^*$. If $a^2 = 0$, we have the desired result. So, we may assume that $a^2 \neq 0$. Moreover, $a^2 \neq a$, otherwise a becomes an non trivial idempotent which is impossible since R is indecomposable. Thus, since $AG(R)$ is complete and a and a^2 are two distinct nonzero zero-divisors of R, we have $\mathrm{ann}_R(a) \cup \mathrm{ann}_R(a^2) \subsetneq \mathrm{ann}_R(a^3)$. Thus, $\mathrm{ann}(a) \subseteq \mathrm{ann}_R(a^2) \subsetneq \mathrm{ann}_R(a^3)$. There exists $r \in R$ such that $ra^3 = 0$ and $ra^2 \neq 0$. Thus, $(ra)a^2 = 0$ and $(ra)a \neq 0$. Hence, $ra \in \mathrm{ann}_R(a^2) \backslash \mathrm{ann}_R(a)$. Consequently, for each $a \in Z(R)^*$, we have $\mathrm{ann}_R(a) \neq \mathrm{ann}_R(a^2)$.
(11) ⇒ (12) Let $a, b \in Z(R)^*$. Clearly, $b^2 \neq b$ since R is indecomposable.
Suppose that $ab = 0$. If $b^2 = 0$, then $b(a + b) = 0$, and so $a + b \in Z(R)$. Suppose now that $b^2 \neq 0$. Since $AG(R)$ is complete, there exists $c \in R$ such that $cb^3 = 0$ and $cb^2 \neq 0$. Thus, $cb^2(a + b) = 0$. Hence, $a + b \in Z(R)$. Assume now that $ab \neq 0$ and let $r \in R^*$ such that $ra = 0$. Clearly $r \neq b$. Thus, there exists $r' \in R$ such that $r'rb = 0$ and $rr' \neq 0$. Hence, $rr'(a + b) = 0$. So, $a + b \in Z(R)$. Consequently, $Z(R)$ is an ideal of R.
(12) ⇒ (11) Let e be an non trivial idempotent of R. Then, $e, 1 - e \in Z(R)$. Thus, $1 = e + (1 - e) \in Z(R)$, a contradiction. Thus, R is indecomposable.
(12) ⇔ (13) First, it is easy to show that $AG(R)$ is complete if and only if $AG(Q)$ is complete. Moreover, Q is a total ring of fractions, and then every proper ideal of Q is contained in $Z(Q)$. So, Q is local if and only if $Z(Q)$ is an ideal. On the other hand, $Z(R)$ is an ideal of R if and only if $Z(Q)$ is an ideal of Q. Thus, we conclude the desired equivalence. □

In [1], Adlifard and Payrovi study when $AG(R)$ is complete. The next result continues in this line of investigation.

Corollary 3.12. *Let R be a non-reduced ring. Then, the following are equivalent.*

1. *Every nonzero proper ideal of R is a strongly r-ideal.*
2. *For each nonzero nonunit element $x \in R$, Rx is a strongly r-ideal.*

3. *For each nonzero nonunit elements* $x, y \in R$, $\operatorname{ann}_R(y) \neq \operatorname{ann}_R(xy)$.
4. *For each nonzero nonunit elements,* $\operatorname{ann}_R(x) \cup \operatorname{ann}_R(y) \neq \operatorname{ann}_R(xy)$.
5. *For each nonzero nonunit elements,* $Rx \cap \operatorname{ann}_R(y) \neq (0)$.
6. *For each ideal nonzero ideal* I *of* R *and each nonzero nonunit element* $y \in R$, $I \cap \operatorname{ann}_R(y) \neq (0)$.
7. *For each ideal* $(0) \neq I \subseteq Z(R)$ *and each nonzero nonunit element* $y \in R$, $I \cap \operatorname{ann}_R(y) \neq (0)$.
8. R *is local with maximal ideal* $Z(R)$ *and* $AG(R)$ *is complete.*

Proof. The implications (1) ⇒ (2) and (5) ⇒ (6) ⇒ (7) are clear.
(2) ⇒ (3) Let x be a nonzero nonunit element of R. Since Rx is a strongly r-ideal, we have $x \in Rx \subseteq Z(R) = Z(Rx)$. Thus, every nonzero nonunit element of R is a zero-divisor. Thus, the desired result follows from the implication (4) ⇒ (5) of Theorem 3.11.
The proof of the implications (3) ⇒ (4) ⇒ (5) is similar to the proof of (5) ⇒ (6) ⇒ (7) in Theorem 3.11.
(7) ⇒ (8) Since $\operatorname{ann}_R(y) \neq (0)$ for each nonzero nonunit element of R, every nonzero nonunit element of R is a zero-divisor. Thus, R is a total quotient ring. Hence, $R = Q(R)$. Thus, the desired result follows from (9) ⇒ (13) of Theorem 3.11.
(8) ⇒ (1) Once again, it is clear that R is a total quotient ring, and so $R = Q(R)$. Thus, the present implication is just (13) ⇒ (1) of Theorem 3.11. □

Recall from [9] that a ring R is said to be an *UN*-ring if every nonunit element a of R is a product of a unit and a nilpotent elements. Following [17, Proposition 2.25], R is a *UN*-ring if and only if every element of R is either nilpotent or unit if and only if $\operatorname{nil}(R)$ is a maximal ideal of R. A simple example of *UN*-rings is $\mathbb{Z}/9\mathbb{Z}$.

Proposition 3.13. *Let* R *be a ring and set* $Q = Q(R)$.

1. *If* $Z(R) = \operatorname{nil}(R)$ *then, every nonzero proper ideal of* Q *is a strongly r-ideal of* Q.
2. *If* R *is a UN-ring then, every nonzero proper ideal of* R *is strongly r-ideal of* R.

Proof. (1) If R is reduced then $Z(R) = (0)$, and so R is a domain and Q is a field. In this case, the desired result follows trivially. Now, assume that R is non-reduced. Using Theorem 3.11, we have to show that $\operatorname{ann}_R(a) \neq \operatorname{ann}_R(ab)$ for each $a, b \in Z(R)^*$. Let $a, b \in Z(R)^*$. If $ab = 0$ then $\operatorname{ann}_R(a) \neq R = \operatorname{ann}_R(ab)$. Hence, assume that $ab \neq 0$. Let n be the smallest integer such $ab^n = 0$. Such integer exists since b is nilpotent, and we have $n > 1$. Then, $b^{n-1} \in \operatorname{ann}_R(ab) \backslash \operatorname{ann}_R(a)$. So, we have the desired result.
(2) Since R is a *UN*-ring, $Z(R) = \operatorname{nil}(R)$ is a maximal ideal. Thus, $R = Q$. So, the desired result follows from (1). □

Theorem 3.14. *Let* R *be a ring with the ascending chain condition on annihilator ideals (in particular if* R *is Noetherian) and set* $Q = Q(R)$. *Then, the following are equivalent.*

1. *Every nonzero proper ideal of* Q *is a strongly r-ideal of* Q.
2. $Z(R) = \operatorname{nil}(R)$.

Proof. (1) ⇒ (2) Suppose that R is reduced. Let $x \in Z(R)^*$. By Theorem 3.11, we have $Z(xR) = Z(R)$. Thus, there exists $r \in R$ such that $rx \neq 0$ and $rx^2 = 0$. Thus, $(rx)^2 = 0$, and so $rx = 0$, a contradiction. Hence, $Z(R) = (0)$. So, $Z(R) = \operatorname{nil}(R) = (0)$, as desired. Assume now that R is not reduced. Let $a \in Z(R)^*$. Consider the increasing sequence of annihilator ideals:

$$\operatorname{ann}_R(a) \subseteq \operatorname{ann}_R(a^2) \subseteq \cdots \subseteq \operatorname{ann}_R(a^n) \subseteq \cdots$$

There exists n such that $\operatorname{ann}_R(a^n) = \operatorname{ann}_R(a^{n+1})$. Hence, by Theorem 3.11, $a^n = 0$. Thus, $a \in \operatorname{nil}(R)$. So, $\operatorname{nil}(R) = Z(R)$.
(2) ⇒ (1) Follows from Proposition 3.13. □

Theorem 3.15. *Let* R *be a ring with the ascending chain condition on annihilator ideals. Then, the following are equivalent.*

1. *Every nonzero proper ideal of R is a strongly r-ideal of R.*
2. *R is a UN-ring.*

Proof. (1) ⇒ (2) If R is reduced, then as in the proof of Theorem 3.14, we can prove that every nonunit element is zero. Thus, R is a field. Thus, R is a UN-ring. Now, if R is non-reduced, by Corollary 3.12, R is local with maximal ideal $Z(R)$, and so $R = Q(R)$. Then, by Theorem 3.14, $Z(R) = \mathrm{nil}(R)$ is maximal, and so R is a UN-ring.
(2) ⇒ (1) Follows from Proposition 3.13. □

Proposition 3.16. *Let R be a ring such that every nonzero prime ideal is maximal. Then, the following are equivalent.*

1. *Every nonzero proper ideal of R is a strongly r-ideal of R.*
2. *R is a UN-ring.*

Proof. (1) ⇒ (2) If R is reduced, then as in the proof of Theorem 3.14, we can prove that every nonunit element is zero. Thus, R is a field. Thus, R is a UN-ring. So, suppose that R is not reduced. By Corollary 3.12, R is local with maximal ideal $Z(R)$. Let P be a minimal prime ideal. Since $P \neq (0)$ (otherwise, R is reduced), P is maximal, and so $P = Z(R)$. Thus, R admits a unique prime ideal which is necessarily $\mathrm{nil}(R)$. Thus, $\mathrm{nil}(R) = Z(R)$ is maximal. So, R is a UN-ring.
(2) ⇒ (1) Follows from Proposition 3.13. □

Remarks 3.17. *It is possible to have a local ring R which is not a UN-ring such that $Z(R)$ is a maximal ideal of R and $AG(R)$ is a complete graph. Let $\mathbb{Z}_{(2)}$ be the ring of integers localized at the prime ideal (2), that is $\mathbb{Z}_{(2)} = \left\{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in 2\mathbb{Z}+1\right\}$. Set $M = \mathbb{Q}/\mathbb{Z}_{(2)}$ and consider $R = \mathbb{Z}(+)M$, the trivial extension of $\mathbb{Z}$ by the $\mathbb{Z}$-module M. Then, R is a local ring with maximal ideal $Z(R) = 2\mathbb{Z}(+)M$, $\mathrm{nil}(R) = \{0\}(+)M \neq Z(R)$, and $AG(R)$ is complete (by [8, Theorem 3.24]).*

Acknowledgement

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through the project (R.G.P.1/101/43). The authors would like also to thank the referee for the careful review and the valuable comments which improved the presentation of the paper.

References

[1] M. Adlifard and Sh. Payrovi, Some remarks on the annihilator graph of a commutative ring, Hokkaido Math. J. 49(2) (2020), 325–332.
[2] A. R. Aliabad and R. Mohamadian, On sz^0-ideals in polynomial rings, Comm. Algebra 39(2) 2011, 701–717.
[3] D.F. Anderson, M. Axtell, and J. Stickles, Zero-divisor graphs in commutative rings. In: Fontana M, Kabbaj SE, Olberding B, Swanson I, editors. Commutative Algebra, Noetherian and Non-Noetherian Perspectives. New York, NY, USA: Springer-Verlag, 2010, pp. 23–45.
[4] D.F. Anderson and A. Badawi, On the zero-divisor graph of a ring, Comm. Algebra 36(8) (2008), 3073–3092.
[5] D.F. Anderson and P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217(2) (1999) 434–447.
[6] J. Arnold and R. Gilmer, On the contents of polynomials. Proc. Am. Math. Soc. 24 (1970), 556–562.
[7] F. Azarpanah, O. A. S. Karamzadeh, and A. Rezaei Aliabad, On ideals consisting entirely of zero-divisors, Comm. Algebra 28(2) (2000), 1061–1073.
[8] A. Badawi, On the annihilator graph of a commutative ring, Comm. Algebra 42 (1) (2014), 108–121.
[9] G. Calugareanu, *UN*-rings, J. Algebra Appl. 15(10) (2016), 1650182.
[10] C. Faith, Annihilator ideals, associated primes and kasch-mccoy commutative rings, Comm. Algebra 19(7) (1991), 1867–1892.
[11] J. A. Huckaba, Commutative Ring with zero-divisors, Marcel Dekker, New York, 1988.
[12] J. Huckaba and J. Keller, Annihilation of ideals in commutative rings, Pacific J. Math. 83(2) (1979), 375–379.
[13] T. Lucas, Two annihilator conditions: Property (A) and (a.c.), Comm. Algebra 14(3) (1986), 557–580.
[14] T. G. Lucas, The diameter of a zero-divisor graph, J. Algebra 301 (2006), 174–193.
[15] R. Mohamadian, *r*-ideals in commutative rings, Turkish J. Math. 39(5) (2015), 733-749.
[16] H. Mostafanasab, Zero-annihilator graphs of commutative rings, Kragujevac J. Math. 42(4) (2018), 517–525.
[17] U. Tekir, S. Koc and K. H. Oral, *n*-Ideals of commutative rings, Filomat 31(10) (2017), 2933–2941.
[18] D.B. West, Introduction to graph theory, 2nd ed, Prentice Hall, 2001.
[19] E. A. Ugurlu, Generalizations of *r*-ideals of commutative rings, Journal of Interdisciplinary Mathematics 24 (8) (2021), 2283-2293.