Filomat 37:15 (2023), 4991–5003
https://doi.org/10.2298/FIL2315991M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nodal solutions for singular semilinear elliptic systems

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Abstract. In this paper, we prove existence of nodal solutions for singular semilinear elliptic systems without variational structure where its both components are of sign changing. Our approach is based on sub-supersolutions method combined with perturbation arguments involving singular terms.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ having a smooth boundary $\partial\Omega$ and a positive measure with $|\Omega| > 1$. Consider the following system of semilinear elliptic equations

$$(\mathrm{P}) \qquad \begin{cases} -\Delta u + h_{\lambda,\phi_1}(u) = a_1(x)\frac{f_1(v)}{|u|^{\alpha_1}} & \text{in } \Omega \\ -\Delta v + h_{\lambda,\phi_1}(v) = a_2(x)\frac{f_2(u)}{|v|^{\alpha_2}} & \text{in } \Omega \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ stands for the Laplace differential operator and h_{λ,ϕ_1} is a linear function defined by

$$h_{\lambda,\phi_1}(s) := \lambda(s + \phi_1), \text{ for } s \in \mathbb{R}, \text{ for } \lambda > 0, \tag{1}$$

where ϕ_1 denotes the positive eigenfunction corresponding to the first eigenvalue λ_1. In the reaction terms (the right hand side) of problem (P), functions $a_i \in L^\infty(\Omega)$, $i = 1, 2$, satisfy

H(a) There exists a constant $1 < \rho_i < |\Omega|$ such that

$$|\Omega_{\rho_i}|, \;\; |\Omega \backslash \overline{\Omega}_{\rho_i}| \neq 0$$

and

$$\begin{cases} a_i(x) > 0 \text{ for a.a. } x \in \Omega_{\rho_i} \\ a_i(x) \leq 0 \text{ for a.e. } x \in \Omega \backslash \overline{\Omega}_{\rho_i}, \end{cases}$$

where

$$\Omega_{\rho_i} = \{x \in \Omega : d(x, \partial\Omega) < \rho_i\}, \text{ for } i = 1, 2,$$

2020 *Mathematics Subject Classification*. 35J75; 35J91; 35B99; 35J61.
Keywords. Laplacian; Singular systems; Nodal solutions; Sub-supersolutions; Truncation; Nonvariational problem.
Received: 23 August 2022; Accepted: 08 January 2023
Communicated by Marko Nedeljkov
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while f_i, $i = 1, 2$, are continuous functions satisfying the growth condition

H(f) There are constants $m_i, M_i > 0$ and $\beta_i \in (0, 1)$ such that

$$m_i \leq f_i(s) \leq M_i(1 + |s|^{\beta_i}), \text{ for all } s \in \mathbb{R}, \ i = 1, 2.$$

We consider the system (P) in a singular case assuming that

$$0 < \alpha_1, \alpha_2 < 1. \tag{2}$$

Our main goal is to provide a nodal solution (u, v) for the singular nonvariational elliptic system (P). This means u, v are both sign changing. According to our knowledge, this topic is a novelty. The virtually nonexistent works in the literature devoted to this subject is partly due to the fact that the study of the existence of nodal solutions for systems is more delicate than in the scalar case. Indeed, [14] is the only paper that has addressed this issue for nonvariational systems through topological degree argument where a different concept of nodal solutions is introduced. Precisely, in [14], the components of the latter are defined as being nontrivial and are not of the same constant sign. This type of solutions has also been studied in [9, 15] for a class of quasilinear systems with variational structure by combining variational methods with suitable truncation. However, in [9], nodal solutions are defined in more subtle way considering that either their components are of the same constant sign or at least one of them is a sign changing function. Thence, even for variational systems, the question of nodal solutions whose two components change sign remains open. Here, it should be noted that system (P) under assumptions above is not in variational form, so the variational methods are not applicable.

Another main technical difficulty in this paper consists in the presence of singularities in system (P) near the origin that occur under assumption (2). These singularities make difficult any study attendant to nodal solutions for (P) which, because of their sign change, necessarily pass through zero. It represents a serious difficulty to overcome and, as far as we know, is never handled in the literature even for singular problems in the scalar case. For more inquiries on the study of constant sign solutions for singular systems we refer to [4, 5, 7, 10–12] and the references therein.

To handle our problem, we show that the sets where u and v vanish are of zero measure. This is an essential point enabling nodal solutions investigation. Thereby, by a solution of problem (P) we mean a couple $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that the set where u (resp. v) vanishes is negligible and

$$\begin{cases} \int_\Omega (\nabla u \nabla \varphi_1 + h_{\lambda,\phi_1}(u)\varphi_1) \, dx = \int_\Omega a_1(x) \frac{f_1(v)}{|u|^{\alpha_1}} \varphi_1 \, dx \\ \int_\Omega (\nabla v \nabla \varphi_2 + h_{\lambda,\phi_1}(v)\varphi_2) \, dx = \int_\Omega a_2(x) \frac{f_2(u)}{|v|^{\alpha_2}} \varphi_2 \, dx, \end{cases} \tag{3}$$

for all $(\varphi_1, \varphi_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ provided the integrals in the right-hand side of the above identities exist.

Our approach is chiefly based on sub-supersolution method. It is applied to a disturbed system (P$_\varepsilon$) depending on parameter $\varepsilon > 0$ whose study is relevant for problem (P). The obtained solution $(u_\varepsilon, v_\varepsilon)$ of (P$_\varepsilon$) is located in the rectangle formed by sub-supersolutions. A significant feature of our result lies in the construction of the sub- and supersolution pair for (P$_\varepsilon$). Indeed, the choice of suitable functions with an adjustment of adequate constants is crucial. Specifically, exploiting spectral properties of the Laplacian operator, the supersolution $(\overline{u}, \overline{v})$, constructed explicitly, is sign-changing and independent of $\varepsilon > 0$, while the subsolution $(\underline{u}_\varepsilon, \underline{v}_\varepsilon)$ which, besides the dependence of ε, does not have an explicit form, admits a limit as $\varepsilon \to 0$ the couple $(\underline{u}, \underline{v})$ where the component $\underline{u}$ (resp. $\underline{v}$) is negative in Ω_{ρ_1} (resp. Ω_{ρ_2}) and positive in $\Omega \backslash \overline{\Omega}_{\rho_1}$ (resp. $\Omega \backslash \overline{\Omega}_{\rho_2}$). Actually, it is worth noting that $(\underline{u}_\varepsilon, \underline{v}_\varepsilon)$ is a solution of an auxiliary problem ($\tilde{\mathrm{P}}_\varepsilon$) related to (P$_\varepsilon$). Then, the general theory of sub-supersolutions for systems of quasilinear equations (see [2]) implies the existence of a solution $(u_\varepsilon, v_\varepsilon)$ of problem (P$_\varepsilon$) with the sets where u_ε and v_ε vanish are negligible. In particular, this establish that u_ε and v_ε cannot be identically zero in Ω_{ρ_1} and Ω_{ρ_2}, respectively. Then, the solution (u,v) of (P), lying in $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$, is derived by passing to the limit as $\varepsilon\to 0$. The argument is based on a priori estimates, dominated convergence Theorem as well as S_+-property of the negative Laplacian. Hence, (u,v) turns out a nodal solution of (P) and u, v are both of sign changing.

The rest of this article is organized as follows. Section 2 contains the proof of the existence of solutions for regularized system (P_ε). Section 3 presents the proof of the existence of nodal solutions of system (P).

2. The regularized system

For $\varepsilon>0$, let consider the auxiliary system

$$(\mathrm{P}_\varepsilon)\qquad \begin{cases} -\Delta u + h_{\lambda,\phi_1}(u) = a_1(x)\frac{f_1(v)}{(|u|+\varepsilon)^{\alpha_1}} & \text{in } \Omega \\ -\Delta v + h_{\lambda,\phi_1}(v) = a_2(x)\frac{f_2(u)}{(|v|+\varepsilon)^{\alpha_2}} & \text{in } \Omega \\ u,v=0 & \text{on } \partial\Omega. \end{cases}$$

Employing sub-supersolution method we shall prove that problem (P_ε) admits a nontrivial solution.

We recall that a sub-supersolution for (P_ε) is any pairs $(\underline{u},\underline{v}),(\overline{u},\overline{v})\in(H^1(\Omega)\cap L^\infty(\Omega))^2$ for which there hold $(\overline{u},\overline{v})\geq(\underline{u},\underline{v})$ a.e. in $\overline{\Omega}$,

$$\begin{cases} \int_\Omega(\nabla\underline{u}\nabla\varphi_1 + h_{\lambda,\phi_1}(\underline{u})\varphi_1)\,\mathrm{d}x - \int_\Omega a_1(x)\frac{f_1(v)}{(|\underline{u}|+\varepsilon)^{\alpha_1}}\varphi_1\,\mathrm{d}x \leq 0 \\ \int_\Omega(\nabla\underline{v}\nabla\varphi_2 + h_{\lambda,\phi_1}(\underline{v})\varphi_2)\,\mathrm{d}x - \int_\Omega a_2(x)\frac{f_2(u)}{(|\underline{v}|+\varepsilon)^{\alpha_2}}\varphi_2\,\mathrm{d}x \leq 0, \end{cases}$$

$$\begin{cases} \int_\Omega(\nabla\overline{u}\nabla\varphi_1 + h_{\lambda,\phi_1}(\overline{u})\varphi_1)\,\mathrm{d}x - \int_\Omega a_1(x)\frac{f_1(v)}{(|\overline{u}|+\varepsilon)^{\alpha_1}}\varphi_1\,\mathrm{d}x \geq 0 \\ \int_\Omega(\nabla\overline{v}\nabla\varphi_2 + h_{\lambda,\phi_1}(\overline{v})\varphi_2)\,\mathrm{d}x - \int_\Omega a_2(x)\frac{f_2(u)}{(|\overline{v}|+\varepsilon)^{\alpha_2}}\varphi_2\,\mathrm{d}x \geq 0, \end{cases}$$

for all $\varphi_1,\varphi_2\in H_0^1(\Omega)$ with $\varphi_1,\varphi_2\geq 0$ a.e. in Ω and for all $u,v\in H_0^1(\Omega)$ satisfying $\underline{u}\leq u\leq\overline{u}$ and $\underline{v}\leq v\leq\overline{v}$ a.e. in Ω.

2.1. A constant sign sub-supersolution pair

In what follows ϕ_1 denotes the positive eigenfunction corresponding to the principal eigenvalue λ_1, that is,

$$-\Delta\phi_1=\lambda_1\phi_1 \text{ in } \Omega,\ \phi_1=0 \text{ on } \partial\Omega.$$

which is well known to verify

$$l^{-1}d(x)\leq\phi_1(x)\leq ld(x) \text{ for all } x\in\Omega, \tag{4}$$

$$|\nabla\phi_1|\geq\eta \ \text{ as } d(x)\to 0, \tag{5}$$

with constants $l>1$ and $\eta>0$. Here, $d(x)$ denotes the distance from a point $x\in\overline{\Omega}$ to the boundary $\partial\Omega$ and $\overline{\Omega}=\Omega\cup\partial\Omega$ is the closure of $\Omega\subset\mathbb{R}^N$.

Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\tilde{\Omega}$ such that $\overline{\Omega}\subset\tilde{\Omega}$. Denote $\tilde{d}(x):=d(x,\partial\tilde{\Omega})$. By the definition of $\tilde{\Omega}$, there exists a constant $\mu>0$ sufficiently small such that

$$\tilde{d}(x)>\mu \text{ in } \overline{\Omega}. \tag{6}$$

Let $\tilde{e}\in C^1(\overline{\tilde{\Omega}})$ be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta\tilde{e}=1 & \text{in } \tilde{\Omega} \\ \tilde{e}=0 & \text{on } \partial\tilde{\Omega}, \end{cases} \tag{7}$$

which is known to satisfy the estimate

$$c^{-1}\tilde{d}(x)\leq\tilde{e}(x)\leq c\tilde{d}(x) \ \text{ in } \tilde{\Omega}, \tag{8}$$

for certain constant $c>1$ (see [3, Proof of Lemma 3.1]).

Theorem 2.1. *Under assumptions* H(f), H(a) *and (2), system* (P$_\varepsilon$) *possesses a solution* $(u_\varepsilon, v_\varepsilon) \in (H_0^1(\Omega) \cap L^\infty(\Omega))^2$ *within* $[-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]$, *for a large constant* $C > 1$ *independent of* $\lambda \geq 0$, *for all* $\varepsilon \in (0, 1)$.

Proof. Using (6)-(8) it follows that

$$-\Delta(C\tilde{e}) + h_{\lambda,\phi_1}(C\tilde{e}) = C + \lambda(C\tilde{e} + \phi_1) \geq C \ \text{ in } \Omega \tag{9}$$

and

$$\begin{aligned} &-\Delta(-C\tilde{e}) + h_{\lambda,\phi_1}(-C\tilde{e}) = -C + \lambda(-C\tilde{e} + \phi_1) \\ &\leq -C + \lambda(-Cc^{-1}\mu + \left\|\phi_1\right\|_\infty) \leq -C \text{ in } \Omega. \end{aligned} \tag{10}$$

for $C > 0$ large enough that does not depend on $\lambda \geq 0$. By (2), (6), (8) and since $Cc^{-1}\mu > 1$ for $C > 0$ sufficiently large, it holds

$$\begin{aligned} &\tfrac{1+|C\tilde{e}|^{\beta_1}}{(|C\tilde{e}|+\varepsilon)^{\alpha_1}} \leq \tfrac{1+|C\tilde{e}|^{\beta_1}}{|C\tilde{e}|^{\alpha_1}} = |C\tilde{e}|^{-\alpha_1} + |C\tilde{e}|^{\beta_1-\alpha_1} \leq (Cc^{-1}\tilde{d}(x))^{-\alpha_1} + |C\tilde{e}|^{\beta_1-\alpha_1} \\ &\leq (Cc^{-1}\mu)^{-\alpha_1} + (C\tilde{e})^{\beta_1-\alpha_1} \leq 1 + (C\tilde{e})^{\beta_1-\alpha_1} \leq \begin{cases} (1 + (C\,\|\tilde{e}\|_\infty)^{\beta_1-\alpha_1}) & \text{if } \alpha_1 < \beta_1 \\ (1 + (Cc^{-1}\tilde{d}(x))^{\beta_1-\alpha_1}) & \text{if } \alpha_1 \geq \beta_1 \end{cases} \\ &\leq \begin{cases} C^{\beta_1-\alpha_1}(C^{-\beta_1+\alpha_1} + \|\tilde{e}\|_\infty^{\beta_1-\alpha_1}) & \text{if } \alpha_1 < \beta_1 \\ (1 + (Cc^{-1}\mu)^{\beta_1-\alpha_1}) & \text{if } \alpha_1 \geq \beta_1 \end{cases} \leq \begin{cases} C^{\beta_1-\alpha_1}(1 + \|\tilde{e}\|_\infty^{\beta_1-\alpha_1}) & \text{if } \alpha_1 < \beta_1 \\ 2 & \text{if } \alpha_1 \geq \beta_1 \end{cases} \ \text{in } \overline{\Omega}, \end{aligned} \tag{11}$$

for all $\varepsilon \in (0, 1)$. On the other hand, on account of H(f) and H(a), we get

$$a_1(x)\tfrac{f_1(u)}{(|C\tilde{e}|+\varepsilon)^{\alpha_1}} \leq M_1\,\|a_1\|_\infty\,\tfrac{1+|v|^{\beta_1}}{(|C\tilde{e}|+\varepsilon)^{\alpha_1}} \leq M_1\,\|a_1\|_\infty\,\tfrac{1+|C\tilde{e}|^{\beta_1}}{(|C\tilde{e}|+\varepsilon)^{\alpha_1}} \ \text{ in } \overline{\Omega} \tag{12}$$

and

$$a_1(x)\tfrac{f_1(v)}{(|-C\tilde{e}|+\varepsilon)^{\alpha_i}} \geq -\|a_1\|_\infty\,\tfrac{f_1(v)}{(|C\tilde{e}|+\varepsilon)^{\alpha_1}} \geq -M_1\,\|a_1\|_\infty\,\tfrac{1+|v|^{\beta_1}}{(|C\tilde{e}|+\varepsilon)^{\alpha_1}} \geq -M_1\,\|a_1\|_\infty\,\tfrac{1+|C\tilde{e}|^{\beta_1}}{(|C\tilde{e}|+\varepsilon)^{\alpha_1}} \ \text{ in } \overline{\Omega}. \tag{13}$$

for all $(u, v) \in [-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]$ and all $\varepsilon \in (0, 1)$. Then, having in mind that $\beta_1 - \alpha_1 < 1$, gathering (9)-(13) together leads to

$$-\Delta(C\tilde{e}) + h_{\lambda,\phi_1}(C\tilde{e}) \geq a_1(x)\tfrac{f_1(v)}{(|C\tilde{e}|+\varepsilon)^{\alpha_1}} \ \text{ in } \Omega$$

and

$$-\Delta(-C\tilde{e}) + h_{\lambda,\phi_1}(-C\tilde{e}) \leq a_1(x)\tfrac{f_1(v)}{(|-C\tilde{e}|+\varepsilon)^{\alpha_1}} \ \text{ in } \Omega,$$

for all $(u, v) \in [-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]$ and all $\varepsilon \in (0, 1)$, provided that $C > 0$ is sufficiently large. A quite similar argument provides

$$-\Delta(C\tilde{e}) + h_{\lambda,\phi_1}(C\tilde{e}) \geq a_2(x)\tfrac{f_2(u)}{(|C\tilde{e}|+\varepsilon)^{\alpha_2}} \ \text{ in } \Omega$$

and

$$-\Delta(-C\tilde{e}) + h_{\lambda,\phi_1}(-C\tilde{e}) \leq a_2(x)\tfrac{f_2(u)}{(|-C\tilde{e}|+\varepsilon)^{\alpha_2}} \ \text{ in } \Omega,$$

for all $(u, v) \in [-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]$ and all $\varepsilon \in (0, 1)$, for $C > 0$ sufficiently large. This proves that $(-C\tilde{e}, -C\tilde{e})$ and $(C\tilde{e}, C\tilde{e})$ are a sub-supersolution pairs for (P$_\varepsilon$) for all $\varepsilon \in (0, 1)$.

Consequently, we may apply the general theory of sub-supersolutions for systems (see, e.g., [2, Section 5.5]) which ensures the existence of solutions $(u_\varepsilon, v_\varepsilon) \in (H_0^1(\Omega) \cap L^\infty(\Omega))^2$ of (P$_\varepsilon$) within $[-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]$, for all $\varepsilon \in (0, 1)$. The proof is completed. □

2.2. A sign-changing sub-supersolution pair

Assume in H(a) that

$$\rho_i < \frac{1}{2}\max_{\overline{\Omega}}\phi_1, \tag{14}$$

and fix $\gamma_i \in (0,1)$ such that

$$\rho_i := \gamma_i^{\frac{-1}{1-\gamma_i}}, \tag{15}$$

which is possible since $\rho_i > 1$. Here, it should be noted that it does not involve any loss of generality by assuming that $\max_{\overline{\Omega}}\phi_1 > 2$.

Setting

$$\overline{u} = \phi_1^{\gamma_1} - \gamma_1\phi_1, \;\; \overline{v} = \phi_1^{\gamma_2} - \gamma_2\phi_1, \tag{16}$$

observe that

$$\overline{u} \geq 0 \text{ (resp. } \leq 0) \text{ if } 0 \leq \phi_1 \leq \rho_1 \text{ (resp. } \phi_1 \geq \rho_1) \tag{17}$$

and

$$\overline{v} \geq 0 \text{ (resp. } \leq 0) \text{ if } 0 \leq \phi_1 \leq \rho_2 \text{ (resp. } \phi_1 \geq \rho_2). \tag{18}$$

Lemma 2.2. *Under assumptions* H(f), H(a) *and (2), there exists* $(\underline{u}_\varepsilon, \underline{v}_\varepsilon) \in (H_0^1(\Omega) \cap L^\infty(\Omega))^2$*, with* $\underline{u}_\varepsilon \leq \overline{u}$, $\underline{v}_\varepsilon \leq \overline{v}$*, such that the couples* $(\underline{u}_\varepsilon, \underline{v}_\varepsilon)$ *and* $(\overline{u}, \overline{v})$ *form a sub-supersolutions pairs to* (P_ε)*, for* $\lambda > 0$ *big enough and for all* $\varepsilon \in (0,1)$*. Moreover,*

$$\underline{u}_\varepsilon, \underline{v}_\varepsilon \neq 0 \textit{ for a.e. } x \in \Omega. \tag{19}$$

Proof. **Existence of a supersolution:**

A direct computations shows that

$$-\Delta(\phi_1^{\gamma_i}) = \gamma_i\lambda_1\phi_1^{\gamma_i} + \gamma_i(1-\gamma_i)\phi_1^{\gamma_i-2}\left|\nabla\phi_1\right|^2 \text{ in } \Omega, \text{ for } i = 1,2.$$

Hence

$$\begin{aligned} &-\Delta(\phi_1^{\gamma_i} - \gamma_i\phi_1) = \gamma_i\lambda_1\phi_1^{\gamma_i} + \gamma_i(1-\gamma_i)\phi_1^{\gamma_i-2}\left|\nabla\phi_1\right|^2 - \gamma_i\lambda_1\phi_1 \\ &= \lambda_1\gamma_i(\phi_1^{\gamma_i} - \phi_1) + \gamma_i(1-\gamma_i)\phi_1^{\gamma_i-2}\left|\nabla\phi_1\right|^2 \text{ in } \Omega, \text{ for } i = 1,2. \end{aligned} \tag{20}$$

We shall prove that $(\overline{u}, \overline{v})$ is a supersolution for problem (P_ε). To this end, set

$$\Omega_\delta := \{x \in \overline{\Omega} : d(x) < \delta\}, \text{ with a constant } \delta > 0.$$

From (16), (17) and for $\delta > 0$ small enough, we have $\overline{u} \geq 0$, $h_{\lambda,\phi_1}(\overline{u}) \geq 0$ as well as $(\phi_1^{\gamma_1} - \phi_1) \geq 0$ in Ω_δ. Thus, by (2), (4), (20), we get

$$\begin{aligned} &(|\overline{u}| + \varepsilon)^{\alpha_1}(-\Delta\overline{u} + h_{\lambda,\phi_1}(\overline{u})) \geq (|\phi_1^{\gamma_1} - \gamma_1\phi_1| + \varepsilon)^{\alpha_1}(-\Delta\overline{u}) \\ &\geq \gamma_1(1-\gamma_1)(|\phi_1^{\gamma_1} - \gamma_1\phi_1| + \varepsilon)^{\alpha_1}\phi_1^{\gamma_i-2}\left|\nabla\phi_1\right|^2 \\ &\geq \gamma_1(1-\gamma_1)(|\phi_1^{\gamma_1}(1 - \gamma_1\phi_1^{1-\gamma_1})|)^{\alpha_1}\phi_1^{\gamma_i-2}\left|\nabla\phi_1\right|^2 \\ &\geq \gamma_1(1-\gamma_1)|1 - \gamma_1(l\delta)^{1-\gamma_1}|^{\alpha_1}\phi_1^{\gamma_1(1+\alpha_1)-2}\left|\nabla\phi_1\right|^2 \text{ in } \Omega_\delta. \end{aligned}$$

Since $0 < \gamma_1 < 1$ we have $\gamma_1 < \frac{2}{1+\alpha_1}$ and so $\gamma_1(1+\alpha_1) - 2 < 0$ for every $\alpha_1 \in (0,1)$. Fix $C > 0$ such that the conclusion of Theorem 2.1 holds true. By (4), (5) and H(f), we infer that

$$\begin{aligned}
&(|\overline{u}| + \varepsilon)^{\alpha_1}(-\Delta\overline{u} + h_{\lambda,\phi_1}(\overline{u}))\\
&\geq \gamma_1 (1-\gamma_1)\, |1-\gamma_1 (l\delta)^{1-\gamma_1}|^{\alpha_1} (ld(x))^{\gamma_1(1+\alpha_1)-2}\eta^2\\
&\geq \gamma_1 (1-\gamma_1)\, |1-\gamma_1 \delta^{1-\gamma_1}|^{\alpha_1} (l\delta)^{\gamma_1(1+\alpha_1)-2}\eta^2\\
&\geq \|a_1\|_\infty\, (1 + (C\,\|\tilde{e}\|_\infty)^{\beta_1}) \geq \|a_1\|_\infty\, f_1(v)\\
&\geq a_1(x) f_1(v) \ \text{ in } \Omega_\delta,
\end{aligned}$$

for all $v \in [-C\tilde{e}, C\tilde{e}]$, for all $\varepsilon \in (0,1)$, provided $\delta > 0$ is sufficiently small. This shows that

$$-\Delta\overline{u} + h_{\lambda,\phi_1}(\overline{u}) \geq a_1(x)\frac{f_1(v)}{(|\overline{u}|+\varepsilon)^{\alpha_1}} \text{ in } \Omega_\delta, \tag{21}$$

for all $v \in [-C\tilde{e}, C\tilde{e}]$, for all $\varepsilon \in (0,1)$.

Next, we examine the case when $x \in \Omega_{\rho_1}\backslash\overline{\Omega}_\delta$. From (1), (20) and (16), we have

$$\begin{aligned}
&-\Delta\overline{u} + h_{\lambda,\phi_1}(\overline{u}) \geq \gamma_1\lambda_1(\phi_1^{\gamma_1} - \phi_1) + \lambda(\phi_1^{\gamma_1} + (1-\gamma_1)\phi_1)\\
&= (\gamma_1\lambda_1 + \lambda)\phi_1^{\gamma_1} + (\lambda(1-\gamma_1) - \gamma_1\lambda_1)\phi_1) \text{ in } \Omega.
\end{aligned} \tag{22}$$

On account of H(f), (4), (22), (2) and recalling that

$$|\phi_1^{\gamma_1} - \gamma_1\phi_1|^{\alpha_1} > 0 \ \text{ in } \Omega_{\rho_1}\backslash\overline{\Omega}_\delta,$$

we get

$$\begin{aligned}
&(|\overline{u}| + \varepsilon)^{\alpha_1}(-\Delta\overline{u} + h_{\lambda,\phi_1}(\overline{u}))\\
&\geq (|\phi_1^{\gamma_1} - \gamma_1\phi_1| + \varepsilon)^{\alpha_1}[(\gamma_1\lambda_1 + \lambda)\phi_1^{\gamma_1} + (\lambda(1-\gamma_1) - \gamma_1\lambda_1)\phi_1)]\\
&\geq (\gamma_1\lambda_1 + \lambda)|\phi_1^{\gamma_1} - \gamma_1\phi_1|^{\alpha_1}\phi_1^{\gamma_1} \geq \lambda|\phi_1^{\gamma_1} - \gamma_1\phi_1|^{\alpha_1}(l^{-1}\delta)^{\gamma_1}\\
&\geq \|a_1\|_\infty\, (1 + (C\,\|\tilde{e}\|_\infty)^{\beta_1}) \geq a_1(x) f_1(v) \ \text{ in } \Omega_{\rho_1}\backslash\overline{\Omega}_\delta,
\end{aligned}$$

for all $v \in [-C\tilde{e}, C\tilde{e}]$, provided $\lambda > 0$ big enough. Hence, it turns out that

$$-\Delta\overline{u} + \lambda\overline{u} \geq a_1(x)\frac{f_1(v)}{(|\overline{u}|+\varepsilon)^{\alpha_1}} \text{ in } \Omega_{\rho_1}\backslash\overline{\Omega}_\delta, \tag{23}$$

for all $v \in [-C\tilde{e}, C\tilde{e}]$, for all $\varepsilon \in (0,1)$.

It remains to prove that the estimate holds true in $\Omega\backslash\overline{\Omega}_{\rho_1}$. Recall from H($a$) that $a_1(x)$ is nonpositive outside $\overline{\Omega}_{\rho_1}$. Then, by (22), H($f$) and for $\lambda > 0$ large, it follows that

$$\begin{aligned}
&(|\overline{u}| + \varepsilon)^{\alpha_1}(-\Delta\overline{u} + h_{\lambda,\phi_1}(\overline{u}))\\
&\geq |\overline{u}|^{\alpha_1}[(\gamma_1\lambda_1 + \lambda)\phi_1^{\gamma_1} + (\lambda(1-\gamma_1) - \gamma_1\lambda_1)\phi_1)]\\
&\geq 0 \geq a_1(x) f_1(v) \ \text{ in } \Omega\backslash\overline{\Omega}_{\rho_1},
\end{aligned}$$

for all $v \in [-C\tilde{e}, C\tilde{e}]$ and all $\varepsilon \in (0,1)$. Thus, it turns out that

$$-\Delta\overline{u} + \lambda\overline{u} \geq a_1(x)\frac{f_1(v)}{(|\overline{u}|+\varepsilon)^{\alpha_1}} \text{ in } \Omega\backslash\overline{\Omega}_{\rho_1}. \tag{24}$$

Gathering together (21), (23) and (24) we deduce that

$$-\Delta\overline{u} + h_{\lambda,\phi_1}(\overline{u}) \geq a_1(x)\frac{f_1(v)}{(|\overline{u}|+\varepsilon)^{\alpha_1}} \text{ in } \Omega, \tag{25}$$

for all v within $[-C\tilde{e}, \overline{v}]$, for all $\varepsilon \in (0,1)$. In the same manner we infer that

$$-\Delta\overline{v} + h_{\lambda,\phi_1}(\overline{v}) \geq a_2(x)\frac{f_2(u)}{(|\overline{v}|+\varepsilon)^{\alpha_2}} \ \text{ in } \Omega, \tag{26}$$

for all u within $[-C\tilde{e}, \overline{u}]$, for all $\varepsilon \in (0,1)$. Consequently, on the basis of (25) and (26) we conclude that $(\overline{u}, \overline{v})$ is a supersolution of (P_ε).

Existence of subsolution:

In what follows, for any $s \in \mathbb{R}$, denote by $s_+ := \max\{s, 0\}$ and $s_- := \max\{-s, 0\}$. Define the truncation

$$\chi_{\phi_1}(x,s) = \frac{1}{\left\|\phi_1\right\|_\infty} \begin{cases} \phi_1(x) & \text{if } s \geq 2\phi_1(x) \\ s - \phi_1(x) & \text{if } \phi_1(x) \leq s \leq 2\phi_1(x) \\ 0 & \text{if } s \leq \phi_1(x) \end{cases}, \text{ for all } x \in \overline{\Omega}, \tag{27}$$

and consider the problem

$$(\tilde{\mathrm{P}}_\varepsilon) \qquad \begin{cases} -\Delta u + h_{\lambda,\phi_1}(u) = \mathcal{F}_{1,\varepsilon}(x,u,v) & \text{in } \Omega \\ -\Delta v + h_{\lambda,\phi_1}(v) = \mathcal{F}_{2,\varepsilon}(x,u,v) & \text{in } \Omega \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}$$

for $\varepsilon \in (0,1)$, where

$$\mathcal{F}_{1,\varepsilon}(x,u,v) = \begin{cases} a_{1,+}(x)\chi_{\phi_1}(x,u_+)\frac{f_1(v)}{(|\overline{u}|+1)^{\alpha_1}} & \text{in } \Omega_{\rho_1} \\ -a_{1,-}(x)\frac{1+|\overline{v}|^{\beta_1}}{(|u|+\varepsilon)^{\alpha_1}} & \text{in } \Omega\backslash\overline{\Omega}_{\rho_1} \end{cases} \tag{28}$$

and

$$\mathcal{F}_{2,\varepsilon}(x,u,v) = \begin{cases} a_{2,+}(x)\chi_{\phi_1}(x,v_+)\frac{f_2(u)}{(|\overline{v}|+1)^{\alpha_2}} & \text{in } \Omega_{\rho_2} \\ -a_{2,-}(x)\frac{1+|\overline{u}|^{\beta_1}}{(|v|+\varepsilon)^{\alpha_2}} & \text{in } \Omega\backslash\overline{\Omega}_{\rho_2}. \end{cases} \tag{29}$$

On the basis of H(a) and H(f), it is a simple matter to see that

$$\mathcal{F}_{1,\varepsilon}(x,u,v) \leq a_1(x)\tfrac{f_1(v)}{(|u|+\varepsilon)^{\alpha_1}} \text{ and } \mathcal{F}_{2,\varepsilon}(x,u,v) \leq a_2(x)\tfrac{f_2(v)}{(|v|+\varepsilon)^{\alpha_2}}, \tag{30}$$

for all $(u,v) \in [-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]$ and all $\varepsilon \in (0,1)$. Then, any solution of $(\tilde{\mathrm{P}}_\varepsilon)$ within $[-C\tilde{e}, C\tilde{e}] \times [-C\tilde{e}, C\tilde{e}]$ is a subsolution of (P_ε).

We claim that $(-C\tilde{e}, -C\tilde{e})$ is a subsolution of $(\tilde{\mathrm{P}}_\varepsilon)$. Indeed, from (10), (6), (8), H(a) and H(f), increasing $C > 0$ if necessary, we have

$$-\Delta(-C\tilde{e}) + h_{\lambda,\phi_1}(-C\tilde{e}) \leq 0 \leq \mathcal{F}_{1,\varepsilon}(x, -C\tilde{e}, v) \text{ in } \Omega_{\rho_1}$$

and

$$\begin{aligned} &-\Delta(-C\tilde{e}) + h_{\lambda,\phi_1}(-C\tilde{e}) \leq -C \\ &\leq -C^{-\alpha_1}\left\|a_{1,-}\right\|_\infty \tfrac{1+\left\|\overline{v}\right\|_\infty^{\beta_1}}{(c^{-1}\mu)^{\alpha_1}} \\ &\leq -a_{1,-}(x)\tfrac{1+|\overline{v}|^{\beta_1}}{(|-C\tilde{e}|+\varepsilon)^{\alpha_1}} = \mathcal{F}_{1,\varepsilon}(x, -C\tilde{e}, v) \text{ in } \Omega\backslash\overline{\Omega}_{\rho_1}, \end{aligned}$$

for all $v \in [-C\tilde{e}, \overline{v}]$, and all $\varepsilon \in (0,1)$. Similarly, one derives that

$$-\Delta(-C\tilde{e}) + h_{\lambda,\phi_1}(-C\tilde{e}) \leq \mathcal{F}_{2,\varepsilon}(x, u, -C\tilde{e}) \text{ in } \overline{\Omega},$$

for all $u \in [-C\tilde{e}, \overline{u}]$ and all $\varepsilon \in (0,1)$. This proves the claim.

On the basis of (25), (26) and (30), $(\overline{u}, \overline{v})$ in (16) is a supersolution of problem $(\tilde{\mathrm{P}}_\varepsilon)$. Consequently, owing to [2, section 5.5], problem $(\tilde{\mathrm{P}}_\varepsilon)$ admits a solution $(\underline{u}_\varepsilon, \underline{v}_\varepsilon)$ within $[-Ce, \overline{u}] \times [-Ce, \overline{v}]$. Moreover, according to (30), $(\underline{u}_\varepsilon, \underline{v}_\varepsilon)$ is a subsolution of (P_ε) for all $\varepsilon \in (0,1)$.

The subsolution $(\underline{u}_\varepsilon, \underline{v}_\varepsilon)$ is nontrivial:

In $\Omega \backslash \overline{\Omega}_{\rho_i}$ $(i=1,2)$, the conclusion is immediate because $\underline{u}_\varepsilon \leq \overline{u} < 0$ in $\Omega \backslash \overline{\Omega}_{\rho_1}$ and $\underline{v}_\varepsilon \leq \overline{v} < 0$ in $\Omega \backslash \overline{\Omega}_{\rho_2}$. Let us show that the sets where $\underline{u}_\varepsilon$ and $\underline{v}_\varepsilon$ vanish in Ω_{ρ_1} and Ω_{ρ_2}, respectively, are negligible. To this end, set

$$\Gamma_1 := \{x \in \Omega_{\rho_1} : \underline{u}_\varepsilon(x) = 0\} \text{ and } \Gamma_2 := \{x \in \Omega_{\rho_2} : \underline{v}_\varepsilon(x) = 0\}.$$

By a classical result of measure theory (see [16, Theorem 3.28]), Γ_1 and Γ_2 are $\mathcal{G}_\gamma$-sets for less than a zero measure set. Thus, one can write

$$\Gamma_i = \mathcal{A}_i - \mathcal{B}_i, \tag{31}$$

where $\mathcal{A}_i - \mathcal{B}_i$ is the relative complement of $\mathcal{A}_i$ in $\mathcal{B}_i$, $|\mathcal{B}_i| = 0$ and $\mathcal{A}_i$ is a $\mathcal{G}_\gamma$-set, that is

$$\mathcal{A}_i = \bigcap_k \mathcal{G}_{i,k}, \quad \mathcal{G}_{i,k} \text{ is open}, \ i = 1,2.$$

Without loss of generality, we may assume that $\mathcal{B}_i$ is not dense in Ω_{ρ_i} $(i=1,2)$. Otherwise, by density and according to [1, Proposition 2.4], the terms $\mathcal{F}_{1,\varepsilon}(x, \underline{u}_\varepsilon, \underline{v}_\varepsilon) - h_{\lambda,\phi_1}(\underline{u}_\varepsilon)$ and $\mathcal{F}_{2,\varepsilon}(x, \underline{u}_\varepsilon, \underline{v}_\varepsilon) - h_{\lambda,\phi_1}(\underline{v}_\varepsilon)$ should vanish in Ω_{ρ_1} and Ω_{ρ_2}, respectively, which is absurd in view of H(f), H(a) and the definition of χ_{ϕ_1} in (27) as well as the fact that $\lambda\phi_1 > 0$ in Ω.

Let $\tilde{\varphi}_i \in C_0^\infty(\Omega)$ such that

$$\tilde{\varphi}_i > 0 \text{ in } \mathcal{A}_i \text{ and } \tilde{\varphi}_i = 0 \text{ in } \Omega \backslash \mathcal{A}_i, \ i = 1,2.$$

Testing with $\tilde{\varphi}_i \in C_0^\infty(\Omega)$ in $(\tilde{\mathrm{P}}_\varepsilon)$ we get

$$\begin{aligned} \textstyle\int_{\mathcal{A}_1} (\nabla \underline{u}_\varepsilon \nabla \tilde{\varphi}_1 + h_{\lambda,\phi_1}(\underline{u}_\varepsilon)\tilde{\varphi}_1) \, \mathrm{d}x &= \textstyle\int_{\mathcal{A}_1} a_{1,+}(x) \chi_{\phi_1}(x, \underline{u}_{\varepsilon,+}) \frac{f_1(\underline{v}_\varepsilon)}{(|\overline{u}|+1)^{\alpha_1}} \tilde{\varphi}_1 \, \mathrm{d}x, \\ \textstyle\int_{\mathcal{A}_2} (\nabla \underline{v}_\varepsilon \nabla \tilde{\varphi}_2 + h_{\lambda,\phi_1}(\underline{v}_\varepsilon)\tilde{\varphi}_2) \, \mathrm{d}x &= \textstyle\int_{\mathcal{A}_2} a_{2,+}(x) \chi_{\phi_1}(x, \underline{v}_{\varepsilon,+}) \frac{f_2(\underline{u}_\varepsilon)}{(|\overline{v}|+1)^{\alpha_2}} \tilde{\varphi}_2 \, \mathrm{d}x. \end{aligned}$$

In view of (31) we infer that

$$\begin{aligned} \textstyle\int_{\Gamma_1} (\nabla \underline{u}_\varepsilon \nabla \tilde{\varphi}_1 + h_{\lambda,\phi_1}(\underline{u}_\varepsilon)\tilde{\varphi}_1) \, \mathrm{d}x &= \textstyle\int_{\Gamma_1} a_{1,+}(x) \chi_{\phi_1}(x, \underline{u}_{\varepsilon,+}) \frac{f_1(\underline{v}_\varepsilon)}{(|\overline{u}|+1)^{\alpha_1}} \tilde{\varphi}_1 \, \mathrm{d}x, \\ \textstyle\int_{\Gamma_2} (\nabla \underline{v}_\varepsilon \nabla \tilde{\varphi}_2 + h_{\lambda,\phi_1}(\underline{v}_\varepsilon)\tilde{\varphi}_2) \, \mathrm{d}x &= \textstyle\int_{\Gamma_2} a_{2,+}(x) \chi_{\phi_1}(x, \underline{v}_{\varepsilon,+}) \frac{f_2(\underline{u}_\varepsilon)}{(|\overline{v}|+1)^{\alpha_2}} \tilde{\varphi}_2 \, \mathrm{d}x. \end{aligned} \tag{32}$$

Since $\underline{u}_\varepsilon, \underline{v}_\varepsilon \in W^{1,1}_{loc}(\Omega)$ it follows, by [6, Lemma 7.7], that $\nabla \underline{u}_\varepsilon = 0$ on Γ_1 and $\nabla \underline{v}_\varepsilon = 0$ on Γ_2. Replacing in (32) we get

$$\int_{\Gamma_1} \lambda \phi_1 \tilde{\varphi}_1 \, \mathrm{d}x = \int_{\Gamma_1} a_{1,+}(x) \chi_{\phi_1}(x, \underline{u}_{\varepsilon,+}) \frac{f_1(\underline{v}_\varepsilon)}{(|\overline{u}|+1)^{\alpha_1}} \tilde{\varphi}_1 \, \mathrm{d}x$$

and

$$\int_{\Gamma_2} \lambda \phi_1 \tilde{\varphi}_2 \, \mathrm{d}x = \int_{\Gamma_2} a_{2,+}(x) \chi_{\phi_1}(x, \underline{v}_{\varepsilon,+}) \frac{f_2(\underline{u}_\varepsilon)}{(|\overline{v}|+1)^{\alpha_2}} \tilde{\varphi}_2 \, \mathrm{d}x.$$

According to the definition of the function χ_{ϕ_1} in (27) it follows that

$$\chi_{\phi_1}(x, \underline{u}_{\varepsilon,+}) = 0 \text{ in } \Gamma_1 \text{ and } \chi_{\phi_1}(x, \underline{v}_{\varepsilon,+}) = 0 \text{ in } \Gamma_2$$

and thus,

$$\int_{\Gamma_1} \lambda \phi_1 \tilde{\varphi}_1 \, \mathrm{d}x = \int_{\Gamma_2} \lambda \phi_1 \tilde{\varphi}_2 \, \mathrm{d}x = 0.$$

Hence, $|\Gamma_1|, |\Gamma_2| = 0$, showing that (19) holds true. $\square$

Remark 2.3. *A careful inspection of the proof of Lemma 2.2 shows that for a fixed $C > 0$ in Theorem 2.1, constants $\delta := \delta(C)$ and $\lambda := \lambda(C, \delta)$ can be precisely estimated.*

Lemma 2.4. *Under the same assumptions as in Lemma 2.2, there exist functions $\underline{u}, \underline{v} \in H_0^1(\Omega)$ such that*

$$\underline{u}_\varepsilon \to \underline{u} \text{ and } \underline{v}_\varepsilon \to \underline{v} \text{ in } H_0^1(\Omega) \text{ as } \varepsilon \to 0, \tag{33}$$

with

$$\underline{u} < 0 \text{ in } \Omega \backslash \overline{\Omega}_{\rho_1}, \quad \underline{u} > 0 \text{ in } \Omega_{\rho_1} \tag{34}$$

and

$$\underline{v} < 0 \text{ in } \Omega \backslash \overline{\Omega}_{\rho_2}, \quad \underline{v} > 0 \text{ in } \Omega_{\rho_2}, \tag{35}$$

Proof. Set $\varepsilon = \frac{1}{n}$ with any positive integer $n > 1$. From Lemma 2.2, there exist $\underline{u}_n := \underline{u}_{\frac{1}{n}}$ and $\underline{v}_n := \underline{v}_{\frac{1}{n}}$ such that

$$\begin{cases} \langle -\Delta \underline{u}_n + h_{\lambda,\phi_1}(\underline{u}_n), \varphi_1 \rangle = \int_\Omega \mathcal{F}_{1,n}(x, \underline{u}_n, \underline{v}_n)\varphi_1 \, \mathrm{d}x \\ \langle -\Delta \underline{v}_n + h_{\lambda,\phi_1}(\underline{v}_n), \varphi_2 \rangle = \int_\Omega \mathcal{F}_{2,n}(x, \underline{u}_n, \underline{v}_n)\varphi_2 \, \mathrm{d}x \end{cases} \tag{36}$$

for all $\varphi_1, \varphi_2 \in H_0^1(\Omega)$ with

$$-C\tilde{e} \leq \underline{u}_n \leq \overline{u} \leq C\tilde{e}, \ -C\tilde{e} \leq \underline{v}_n \leq \overline{v} \leq C\tilde{e} \text{ in } \Omega. \tag{37}$$

Acting with $\varphi_1 = \underline{u}_n$ in (36), by (28), H(a) and (2), bearing in mind (37), it follows that

$$\begin{aligned} &\int_\Omega (|\nabla \underline{u}_n|^2 + h_{\lambda,\phi_1}(\underline{u}_n)\underline{u}_n) \, \mathrm{d}x \\ &= \int_\Omega (|\nabla \underline{u}_n|^2 + \lambda(|\underline{u}_n|^2 + \phi_1 \underline{u}_n)) \, \mathrm{d}x \\ &= \int_{\Omega_{\rho_1}} a_{1,+}(x)\chi_{\phi_1}(x, \underline{u}_{n,+}) \tfrac{f_1(\underline{v}_n)}{(|\overline{u}|+1)^{\alpha_1}} \underline{u}_n \, \mathrm{d}x - \int_{\Omega \backslash \overline{\Omega}_{\rho_1}} a_{1,-}(x) \tfrac{1+|\overline{v}|^{\beta_1}}{(|\underline{u}_n|+\varepsilon)^{\alpha_1}} \underline{u}_n \, \mathrm{d}x \\ &\leq \int_{\Omega_{\rho_1}} \|a_1\|_\infty (1 + (C\tilde{e})^{\beta_1}) \underline{u}_n \, \mathrm{d}x - \int_{\Omega \backslash \overline{\Omega}_{\rho_1}} a_{1,-}(x) \tfrac{1}{(C\tilde{e}+1)^{\alpha_1}} \underline{u}_n \, \mathrm{d}x \\ &\leq |\Omega| \, \|a_1\|_\infty \left(1 + (C \|\tilde{e}\|_\infty)^{\beta_1}\right) C \|\tilde{e}\|_\infty < \infty. \end{aligned}$$

This shows that $\{\underline{u}_n\}$ is bounded in $H_0^1(\Omega)$. Similarly, we derive that $\{\underline{v}_n\}$ is bounded in $H_0^1(\Omega)$. We are thus allowed to extract subsequences (still denoted by $\{\underline{u}_n\}$ and $\{\underline{v}_n\}$) such that

$$\underline{u}_n \rightharpoonup \underline{u} \text{ and } \underline{v}_n \rightharpoonup \underline{v} \text{ in } H_0^1(\Omega)$$

and

$$-C\tilde{e} \leq \underline{u} \leq \overline{u} \leq C\tilde{e}, \ -C\tilde{e} \leq \underline{v} \leq \overline{v} \leq C\tilde{e} \text{ in } \Omega. \tag{38}$$

Inserting $(\varphi_1, \varphi_2) = (\underline{u}_n - \underline{u}, \underline{v}_n - \underline{v})$ in (36) yields

$$\begin{aligned} \langle -\Delta \underline{u}_n + h_{\lambda,\phi_1}(\underline{u}_n), \underline{u}_n - \underline{u} \rangle &= \int_\Omega \mathcal{F}_{1,n}(x, \underline{u}_n, \underline{v}_n)(\underline{u}_n - \underline{u}) \, \mathrm{d}x, \\ \langle -\Delta \underline{v}_n + h_{\lambda,\phi_1}(\underline{v}_n), \underline{v}_n - \underline{v} \rangle &= \int_\Omega \mathcal{F}_{2,n}(x, \underline{u}_n, \underline{v}_n)(\underline{v}_n - \underline{v}) \, \mathrm{d}x. \end{aligned}$$

By (27), (37), H(f), H(a) and (2), we have

$$|a_{1,+}(x)\chi_{\phi_1}(x, \underline{u}_{n,+}) \tfrac{f_1(\underline{v}_n)}{(|\overline{u}|+1)^{\alpha_1}} (\underline{u}_n - \underline{u})| \leq 2 \|a_1\|_\infty (1 + C \|\tilde{e}\|_\infty^{\beta_1}) C \|\tilde{e}\|_\infty$$

as well as

$$\begin{aligned} &|a_{1,-}(x) \tfrac{1+|\overline{v}|^{\beta_1}}{(|\underline{u}_n|+\varepsilon)^{\alpha_1}} (\underline{u}_n - \underline{u})| \leq \|a_1\|_\infty |\underline{u}_n|^{1-\alpha_1} (1 + |\overline{v}|^{\beta_1}) \\ &\leq \|a_1\|_\infty (C \|\tilde{e}\|_\infty)^{1-\alpha_1} (1 + (C \|\tilde{e}\|_\infty)^{\beta_1}). \end{aligned}$$

Then, applying Fatou's Lemma, it follows that

$$\limsup_{n\to\infty} \int_\Omega \mathcal{F}_{1,n}(x, \underline{u}_n, \underline{v}_n)(\underline{u}_n - \underline{u})\, dx$$
$$\leq \int_\Omega \limsup_{n\to\infty} (\mathcal{F}_{1,n}(x, \underline{u}_n, \underline{v}_n)(\underline{u}_n - \underline{u}))\, dx \to 0 \text{ as } n \to +\infty,$$

showing that $\lim\limits_{n\to\infty} \left\langle -\Delta \underline{u}_n, \underline{u}_n - \underline{u} \right\rangle \leq 0$. Then, the S_+-property of $-\Delta$ on $H_0^1(\Omega)$ guarantees that $\underline{u}_n \to \underline{u}$ in $H_0^1(\Omega)$. Similarly, we prove that $\underline{v}_n \to \underline{v}$ in $H_0^1(\Omega)$.

Let us verify that

$$\lim_{n\to\infty} \int_\Omega \mathcal{F}_{i,n}(x, \underline{u}_n, \underline{v}_n)\varphi_i \, dx = \int_\Omega \mathcal{F}_{i,0}(x, \underline{u}, \underline{v})\varphi_i \, dx \tag{39}$$

for all $\varphi_i \in H_0^1(\Omega)$, where $\mathcal{F}_{i,0}(x, \underline{u}, \underline{v}) := \mathcal{F}_{i,\varepsilon}(x, \underline{u}, \underline{v})$ for $\varepsilon = 0$ in (28) and (29), $i = 1, 2$. From H(f), H(a), (2), (28) and (29), it holds

$$|\mathcal{F}_{1,n}(x, \underline{u}_n, \underline{v}_n)\varphi_1| \leq \|a_1\|_\infty \frac{1 + \|\overline{v}\|_\infty^{\beta_1}}{|\underline{u}_n|^{\alpha_1}} |\varphi_1|$$

and

$$|\mathcal{F}_{2,n}(x, \underline{u}_n, \underline{v}_n)\varphi_2| \leq \|a_2\|_\infty \frac{1 + \|\overline{u}\|_\infty^{\beta_2}}{|\underline{v}_n|^{\alpha_2}} |\varphi_2|,$$

for all $\varphi_1, \varphi_2 \in H_0^1(\Omega)$. Since, by (19), $\underline{u}_n, \underline{v}_n \neq 0$ for a.e. $x \in \Omega$, the assertion (39) stems from Lebesgue's dominated convergence Theorem. Hence, we may pass to the limit in (36) to conclude that $(\underline{u}, \underline{v})$ is a solution of problem $(\tilde{\mathrm{P}}_0)$ (That is $(\tilde{\mathrm{P}}_\varepsilon)$ with $\varepsilon = 0$).

It remains to prove (34) and (35). Since $\underline{u} \leq \overline{u}$ (resp. $\underline{v} \leq \overline{v}$) and $\overline{u}$ (resp. $\overline{v}$) is negative in $\Omega \backslash \overline{\Omega}_{\rho_1}$ (resp. $\Omega \backslash \overline{\Omega}_{\rho_2}$), it turns out that

$$\underline{u} < 0 \text{ in } \Omega \backslash \overline{\Omega}_{\rho_1} \text{ and } \underline{v} < 0 \text{ in } \Omega \backslash \overline{\Omega}_{\rho_2}. \tag{40}$$

Moreover, according to the definition of $\mathcal{F}_{1,0}$ and $\mathcal{F}_{2,0}$ in (28)-(29), H(f), (27), (38) and in view of (40), we can find a constant $L_0 > 0$, independent of $\underline{u}$ and $\underline{v}$, such that

$$\max\{|\mathcal{F}_{1,0}(x, \underline{u}, \underline{v})|, |\mathcal{F}_{2,0}(x, \underline{u}, \underline{v})|\} < L_0, \text{ for a.e. } x \in \overline{\Omega}.$$

Then, bearing in mind the boundedness of $(\underline{u}, \underline{v})$ (see (38)), the regularity theory up to the boundary in [8] implies that $(\underline{u}, \underline{v}) \in C^{1,\sigma}(\overline{\Omega})^2$ for certain $\sigma \in (0, 1)$.

Let $\hat{\Omega}_{\rho_1} \subset \Omega_{\rho_1}$ and $\hat{\Omega}_{\rho_2} \subset \Omega_{\rho_2}$ be nodal domains of $\underline{u}_-$ and $\underline{v}_-$, respectively (see [13, Definition 1.60]). Then, $\mathbb{1}_{\Omega_{\rho_1}} \underline{u}_-, \mathbb{1}_{\Omega_{\rho_2}} \underline{v}_- \in H_0^1(\Omega)$ (see [13, Proposition 1.61]), where $\mathbb{1}_{\hat{\Omega}_{\rho_i}}$ denotes the characteristic function of the subset $\hat{\Omega}_{\rho_i}$, for $i = 1, 2$. Acting on $(\tilde{\mathrm{P}}_0)$ with $(\mathbb{1}_{\hat{\Omega}_{\rho_1}} \underline{u}_-, \mathbb{1}_{\hat{\Omega}_{\rho_2}} \underline{v}_-) \in H_0^1(\Omega) \times H_0^1(\Omega)$ we obtain

$$\int_\Omega (\nabla \underline{u} \nabla (\mathbb{1}_{\hat{\Omega}_{\rho_1}} \underline{u}_-) + h_{\lambda,\phi_1}(\underline{u}) \mathbb{1}_{\hat{\Omega}_{\rho_1}} \underline{u}_-)\, dx = \int_\Omega \mathcal{F}_{1,0}(x, \underline{u}, \underline{v})(\mathbb{1}_{\hat{\Omega}_{\rho_1}} \underline{u}_-)\, dx$$

and

$$\int_\Omega (\nabla \underline{v} \nabla (\mathbb{1}_{\hat{\Omega}_{\rho_2}} \underline{v}_-) + h_{\lambda,\phi_1}(\underline{v}) \mathbb{1}_{\hat{\Omega}_{\rho_2}} \underline{v}_-)\, dx = \int_\Omega \mathcal{F}_{2,0}(x, \underline{u}, \underline{v})(\mathbb{1}_{\hat{\Omega}_{\rho_2}} \underline{v}_-)\, dx.$$

By the definition of functions h_{λ,ϕ_1} in (1) and χ_{ϕ_1} in (27), it follows that

$$\int_\Omega (|\nabla (\mathbb{1}_{\hat{\Omega}_{\rho_1}} \underline{u}_-)|^2 + \lambda (|\mathbb{1}_{\hat{\Omega}_{\rho_1}} \underline{u}_-|^2 + \phi_1 \mathbb{1}_{\hat{\Omega}_{\rho_1}} \underline{u}_-))\, dx$$
$$= \int_{\hat{\Omega}_{\rho_1}} a_{1,+}(x) \chi_{\phi_1}(x, \underline{u}_+) \tfrac{f_1(\underline{v})}{(|\overline{u}|+1)^{\alpha_1}} \underline{u}_- \, dx = 0$$

and

$$\begin{aligned}&\int_\Omega (|\nabla(\mathbb{1}_{\hat{\Omega}_{\rho_2}}\underline{v}_-)|^2+\lambda(|\mathbb{1}_{\hat{\Omega}_{\rho_2}}\underline{v}_-|^2+\phi_1\mathbb{1}_{\hat{\Omega}_{\rho_2}}\underline{v}_-)))\,\mathrm{d}x\\&=\int_{\hat{\Omega}_{\rho_2}} a_{2,+}(x)\chi_{\phi_1}(x,\underline{v}_+)\tfrac{f_2(u)}{(|\overline{v}|+1)^{\alpha_2}}\underline{v}_-\,\mathrm{d}x=0.\end{aligned}$$

Consequently, we deduce that $\underline{u}\geq 0$ in Ω_{ρ_1} and $\underline{v}\geq 0$ in Ω_{ρ_2}.

Finally, by exactly a similar argument which proves (19) in Lemma 2.2, we infer that $\underline{u}>0$ in Ω_{ρ_1} and $\underline{v}>0$ in Ω_{ρ_2}. This completes the proof. □

3. Existence of nodal solutions

Our main result is formulated as follows.

Theorem 3.1. *Under assumptions* H(f), H(a) *and (2), problem* (P) *possesses a nodal solution* $(\breve{u},\breve{v})\in(H_0^1(\Omega)\cap L^\infty(\Omega))^2$*, for* $\lambda>0$ *large.*

Proof. Set $\varepsilon=\frac{1}{n}$ with any positive integer $n>1$. According to Lemma 2.2 and thanks to [2, Section 5.5], there exists $(u_n,v_n):=(u_{\frac{1}{n}},v_{\frac{1}{n}})\in(H_0^1(\Omega)\cap L^\infty(\Omega))^2$ such that

$$\begin{cases}\langle -\Delta u_n+h_{\lambda,\phi_1}(u_n),\varphi_1\rangle=\int_\Omega a_1(x)\frac{f_1(v_n)}{(|u_n|+\frac{1}{n})^{\alpha_1}}\varphi_1\,\mathrm{d}x\\ \langle -\Delta v_n+h_{\lambda,\phi_1}(v_n),\varphi_2\rangle=\int_\Omega a_2(x)\frac{f_2(u_n)}{(|v_n|+\frac{1}{n})^{\alpha_2}}\varphi_2\,\mathrm{d}x\end{cases}\tag{41}$$

for all $\varphi_1,\varphi_2\in H_0^1(\Omega)$ and

$$\underline{u}_n\leq u_n\leq\overline{u},\ \underline{v}_n\leq v_n\leq\overline{v},\ \forall n.\tag{42}$$

We claim that

$$u_n,v_n\neq 0\ \text{ for a.e. } x\in\Omega.\tag{43}$$

Indeed, in $\Omega\backslash\overline{\Omega}_{\rho_i}$ $(i=1,2)$, the conclusion is immediate because $u_n\leq\overline{u}<0$ in $\Omega\backslash\overline{\Omega}_{\rho_1}$ and $v_n\leq\overline{v}<0$ in $\Omega\backslash\overline{\Omega}_{\rho_2}$. Define the measurable set

$$\Gamma_n:=\{x\in\Omega_{\rho_1}:u_n(x)=0\},\ \forall n.$$

Testing with $\tilde{\varphi}\in C_0^\infty(\Omega)$ in (41) such that

$$\tilde{\varphi}>0 \text{ in } \mathcal{A}_n\ \text{ and }\ \tilde{\varphi}=0 \text{ in } \Omega\backslash\mathcal{A}_n,$$

where $\mathcal{A}_n$ is a $\mathcal{G}_\gamma$-set, analysis similar to that in the proof of Lemma 2.2 shows that

$$\int_{\Gamma_n}(\nabla u_n\nabla\tilde{\varphi}+h_{\lambda,\phi_1}(u_n)\tilde{\varphi})\,\mathrm{d}x=\int_{\Gamma_n}a_1(x)\frac{f_1(v_n)}{(|u_n|+\frac{1}{n})^{\alpha_1}}\tilde{\varphi}\,\mathrm{d}x.$$

Since $\nabla u_n=0$ on Γ_n (see [6, Lemma 7.7]) it follows that

$$\int_{\Gamma_n}\lambda\phi_1\tilde{\varphi}\,\mathrm{d}x=n^{\alpha_1}\int_{\Gamma_n}a_1(x)f_1(v_n)\tilde{\varphi}\,dx,\ \forall n\geq 1,$$

which, by H(f), H(a), forces that $|\Gamma_n|=0$, $\forall n\geq 1$. The same conclusion can be drawn for the set

$$\breve{\Gamma}_n:=\{x\in\Omega_{\rho_2}:v_n(x)=0\},$$

showing that $|\breve{\Gamma}_n| = 0$, $\forall n \geq 1$. Consequently, u_n and v_n cannot be identically zero in non negligible measurable sets. This proves the claim.

Acting with $\varphi_1 = u_n$ in (41), by H(f), H(a) and since

$$|u_n|, |v_n| \leq C\tilde{e} \ \text{ in } \Omega, \tag{44}$$

we get

$$\begin{aligned}
&\int_\Omega (|\nabla u_n|^2 + h_{\lambda,\phi_1}(u_n)u_n)\, dx \\
&\leq \int_\Omega \|a_1\|_\infty \tfrac{1+|v_n|^{\beta_1}}{(|u_n|+\frac{1}{n})^{\alpha_1}} u_n\, dx \\
&\leq \|a_1\|_\infty \int_\Omega |u_n|^{1-\alpha_1}(1 + |v_n|^{\beta_1})\, dx \\
&\leq \|a_1\|_\infty \int_\Omega (C\tilde{e})^{1-\alpha_1}(1 + (C\tilde{e})^{\beta_1})\, dx < \infty,
\end{aligned} \tag{45}$$

showing that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Similarly, we prove that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. We are thus allowed to extract subsequences (still denoted by $\{u_n\}$ and $\{v_n\}$) such that

$$u_n \rightharpoonup \hat{u} \ \text{ and } v_n \rightharpoonup \hat{v} \text{ in } H_0^1(\Omega) \tag{46}$$

Moreover, on account of (42), (46) and Lemma 2.4 (see (33)), one has

$$\underline{u} \leq \hat{u} \leq \overline{u}, \ \ \underline{v} \leq \hat{v} \leq \overline{v} \ \text{ in } \Omega. \tag{47}$$

Inserting $(\varphi_1, \varphi_2) = (u_n - \hat{u}, v_n - \hat{v})$ in (41) yields

$$\begin{cases}
\langle -\Delta u_n + h_{\lambda,\phi_1}(u_n), u_n - \hat{u}\rangle = \int_\Omega a_1(x) \frac{f_1(v_n)}{(|u_n|+\frac{1}{n})^{\alpha_1}}(u_n - \hat{u})\, \mathrm{d}x \\
\langle -\Delta v_n + h_{\lambda,\phi_1}(v_n), v_n - \hat{v}\rangle = \int_\Omega a_2(x) \frac{f_2(u_n)}{(|v_n|+\frac{1}{n})^{\alpha_2}}(v_n - \hat{v})\, \mathrm{d}x
\end{cases}$$

By (44), H(f), H(a) and (2), we have

$$|a_1(x) \tfrac{f_1(v_n)}{(|u_n|+\frac{1}{n})^{\alpha_1}}(u_n - \hat{u})| \leq \|a_1\|_\infty (C\|\tilde{e}\|_\infty)^{1-\alpha_1}(1 + C\|\tilde{e}\|_\infty^{\beta_1})$$

and

$$|a_2(x) \tfrac{f_2(u_n)}{(|v_n|+\frac{1}{n})^{\alpha_2}}(v_n - \hat{v})| \leq \|a_1\|_\infty (C\|\tilde{e}\|_\infty)^{1-\alpha_2}(1 + (C\|\tilde{e}\|_\infty)^{\beta_2}).$$

Then, Fatou's Lemma gives

$$\lim_{n\to\infty} \langle -\Delta u_n, u_n - \hat{u}\rangle \leq 0 \ \text{ and } \ \lim_{n\to\infty} \langle -\Delta v_n, v_n - \hat{v}\rangle \leq 0,$$

while the S_+-property of $-\Delta$ on $H_0^1(\Omega)$ ensures that

$$u_n \to \hat{u} \ \text{ and } v_n \to \hat{v} \text{ in } H_0^1(\Omega). \tag{48}$$

On the other hand, by H(f), H(a), (2) and (44), it holds

$$|a_1(x) \tfrac{f_1(v_n)}{(|u_n|+\frac{1}{n})^{\alpha_1}} \varphi_1| \leq \|a_1\|_\infty \tfrac{1+(C\|\tilde{e}\|_\infty)^{\beta_1}}{|u_n|^{\alpha_1}} |\varphi_1| \tag{49}$$

and

$$|a_2(x) \tfrac{f_2(u_n)}{(|v_n|+\frac{1}{n})^{\alpha_2}} \varphi_2| \leq \|a_2\|_\infty \tfrac{1+(C\|\tilde{e}\|_\infty)^{\beta_2}}{|v_n|^{\alpha_2}} |\varphi_2|, \tag{50}$$

for all $\varphi_1, \varphi_2 \in H_0^1(\Omega)$.

Then, on the basis of (43), Lebesgue's dominated convergence Theorem entails

$$\lim_{n\to\infty}\int_\Omega a_1(x)\frac{f_1(v_n)}{(|u_n|+\frac{1}{n})^{\alpha_1}}\varphi_1\,\mathrm{d}x = \int_\Omega a_1(x)\frac{f_1(\hat{v})}{|\hat{u}|^{\alpha_1}}\varphi_1\,\mathrm{d}x,$$
$$\lim_{n\to\infty}\int_\Omega a_2(x)\frac{f_2(u_n)}{(|v_n|+\frac{1}{n})^{\alpha_2}}\varphi_2\,\mathrm{d}x = \int_\Omega a_2(x)\frac{f_2(\hat{u})}{|\hat{v}|^{\alpha_2}}\varphi_2\,\mathrm{d}x,$$

for all $\varphi_1, \varphi_2 \in H_0^1(\Omega)$. Hence, we may pass to the limit in (41) to conclude that $(\hat{u}, \hat{v})$ is a solution of problem (P) within $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$.

Finally, on account of (14)-(18) together with Lemma 2.4, we infer that $\hat{u}$ and $\hat{v}$ are both of sign-changing and satisfy

$$\hat{u} \le \overline{u} < 0 \text{ in } \Omega\backslash\overline{\Omega}_{\rho_1} \text{ and } \hat{u} \ge \underline{u} > 0 \text{ in } \Omega_{\rho_1},$$

$$\hat{v} \le \overline{v} < 0 \text{ in } \Omega\backslash\overline{\Omega}_{\rho_2} \text{ and } \hat{v} \ge \underline{v} > 0 \text{ in } \Omega_{\rho_2}.$$

This completes the proof. □

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