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On the multi-parameterized inequalities involving the tempered fractional integral operators

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Abstract. In virtue of the conception of the tempered fractional integrals, put forward by Sabzikar et al. in the published article [J. Comput. Phys., 293: 14–28, 2015], we present a fractional integral identity together with multi-parameter. Based on it, we develop certain parameterized integral inequalities in association with differentiable mappings. Furthermore, we give two examples to verify the correctness of the derived findings.

1. Introduction

The topic of the integral inequalities taking advantage of function convexity has been considered as an emerging subject in the last decades. The researchers are attempting to discover new generalizations and extensions of convex functions, and as a consequence new outcomes are being enriching to the theory of inequalities. Recently, a large number of researchers have committed themselves to exploring the properties as well as inequalities in accordance with convexity in various orientations, please see the published articles [1, 6, 10, 25, 26, 40] as well as the bibliographies quoted therein. Among them, the Hermite–Hadamard's (HH) integral inequality, one of the most distinguished integral inequalities giving thought to convexity, is employed widely in plenty of other disciplines of applied mathematics. Let us retrospect it as below:

Suppose that $\rho : \mathcal{U} \subseteq \mathbb{R} \to \mathbb{R}$, defined on the real-valued interval $\mathcal{U}$, is a convex mapping, where $\tau_1, \tau_2 \in \mathcal{U}$ with $\tau_1 < \tau_2$. The successive inequality is named as the HH integral inequality:

$$\rho\Big(\frac{\tau_1+\tau_2}{2}\Big) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\rho(h)\mathrm{d}h \leq \frac{\rho(\tau_1)+\rho(\tau_2)}{2}. \tag{1}$$

Due to the fact that the inequality (1) plays a critical role in convex analysis, it obtained numerous researchers' attention. There have been a mass of discussions in relation with the HH-type inequalities for other families of convex mappings. For instance, one could refer to Ref. [29] for convex mappings, to Ref. [38] for s-convex mappings, to Ref. [21] for h-convex mappings, to Ref. [42] for h-preinvex mappings, to Ref. [51] for generalized harmonically convex mappings, to Ref. [2] for N-quasiconvex mappings, to [52]

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for s-type preinvex convex mappings, and to Ref. [14] for exponential-type convex functions and so on. For more findings in connection with such types of inequalities, the reader may consult Refs. [3, 17, 20, 30, 39, 41] and the bibliographies quoted therein.

In order to meet the need of the later exploration, let us look back some indispensable definitions as below.

Definition 1.1. *[53] Considering the mapping* $\eta : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$*, if for each* $\tau_1, \tau_2 \in \mathcal{U}$ *as well as* $\xi \in [0,1]$, $\tau_1 + \xi\eta(\tau_2, \tau_1) \in \mathcal{U}$*, the set* $\mathcal{U} \subseteq \mathbb{R}^n$ *is named as an invex set in relation with the mapping* η*.*

Definition 1.2. *[53] Given that* $\mathcal{U} \subseteq \mathbb{R}^n$ *is an invex set in association with* $\eta : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$*. The mapping* ρ *defined on the invex set* $\mathcal{U} \subseteq \mathbb{R}^n$ *is called to be a preinvex mapping regarding the mapping* η*, if for each* $\tau_1, \tau_2 \in \mathcal{U}$ *as well as* $\xi \in [0,1]$ *one gains that*

$$\rho\left(\tau_1 + \xi\eta(\tau_2, \tau_1)\right) \leq (1-\xi)\rho(\tau_1) + \xi\rho(\tau_2).$$

If we consider taking the mapping $\eta(\tau_2, \tau_1) = \tau_2 - \tau_1$, then the preinvex mapping transfers to the classically convex mapping.

Definition 1.3. *[24] The set* $\mathcal{U} \subseteq \mathbb{R}^n$ *is called* w*-invex with regard to the mapping* $\eta : \mathcal{U} \times \mathcal{U} \times (0,1] \to \mathbb{R}^n$ *regarding certain fixed* $w \in (0,1]$ *if* $w\tau_1 + \xi\eta(\tau_2, \tau_1, w) \in \mathcal{U}$ *holds true for every* $\tau_1, \tau_2 \in \mathcal{U}$ *as well as* $\xi \in [0,1]$*.*

Definition 1.4. *[56] Assume that* $\mathcal{U} \subseteq \mathbb{R}^n$ *is an* w*-invex set pertaining to* $\eta : \mathcal{U} \times \mathcal{U} \times (0,1] \to \mathbb{R}^n$*. For each* $\tau_1, \tau_2 \in \mathcal{U}$ *as well as* $w \in (0,1]$*, the* η_w*-path* $P_{\tau_1\tau_3}$ *linking the points* $w\tau_1$ *with* $\tau_3 = w\tau_1 + \eta(\tau_2, \tau_1, w)$ *is defined as*

$$P_{\tau_1\tau_3} = \left\{\theta | \theta = w\tau_1 + \xi\eta(\tau_2, \tau_1, w), \xi \in [0,1]\right\}.$$

Definition 1.5. *[43] For any real numbers* $\alpha > 0$ *and* $\lambda, \mu \geq 0$*, the* μ*-incomplete Gamma function is defined by:*

$$\gamma_\mu(\alpha, \lambda) = \int_0^\lambda h^{\alpha-1} e^{-\mu h} \mathrm{d}h.$$

Apparently, if we consider taking $\mu = 1$, then the μ-incomplete Gamma function transfers to the incomplete Gamma function [19]:

$$\gamma(\alpha, \lambda) = \int_0^\lambda h^{\alpha-1} e^{-h} \mathrm{d}h, \alpha > 0.$$

Definition 1.6. *[18] It is assumed that the mapping* $\eta : \mathcal{U} \times \mathcal{U} \times (0,1] \to \mathbb{R}$*, where* $\mathcal{U} \subseteq \mathbb{R}$ *be an open* w*-invex subset with some fixed* $w \in (0,1]$*. For any real numbers* $\alpha > 0$ *along with* $\lambda, \mu \geq 0$*, the* (μ, η)*-incomplete Gamma function is defined by:*

$$\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda) = \int_0^\lambda h^{\alpha-1} e^{-\mu\eta(z,u,w)h} \mathrm{d}h.$$

Definition 1.7. *[36] Given that* $[u, z]$ *is a real-valued interval and* $u \geq 0, \alpha > 0$*. For a mapping* $\rho \in L^1([u, z])$*, the left- as well as right-sided Riemann–Liouville (RL) fractional integrals, correspondingly are defined as:*

$$\mathcal{J}_{u^+}^{\alpha}\rho(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-h)^{\alpha-1} \rho(h) \mathrm{d}h, \ x > u,$$

and

$$\mathcal{J}_{z^-}^{\alpha}\rho(x) = \frac{1}{\Gamma(\alpha)} \int_x^z (h-x)^{\alpha-1} \rho(h) \mathrm{d}h, \ x < z,$$

in which the Gamma function is defined as $\Gamma(\alpha) = \int_0^\infty e^{-h} h^{\alpha-1} \mathrm{d}h$*, as well as* $\mathcal{J}_{u^+}^{0}\rho(x) = \mathcal{J}_{z^-}^{0}\rho(x) = \rho(x)$*.*

By virtue of the RL-fractional integrals above, Sarikaya et al. generalized and extended the classical HH integral inequality to the form of fractional integrals as below.

Theorem 1.8. *[49] Suppose that $\rho : [u, z] \to \mathbb{R}$ is a mapping together with $u < z$ as well as $\rho \in L^1([u, z])$. If the mapping ρ is convex defined on $[u, z]$ and $\alpha > 0$, then one acquires the undermentioned fractional integral inequalities*

$$\rho\left(\frac{u+z}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(z-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\rho(z) + \mathcal{J}_{z^-}^{\alpha}\rho(u)\right] \leq \frac{\rho(u)+\rho(z)}{2}. \tag{2}$$

As a quite powerful approach, the fractional calculus is widely acknowledged to be a crucial cornerstone of mathematics and applied science. Many academics have paid attention to a series of researches by virtue of fractional calculus. For example, taking advantage of conformable fractional integrals, Khan et al. [35] deduced the left parts of the HH-type integral inequalities. And they applied the derived outcomes to special means as well as midpoint formula as applications. İşcan et al. [31] inferred certain HH-type as well as Bullen-type integral inequalities with relation to Lipschitzian mappings in terms of conformable fractional integrals. The outcomes acquired in the work extended and generalized the original study. And in the sense of RL-type fractional integrals, Set et al. [50] and Nasir et al. [44] developed the Simpson-type integral inequalities. They illustrated the relationship between the outcomes obtained in their study and previous findings in special cases. In the sense of multiplicative differentiable functions, Ali et al. [4] obtained two identities to achieve Ostrowski- and Simpson-type multiplicative integral inequalities, and they gave detail applications of their main outcomes. Baleanu et al. [8] explored the approximations on trapezoidal type integral inequalities regarding the classical integrals as well as the RL-fractional integral operators. And they constructed inequalities involving moments in correlation with a continuous variable by applying the obtained inequalities. Also by means of the AB-fractional integral operators, Butt et al. [16] investigated integral inequalities of various Hadamard types with the help of a general integral identity. For more interesting findings with relation to the fractional integrals by different approaches, we recommend the published literatures [7, 9, 13, 15, 22, 27, 32] for reference. Also, we mentioned some papers devoted to parameterized inequalities, please see the published papers [11, 12, 33, 34, 55] and the bibliographies quoted therein.

Recently, Sabzikar et al. [48] introduced the notion of the tempered fractional integrals.

Definition 1.9. *Assume that $[u, z]$ is a real-valued interval and $\mu \geq 0, \alpha > 0$. For a mapping $\rho \in L^1([u, z])$, the left- as well as right-sided tempered fractional integral operators, respectively, are defined as:*

$$\mathcal{J}_{u^+}^{(\alpha,\mu)}\rho(x) = \frac{1}{\Gamma(\alpha)}\int_u^x (x-h)^{\alpha-1}e^{-\mu(x-h)}\rho(h)\mathrm{d}h,\ x > u,$$

and

$$\mathcal{J}_{z^-}^{(\alpha,\mu)}\rho(x) = \frac{1}{\Gamma(\alpha)}\int_x^z (h-x)^{\alpha-1}e^{-\mu(h-x)}\rho(h)\mathrm{d}h,\ x < z.$$

If one attempts to take $\mu = 0$, then the tempered fractional integrals turns to the RL-fractional integrals.

For recent relevant development pertaining to the tempered fractional integrals, please see the published articles [28, 43, 46] and the literatures cited therein.

Enlightened by the above-mentioned works, particularly the outcomes displayed in the papers [18, 43], the current paper is mainly committed to investigating some HH-type integral inequalities in association with the established multi-parameterized integral identity, which involves with the tempered fractional integral operators. Given this goal, we analyze the coming two cases: (i) the derivative of the disquisitive function is generalized (s, w)-type preinvex; (ii) the derivative of the disquisitive function is bounded. The obtained findings here can be transfered to the RL-fractional integral inequalities for $\mu = 0$ and the Riemann integral inequalities for $\alpha = 1$ along with $\mu = 0$.

2. Main Results

Throughout this study, we assume that $\mathcal{U} \subseteq \mathbb{R}$ is an open w-invex set regarding $\eta : \mathcal{U} \times \mathcal{U} \times (0,1] \to \mathbb{R}$ for certain fixed $w \in (0,1]$, $u, z \in \mathcal{U}$ together with $wu < wu + \eta(z,u,w)$, as well as $\rho : \mathcal{U} \to \mathbb{R}$ is differentiable satisfying that ρ' is integrable on η_w-path $P_{\tau_1\tau_3} : \theta = w\tau_1 + \lambda\eta(\tau_2, \tau_1, w)$ for any $\tau_1, \tau_2 \in [u, z]$.

2.1. A new definition and a lemma

Now, we come up with the conception of the generalized (s,w)-type preinvex mappings, which is the extension of preinvex functions, s-type convex functions as well as convex functions.

Definition 2.1. *It is assumed that $\mathcal{U} \subseteq \mathbb{R}^n$ is an open w-invex set with regard to $\eta : \mathcal{U} \times \mathcal{U} \times (0,1] \to \mathbb{R}^n$. The mapping $\rho : \mathcal{U} \to \mathbb{R}$ is named as the generalized (s,w)-type preinvex functions if the subsequent inequality*

$$\rho\Big(w\tau_1 + \xi\eta(\tau_2, \tau_1, w)\Big) \le w(1 - s\xi)\rho(\tau_1) + [1 - s(1 - \xi)]\rho(\tau_2)$$

is available for all $\tau_1, \tau_2 \in \mathcal{U}$, certain fixed $s, w \in (0,1]$ and $\xi \in [0,1]$.

Remark 2.2. *Certain special cases concerning with Def. 2.1 are stated below.*
(i) If the mapping $\eta(\tau_2, \tau_1, w)$ degenerates to $\eta(\tau_2, \tau_1)$ along with $w = 1$, then we attain the conception of the s-type preinvex functions explored by Tariq et al. in [52]. Furthermore, if we take $s = 1$, then we attain the preinvex functions given by Weir and Mond in [53].
(ii) If we consider taking $\eta(\tau_2, \tau_1, w) = \tau_2 - \tau_1 w$ together with $w = 1$, then we attain the conception of the s-type convex functions presented by Rashid et al. in [47]. Furthermore, if we consider taking $s = 1$, then we have the conception of the convex mappings.

We next give subsequent lemma, which is of importance in deducing the parameterized integral inequalities.

Lemma 2.3. *The undermentioned tempered fractional integral equality along with $0 \le \lambda, \theta \le 1, \alpha > 0, \mu \ge 0$ holds true:*

$$\begin{aligned}
&\mathcal{T}_\rho(\alpha, \mu, \lambda, \theta; \eta, w) \\
&= \eta(z,u,w)\Big[\int_0^\lambda \Big(\theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha, \lambda - t)\Big)\rho'\Big(wu + t\eta(z,u,w)\Big)\mathrm{d}t \\
&\quad + \int_{1-\lambda}^1 \Big(\gamma_{\mu\eta(z,u,w)}(\alpha, t + \lambda - 1) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda)\Big)\rho'\Big(wu + t\eta(z,u,w)\Big)\mathrm{d}t\Big],
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
&\mathcal{T}_\rho(\alpha, \mu, \lambda, \theta; \eta, w) \\
&:= \gamma_{\mu\eta(z,u,w)}(\alpha, \lambda)\Big[\theta\rho\Big(wu + \lambda\eta(z,u,w)\Big) + (1-\theta)\rho(wu)\Big] \\
&\quad + \gamma_{\mu\eta(z,u,w)}(\alpha, \lambda)\Big[\theta\rho\Big(wu + (1-\lambda)\eta(z,u,w)\Big) + (1-\theta)\rho\Big(wu + \eta(z,u,w)\Big)\Big] \\
&\quad - \frac{\Gamma(\alpha)}{\eta^\alpha(z,u,w)}\Big[\mathcal{J}_{wu^+}^{(\alpha,\mu)}\rho\Big(wu + \lambda\eta(z,u,w)\Big) + \mathcal{J}_{(wu+\eta(z,u,w))^-}^{(\alpha,\mu)}\rho\Big(wu + (1-\lambda)\eta(z,u,w)\Big)\Big].
\end{aligned}$$

Proof. Integrating by part and changing variable of definite integral yield that

$$
\begin{aligned}
I_1 &= \int_0^\lambda \Big(\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big)\rho'\Big(wu + t\eta(z,u,w)\Big)\mathrm{d}t \\
&= \frac{1}{\eta(z,u,w)}\Big[\Big(\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big)\rho\Big(wu + t\eta(z,u,w)\Big)\Big|_0^\lambda \\
&\quad - \int_0^\lambda (\lambda-t)^{\alpha-1}e^{-\mu\eta(z,u,w)(\lambda-t)}\rho\Big(wu + t\eta(z,u,w)\Big)\mathrm{d}t\Big] \\
&= \frac{1}{\eta(z,u,w)}\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big[\theta\rho\Big(wu + \lambda\eta(z,u,w)) + (1-\theta)\rho(wu)\Big] \\
&\quad - \frac{1}{\eta^{\alpha+1}(z,u,w)}\int_{wu}^{wu+\lambda\eta(z,u,w)}\Big(wu + \lambda\eta(z,u,w) - v\Big)^{\alpha-1}e^{-\mu\left(wu+\lambda\eta(z,u,w)-v\right)}\rho(u)\mathrm{d}v \\
&= \frac{1}{\eta(z,u,w)}\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big((1-\theta)\rho(wu) + \theta\rho\Big(wu + \lambda\eta(z,u,w)\Big)\Big) \\
&\quad - \frac{\Gamma(\alpha)}{\eta^{\alpha+1}(z,u,w)}\mathcal{J}_{wu^+}^{(\alpha,\mu)}\rho\Big(wu + \lambda\eta(z,u,w)\Big).
\end{aligned}
$$

Similarly, we obtain that

$$
\begin{aligned}
I_2 &= \int_{1-\lambda}^1 \Big(\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big)\rho'\Big(wu + t\eta(z,u,w)\Big)\mathrm{d}t \\
&= \frac{1}{\eta(z,u,w)}\Big[\Big(\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big)\rho\Big(wu + t\eta(z,u,w)\Big)\Big|_{1-\lambda}^1 \\
&\quad - \int_{1-\lambda}^1 (t+\lambda-1)^{\alpha-1}e^{-\mu\eta(z,u,w)(t+\lambda-1)}\rho\Big(wu + t\eta(z,u,w)\Big)\mathrm{d}t\Big] \\
&= \frac{1}{\eta(z,u,w)}\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big[\theta\rho\Big(wu + (1-\lambda)\eta(z,u,w)\Big) + (1-\theta)\rho\Big(wu + \eta(z,u,w)\Big)\Big] \\
&\quad - \frac{1}{\eta^{\alpha+1}(z,u,w)}\int_{wu+(1-\lambda)\eta(z,u,w)}^{wu+\eta(z,u,w)}\Big[v - \Big(wu + (1-\lambda)\,\eta(z,u,w)\Big)\Big]^{\alpha-1}e^{-\mu\left(v-\left(wu+(1-\lambda)\eta(z,u,w)\right)\right)}\rho(u)\mathrm{d}v \\
&= \frac{1}{\eta(z,u,w)}\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big[\theta\rho\Big(wu + (1-\lambda)\eta(z,u,w)\Big) + (1-\theta)\rho\Big(wu + \eta(z,u,w)\Big)\Big] \\
&\quad - \frac{\Gamma(\alpha)}{\eta^{\alpha+1}(z,u,w)}\mathcal{J}_{(wu+\eta(z,u,w))^-}^{(\alpha,\mu)}\rho\Big(wu + (1-\lambda)\,\eta(z,u,w)\Big).
\end{aligned}
$$

Adding $I_1 + I_2$ and then multiplying the resulting by the factor $\eta(z,u,w)$, we can acquire the desired equality. This accomplishes the proof.

Corollary 2.4. *If one attempts to take $\mu = 0$ and $\eta(z,u,w) = z - wu$ together with $w = 1$ in Lemma 2.3, then the equality* (3) *transfers to*

$$
\begin{aligned}
&\lambda^\alpha\Big[\theta\rho\Big((1-\lambda)u + \lambda z\Big) + (1-\theta)\rho(u)\Big] + \lambda^\alpha\Big[\theta\rho\Big(\lambda u + (1-\lambda)z\Big) + (1-\theta)\rho(z)\Big] \\
&\quad - \frac{\Gamma(\alpha+1)}{(z-u)^\alpha}\Big[\mathcal{J}_{u^+}^{\alpha}\rho\Big((1-\lambda)u + \lambda z\Big) + \mathcal{J}_{z^-}^{\alpha}\rho\Big(\lambda u + (1-\lambda)z\Big)\Big] \\
&= (z-u)\Big[\int_0^\lambda \Big(\theta\lambda^\alpha - (\lambda-t)^\alpha\Big)\rho'\Big((1-t)u + tz\Big)\mathrm{d}t + \int_{1-\lambda}^1 \Big((t+\lambda-1)^\alpha - \theta\lambda^\alpha\Big)\rho'\Big((1-t)u + tz\Big)\mathrm{d}t\Big].
\end{aligned}
$$

Remark 2.5. *From Corollary 2.4, one could see that the next identities are correct clearly:*

(1) If we consider taking $\lambda = 1$ as well as $\theta = 1$, then we gain that

$$\frac{\rho(u)+\rho(z)}{2} - \frac{\Gamma(\alpha+1)}{2(z-u)^{\alpha}}\Big[\mathcal{J}^{\alpha}_{u^+}\rho(z) + \mathcal{J}^{\alpha}_{z^-}\rho(u)\Big] = \frac{z-u}{2}\int_0^1 [t^{\alpha} - (1-t)^{\alpha}]\rho'\Big((1-t)u+tz\Big)\mathrm{d}t,$$

which is given by Sarikaya et al. in [49]. In particular, if we consider taking $\alpha = 1$, then we get that

$$\frac{\rho(u)+\rho(z)}{2} - \frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t = \frac{z-u}{2}\int_0^1 \Big(2t-1\Big)\rho'\Big((1-t)u+tz\Big)\mathrm{d}t,$$

which is demonstrated by Dragomir and Agarwal in [23].
(2) If we consider taking $\lambda = \frac{1}{2}$ as well as $\theta = 1$, then we obtain that

$$\begin{aligned}&\frac{1}{2^{\alpha-1}}\rho\Big(\frac{u+z}{2}\Big) - \frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\Big[\mathcal{J}^{\alpha}_{u^+}\rho\Big(\frac{u+z}{2}\Big) + \mathcal{J}^{\alpha}_{z^-}\rho\Big(\frac{u+z}{2}\Big)\Big]\\ &= (z-u)\Big\{\int_0^{\frac{1}{2}}\Big[\Big(\frac{1}{2}\Big)^{\alpha} - \Big(\frac{1}{2}-t\Big)^{\alpha}\Big]\rho'\Big((1-t)u+tz\Big)\mathrm{d}t + \int_{\frac{1}{2}}^1\Big[\Big(t-\frac{1}{2}\Big)^{\alpha} - \Big(\frac{1}{2}\Big)^{\alpha}\Big]\rho'\Big((1-t)u+tz\Big)\mathrm{d}t\Big\}.\end{aligned}$$

Particularly, if we attempt to take $\alpha = 1$, then we get that

$$\rho\Big(\frac{u+z}{2}\Big) - \frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t = (z-u)\Big[\int_0^{\frac{1}{2}} t\rho'\Big((1-t)u+tz\Big)\mathrm{d}t + \int_{\frac{1}{2}}^1 (t-1)\rho'\Big((1-t)u+tz\Big)\mathrm{d}t\Big],$$

which is given by Kirmaci et al. in [37].
(3) If we consider taking $\lambda = \frac{1}{2}$ as well as $\theta = \frac{1}{2}$, then we achieve that

$$\begin{aligned}&\frac{1}{2^{\alpha}}\Big[\frac{\rho(u)+\rho(z)}{2} + \rho\Big(\frac{u+z}{2}\Big)\Big] - \frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\Big[\mathcal{J}^{\alpha}_{u^+}\rho\Big(\frac{u+z}{2}\Big) + \mathcal{J}^{\alpha}_{z^-}\rho\Big(\frac{u+z}{2}\Big)\Big]\\ &= (z-u)\Big\{\int_0^{\frac{1}{2}}\Big[\frac{1}{2}\Big(\frac{1}{2}\Big)^{\alpha} - \Big(\frac{1}{2}-t\Big)^{\alpha}\Big]\rho'\Big((1-t)u+tz\Big)\mathrm{d}t + \int_{\frac{1}{2}}^1\Big[\Big(t-\frac{1}{2}\Big)^{\alpha} - \frac{1}{2}\Big(\frac{1}{2}\Big)^{\alpha}\Big]\rho'\Big((1-t)u+tz\Big)\mathrm{d}t\Big\}.\end{aligned}$$

Particularly, if we consider taking $\alpha = 1$, then we get that

$$\begin{aligned}&\frac{1}{2}\Big[\frac{\rho(u)+\rho(z)}{2} + \rho\Big(\frac{u+z}{2}\Big)\Big] - \frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t\\ &= (z-u)\Big[\int_0^{\frac{1}{2}}\Big(t-\frac{1}{4}\Big)\rho'\Big((1-t)u+tz\Big)\mathrm{d}t + \int_{\frac{1}{2}}^1\Big(t-\frac{3}{4}\Big)\rho'\Big((1-t)u+tz\Big)\mathrm{d}t\Big],\end{aligned}$$

which is established by Qi et al. in [45].
(4) If we attempt to take $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we get that

$$\begin{aligned}&\frac{1}{3\cdot 2^{\alpha}}\Big[\rho(u) + 4\rho\Big(\frac{u+z}{2}\Big) + \rho(z)\Big] - \frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\Big[\mathcal{J}^{\alpha}_{u^+}\rho\Big(\frac{u+z}{2}\Big) + \mathcal{J}^{\alpha}_{z^-}\rho\Big(\frac{u+z}{2}\Big)\Big]\\ &= (z-u)\Big\{\int_0^{\frac{1}{2}}\Big[\frac{2}{3}\Big(\frac{1}{2}\Big)^{\alpha} - \Big(\frac{1}{2}-t\Big)^{\alpha}\Big]\rho'\Big((1-t)u+tz\Big)\mathrm{d}t + \int_{\frac{1}{2}}^1\Big[\Big(t-\frac{1}{2}\Big)^{\alpha} - \frac{2}{3}\Big(\frac{1}{2}\Big)^{\alpha}\Big]\rho'\Big((1-t)u+tz\Big)\mathrm{d}t\Big\}.\end{aligned}$$

In particular, if we consider taking $\alpha = 1$, then we obtain that

$$\begin{aligned}&\frac{1}{6}\Big[\rho(u) + 4\rho\Big(\frac{u+z}{2}\Big) + \rho(z)\Big] - \frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t\\ &= (z-u)\Big[\int_0^{\frac{1}{2}}\Big(t-\frac{1}{6}\Big)\rho'\Big((1-t)u+tz\Big)\mathrm{d}t + \int_{\frac{1}{2}}^1\Big(t-\frac{5}{6}\Big)\rho'\Big((1-t)u+tz\Big)\mathrm{d}t\Big],\end{aligned}$$

which is provided by Alomari et al. in [5].

2.2. $|\rho'|$ and $|\rho'|^q$ are generalized (s, w)-type preinvex

Considering the mappings whose the absolute values of the first-order derivative are the generalized (s, w)-type preinvexity, we are capable of establishing certain tempered fractional integral inequalities with regard to such class of mapping.

Theorem 2.6. *If $|\rho'|$ is a generalized (s, w)-type preinvex mapping along with certain fixed $s, w \in (0, 1]$, then the following inequality for tempered fractional integral operators with $0 < \lambda, \theta \leq 1$, $\alpha > 0$ and $\mu \geq 0$ holds:*

$$\left|\mathcal{T}_\rho(\alpha, \mu, \lambda, \theta; \eta, w)\right| \leq \eta(z, u, w)(2 - s)\Big(w|\rho'(u)| + |\rho'(z)|\Big)\delta_1, \tag{4}$$

in which

$$\delta_1 = \int_0^\lambda \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha, \lambda - t)\right| dt.$$

Proof. In terms of Lemma 2.3 as well as the generalized (s, w)-type preinvexity of $|\rho'|$, one deduces that

$$\begin{aligned}
&\left|\mathcal{T}_\rho(\alpha, \mu, \lambda, \theta; \eta, w)\right| \\
&\leq \eta(z, u, w)\bigg[\int_0^\lambda \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha, \lambda - t)\right| \left|\rho'\Big(wu + t\eta(z, u, w)\Big)\right| dt \\
&\quad + \int_{1-\lambda}^1 \left|\gamma_{\mu\eta(z,u,w)}(\alpha, t + \lambda - 1) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda)\right| \left|\rho'\Big(wu + t\eta(z, u, w)\Big)\right| dt\bigg] \\
&\leq \eta(z, u, w)\bigg[\int_0^\lambda \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha, \lambda - t)\right| \Big|w(1 - st)|\rho'(u)| + [1 - s(1 - t)]|\rho'(z)|\Big| dt \\
&\quad + \int_{1-\lambda}^1 \left|\gamma_{\mu\eta(z,u,w)}(\alpha, t + \lambda - 1) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda)\right| \Big|w(1 - st)|\rho'(u)| + [1 - s(1 - t)]|\rho'(z)|\Big| dt\bigg] \\
&= \eta(z, u, w)\bigg[\int_0^\lambda \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha, \lambda - t)\right| \Big|w(1 - st)|\rho'(u)| + [1 - s(1 - t)]|\rho'(z)|\Big| dt \\
&\quad + \int_0^\lambda \left|\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda - t) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda)\right| \Big|w[1 - s(1 - t)]|\rho'(u)| + (1 - st)|\rho'(z)|\Big| dt\bigg] \\
&= \eta(z, u, w)(2 - s)\Big(|w\rho'(u)| + |\rho'(z)|\Big) \int_0^\lambda \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha, \lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha, \lambda - t)\right| dt.
\end{aligned} \tag{5}$$

As a consequence, the proof of Theorem 2.6 is finished.

Corollary 2.7. *If we consider taking $\mu = 0$ as well as $\eta(z, u, w) = z - wu$ alone with $w = 1$ in Theorem 2.6, then we derive that*

$$\begin{aligned}
&\bigg|\lambda^\alpha\Big[\theta\rho\Big((1 - \lambda)u + \lambda z\Big) + (1 - \theta)\rho(u)\Big] \\
&\quad + \lambda^\alpha\Big[\theta\rho\Big(\lambda u + (1 - \lambda)z\Big) + (1 - \theta)\rho(z)\Big] \\
&\quad - \frac{\Gamma(\alpha + 1)}{(z - u)^\alpha}\Big[\mathcal{J}^\alpha_{u^+}\rho\Big((1 - \lambda)u + \lambda z\Big) + \mathcal{J}^\alpha_{z^-}\rho\Big(\lambda u + (1 - \lambda)z\Big)\Big]\bigg| \\
&\leq (z - u)(2 - s)\Big(|\rho'(u)| + |\rho'(z)|\Big)\Omega(\lambda, \theta, \alpha),
\end{aligned}$$

where

$$\Omega(\lambda, \theta, \alpha) = \lambda^{\alpha+1}\left(\frac{1}{\alpha + 1} - \theta + \frac{2\alpha}{\alpha + 1}\theta^{\frac{\alpha+1}{\alpha}}\right). \tag{6}$$

Remark 2.8. *From Corollary 2.7, we see that the next inequalities are correct clearly:*
(1) If we attempt to take $\lambda = \frac{1}{2}$ as well as $\theta = 0$, then we achieve that

$$\begin{aligned}&\left|\frac{\rho(u)+\rho(z)}{2^{\alpha}}-\frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\left[\mathcal{J}_{u^{+}}^{\alpha}\rho\left(\frac{u+z}{2}\right)+\mathcal{J}_{z^{-}}^{\alpha}\rho\left(\frac{u+z}{2}\right)\right]\right|\\&\quad\leq\frac{1}{(\alpha+1)2^{\alpha+1}}(z-u)(2-s)\Big(\left|\rho'(u)\right|+\left|\rho'(z)\right|\Big).\end{aligned}$$

In particular, if we take $\alpha = 1 = s$, then we get that

$$\left|\frac{\rho(u)+\rho(z)}{2}-\frac{1}{z-u}\int_{u}^{z}\rho(t)\mathrm{d}t\right|\leq\frac{(z-u)}{8}\Big(\left|\rho'(u)\right|+\left|\rho'(z)\right|\Big),$$

which is the same result established by Dragomir and Agarwal in [23, Theorem 2.2].

(2) If we consider taking $\lambda = \frac{1}{2}$ and $\theta = 1$, then we deduce that

$$\begin{aligned}&\left|\frac{1}{2^{\alpha-1}}\rho\left(\frac{u+z}{2}\right)-\frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\left[\mathcal{J}_{u^{+}}^{\alpha}\rho\left(\frac{u+z}{2}\right)+\mathcal{J}_{z^{-}}^{\alpha}\rho\left(\frac{u+z}{2}\right)\right]\right|\\&\quad\leq\frac{\alpha}{(\alpha+1)2^{\alpha+1}}(z-u)(2-s)\Big(\left|\rho'(u)\right|+\left|\rho'(z)\right|\Big).\end{aligned}$$

In particular, if we attempt to take $\alpha = 1 = s$, then we get that

$$\left|\rho\left(\frac{u+z}{2}\right)-\frac{1}{z-u}\int_{u}^{z}\rho(t)\mathrm{d}t\right|\leq\frac{(z-u)}{8}\Big(\left|\rho'(u)\right|+\left|\rho'(z)\right|\Big).$$

This is the same result given in [37, Theorem 2.2].

(3) If we attempt to take $\lambda = \frac{1}{2}$ and $\theta = \frac{1}{2}$, then we obtain that

$$\begin{aligned}&\left|\frac{1}{2^{\alpha}}\left[\frac{\rho(u)+\rho(z)}{2}+\rho\left(\frac{u+z}{2}\right)\right]-\frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\left[\mathcal{J}_{u^{+}}^{\alpha}\rho\left(\frac{u+z}{2}\right)+\mathcal{J}_{z^{-}}^{\alpha}\rho\left(\frac{u+z}{2}\right)\right]\right|\\&\quad\leq(z-u)(2-s)\frac{1}{2^{\alpha+1}}\left[\frac{1}{\alpha+1}-\frac{1}{2}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}}\right]\Big(\left|\rho'(u)\right|+\left|\rho'(z)\right|\Big).\end{aligned}$$

In particular, if we consider taking $\alpha = 1 = s$, then we gain that

$$\left|\frac{1}{2}\left[\frac{\rho(u)+\rho(z)}{2}+\rho\left(\frac{u+z}{2}\right)\right]-\frac{1}{z-u}\int_{u}^{z}\rho(t)\mathrm{d}t\right|\leq\frac{z-u}{16}\Big(\left|\rho'(u)\right|+\left|\rho'(z)\right|\Big),$$

which is the same result provided by Xi and Qi in [54, Corollary 3.4].
(4) If we attempt to take $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we infer that

$$\begin{aligned}&\left|\frac{1}{3\cdot 2^{\alpha}}\left[\rho(u)+4\rho\left(\frac{u+z}{2}\right)+\rho(z)\right]-\frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\left[\mathcal{J}_{u^{+}}^{\alpha}\rho\left(\frac{u+z}{2}\right)+\mathcal{J}_{z^{-}}^{\alpha}\rho\left(\frac{u+z}{2}\right)\right]\right|\\&\quad\leq(z-u)(2-s)\frac{1}{2^{\alpha+1}}\left[\frac{1}{\alpha+1}-\frac{2}{3}+\frac{2\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}\right]\Big(\left|\rho'(u)\right|+\left|\rho'(z)\right|\Big).\end{aligned}$$

In particular, if we take $\alpha = 1 = s$, then we derive that

$$\left|\frac{1}{6}\Big[\rho(u) + 4\rho\Big(\frac{u+z}{2}\Big) + \rho(z)\Big] - \frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t\right| \le \frac{5(z-u)}{72}\Big(\big|\rho'(u)\big| + \big|\rho'(z)\big|\Big).$$

This is the same result presented in [5, Corollary 1].

Theorem 2.9. *For $q > 1$ along with $q^{-1} + p^{-1} = 1$, if $|\rho'|^q$ belongs to the generalized (s, w)-preinvexity together with certain fixed $s, w \in (0, 1]$, then the following inequality in correlation with tempered fractional integral operators with $0 < \lambda, \theta \le 1$, $\alpha > 0$ and $\mu \ge 0$ holds:*

$$\begin{aligned}&\Big|\mathcal{T}_\rho(\alpha,\mu,\lambda,\theta;\eta,w)\Big|\\ &\le \big|\eta(z,u,w)\big|\Bigg\{\mathcal{L}_1^{\frac{1}{p}}\Big[w\Big(\lambda - \frac{s}{2}\lambda^2\Big)|\rho'(u)|^q + \Big(\lambda - s\lambda + \frac{s}{2}\lambda^2\Big)|\rho'(z)|^q\Big]^{\frac{1}{q}}\\ &\qquad + \mathcal{L}_2^{\frac{1}{p}}\Big[w\Big(\lambda - s\lambda + \frac{s}{2}\lambda^2\Big)|\rho'(u)|^q + \Big(\lambda - \frac{s}{2}\lambda^2\Big)|\rho'(z)|^q\Big]^{\frac{1}{q}}\Bigg\},\end{aligned} \tag{7}$$

where

$$\mathcal{L}_1 = \int_0^\lambda \Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha,\lambda - t)\Big|^p \mathrm{d}t,$$

and

$$\mathcal{L}_2 = \int_{1-\lambda}^1 \Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|^p \mathrm{d}t.$$

Proof. Making use of Lemma 2.3 as well as the Hölder's inequality, it finds out that

$$\begin{aligned}&\Big|\mathcal{T}_\rho(\alpha,\mu,\lambda,\theta;\eta,w)\Big|\\ &\le \big|\eta(z,u,w)\big|\Bigg[\int_0^\lambda \Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha,\lambda - t)\Big|\Big|\rho'\big(wu + t\eta(z,u,w)\big)\Big|\mathrm{d}t\\ &\qquad + \int_{1-\lambda}^1 \Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|\Big|\rho'\big(wu + t\eta(z,u,w)\big)\Big|\mathrm{d}t\Bigg]\\ &\le \big|\eta(z,u,w)\big|\Bigg\{\Big(\int_0^\lambda \Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda) - \gamma_{\mu\eta(z,u,w)}(\alpha,\lambda - t)\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^\lambda \Big|\rho'\big(wu + t\eta(z,u,w)\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\ &\qquad + \Big(\int_{1-\lambda}^1 \Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1) - \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_{1-\lambda}^1 \Big|\rho'\big(wu + t\eta(z,u,w)\big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\}.\end{aligned} \tag{8}$$

Due to the generalized (s, w)-type preinvexity of $|\rho'|^q$, one acquires that

$$\begin{aligned}\int_0^\lambda \Big|\rho'\big(wu + t\eta(z,u,w)\big)\Big|^q\mathrm{d}t &\le \int_0^\lambda \Big\{w(1-st)|\rho'(u)|^q + [1 - s(1-t)]|\rho'(z)|^q\Big\}\mathrm{d}t\\ &= w\Big(\lambda - \frac{s}{2}\lambda^2\Big)|\rho'(u)|^q + \Big(\lambda - s\lambda + \frac{s}{2}\lambda^2\Big)|\rho'(z)|^q,\end{aligned} \tag{9}$$

and

$$\begin{aligned}\int_{1-\lambda}^1 \Big|\rho'\big(wu + t\eta(z,u,w)\big)\Big|^q\mathrm{d}t &\le \int_{1-\lambda}^1 \Big\{w(1-st)|\rho'(u)|^q + [1 - s(1-t)]|\rho'(z)|^q\Big\}\mathrm{d}t\\ &= w\Big(\lambda - s\lambda + \frac{s}{2}\lambda^2\Big)|\rho'(u)|^q + \Big(\lambda - \frac{s}{2}\lambda^2\Big)|\rho'(z)|^q.\end{aligned} \tag{10}$$

Applying (9) and (10) to (8), we deduce the desired result. This accomplishes the proof.

Corollary 2.10. *If we attempt to take $\mu = 0$ as well as $\eta(z,u,w) = z - wu$ together with $w = 1$ in Theorem 2.9, then the inequality (7) transfers to*

$$\begin{aligned}
&\bigg|\lambda^{\alpha}\Big[\theta\rho\Big((1-\lambda)u+\lambda z\Big)+(1-\theta)\rho(u)\Big] \\
&\quad + \lambda^{\alpha}\Big[\theta\rho\Big(\lambda u+(1-\lambda)z\Big)+(1-\theta)\rho(z)\Big] \\
&\quad - \frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\Big[\mathcal{J}_{u^+}^{\alpha}\rho\Big((1-\lambda)u+\lambda z\Big)+\mathcal{J}_{z^-}^{\alpha}\rho\Big(\lambda u+(1-\lambda)z\Big)\Big]\bigg| \\
&\leq (z-u)\bigg\{\Big(\int_0^{\lambda}\Big|\theta\lambda^{\alpha}-(\lambda-t)^{\alpha}\Big|^{p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\Big(\lambda-\frac{s}{2}\lambda^2\Big)|\rho'(u)|^{q}+\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}} \\
&\quad + \Big(\int_{1-\lambda}^{1}\Big|(t+\lambda-1)^{\alpha}-\theta\lambda^{\alpha}\Big|^{p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big)|\rho'(u)|^{q}+\Big(\lambda-\frac{s}{2}\lambda^2\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}}\bigg\}.
\end{aligned}$$

Remark 2.11. *In terms of Corollary 2.10, we see that the next inequalities are correct clearly:*
(1) If we attempt to take $\lambda = 1$ as well as $\theta = 1$, then we can gain that

$$\begin{aligned}
&\bigg|\frac{\rho(u)+\rho(z)}{2}-\frac{\Gamma(\alpha+1)}{2(z-u)^{\alpha}}\Big[\mathcal{J}_{u^+}^{\alpha}\rho(z)+\mathcal{J}_{z^-}^{\alpha}\rho(u)\Big]\bigg| \\
&\leq \frac{z-u}{2}\bigg\{\Big(\int_0^1\Big|1-(1-t)^{\alpha}\Big|^{p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\Big(1-\frac{s}{2}\Big)|\rho'(u)|^{q}+\Big(1-\frac{s}{2}\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}} \\
&\quad + \Big(\int_0^1\Big|t^{\alpha}-1\Big|^{p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\Big(1-\frac{s}{2}\Big)|\rho'(u)|^{q}+\Big(1-\frac{s}{2}\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}}\bigg\} \\
&\leq (z-u)\Big(1-\frac{1}{\alpha p+1}\Big)^{\frac{1}{p}}\Big[\Big(1-\frac{s}{2}\Big)|\rho'(u)|^{q}+\Big(1-\frac{s}{2}\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}}.
\end{aligned}$$

In particular, if we take $\alpha = 1 = s$, then we obtain that

$$\left|\frac{\rho(u)+\rho(z)}{2}-\frac{1}{z-u}\int_u^z\rho(t)\mathrm{d}t\right| \leq (z-u)\Big(\frac{p}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}|\rho'(u)|^{q}+\frac{1}{2}|\rho'(z)|^{q}\Big)^{\frac{1}{q}}.$$

(2) If we consider taking $\lambda = \frac{1}{2}$ and $\theta = 1$, then we achieve that

$$\begin{aligned}
&\bigg|\frac{1}{2^{\alpha-1}}\rho\Big(\frac{u+z}{2}\Big)-\frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\Big[\mathcal{J}_{u^+}^{\alpha}\rho\Big(\frac{u+z}{2}\Big)+\mathcal{J}_{z^-}^{\alpha}\rho\Big(\frac{u+z}{2}\Big)\Big]\bigg| \\
&\leq (z-u)\bigg\{\Big(\int_0^{\frac{1}{2}}\Big|\Big(\frac{1}{2}\Big)^{\alpha}-\Big(\frac{1}{2}-t\Big)^{\alpha}\Big|^{p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\Big(\frac{1}{2}-\frac{1}{8}s\Big)|\rho'(u)|^{q}+\Big(\frac{1}{2}-\frac{3}{8}s\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}} \\
&\quad + \Big(\int_{\frac{1}{2}}^{1}\Big|\Big(t-\frac{1}{2}\Big)^{\alpha}-\Big(\frac{1}{2}\Big)^{\alpha}\Big|^{p}\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\Big(\frac{1}{2}-\frac{3}{8}s\Big)|\rho'(u)|^{q}+\Big(\frac{1}{2}-\frac{1}{8}s\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}}\bigg\} \\
&\leq (z-u)\Big[\frac{\alpha p}{\alpha p+1}\Big(\frac{1}{2}\Big)^{\alpha p+1}\Big]^{\frac{1}{p}}\bigg\{\Big[\Big(\frac{1}{2}-\frac{1}{8}s\Big)|\rho'(u)|^{q}+\Big(\frac{1}{2}-\frac{3}{8}s\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}} \\
&\quad + \Big[\Big(\frac{1}{2}-\frac{3}{8}s\Big)|\rho'(u)|^{q}+\Big(\frac{1}{2}-\frac{1}{8}s\Big)|\rho'(z)|^{q}\Big]^{\frac{1}{q}}\bigg\}.
\end{aligned}$$

In particular, if we attempt to take $\alpha = 1 = s$, then we derive that

$$\left|\rho\left(\frac{u+z}{2}\right) - \frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t\right| \\ \leq \frac{z-u}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3}{4}|\rho'(u)|^q + \frac{1}{4}|\rho'(z)|^q\right)^{\frac{1}{q}} \\ + \left(\frac{1}{4}|\rho'(u)|^q + \frac{3}{4}|\rho'(z)|^q\right)^{\frac{1}{q}}\right].$$

(3) If we consider taking $\lambda = \frac{1}{2}$ as well as $\theta = \frac{1}{2}$, then one deduces that

$$\left|\frac{1}{2^\alpha}\left[\frac{\rho(u)+\rho(z)}{2} + \rho\left(\frac{u+z}{2}\right)\right] - \frac{\Gamma(\alpha+1)}{(z-u)^\alpha}\left[\mathcal{J}^\alpha_{u^+}\rho\left(\frac{u+z}{2}\right) + \mathcal{J}^\alpha_{z^-}\rho\left(\frac{u+z}{2}\right)\right]\right| \\ \leq (z-u)\left\{\left(\int_0^{\frac{1}{2}}\left|\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^\alpha - \left(\frac{1}{2}-t\right)^\alpha\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}-\frac{1}{8}s\right)|\rho'(u)|^q + \left(\frac{1}{2}-\frac{3}{8}s\right)|\rho'(z)|^q\right]^{\frac{1}{q}} \\ + \left(\int_{\frac{1}{2}}^1\left|\left(t-\frac{1}{2}\right)^\alpha - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^\alpha\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}-\frac{3}{8}s\right)|\rho'(u)|^q + \left(\frac{1}{2}-\frac{1}{8}s\right)|\rho'(z)|^q\right]^{\frac{1}{q}}\right\}.$$

In particular, if we attempt to take $\alpha = 1 = s$, then we get that

$$\left|\frac{1}{2}\left[\frac{\rho(u)+\rho(z)}{2} + \rho\left(\frac{u+z}{2}\right)\right] - \frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t\right| \\ \leq \frac{z-u}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3}{4}|\rho'(u)|^q + \frac{1}{4}|\rho'(z)|^q\right)^{\frac{1}{q}} \\ + \left(\frac{1}{4}|\rho'(u)|^q + \frac{3}{4}|\rho'(z)|^q\right)^{\frac{1}{q}}\right].$$

(4) If we consider taking $\lambda = \frac{1}{2}$ as well as $\theta = \frac{2}{3}$, then we can get that

$$\left|\frac{1}{3\cdot 2^\alpha}\left[\rho(u) + 4\rho\left(\frac{u+z}{2}\right) + \rho(z)\right] - \frac{\Gamma(\alpha+1)}{(z-u)^\alpha}\left[\mathcal{J}^\alpha_{u^+}\rho\left(\frac{u+z}{2}\right) + \mathcal{J}^\alpha_{z^-}\rho\left(\frac{u+z}{2}\right)\right]\right| \\ \leq (z-u)\left\{\left(\int_0^{\frac{1}{2}}\left|\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^\alpha - \left(\frac{1}{2}-t\right)^\alpha\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}-\frac{1}{8}s\right)|\rho'(u)|^q + \left(\frac{1}{2}-\frac{3}{8}s\right)|\rho'(z)|^q\right]^{\frac{1}{q}} \\ + \left(\int_{\frac{1}{2}}^1\left|\left(t-\frac{1}{2}\right)^\alpha - \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^\alpha\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}-\frac{3}{8}s\right)|\rho'(u)|^q + \left(\frac{1}{2}-\frac{1}{8}s\right)|\rho'(z)|^q\right]^{\frac{1}{q}}\right\}.$$

In particular, if we take $\alpha = 1 = s$, then we derive that

$$\left|\frac{1}{6}\left[\rho(u) + 4\rho\left(\frac{u+z}{2}\right) + \rho(z)\right] - \frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t\right| \\ \leq (z-u)\left[\frac{(\frac{1}{6})^{p+1} + (\frac{1}{3})^{p+1}}{p+1}\right]^{\frac{1}{p}}\left[\left(\frac{3}{8}|\rho'(u)|^q + \frac{1}{8}|\rho'(z)|^q\right)^{\frac{1}{q}} + \left(\frac{1}{8}|\rho'(u)|^q + \frac{3}{8}|\rho'(z)|^q\right)^{\frac{1}{q}}\right].$$

Theorem 2.12. *For $q > 1$, some fixed $s,w \in (0,1]$, if the mapping $|\rho'|^q$ belongs to the generalized (s, w)-type preinvexity, then the undermentioned integral inequality for tempered fractional integral operators together with $0 < \lambda, \theta \leq 1$, $\alpha > 0$ and $\mu \geq 0$ holds:*

$$\begin{aligned}&\Big|\mathcal{T}_\rho(\alpha,\mu,\lambda,\theta;\eta,w)\Big|\\&\leq \Big|\eta(z,u,w)\Big|\left\{\delta_1^{1-\frac{1}{q}}\Big[w\Delta_1|\rho'(u)|^q+\Delta_2|\rho'(z)|^q\Big]^{\frac{1}{q}}+\delta_2^{1-\frac{1}{q}}\Big[w\Delta_2|\rho'(u)|^q+\Delta_1|\rho'(z)|^q\Big]^{\frac{1}{q}}\right\},\end{aligned}\tag{11}$$

in which

$$\delta_2=\int_{1-\lambda}^1\Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)-\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|\mathrm{d}t,$$

$$\Delta_1=\Big(\lambda-\frac{s}{2}\lambda^2\Big)(\theta+1)\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)+(s\lambda-1)\gamma_{\mu\eta(z,u,w)}(\alpha+1,\lambda)-\frac{s}{2}\gamma_{\mu\eta(z,u,w)}(\alpha+2,\lambda),$$

and

$$\Delta_2=\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big)(\theta+1)\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)+(s-s\lambda-1)\gamma_{\mu\eta(z,u,w)}(\alpha+1,\lambda)+\frac{s}{2}\gamma_{\mu\eta(z,u,w)}(\alpha+2,\lambda),$$

where δ_1 is the same as in Theorem 2.6.

Proof. Taking advantage of Lemma 2.3 as well as the power-mean integral inequality, we achieve that

$$\begin{aligned}\Big|\mathcal{T}_\rho(\alpha,\mu,\lambda,\theta;\eta,w)\Big| &\leq \Big|\eta(z,u,w)\Big|\bigg[\int_0^\lambda\Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)-\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big|\Big|\rho'\big(wu+t\eta(z,u,w)\big)\Big|\mathrm{d}t\\&\qquad+\int_{1-\lambda}^1\Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)-\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|\Big|\rho'\big(wu+t\eta(z,u,w)\big)\Big|\mathrm{d}t\bigg]\\&\leq \Big|\eta(z,u,w)\Big|\bigg\{\bigg(\int_0^\lambda\Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)-\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big|\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\quad\times\bigg[\int_0^\lambda\Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)-\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big|\Big|\rho'\big(wu+t\eta(z,u,w)\big)\Big|^q\mathrm{d}t\bigg]^{\frac{1}{q}}\\&\quad+\bigg(\int_{1-\lambda}^1\Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)-\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\quad\times\bigg[\int_{1-\lambda}^1\Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)-\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|\Big|\rho'\big(wu+t\eta(z,u,w)\big)\Big|^q\mathrm{d}t\bigg]^{\frac{1}{q}}\bigg\}\\&\leq \Big|\eta(z,u,w)\Big|\bigg\{\bigg(\int_0^\lambda\Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)-\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big|\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\quad\times\bigg[\int_0^\lambda\Big(\Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|+\Big|\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big|\Big)\Big|\rho'\big(wu+t\eta(z,u,w)\big)\Big|^q\mathrm{d}t\bigg]^{\frac{1}{q}}\\&\quad+\bigg(\int_{1-\lambda}^1\Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)-\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|\mathrm{d}t\bigg)^{1-\frac{1}{q}}\\&\quad\times\bigg[\int_{1-\lambda}^1\Big(\Big|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)\Big|+\Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big|\Big)\Big|\rho'\big(wu+t\eta(z,u,w)\big)\Big|^q\mathrm{d}t\bigg]^{\frac{1}{q}}\bigg\}.\end{aligned}\tag{12}$$

Using the generalized (s,w)-type preinvexity of $|\rho'|^q$ and changing the order of the integration, we deduce that

$$\begin{aligned}
&\int_0^\lambda \left|\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\right|\left|\rho'\big(wu+t\eta(z,u,w)\big)\right|^q \mathrm{d}t \\
&\leq \int_0^\lambda \left|\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\right|\left|w(1-st)|\rho'(u)|^q+[1-s(1-t)]|\rho'(z)|^q\right|\mathrm{d}t \\
&= w|\rho'(u)|^q\int_0^\lambda (1-st)\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\mathrm{d}t+|\rho'(z)|^q\int_0^\lambda [1-s(1-t)]\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\mathrm{d}t \\
&= w|\rho'(u)|^q\int_0^\lambda\int_0^{\lambda-t}(1-st)v^{\alpha-1}e^{-\mu\eta(z,u,w)v}\mathrm{d}v\mathrm{d}t \\
&\quad+|\rho'(z)|^q\int_0^\lambda\int_0^{\lambda-t}[1-s(1-t)]v^{\alpha-1}e^{-\mu\eta(z,u,w)v}\mathrm{d}v\mathrm{d}t \qquad (13)\\
&= w|\rho'(u)|^q\int_0^\lambda\int_0^{\lambda-v}(1-st)v^{\alpha-1}e^{-\mu\eta(z,u,w)v}\mathrm{d}t\mathrm{d}v \\
&\quad+|\rho'(z)|^q\int_0^\lambda\int_0^{\lambda-v}[1-s(1-t)]v^{\alpha-1}e^{-\mu\eta(z,u,w)v}\mathrm{d}t\mathrm{d}v \\
&= w|\rho'(u)|^q\Big[\Big(\lambda-\frac{s}{2}\lambda^2\Big)\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)+(s\lambda-1)\gamma_{\mu\eta(z,u,w)}(\alpha+1,\lambda)-\frac{s}{2}\gamma_{\mu\eta(z,u,w)}(\alpha+2,\lambda)\Big] \\
&\quad+|\rho'(z)|^q\Big[\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big)\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)+(s-s\lambda-1)\gamma_{\mu\eta(z,u,w)}(\alpha+1,\lambda)+\frac{s}{2}\gamma_{\mu\eta(z,u,w)}(\alpha+2,\lambda)\Big].
\end{aligned}$$

Analogously, we can get that

$$\begin{aligned}
&\int_{1-\lambda}^1 \left|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)\right|\left|\rho'\big(wu+t\eta(z,u,w)\big)\right|^q \mathrm{d}t \\
&\leq \int_{1-\lambda}^1 \left|\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)\right|\left|w(1-st)|\rho'(u)|^q+[1-s(1-t)]|\rho'(z)|^q\right|\mathrm{d}t \qquad (14)\\
&= w|\rho'(u)|^q\Big[\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big)\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)+(s-s\lambda-1)\gamma_{\mu\eta(z,u,w)}(\alpha+1,\lambda)+\frac{s}{2}\gamma_{\mu\eta(z,u,w)}(\alpha+2,\lambda)\Big] \\
&\quad+|\rho'(z)|^q\Big[\Big(\lambda-\frac{s}{2}\lambda^2\Big)\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)+(s\lambda-1)\gamma_{\mu\eta(z,u,w)}(\alpha+1,\lambda)-\frac{s}{2}\gamma_{\mu\eta(z,u,w)}(\alpha+2,\lambda)\Big].
\end{aligned}$$

Employing the generalized (s,w)-type preinvexity of $|\rho'|^q$ and integrating by part yield that

$$\begin{aligned}
&\int_0^\lambda \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\right|\left|\rho'\big(wu+t\eta(z,u,w)\big)\right|^q \mathrm{d}t \\
&\leq \int_0^\lambda \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\right|\left|w(1-st)|\rho'(u)|+[1-s(1-t)]|\rho'(z)|\right|\mathrm{d}t \qquad (15)\\
&= \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)w|\rho'(u)|\Big(t-s\frac{t^2}{2}\Big)\Big|_0^\lambda+\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)|\rho'(z)|\Big(t-st+s\frac{t^2}{2}\Big)\Big|_0^\lambda \\
&= \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)w|\rho'(u)|\Big(\lambda-\frac{s}{2}\lambda^2\Big)+\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)|\rho'(z)|\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{1-\lambda}^{1} \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\right|\left|\rho'\Big(wu+t\eta(z,u,w)\Big)\right|^{q}\mathrm{d}t \\
&\leq \int_{1-\lambda}^{1} \left|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\right|\left|(1-st)w|\rho'(u)|+[1-s(1-t)]|\rho'(z)|\right|\mathrm{d}t \\
&= \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)w|\rho'(u)|\Big(t-s\frac{t^2}{2}\Big)\Big|_{1-\lambda}^{1}+\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)|\rho'(z)|\Big(t-st+s\frac{t^2}{2}\Big)\Big|_{1-\lambda}^{1} \\
&= \theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)w|\rho'(u)|\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big)+\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)|\rho'(z)|\Big(\lambda-\frac{s}{2}\lambda^2\Big).
\end{aligned} \tag{16}$$

As a result, the desired outcome can be obtained by combining inequalities (13), (14), (15) and (16). The proof is completed.

Our next purpose is to gain the error bounds involving with the tempered fractional integral operators when the derivative of the researchful functions ρ' is bounded.

Theorem 2.13. *If there exist constants $m < M$ satisfying that $-\infty < m \leq \rho'(x) \leq M < +\infty$ for every $x \in [wu, wu+\eta(z,u,w)]$, then the successive inequality is effective:*

$$\left|\mathcal{T}_\rho(\alpha,\mu,\lambda,\theta;\eta,w)\right| \leq \eta(z,u,w)(M-m)\delta_1, \tag{17}$$

where δ_1 is the same as in Theorem 2.6.

Proof. It yields from Lemma 2.3 that

$$\begin{aligned}
&\mathcal{T}_\rho(\alpha,\mu,\lambda,\theta;\eta,w) \\
&= \eta(z,u,w)\Big[\int_0^{\lambda}\Big(\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)-\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big)\rho'\Big(wu+t\eta(z,u,w)\Big)\mathrm{d}t \\
&\quad+\int_{1-\lambda}^{1}\Big(\gamma_{\mu\eta(z,u,w)}(\alpha,t+\lambda-1)-\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big)\rho'\Big(wu+t\eta(z,u,w)\Big)\mathrm{d}t\Big] \\
&= \eta(z,u,w)\Big\{\int_0^{\lambda}\Big(\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)-\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big)\Big[\rho'\Big(wu+t\eta(z,u,w)\Big)-\frac{m+M}{2}+\frac{m+M}{2}\Big]\mathrm{d}t \\
&\quad+\int_0^{\lambda}\Big(\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)-\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big)\Big[\rho'\Big(wu+(1-t)\eta(z,u,w)\Big)-\frac{m+M}{2}+\frac{m+M}{2}\Big]\mathrm{d}t\Big\} \\
&= \eta(z,u,w)\Big\{\int_0^{\lambda}\Big(\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)-\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big)\Big[\rho'\Big(wu+t\eta(z,u,w)\Big)-\frac{m+M}{2}\Big]\mathrm{d}t \\
&\quad+\int_0^{\lambda}\Big(\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)-\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)\Big)\Big[\rho'\Big(wu+(1-t)\eta(z,u,w)\Big)-\frac{m+M}{2}\Big]\mathrm{d}t\Big\}.
\end{aligned}$$

Due to the function ρ' meets that $-\infty < m \leq \rho'(x) \leq M < +\infty$, we gain that

$$m-\frac{m+M}{2} \leq \rho'(x)-\frac{m+M}{2} \leq M-\frac{m+M}{2},$$

which implies that

$$\left|\rho'(x)-\frac{m+M}{2}\right| \leq \frac{M-m}{2}.$$

Taking modulus the identity above, one derives that

$$\begin{aligned}&\Big|\mathcal{T}_\rho(\alpha,\mu,\lambda,\theta;\eta,w)\Big|\\&\leq \eta(z,u,w)(M-m)\int_0^\lambda\Big|\theta\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda)-\gamma_{\mu\eta(z,u,w)}(\alpha,\lambda-t)\Big|\mathrm{d}t\\&=\eta(z,u,w)(M-m)\delta_1.\end{aligned}$$

As a consequence, the proof of Theorem 2.13 is fulfilled.

Corollary 2.14. *If we attempt to take $\mu = 0$ and $\eta(z,u,w) = z - wu$ along with $w = 1$ in Theorem 2.13, then we acquire that*

$$\begin{aligned}&\Bigg|\lambda^\alpha\Big[\theta\rho\Big((1-\lambda)u+\lambda z\Big)+(1-\theta)\rho(u)\Big]+\lambda^\alpha\Big[\theta\rho\Big(\lambda u+(1-\lambda)z\Big)+(1-\theta)\rho(z)\Big]\\&\quad-\frac{\Gamma(\alpha+1)}{(z-u)^\alpha}\Big[\mathcal{J}^\alpha_{u^+}\rho\Big((1-\lambda)u+\lambda z\Big)+\mathcal{J}^\alpha_{z^-}\rho\Big(\lambda u+(1-\lambda)z\Big)\Big]\Bigg|\\&\leq (M-m)(z-u)\Omega(\lambda,\theta,\alpha),\end{aligned}$$

where $\Omega(\lambda,\theta,\alpha)$ is defined by (6) in Corollary 2.7.

Remark 2.15. *In accordance with Corollary 2.14, we clearly see that the following inequalities are true:*
(1) If we take $\lambda = \frac{1}{2}$ and $\theta = 0$, then we deduce that

$$\begin{aligned}&\left|\frac{\rho(u)+\rho(z)}{2^\alpha}-\frac{\Gamma(\alpha+1)}{(z-u)^\alpha}\left[\mathcal{J}^\alpha_{u^+}\rho\left(\frac{u+z}{2}\right)+\mathcal{J}^\alpha_{z^-}\rho\left(\frac{u+z}{2}\right)\right]\right|\\&\leq\frac{(M-m)(z-u)}{(\alpha+1)2^{\alpha+1}}.\end{aligned}$$

In particular, if we consider taking $\alpha = 1$, then we can achieve that

$$\left|\frac{\rho(u)+\rho(z)}{2}-\frac{1}{z-u}\int_u^z\rho(t)\mathrm{d}t\right|\leq\frac{(M-m)(z-u)}{8}.$$

(2) If we attempt to take $\lambda = \frac{1}{2}$ as well as $\theta = 1$, then we deduce that

$$\begin{aligned}&\left|\frac{1}{2^{\alpha-1}}\rho\Big(\frac{u+z}{2}\Big)-\frac{\Gamma(\alpha+1)}{(z-u)^\alpha}\Big[\mathcal{J}^\alpha_{u^+}\rho\Big(\frac{u+z}{2}\Big)+\mathcal{J}^\alpha_{z^-}\rho\Big(\frac{u+z}{2}\Big)\Big]\right|\\&\leq\frac{\alpha(M-m)(z-u)}{(\alpha+1)2^{\alpha+1}}.\end{aligned}$$

In particular, if we attempt to take $\alpha = 1$, then we infer that

$$\left|\rho\Big(\frac{u+z}{2}\Big)-\frac{1}{z-u}\int_u^z\rho(t)\mathrm{d}t\right|\leq\frac{(M-m)(z-u)}{8}.$$

(3) If we attempt to take $\lambda = \frac{1}{2}$ as well as $\theta = \frac{1}{2}$, then one gains that

$$\begin{aligned}&\left|\frac{1}{2^\alpha}\Big[\frac{\rho(u)+\rho(z)}{2}+\rho\Big(\frac{u+z}{2}\Big)\Big]-\frac{\Gamma(\alpha+1)}{(z-u)^\alpha}\Big[\mathcal{J}^\alpha_{u^+}\rho\Big(\frac{u+z}{2}\Big)+\mathcal{J}^\alpha_{z^-}\rho\Big(\frac{u+z}{2}\Big)\Big]\right|\\&\qquad\leq\frac{1}{2^{\alpha+1}}\left[\frac{1}{\alpha+1}-\frac{1}{2}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{\alpha+1}{\alpha}}\right](M-m)(z-u).\end{aligned}$$

In particular, if we take $\alpha = 1$, then we derive that

$$\left|\frac{1}{2}\left[\frac{\rho(u)+\rho(z)}{2}+\rho\left(\frac{u+z}{2}\right)\right]-\frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t\right| \leq \frac{(M-m)(z-u)}{16}.$$

(4) If we take $\lambda = \frac{1}{2}$ as well as $\theta = \frac{2}{3}$ in Corollary 2.14, then we get that

$$\left|\frac{1}{3\cdot 2^{\alpha}}\left[\rho(u)+4\rho\left(\frac{u+z}{2}\right)+\rho(z)\right]-\frac{\Gamma(\alpha+1)}{(z-u)^{\alpha}}\left[\mathcal{J}^{\alpha}_{u^+}\rho\left(\frac{u+z}{2}\right)+\mathcal{J}^{\alpha}_{z^-}\rho\left(\frac{u+z}{2}\right)\right]\right|$$
$$\leq \frac{1}{2^{\alpha+1}}\left[\frac{1}{\alpha+1}-\frac{2}{3}+\frac{2\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}\right](M-m)(z-u).$$

In particular, if we take $\alpha = 1$, then we have that

$$\left|\frac{1}{6}\left[\rho(u)+4\rho\left(\frac{u+z}{2}\right)+\rho(z)\right]-\frac{1}{z-u}\int_u^z \rho(t)\mathrm{d}t\right| \leq \frac{5}{72}(M-m)(z-u).$$

3. Examples

To help readers understand the findings established in the previous section more intuitively, we provide two examples to illustrate the correctness of Theorem 2.6 and Theorem 2.9 in this part.

Example 3.1. *Let us consider the mapping $\rho(x) = \frac{1}{3}x^3$, for $x \in (-\infty, \infty)$. Then $|\rho'| = x^2$ is an (s, w)-type preinvex mapping in association with $\eta(y, x, 1) = y - x$ with $s = 1$ and $w = 1$. If we take $u = 0$, $z = 1$, $\alpha = \frac{1}{2}$, $\mu = 1$, $\theta = \frac{1}{2}$ as well as $\lambda = \frac{1}{2}$, then all hypotheses mentioned in Theorem 2.6 are met.*
Clearly, the left-hand part of the inequality (4) *is:*

$$\left|\mathcal{T}_{\rho}(\alpha,\mu,\lambda,\theta;\eta,w)\right|$$
$$=\left|\frac{5}{24}\int_0^{\frac{1}{2}} t^{-\frac{1}{2}}e^{-t}\mathrm{d}t-\frac{1}{3}\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^{-\frac{1}{2}}e^{-\left(\frac{1}{2}-t\right)}t^3\mathrm{d}t+\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)^{\frac{1}{2}}e^{-\left(t-\frac{1}{2}\right)}t^3\mathrm{d}t\right]\right|$$
$$\approx 0.101492.$$

For the right-hand part of the inequality (4)*, we have that*

$$\eta(z,u,w)(2-s)\Big(w|\rho'(u)|+|\rho'(z)|\Big)\delta_1$$
$$=\int_0^{\frac{1}{2}}\left|\frac{1}{2}\int_0^{\frac{1}{2}} t^{-\frac{1}{2}}e^{-t}\mathrm{d}t-\int_0^{\frac{1}{2}-t} t^{-\frac{1}{2}}e^{-t}\mathrm{d}t\right|\mathrm{d}t$$
$$\approx 0.164717.$$

Obviously to see that 0.101491 < 0.164717*, which shows the correctness of the outcome described in Theorem 2.6.*

Remark 3.2. *Case 1: Suppose that the parameter μ is not a fixed constant in Example 3.1. For instance, we consider taking $\mu \in [1, 3]$, by virtue of Theorem 2.6, we can obtain the outcome for the parameter μ as below:*

$$-\int_0^{\frac{1}{2}}\left|\frac{1}{2}\int_0^{\frac{1}{2}} x^{-\frac{1}{2}}e^{-\mu x}\mathrm{d}x-\int_0^{\frac{1}{2}-t} x^{-\frac{1}{2}}e^{-\mu x}\mathrm{d}x\right|\mathrm{d}t$$
$$\leq \frac{5}{24}\int_0^{\frac{1}{2}} t^{-\frac{1}{2}}e^{-\mu t}\mathrm{d}t-\frac{1}{3}\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^{-\frac{1}{2}}e^{-\mu\left(\frac{1}{2}-t\right)}t^3\mathrm{d}t+\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)^{-\frac{1}{2}}e^{-\mu\left(t-\frac{1}{2}\right)}t^3\mathrm{d}t\right] \tag{18}$$
$$\leq \int_0^{\frac{1}{2}}\left|\frac{1}{2}\int_0^{\frac{1}{2}} x^{-\frac{1}{2}}e^{-\mu x}\mathrm{d}x-\int_0^{\frac{1}{2}-t} x^{-\frac{1}{2}}e^{-\mu x}\mathrm{d}x\right|\mathrm{d}t.$$

Three functions given by the double inequalities on the left-, middle- and right-hand parts (18) *are plotted in Figure 1 against* $\mu \in [1,3]$. *The graphs of the functions illustrate the validity of dual inequalities.*

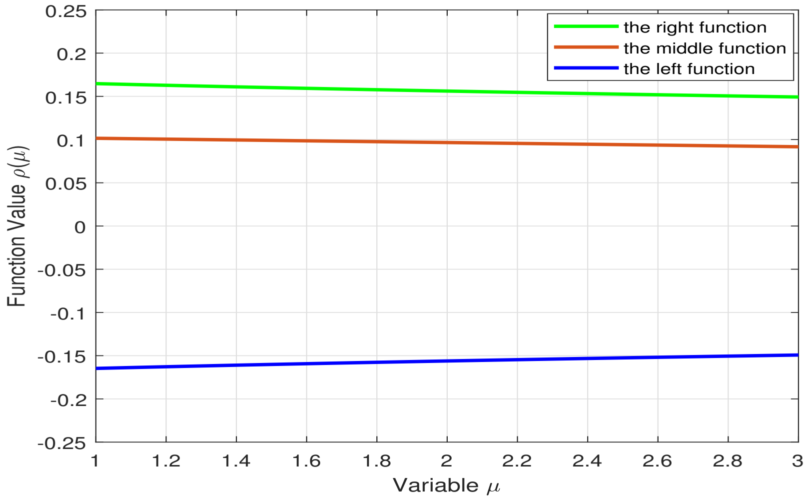

Figure 1: Graphical representation of Example 3.1 for variable $\alpha = \frac{1}{2}$ and $\mu \in [1,3]$

Case 2: Suppose that the parameter α *is not a fixed constant in Example 3.1. For instance,* $\alpha \in [1,3]$, *on the basis of Theorem 2.6, we achieve the finding for the parameter* α *as below:*

$$\begin{aligned} &-\int_0^{\frac{1}{2}} \left| \frac{1}{2} \int_0^{\frac{1}{2}} x^{\alpha-1} e^{-x} \mathrm{d}x - \int_0^{\frac{1}{2}-t} x^{\alpha-1} e^{-x} \mathrm{d}x \right| \mathrm{d}t \\ &\leq \frac{5}{24} \int_0^{\frac{1}{2}} t^{\alpha-1} e^{-t} \mathrm{d}t - \frac{1}{3} \Big[\int_0^{\frac{1}{2}} \left(\frac{1}{2} - t\right)^{\alpha-1} e^{-(\frac{1}{2}-t)} t^3 \mathrm{d}t + \int_{\frac{1}{2}}^1 \left(t - \frac{1}{2}\right)^{\alpha-1} e^{-(t-\frac{1}{2})} t^3 \mathrm{d}t \Big] \\ &\leq \int_0^{\frac{1}{2}} \left| \frac{1}{2} \int_0^{\frac{1}{2}} x^{\alpha-1} e^{-x} \mathrm{d}x - \int_0^{\frac{1}{2}-t} x^{\alpha-1} e^{-x} \mathrm{d}x \right| \mathrm{d}t. \end{aligned} \tag{19}$$

The visualization findings of three functions given by the double inequalities on the left-, middle- as well as right-hand parts (19) *are plotted in Figure 2 against* $\alpha \in [1,3]$. *As we can see from Figure 2, the findings presented in Theorem 2.6 are always hold true regarding the parameters* $\mu = 1$ *as well as* $\alpha \in [1,3]$ *are given any value.*

Case 3: Suppose that the parameters α *as well as* μ *are not two fixed constants in Example 3.1. For instance,* $\alpha, \mu \in [1,3]$, *according to Theorem 2.6, we can achieve the outcome for the parameters* α *and* μ *as follows:*

$$\begin{aligned} &-\int_0^{\frac{1}{2}} \left| \frac{1}{2} \int_0^{\frac{1}{2}} x^{\alpha-1} e^{-\mu x} \mathrm{d}x - \int_0^{\frac{1}{2}-t} x^{\alpha-1} e^{-\mu x} \mathrm{d}x \right| \mathrm{d}t \\ &\leq \frac{5}{24} \int_0^{\frac{1}{2}} t^{\alpha-1} e^{-\mu t} \mathrm{d}t - \frac{1}{3} \Big[\int_0^{\frac{1}{2}} \left(\frac{1}{2} - t\right)^{\alpha-1} e^{-\mu(\frac{1}{2}-t)} t^3 \mathrm{d}t + \int_{\frac{1}{2}}^1 \left(t - \frac{1}{2}\right)^{\alpha-1} e^{-\mu(t-\frac{1}{2})} t^3 \mathrm{d}t \Big] \\ &\leq \int_0^{\frac{1}{2}} \left| \frac{1}{2} \int_0^{\frac{1}{2}} x^{\alpha-1} e^{-\mu x} \mathrm{d}x - \int_0^{\frac{1}{2}-t} x^{\alpha-1} e^{-\mu x} \mathrm{d}x \right| \mathrm{d}t. \end{aligned} \tag{20}$$

Let us discretize the region of $[1,3] \times [1,3]$. *From the visual perspective of graphics, Figure 3 vividly describes the outcome exhibited in Example 3.1.*

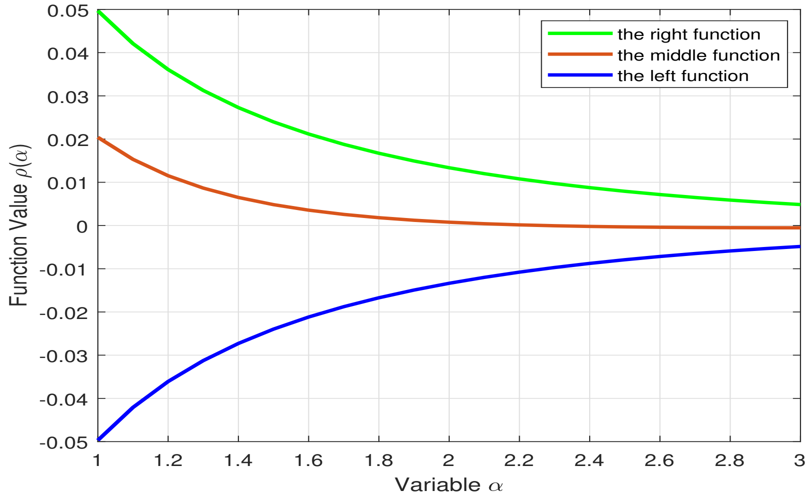

Figure 2: Graphical representation of Example 3.1 for Variable $\mu = 1$ and $\alpha \in [1, 3]$

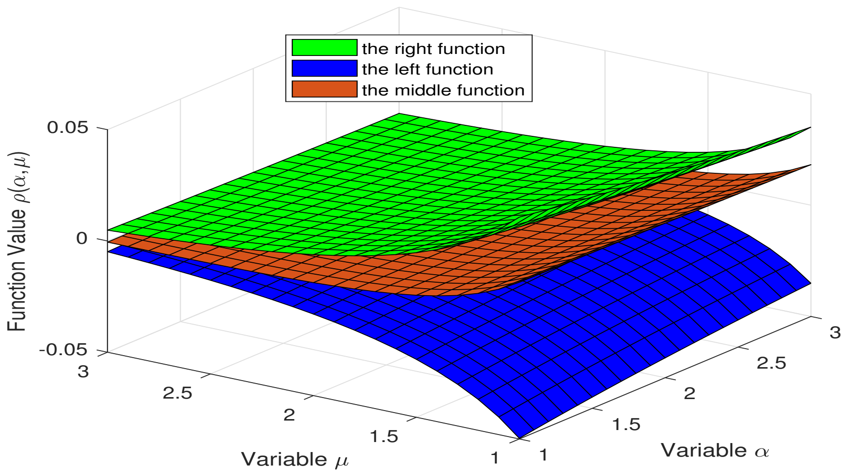

Figure 3: Graphical representation of Example 3.1 for Three-dimensional

Example 3.3. *Let us consider the mapping $\rho(x) = e^x$, for $x \in (-\infty, \infty)$. Then $|\rho'(x)|^q$ is a generalized (s, w)-type preinvex mapping in association with $\eta(y, x, 1) = y - x$ with $s = 1$ and $w = 1$. If we consider taking $u = 0$, $z = 1$, $\theta = \frac{1}{2}$, $p = 2 = q$, $\alpha = \frac{1}{2}$, $\mu = 0$ and $\lambda = \frac{1}{2}$, then all hypotheses mentioned in Theorem 2.9 are met.*
Clearly, the left-hand part of the inequality (7) *becomes:*

$$\left|\mathcal{T}_\rho(\alpha, \mu, \lambda, \theta; \eta, w)\right| = \frac{\sqrt{2}}{2}\left(e^{\frac{1}{2}} + \frac{1}{2} + \frac{1}{2}e\right) - \frac{1}{2}\left(\int_0^{\frac{1}{2}} \left(\frac{1}{2} - t\right)^{-\frac{1}{2}} e^t \mathrm{d}t + \int_{\frac{1}{2}}^1 \left(t - \frac{1}{2}\right)^{-\frac{1}{2}} e^t \mathrm{d}t\right)$$
$$\approx 0.089819.$$

The right-hand part of the inequality (7) *becomes:*

$$\begin{aligned}
&\left|\eta(z,u,w)\right|\bigg\{\mathcal{L}_1^{\frac{1}{p}}\Big[w\Big(\lambda-\frac{s}{2}\lambda^2\Big)|\rho'(u)|^q+\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big)|\rho'(z)|^q\Big]^{\frac{1}{q}}\\
&\qquad+\mathcal{L}_2^{\frac{1}{p}}\Big[w\Big(\lambda-s\lambda+\frac{s}{2}\lambda^2\Big)|\rho'(u)|^q+\Big(\lambda-\frac{s}{2}\lambda^2\Big)|\rho'(z)|^q\Big]^{\frac{1}{q}}\bigg\}\\
&=\left(\int_0^{\frac{1}{2}}\left|\frac{\sqrt{2}}{4}-\left(\frac{1}{2}-t\right)^{\frac{1}{2}}\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left\{\left(\frac{3}{8}+\frac{1}{8}e^2\right)+\left(\frac{1}{8}+\frac{3}{8}e^2\right)\right\}\\
&\approx 0.8843501.
\end{aligned}$$

Obviously to see that $0.089819 < 0.8843501$, *which demonstrates the correctness of the outcome derived in Theorem 2.9.*

Remark 3.4. *Case 1: Suppose that the parameter α is not a fixed constant in Example 3.3. For instance, $\alpha \in [1,3]$, on the basis of Theorem 2.9, we infer the result for the parameter α as below:*

$$\begin{aligned}
&-\left(\int_0^{\frac{1}{2}}\left|\frac{1}{2}\left(\frac{1}{2}\right)^\alpha-\left(\frac{1}{2}-t\right)^\alpha\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left\{\left(\frac{3}{8}+\frac{1}{8}e^2\right)^{\frac{1}{2}}+\left(\frac{1}{8}+\frac{3}{8}e^2\right)^{\frac{1}{2}}\right\}\\
&\quad\leq\left(\frac{1}{2}\right)^\alpha\left[2e^{\frac{1}{2}}+\frac{1}{2}+\frac{1}{2}e\right]-\alpha\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^{\alpha-1}e^t\mathrm{d}t+\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)^{\alpha-1}e^t\mathrm{d}t\right) \qquad (21)\\
&\quad\leq\left(\int_0^{\frac{1}{2}}\left|\frac{1}{2}\left(\frac{1}{2}\right)^\alpha-\left(\frac{1}{2}-t\right)^\alpha\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left\{\left(\frac{3}{8}+\frac{1}{8}e^2\right)^{\frac{1}{2}}+\left(\frac{1}{8}+\frac{3}{8}e^2\right)^{\frac{1}{2}}\right\}.
\end{aligned}$$

Three functions given by the double inequalities on the left-, middle- as well as right-sides (21) *are plotted in Figure 4 against $\alpha \in [1,3]$. The graphs of the mappings show the validity of dual inequalities.*

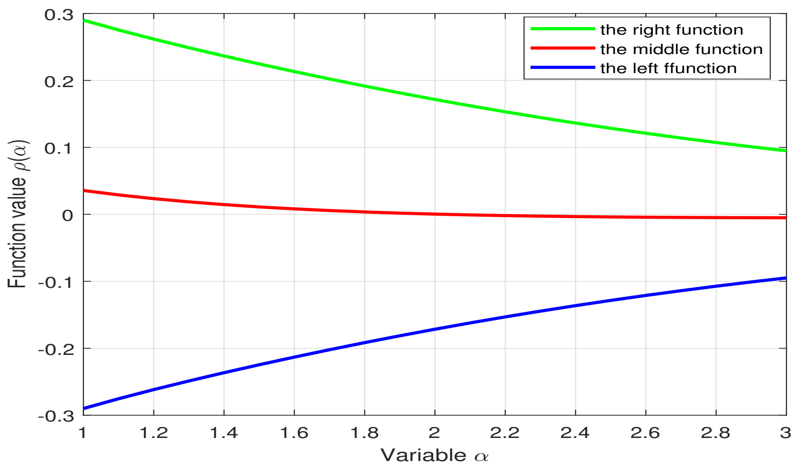

Figure 4: Graphical representation of Example 3.3 for the variable $\theta = \frac{1}{2}$ and $\alpha \in [1,3]$

Case 2: Suppose that the parameter θ is not a fixed constant in Example 3.3. For instance, $\theta \in [1,3]$, based upon

Theorem 2.9, we get the result for the parameter θ as below:

$$\begin{aligned}&-\left(\int_0^{\frac{1}{2}}\left|\frac{\sqrt{2}}{2}\theta-\left(\frac{1}{2}-t\right)^{\frac{1}{2}}\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left\{\left(\frac{3}{8}+\frac{1}{8}e^2\right)^{\frac{1}{2}}+\left(\frac{1}{8}+\frac{3}{8}e^2\right)^{\frac{1}{2}}\right\}\\&\le\frac{\sqrt{2}}{2}\left[2\theta e^{\frac{1}{2}}+(1-\theta)+(1-\theta)e\right]-\frac{1}{2}\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^{-\frac{1}{2}}e^t\mathrm{d}t+\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)^{-\frac{1}{2}}e^t\mathrm{d}t\right)\\&\le\left(\int_0^{\frac{1}{2}}\left|\frac{\sqrt{2}}{2}\theta-\left(\frac{1}{2}-t\right)^{\frac{1}{2}}\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left\{\left(\frac{3}{8}+\frac{1}{8}e^2\right)^{\frac{1}{2}}+\left(\frac{1}{8}+\frac{3}{8}e^2\right)^{\frac{1}{2}}\right\}.\end{aligned}\tag{22}$$

The visualization results of three functions given by the double inequalities on the left-, middle- as well as right-sides (22) *are plotted in Figure 5 against $\theta \in [1,3]$. As can be seen from Figure 5, the results given in Theorem 2.9 are consistently correct if the parameters $\alpha = \frac{1}{2}$ as well as $\theta \in [1,3]$ are given any value.*

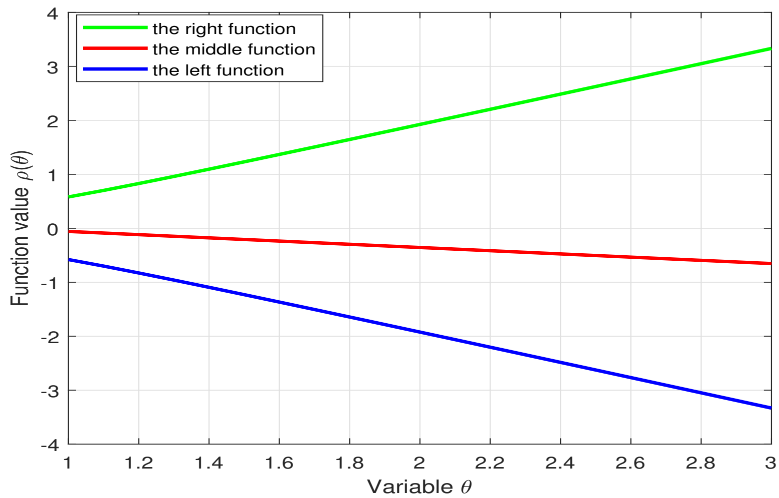

Figure 5: Graphical representation of Example 3.3 for Variable $\alpha = \frac{1}{2}$ and $\theta \in [1,3]$

Case 3: Suppose that the parameters α as well as θ are not two fixed constants in Example 3.3. For instance, $\alpha, \theta \in [1,3]$, based upon Theorem 2.9, one gains the result for the parameters α and θ as below:

$$\begin{aligned}&-\left(\int_0^{\frac{1}{2}}\left|\theta\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{1}{2}-t\right)^{\alpha}\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left\{\left(\frac{3}{8}+\frac{1}{8}e^2\right)^{\frac{1}{2}}+\left(\frac{1}{8}+\frac{3}{8}e^2\right)^{\frac{1}{2}}\right\}\\&\le\left(\frac{1}{2}\right)^{\alpha}\left[2\theta e^{\frac{1}{2}}+(1-\theta)+(1-\theta)e\right]-\alpha\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)^{\alpha-1}e^t\mathrm{d}t+\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)^{\alpha-1}e^t\mathrm{d}t\right)\\&\le\left(\int_0^{\frac{1}{2}}\left|\theta\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{1}{2}-t\right)^{\alpha}\right|^2\mathrm{d}t\right)^{\frac{1}{2}}\left\{\left(\frac{3}{8}+\frac{1}{8}e^2\right)^{\frac{1}{2}}+\left(\frac{1}{8}+\frac{3}{8}e^2\right)^{\frac{1}{2}}\right\}.\end{aligned}\tag{23}$$

Let us discretize the region of $[1,3]\times[1,3]$. From the visual perspective of graphics, Figure 6 vividly describes the result exhibited in Example 3.3.

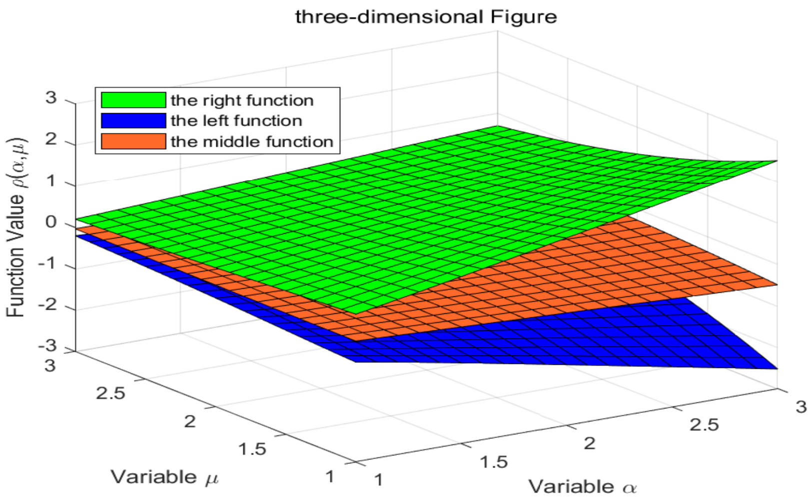

Figure 6: Graphical representation of Example 3.3 for Three-dimensional

4. Conclusion

In the current study, we apply the (μ, η)-incomplete gamma functions to generalize a series of findings, which involve the HH-type integral inequalities in relation with the generalized (s, w)-type previnexity and bounded functions, respectively. To obtain the novel outcomes in the investigation, we propose a multi-parameterized identity through the tempered fractional integrals. Furthermore, the visualization results of two interesting examples are enumerated to identify the correctness of the acquired inequalities. With these contributions, we are convinced that the outcomes of the present study could be a source of enlightenment for researchers working in the fractional integral inequalities field.

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