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On a Karhunen-Loève expansion based on Krawtchouk polynomials with application to Bahadur optimality for the binomial location family

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Abstract. In this paper we extend to the discrete case a Karhunen-Loève expansion already known for continuous families of classical orthogonal polynomials. This expansion involves Krawtchouk polynomials. It provides us with the orthogonal decomposition of the covariance function of a weighted discrete Brownian bridge process. We introduce a discrete Cramér-von Mises statistic associated with this covariance function. We show that this statistic satisfies a property of Bahadur local optimality for a statistical test in the location family for binomial distributions. Our statistic and the goodness-of-fit problem we deal with can be seen as a discrete version of a problem stated by Y. Nikitin about the statistic of de Wet and Venter. Our proofs make use of the formulas valid for all classical orthogonal families of polynomials, so that the way most of our results can be extended to Meixner, Hahn, and Charlier polynomials and the associated distributions is clearly outlined.

1. Introduction

An open problem motivating the present paper was stated by Nikitin [18, p. 79-80], concerning a statistic, associated with Hermite polynomials, introduced by de Wet and Venter (W-V) in [10]. The problem can be stated as follows: is the W-V statistic locally Bahadur optimal for the mean-shift problem with the normal distribution?

The present paper discusses the issue of Bahadur efficiency for a statistic associated with Krawtchouk polynomials and the binomial distribution, in the same way as W-V's statistic is associated with Hermite polynomials and the normal law.

The binomial distribution can be seen as a finite discrete version of the normal law, and our statistic as a discrete version of W-V statistic. It will happen that the property of local optimality holds for our statistic against a mean-shift alternative for a binomial distribution, see Section 6. Thus our paper provides a positive answer to a discrete version of the Nikitin-W-V problem.

Both W-V and our statistics belong to the family of Cramér-von Mises statistics. Orthogonal decompositions provide a powerful tool in the study of these statistics, see [25, Chapter 5]. In this reference, formulas (1) – (2) p. 201 give the general form of the Karhunen-Loève (K-L) expansion of a covariance function, and the K-L representation of the associated centred Gaussian process.

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	(a,b)	$\sigma(x)$	$\omega(x)$	$\lambda_k,\ k \in \mathbb{N}$
Jacobi	$(-1,1)$	$1-x^2$	$(1-x)^\alpha(1+x)^\beta,\ \alpha,\beta > -1$	$k(k+\alpha+\beta+1)$
Laguerre	$(0,+\infty)$	x	$x^\alpha e^{-x},\ \alpha > -1$	k
Hermite	$(-\infty,+\infty)$	1	$e^{-x^2/2}$	k

Table 1: Data for classical continuous orthogonal polynomials.

The well-known Cramér-von Mises and Anderson-Darling statistics belong to the same family. The K-L expansions associated with these two statistics (see Proposition 1 p. 213 and Theorem 1 p. 225 in [25]) allow to derive most of their basic properties.

Our recent result [20, Theorem 6.3] is a generalization of these two historic K-L expansions to all classical orthogonal polynomials in the continuous case. More precisely, our theorem provides the Karhunen-Loève representation of a centred Gaussian process, equivalent to the Karhunen-Loève expansion of its covariance function (as expansion (1) and representation (2) p. 201 in [25] are equivalent), given by

$$\frac{\Omega(x_1)\overline{\Omega}(x_2)}{\sqrt{\sigma(x_1)}\omega(x_1)\sqrt{\sigma(x_2)}\omega(x_2)} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \cdot \frac{d_0 \sqrt{\sigma(x_1)}C_k'(x_1)\sqrt{\sigma(x_2)}C_k'(x_2)}{d_k\lambda_k}, \tag{1}$$

holding for $a < x_1 \le x_2 < b$, whenever $(C_k)_{k\ge 0}$ denotes one of the classical Jacobi, Laguerre or Hermite sequences of orthogonal polynomials, the auxiliary functions associated with these polynomials being given by Table 1 and

$$\Omega(x) = \int_a^x \omega(y)dy,\ \overline{\Omega}(x) = \Omega(b) - \Omega(x) \quad (a < x < b) \tag{2}$$

(see Chapter 22 in [1], Appendix Tables B2, B5 and B8 in [23], or [15] p. 209).

Recall that these polynomials satisfy the orthogonality relations and a differential equation of the form

$$\int_a^b \omega(y)C_k(y)C_\ell(y)dy = d_k\delta_{k,\ell},\ [\sigma(x)\omega(x)C_k'(x)]' = -\lambda_k\omega(x)C_k(x) \tag{3}$$

for $a < x < b$ and $k,\ell \in \mathbb{N}$, where for $k,\ell \in \mathbb{Z}$, $\delta_{k,\ell}$ denotes the Kronecker symbol

$$\delta_{k,\ell} = \begin{cases} 1 & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell. \end{cases}$$

The aim of the present paper is to state a discrete analogue of (1) involving Krawtchouk polynomials, and then outline applications to hypothesis testing and Bahadur efficiency.

Concerning Bahadur efficiency the best introduction remains the original [5]. See also [18] for a survey, more recent advances and a gallery of applications. About recent advances related to Bahadur efficiency, see, among others, [2],[3],[6],[7], [9], [16],[19],[21],[26].

Our paper is organized as follows. In Section 2 we introduce some notations concerning binomial distributions and a discrete Brownian bridge process (13) associated with this distribution.

In Section 3 our first main result is Theorem 3.2, where development (19) is an analogue, for Krawtchouk polynomials, of development (1).

It can be checked that development (1), in the case of Jacobi polynomials with $(\alpha,\beta) = (0,0)$ or $(-1/2, 1/2)$, leads to the K-L expansions associated with Anderson-Darling and Cramer-von Mises statistics mentioned above. Therefore a natural development of this first result lies in the study of the associated statistic.

To this end, the rest of the paper is organized as follows.

In Section 4 we introduce our new statistic defined by (31) − (32). It belongs to the class of the so-called discrete Cramér-von Mises statistics, which were introduced and discussed by [8] and [4].

In Section 5 we derive some of its properties under the null hypothesis of binomial distribution. In particular Proposition 5.2 states a result about the probability of large deviations, a key-result for the subsequent study of Bahadur efficiency.

In Section 6 we study some properties of our statistic under the general alternative and prove its local Bahadur optimality in the case of the mean-shift alternative within the binomial family.

2. A discrete Brownian bridge associated with the binomial distribution

Assume $p = 1 - q \in (0,1)$ and let N be a positive integer. Consider a random variable $\mathbf{X}^{(p,N)}$ with a binomial probability mass function (p.m.f.)

$$P(\mathbf{X}^{(p,N)} = i) = \omega^{(p,N)}(i) = \begin{cases} \binom{N}{i} p^i q^{N-i} & \text{for } i \in \{0,1,...,N\}, \\ 0 & \text{for } i \in \mathbb{Z} \setminus \{0,1,...,N\}. \end{cases} \tag{4}$$

The associated cumulative distribution function (c.d.f.) is given by

$$\Omega^{(p,N)}(i) := P(\mathbf{X}^{(p,N)} \le i) = \sum_{j \le i} \omega^{(p,N)}(j) \quad (i \in \mathbb{Z}), \tag{5}$$

$$\Omega^{(p,N)}(i) = \frac{\Gamma(N+1)}{\Gamma(N-i)\Gamma(i+1)} \int_0^q x^{N-i-1}(1-x)^i dx \quad (0 \le i \le N) \tag{6}$$

(for the last equality, where Γ denotes the gamma function, see formulas (1.35) p. 8, (1.82) – (1.83) p. 17 and (3.18) p. 113 in [12]). The tail distribution is denoted by

$$\overline{\Omega}^{(p,N)}(i) := \sum_{j>i} \omega^{(p,N)}(j) = 1 - \Omega^{(p,N)}(i) \quad (i \in \mathbb{Z}).$$

For us, Krawtchouk polynomials, denoted by $K_k^{(p,N)}(x)$ for $0 \le k \le N$, will be those denoted by $k_k^{(p)}(x,N)$ in [17, Table 2.3-4], i.e.

$$K_k^{(p,N)}(x) = (-1)^k \binom{N}{k} p^k \sum_{j=0}^{k} \frac{(-k)_j (-x)_j}{(-N)_j \, j!} p^{-j} \quad (k = 0,1,...,N),$$

where the Pochhammer symbol $(a)_k$ is defined by

$$(a)_0 = 1, \ (a)_k = a(a+1)\cdots(a+k-1) \quad (k = 1,2,3,...).$$

The first few Krawtchouk polynomials are given by

$$K_0^{(p,0)}(x) = 1, \tag{7}$$

$$K_0^{(p,1)}(x) = 1, \ K_1^{(p,1)}(x) = x - p, \tag{8}$$

$$K_0^{(p,2)}(x) = 1, \ K_1^{(p,2)}(x) = x - 2p, \ K_2^{(p,2)}(x) = p^2 - px + \frac{x(x-1)}{2}. \tag{9}$$

Krawtchouk polynomials satisfy the orthogonality relations

$$\sum_{i=0}^{N} \omega^{(p,N)}(i) K_k^{(p,N)}(i) K_\ell^{(p,N)}(i) = \delta_{k,\ell} d_k^{(p,N)}, \ d_k^{(p,N)} = \binom{N}{k} (pq)^k. \tag{10}$$

They also satisfy the difference equation

$$\Delta[\sigma(i)\omega^{(p,N)}(i) \nabla K_k^{(p,N)}(i)] = -\lambda_k^{(p)} \omega^{(p,N)}(i) K_k^{(p,N)}(i) \quad (0 \le k \le N, i \in \mathbb{Z}) \tag{11}$$

(see [17, (2.1.18) p. 21]), where for any function $f : \mathbb{Z} \to \mathbb{R}$, the forward and backward shift operators are defined by

$$\Delta f(i) = f(i+1) - f(i),\ \nabla f(i) = f(i) - f(i-1) = \Delta f(i-1) \quad (i \in \mathbb{Z}),$$

and with

$$\lambda_k^{(p)} = k/q \quad (0 \le k \le N), \qquad\qquad \sigma(i) = i \quad (0 \le i \le N). \tag{12}$$

Given a Brownian bridge process, i.e. a centred Gaussian process $\mathbf{B} = \{\mathbf{B}(t) : 0 \le t \le 1\}$, with covariance function $(s,t) \mapsto \min(s,t) - st$, we define a discrete Brownian bridge process $\mathbf{D} = \{\mathbf{D}(i) : 1 \le i \le N\}$ by setting

$$\mathbf{D}(i) = \frac{\mathbf{B}\left\{\Omega^{(p,N)}(i-1)\right\}}{\sqrt{\sigma(i)}\,\omega^{(p,N)}(i)}, \quad i \in \{1, ..., N\}. \tag{13}$$

The process is Gaussian centred, with covariance kernel

$$\mathbf{\Gamma}_{i,j}^{(p,N)} = \mathbf{\Gamma}_{j,i}^{(p,N)} = \frac{\Omega^{(p,N)}(i-1)\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(i)\sigma(j)}\,\omega^{(p,N)}(i)\omega^{(p,N)}(j)} \quad (1 \le i \le j \le N). \tag{14}$$

Let us give an orthogonal decomposition of this kernel.

3. Orthogonal decomposition of the weighted discrete Brownian bridge

First, we shall prove the following auxiliary properties of Krawtchouk polynomials.

Proposition 3.1. *Krawtchouk polynomials satisfy*

$$\sum_{i=1}^{N} \nabla K_k^{(p,N)}(i) \nabla K_\ell^{(p,N)}(i) \sigma(i) \omega^{(p,N)}(i) = \delta_{k,\ell}\, d_k^{(p,N)} \lambda_k^{(p)} \quad (1 \le k, \ell \le N). \tag{15}$$

Proof. From [17, (2.4.15) p. 36], we have the first equality

$$\Delta K_k^{(p,N)} = K_{k-1}^{(p,N-1)} \quad (1 \le k \le N).$$

Then from relations, valid for $0 \le i \le N-1$ and $1 \le k \le N$,

$$\lambda_k^{(p)} = \frac{k}{q}, \frac{d_k^{(p,N)}}{d_{k-1}^{(p,N-1)}} = \frac{Npq}{k}, \frac{\omega^{(p,N-1)}(i)}{\sigma(i+1)\,\omega^{(p,N)}(i+1)} = \frac{1}{Np}$$

we obtain the second equality

$$\lambda_k^{(p)} d_k^{(p,N)} \omega^{(p,N-1)}(i) = d_{k-1}^{(p,N-1)} \sigma(i+1) \omega^{(p,N)}(i+1).$$

These two equalities allow us, in turn, to write, given $1 \le k, \ell \le N$,

$$\begin{aligned}
&\sum_{i=0}^{N-1} \Delta K_k^{(p,N)}(i) \Delta K_\ell^{(p,N)}(i) \sigma(i+1) \omega^{(p,N)}(i+1) \\
&= \sum_{i=0}^{N-1} \lambda_k^{(p)} d_k^{(p,N)} \frac{K_{k-1}^{(p,N-1)}(i) K_{\ell-1}^{(p,N-1)}(i) \omega^{(p,N-1)}(i)}{d_{k-1}^{(p,N-1)}} = \lambda_k^{(p)} d_k^{(p,N)} \delta_{k,\ell},
\end{aligned}$$

which is equivalent to (15), since $\Delta K_k^{(p,N)}(i) = \nabla K_k^{(p,N)}(i+1)$. □

Note that the preceding Proposition establishes that the functions defined by

$$\phi_k^{(p,N)}(0) := 0,\ \phi_k^{(p,N)}(i) := \frac{\sqrt{\sigma(i)}\,\nabla K_k^{(p,N)}(i)}{\sqrt{\lambda_k^{(p)} d_k^{(p,N)}}} \quad (1 \le i \le N,\ 1 \le k \le N) \tag{16}$$

form an orthonormal system of functions of $\mathbb{R}^{N+1}$ endowed with the scalar product

$$\langle u|v\rangle_\omega := \sum_{i=0}^{N} u(i)v(i)\omega^{(p,N)}(i) \quad (u, v \in \mathbb{R}^{N+1}),$$

and that (10), (15) and (16) can be rewritten as

$$\langle K_k^{(p,N)}|K_\ell^{(p,N)}\rangle_\omega = d_k^{(p,N)}\,\delta_{k,\ell},\ \langle \phi_k^{(p,N)}|\phi_\ell^{(p,N)}\rangle_\omega = \delta_{k,\ell}. \tag{17}$$

We are now in a position to state our first main result.

Theorem 3.2. *The spectral decomposition*

$$\mathbf{\Gamma}_{i,j}^{(p,N)} = \sum_{k=1}^{N} \frac{1}{\lambda_k^{(p)}}\,\phi_k^{(p,N)}(i)\phi_k^{(p,N)}(j) \quad (1 \le i, j \le N) \tag{18}$$

holds. In other words, Krawtchouk polynomials satisfy the identity

$$\frac{\Omega^{(p,N)}(i-1)\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(i)\sigma(j)\omega^{(p,N)}(i)\omega^{(p,N)}(j)}} = \sum_{k=1}^{N} \frac{1}{\lambda_k^{(p)}} \cdot \frac{\sqrt{\sigma(i)}\nabla K_k^{(p,N)}(i) \cdot \sqrt{\sigma(j)}\nabla K_k^{(p,N)}(j)}{d_k^{(p,N)}\lambda_k^{(p)}}$$
$$(1 \le i \le j \le N). \tag{19}$$

Proof. The first equality is a spectral decomposition in terms of the eigenfunctions provided by Lemma 7.1 in Section 7. Development (19) then follows immediately from (16). □

Note that (19) can written with matrices in the form

$$\Big(\frac{\Omega^{(p,N)}(i-1)\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(i)\sigma(j)\omega^{(p,N)}(i)\omega^{(p,N)}(j)}}\Big)_{\substack{1\le i\le N\\ 1\le j\le N}} = \sum_{k=1}^{N} \frac{1}{\lambda_k^{(p)}} \cdot \frac{1}{d_k^{(p,N)}\lambda_k^{(p)}}\Big(\sqrt{\sigma(i)}\nabla K_k^{(p,N)}(i) \cdot \sqrt{\sigma(j)}\nabla K_k^{(p,N)}(j)\Big)_{\substack{1\le i\le N\\ 1\le j\le N}}. \tag{20}$$

Example 3.3. *Let us write explicitly the first spectral expansions, using relations* (9), (10), *and* (12).
For $N = 1$, (20) *reduces to*

$$\frac{\Omega^{(p,1)}(0)\overline{\Omega}^{(p,N)}(0)}{\sqrt{\sigma(1)\sigma(1)\omega^{(p,N)}(1)\omega^{(p,N)}(1)}} = \frac{1}{\lambda_1^{(p)}} \cdot \frac{\sqrt{\sigma(i)}\nabla K_1^{(p,1)}(1) \cdot \sqrt{\sigma(1)}\nabla K_1^{(p,N)}(1)}{d_1^{(p,1)}\lambda_1^{(p)}},$$

which with numerical values reads

$$\frac{q \times p}{\sqrt{1 \times 1} \times p \times p} = \frac{1}{q^{-1}} \cdot \frac{\sqrt{1} \times 1 \cdot \sqrt{1} \times 1}{pq \times q^{-1}},$$

an obviously true relation.

For $N = 2$, (20) *yields*

$$
\begin{aligned}
&\Big(\frac{\Omega^{(p,2)}(i-1)\overline{\Omega}^{(p,2)}(j-1)}{\sqrt{\sigma(i)\sigma(j)\omega^{(p,2)}(i)\omega^{(p,2)}(j)}}\Big)_{\substack{1\le i\le 2\\ 1\le j\le 2}}\\
&= \frac{1}{\lambda_1^{(p)}}\cdot\frac{1}{d_1^{(p,2)}\lambda_1^{(p)}}\Big(\sqrt{\sigma(i)}\nabla K_1^{(p,N)}(i)\cdot\sqrt{\sigma(j)}\nabla K_1^{(p,N)}(j)\Big)_{\substack{1\le i\le 2\\ 1\le j\le 2}}\\
&\qquad\qquad + \frac{1}{\lambda_2^{(p)}}\cdot\frac{1}{d_2^{(p,2)}\lambda_2^{(p)}}\Big(\sqrt{\sigma(i)}\nabla K_2^{(p,N)}(i)\cdot\sqrt{\sigma(j)}\nabla K_2^{(p,2)}(j)\Big)_{\substack{1\le i\le 2,\\ 1\le j\le 2}}
\end{aligned}
$$

which becomes, with numerical values,

$$
\begin{pmatrix} \frac{1-q^2}{4p^2} & \frac{q}{2\sqrt{2}p} \\ \frac{q}{2\sqrt{2}p} & \frac{1-p^2}{2p^2} \end{pmatrix} = \frac{1}{q^{-1}}\cdot\frac{1}{2pq\times q^{-1}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} + \frac{1}{2q^{-1}}\cdot\frac{1}{p^2q^2\times 2q^{-1}}\begin{pmatrix} p^2 & -\sqrt{2}pq \\ -\sqrt{2}pq & 2q^2 \end{pmatrix}
$$

which is easily checked to be valid.

Remark 3.4. *As classical discrete orthogonal polynomials, Hahn, Meixner and Charlier polynomials (see [17, chapter 2]) also satisfy a difference equation of type* (11)*. Since all our proofs are based on properties shared by these polynomials, developments* (15) *and* (19) *are valid for these three other families as well. We omit details.*

4. A family of discrete Cramér-von Mises statistics

Let $X_1, ..., X_n$ be a sample of size $n \ge 1$ from a population whose distribution, with support $\{0, 1, ..., N\}$, has p.m.f. and c.d.f. denoted by ω and Ω. The observed frequencies associated with our sample are, for $0 \le i \le N$,

$$\hat{n}_i := \sum_{m=1}^{n} \mathbf{1}_{\{X_m=i\}} \quad (0 \le i \le N),$$

the empirical p.m.f. and c.d.f. being denoted and given by

$$\hat{\omega}_n(i) := \frac{\hat{n}_i}{n},\ \hat{\Omega}_n(i) := \sum_{j=0}^{i} \hat{\omega}_n(j) \quad (0 \le i \le N), \tag{21}$$

respectively. Let E denote the expectation operator under the null hypothesis

$$H_0 : \omega = \omega^{(p,N)}. \tag{22}$$

For $1 \le m \le n$, one can associate with X_m the random $(N+1)-$vector V_m, whose components are the random variables

$$V_m(i) = \mathbf{1}_{\{X_m \le i\}} - \Omega^{(p,N)}(i) \quad (0 \le i \le N). \tag{23}$$

For $0 \le i, j \le N$, we clearly have

$$E[\mathbf{1}_{\{X_m \le i\}}] = \Omega^{(p,N)}(i), \tag{24}$$

$$E[\mathbf{1}_{\{X_m \le i\}}\mathbf{1}_{\{X_m \le j\}}] = E[\mathbf{1}_{\{X_m \le \min(i,j)\}}] = \Omega^{(p,N)}(\min(i, j)). \tag{25}$$

These relations imply in turn

$$E[V_m(i)V_m(j)] = \Omega^{(p,N)}(\min[i, j]) - \Omega^{(p,N)}(i)\Omega^{(p,N)}(j) \quad (0 \le i, j \le N). \tag{26}$$

If $1 \le \ell \ne m \le n$, then V_ℓ and V_m are independent, so that

$$E[V_\ell(i)V_m(j)] = E[V_\ell(i)]E[V_m(j)] = 0. \tag{27}$$

Now, consider the empirical process defined over $\{0,1,...,N\}$ by

$$\mathbb{X}_n^{(p,N)}(0) = 0,\ \mathbb{X}_n^{(p,N)}(i) = \mathbb{X}_n^{(p,N)}[X_1,...,X_n](i) = \sqrt{n}\cdot\frac{\hat{\Omega}_n(i-1)-\Omega^{(p,N)}(i-1)}{\sqrt{\sigma(i)}\omega^{(p,N)}(i)} \tag{28}$$

$$= n^{-1/2}\sum_{m=1}^{n}\frac{V_m(i-1)}{\sqrt{\sigma(i)}\,\omega^{(p,N)}(i)} = n^{-1/2}\sum_{m=1}^{n}\mathbb{X}_1^{(p,N)}[X_m](i) \quad (1\le i\le N). \tag{29}$$

Proposition 4.1. *One has, under H_0,*

$$E\{\mathbb{X}_n^{(p,N)}(i)\} = 0,\ E\{\mathbb{X}_n^{(p,N)}(i)\mathbb{X}_n^{(p,N)}(j)\} = \mathbf{\Gamma}_{i,j}^{(p,N)} \quad (1\le i,j\le N). \tag{30}$$

Proof. The first equality is straightforward. The second equality follows, keeping (14) in mind, from (26) − (27), combined with the first expression in (29). □

The discrete Cramér- von Mises statistic, say $T_n^{(p,N)}$, associated with our empirical process, is defined by

$$T_n^{(p,N)} = \|\mathbb{X}_n^{(p,N)}\|_\omega = \left(\sum_{i=1}^{N}\{\mathbb{X}_n^{(p,N)}(i)\}^2\omega^{(p,N)}(i)\right)^{1/2}. \tag{31}$$

The statistic $T_n^{(p,N)}$ has to be thought of as a test statistic, large values of $T_n^{(p,N)}$ being significant, i.e. leading to the rejection of H_0. For computations one can use the equality

$$n(T_n^{(p,N)})^2 = \sum_{i=1}^{N}\frac{\{n\hat{\Omega}_n(i-1)-n\Omega^{(p,N)}(i-1)\}^2}{\sigma(i)\omega^{(p,N)}(i)}, \tag{32}$$

to be compared with the widely used Chi-squared statistic

$$(D_n^{(p,N)})^2 = \sum_{i=0}^{N}\frac{\{n\,\hat{\omega}_n(i)-n\,\omega^{(p,N)}(i)\}^2}{n\omega^{(p,N)}(i)}, \tag{33}$$

Example 4.2. *For $N\le 3$, the first binomial p.m.f. are given by*

$$\begin{array}{llll}\omega^{(p,1)}(0)=q, & \omega^{(p,1)}(1)=p, & & \\ \omega^{(p,2)}(0)=q^2, & \omega^{(p,2)}(1)=2pq & \omega^{(p,2)}(2)=p^2, & \\ \omega^{(p,3)}(0)=q^3, & \omega^{(p,3)}(1)=3q^2p, & \omega^{(p,3)}(2)=3qp^2, & \omega^{(p,3)}(3)=p^3,\end{array}$$

and the associated c.d.f. by

$$\begin{array}{llll}\Omega^{(p,1)}(0)=q, & \Omega^{(p,1)}(1)=1, & & \\ \Omega^{(p,2)}(0)=q^2, & \Omega^{(p,2)}(1)=q^2+2pq, & \Omega^{(p,2)}(2)=1 & \\ \Omega^{(p,3)}(0)=q^3, & \Omega^{(p,3)}(1)=q^3+3q^2p, & \Omega^{(p,3)}(2)=q^3+3q^2p+3qp^2, & \Omega^{(p,3)}(3)=1\end{array}.$$

With the notation (21) for $\hat{n}_i$, the number of times the value $i\in\{0,1,...,N\}$ occurs in our sample, we obtain the following first expressions for our statistic:

$$n(T_n^{(p,1)})^2 = \frac{(\hat{n}_0-nq)^2}{1\times p},$$

$$n(T_n^{(p,2)})^2 = \frac{(\hat{n}_0-nq^2)^2}{1\times 2pq}+\frac{(\hat{n}_0+\hat{n}_1-n[q^2+2pq])^2}{2\times p^2},$$

$$n(T_n^{(p,3)})^2 = \frac{(\hat{n}_0-nq^3)^2}{1\times 3q^2p}+\frac{(\hat{n}_0+\hat{n}_1-n[q^3+3q^2p])^2}{2\times 3qp^2}+\frac{(\hat{n}_0+\hat{n}_1+\hat{n}_2-n[q^3+3q^2p+3qp^2])^2}{3\times p^3}.$$

5. Convergence and large deviations under H_0

Let us first provide an alternative expression defining $T_n^{(p,N)}$. On the first hand, Pythagorean theorem with respect to the orthonormal system $(\phi_k^{(p,N)})_{1\leq k\leq N}$ gives

$$\{T_n^{(p,N)}\}^2 = \|\mathbb{X}_n^{(p,N)}\|_\omega^2 = \sum_{k=1}^{N} \langle \mathbb{X}_n^{(p,N)} | \phi_k^{(p,N)} \rangle_\omega^2 = \sum_{k=1}^{n} \frac{(Z_{k,n}^{(p,N)})^2}{\lambda_k^{(p)}}, \tag{34}$$

where the so-called principal components, introduced by [11] for the continuous case, are given, for $1 \leq k \leq N$, by

$$Z_{k,n}^{(p,N)} := \sqrt{\lambda_k^{(p)}}\, \langle \phi_k^{(p,N)} | \mathbb{X}_n^{(p,N)} \rangle_\omega = \sqrt{\lambda_k^{(p)}} \sum_{i=1}^{N} \phi_k^{(p,N)}(i) \mathbb{X}_n^{(p,N)}(i) \omega^{(p,N)}(i). \tag{35}$$

A fruitful expression for the principal components is available in terms of Krawtchouk polynomials. Convergence in distribution is denoted by the sign $\Rightarrow$, and almost sure convergence by $\xrightarrow{a.s.}$.

Proposition 5.1. *The principal components admit the expressions*

$$Z_{k,n}^{(p,N)} = -\left(nd_k^{(p,N)}\right)^{-1/2} \sum_{m=1}^{n} K_k^{(p,N)}(X_m) \quad (1 \leq k \leq N). \tag{36}$$

Furthermore, under H_0, the equalities and the convergence in law

$$E(Z_{k,n}^{(p,N)}) = 0,\ E(Z_{k,n}^{(p,N)} Z_{\ell,n}^{(p,N)}) = \delta_{k,\ell},\ Z_{k,n}^{(p,N)} \Rightarrow \xi_k \quad (1 \leq k \leq N,\ n \in \mathbb{N}^*) \tag{37}$$

and

$$\hat{\omega}_n \xrightarrow{a.s.} \omega^{(p,N)}, \quad \sqrt{n/(Npq)}(\hat{\omega}_n - \omega^{(p,N)}) \Rightarrow \mathcal{N}(0, \mathbf{I}),\ (T_n^{(p,N)})^2 \Rightarrow \sum_{k=1}^{N} \frac{\xi_k^2}{\lambda_k^{(p)}}, \tag{38}$$

hold, where $\mathbf{I}$ denotes the unit matrix of order $(N+1)$, and $\xi_1, ..., \xi_N$ are independent standard normal random variables .

Proof. As for (36), note first that for $n = 1$, in view of (17) (repeatedly used in the present proof) and (28), we have

$$\begin{aligned}
&\sqrt{\lambda_k^{(p)} d_k^{(p,N)}}\, \langle \phi_k^{(p,N)} | \mathbb{X}_1^{(p,N)} \rangle_\omega = \sum_{i=1}^{N} [\mathbf{1}_{\{X_1 \leq i-1\}} - \Omega^{(p,N)}(i-1)] \nabla K_k^{(p,N)}(i) \\
&= \sum_{i\in\mathbb{Z}} [\mathbf{1}_{\{X_1 \leq i-1\}} - \Omega^{(p,N)}(i-1)] \nabla K_k^{(p,N)}(i) \\
&= \sum_{i\in\mathbb{Z}} [\mathbf{1}_{\{X_1 \leq i-1\}} - \mathbf{1}_{\{X_1 \leq i\}}]\, K_k^{(p,N)}(i) + \sum_{i\in\mathbb{Z}} \left[-\Omega^{(p,N)}(i-1) + \Omega^{(p,N)}(i)\right] K_k^{(p,N)}(i) \\
&= -K_k^{(p,N)}(X_1) + \langle K_0^{(p,N)} | K_k^{(p,N)} \rangle_\omega = -K_k^{(p,N)}(X_1)
\end{aligned}$$

Then (36) follows from the second equality of (29).

The two equalities in (37) are straightforward consequences of (36), combined with the equalities, valid for $1 \leq m \leq n$,

$$E\{K_k^{(p,N)}(X_m)\} = \langle K_k^{(p,N)} | K_0^{(p,N)} \rangle_\omega = 0,\ E\{[K_k^{(p,N)}]^2(X_m)\} = \langle K_k^{(p,N)} | K_k^{(p,N)} \rangle_\omega = d_k^{(p,N)}.$$

Relations (38) are direct consequences of the law of large numbers and the central limit theorem. □

We are in the case (discrete finite support) where an important result about the probability of large deviations can be obtained by simply applying Sethuraman's Theorem [24], completed by [22, Lemma 2.2]. We will use, for reference to their assumptions, their restatement [18, Theorem 1.6.3]. Recall that as $t \to t_0$, the notation $a(t) \sim b(t)$ means that functions a and b satisfy $\lim_{t\to t_0}[a(t)/b(t)] = 1$.

Proposition 5.2. *Under H_0, there exists a function f, such that in a neighbourhood of 0, f is continuous and*

$$\lim_{n\to\infty} n^{-1}\log[P(T_n^{(p,N)} > n^{1/2}t)] = -f(t), with f(t) \sim \frac{\lambda_1^{(p)}}{2}t^2 \quad (t \to 0). \tag{39}$$

Proof. Consider, for $1 \le m \le n$, the random vector $\Upsilon_m \in \mathbb{R}^{N+1}$, equal in law to $\mathbb{X}_1^{(p,N)}$, with components

$$\Upsilon_m(0) = 0,\ \Upsilon_m(i) = \frac{V_m(i-1)}{\sqrt{\sigma(i)}\omega^{(p,N)}(i)} \quad (1 \le i \le N)$$

with reference to definition (23). Note first that for every $t \in \mathbb{R}$

$$E\exp\{t\|\Upsilon_1\|_\omega\} < \infty, \tag{40}$$

since this expectation reduces to a finite sum. Thus [18, (1.6.9)] is satisfied.

Let us then prove that

$$\sup\{\mathrm{Var}\, y^*(\Upsilon_1) : y^* \in (\mathbb{R}^{N+1})^*, \|y^*\|_\omega = 1\} = \frac{1}{\lambda_1^{(p)}},$$

where $(\mathbb{R}^{N+1})^*$ is the dual space of $\mathbb{R}^{N+1}$. Any unit element $y^* \in (\mathbb{R}^{N+1})^*$ is associated with a vector $y \in \mathbb{R}^{N+1}$, such that $\|y\|_\omega = 1$, whose action on Υ_1 is given by $y^*(\Upsilon_1) = \sum_{i=0}^N y(i)\Upsilon_1(i)$.

First we have $E[y^*(\Upsilon_1)] = \sum_{i=0}^N y(i)E[\mathbb{X}_1^{(p,N)}(i)] = 0$ (i.e. [18, (1.6.8)] is satisfied) and then

$$\begin{aligned}\mathrm{Var}[y^*(\Upsilon_1)] &= E[y^*(\Upsilon_1)^2] = E\{[\sum_{i=0}^N y(i)\Upsilon_1(i)]^2\} = \sum_{i,j} y(i)y(j)E[\Upsilon_1(i)\Upsilon_1(j)]\\ &= \sum_{i,j} y(i)y(j)E[\mathbb{X}_1^{(p,N)}(i)\mathbb{X}_1^{(p,N)}(j)] = \sum_{i,j} y(i)y(j)\mathbf{\Gamma}_{i,j}^{(p,N)}.\end{aligned}$$

In other words the maximal variance, say σ^2, we are looking for, is the maximal value of the quadratic form with matrix $\mathbf{\Gamma}_{i,j}^{(p,N)}$, over the unit ball. The positive eigenvalues of $\mathbf{\Gamma}_{i,j}^{(p,N)}$ are, in decreasing order, $1/\lambda_1^{(p)}$, ...,$1/\lambda_N^{(p)}$. The maximal variance σ^2 will therefore be $1/\lambda_1^{(p)}$. From [18, 1.6.3] we infer the existence of f such that

$$\lim_{n\to\infty} n^{-1}\log P(\|\Upsilon_1 + \cdots + \Upsilon_n\|_\omega > nt) = -f(t) \sim -\frac{t^2}{2\sigma^2} = -\frac{\lambda_1^{(p)}t^2}{2} \quad (t \to 0)$$

and (39) follows from the equality $\|\Upsilon_1 + \cdots + \Upsilon_n\|_\omega = \|\sqrt{n}\,\mathbb{X}_n^{(p,N)}\|_\omega = \sqrt{n}\,T_n^{(p,N)}$. $\square$

6. Exact slope under H_1 and Bahadur local optimality under the location alternative

Let us apply Bahadur's fundamental result [5, §7] to the sequence of statistics $T = (T_n^{(p,N)})_{n\ge1}$.

Recall the following basic principles of Bahadur's efficiency.

Assume $(X_m)_{m\ge1}$ is a sequence of i.i.d. random variables following a distribution determined by a set of parameter, say $\theta \in \Theta$. Let $\theta_0 \in \Theta$. The efficiency of a test based on the rejection of $H_0 : \theta = \theta_0$ against $H_1 : \theta \ne \theta_0$ for large values of the statistic $T_n = T_n(X_1, ..., X_n)$, is measured by the magnitude of a positive coefficient called the slope of the sequence $T = (T_n)$, denoted by c_T. High values of c_T correspond to a good efficiency of the test. The main way to compute c_T is provided by [5, Theorem 7.2]. Furthermore, an upper bound for c_T is the Kullback-Leibler information number $K(\theta, \theta_0)$, see [5, Theorem 7.5]. The test is asymptotically optimal whenever this upper bound is reached, or locally asymptotically optimal if

$$c_T(\theta) \sim 2K(\theta, \theta_0), \quad \theta \to \theta_0.$$

Given the null hypothesis (22) we will first consider an alternative hypothesis H_1 under which the distribution is a p.m.f., supported, as under H_0, by $\{0, 1, ..., N\}$, and denoted by $(\omega(i))_{0 \le i \le N}$, the associated c.d.f. being $(\Omega(i))_{0 \le i \le N}$ with $\Omega(i) = \sum_{j=0}^{i} \omega(j)$, $0 \le i \le N$.

Let us state a first result, recalling that function f was defined above in Proposition 5.2.

Proposition 6.1. *If the alternative hypothesis $H_1 : \omega \neq \omega_0 = \omega^{(p,N)}$ obtains, then the convergence in probability*

$$\lim_{n\to\infty} n^{-1/2} T_n^{(p,N)} = \{\sum_{i=1}^{N} \frac{[\Omega(i-1) - \Omega^{(p,N)}(i-1)]^2}{\sigma(i)\omega^{(p,N)}(i)}\}^{1/2} =: b^{(p,N)}(\omega) \tag{41}$$

takes place and the exact slope of $T_n^{(p,N)}$ satisfies

$$c_T(\omega) = 2f(b^{(p,N)}(\omega)) \sim \lambda_1^{(p)} b^{(p,N)}(\omega)^2 \quad (\omega \Rightarrow \omega^{(p,N)}). \tag{42}$$

Proof. Relation (41) follows from the law of large numbers applied to (28). Then (39) allows us to use [5, Theorem 7.2], and conclude that (42) holds. □

Let us now focus on the case of a binomial alternative, i.e. $H_1 : \omega = \omega^{(p+\theta,N)}$ for $\theta \neq 0$. In other words, we wish to test the null hypothesis

$$H_0 : \theta = 0 \tag{43}$$

against the alternative

$$H_1 : \theta \neq 0. \tag{44}$$

Let us simplify the notation of function b defined in (41) by putting

$$b^{(p,N)}(\omega^{(p+\theta,N)}) = \tilde{b}(\theta).$$

Recall the Kullback-Leibler information number (introduced by [14]) of $\omega^{(p,N)}$ and $\omega^{(p+\theta,N)}$ is defined to be

$$K(\omega^{(p+\theta,N)}, \omega^{(p,N)}) = \sum_{i=0}^{N} \omega^{(p+\theta,N)}(i) \log \frac{\omega^{(p+\theta,N)}(i)}{\omega^{(p,N)}(i)}.$$

Theorem 6.2. *If H_1 holds then*

$$\lim_{n\to\infty} n^{-1/2} T_n^{(p,N)} = \tilde{b}(\theta) := (\sum_{i=1}^{N} \frac{\{\Omega^{(p+\theta,N)}(i-1) - \Omega^{(p,N)}(i-1)\}^2}{\sigma(i)\omega^{(p,N)}(i)})^{1/2} \tag{45}$$

$$\sim (\frac{N\theta^2}{p})^{1/2} \quad (\theta \to 0). \tag{46}$$

Therefore, the exact slope of $T_n^{(p,N)}$ satisfies

$$c_T(\theta) = 2f(\tilde{b}(\theta)) \sim \frac{N\theta^2}{pq} \sim 2K(\omega^{(p+\theta,N)}, \omega^{(p,N)}) \quad (\theta \to 0), \tag{47}$$

so that the statistic $T_n^{(p,N)}$ is locally asymptotically optimal in the sense of Bahadur, with respect to the statistical problem (43) – (44).

Proof. The first result is a direct consequence of (41), combined with Lemma 7.2. Then (42), rewritten with $\lambda_1^{(p)} = q^{-1}$ and $b(\omega)$ replaced by $\tilde{b}(\theta)$ given by (45), leads to the first equality and equivalence in (47).

Finally, Fisher information being given by $\mathcal{I}^{(N,p)} = N/(pq)$ (see, e.g., Theorem 9.17 and Example 9.20 in [27]), Kullback-Leibler divergence satisfies

$$K(\omega^{(p+\theta,N)}, \omega^{(p,N)}) \sim \frac{1}{2}\mathcal{I}^{(N,p)}\theta^2 = \frac{N\theta^2}{2pq} \quad (\theta \to 0) \tag{48}$$

(see (2.7) in [14]), and the last equivalence in (47) readily follows. □

Remark 6.3. *A new goodness-of-fit test for a distribution as standard as the binomial might seem of little use. It is, however, not the case. As mentioned already by [13], and more recently by [12, §3.8.4], goodness-of-fit tests for discrete distributions are, for long, not researched as extensively as those for continuous distributions. Therefore, in view of its fairly good theoretical asymptotic properties, our statistic might prove a useful tool in this field. This aspect should be discussed in a paper devoted to this issue, and most of all including simulations for comparisons with other widely used tests.*

7. Useful technical results

We shall repeatedly use the summation by part formula,

$$\sum_{i=a}^{b-1} f(i)\Delta g(i) = [f(i)g(i)]_a^b - \sum_{i=a}^{b-1} \Delta f(i) g(i+1),$$

with integers $a < b$.

Lemma 7.1. *One has, for $j, k \in \{1, ..., N\}$,*

$$\sum_{i=1}^{N} \mathbf{\Gamma}_{i,j}^{(p,N)}[\sqrt{\sigma(i)}\nabla K_k^{(p,N)}(i)]\omega^{(p,N)}(i) = \frac{1}{\lambda_k^{(p)}}\sqrt{\sigma(j)}\nabla K_k^{(p,N)}(j).$$

Proof. On the first hand,

$$\begin{aligned}
&\sum_{i=1}^{j-1} \mathbf{\Gamma}_{i,j}^{(p,N)}[\sqrt{\sigma(i)}\nabla K_k^{(p,N)}(i)]\omega^{(p,N)}(i) = \frac{\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)} \sum_{i=1}^{j-1} \Omega^{(p,N)}(i-1)\nabla K_k^{(p,N)}(i) \\
&= \frac{\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)} \sum_{i=0}^{j-1} \Omega^{(p,N)}(i-1)\nabla K_k^{(p,N)}(i) = \frac{\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)} \sum_{i=-1}^{j-2} \Omega^{(p,N)}(i)\Delta K_k^{(p,N)}(i) \\
&= \frac{\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)} \left\{ [\Omega^{(p,N)}(i)K_k^{(p,N)}(i)]_{-1}^{j-1} - \sum_{i=-1}^{j-2} [\Delta\Omega^{(p,N)}(i)K_k^{(p,N)}(i+1)] \right\} \\
&= \frac{\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)} \left\{ [\Omega^{(p,N)}(i)K_k^{(p,N)}(i)]_{-1}^{j-1} - \sum_{i=-1}^{j-2} [\omega^{(p,N)}(i+1)K_k^{(p,N)}(i+1)] \right\} \\
&= \frac{\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)} \left\{ [\Omega^{(p,N)}(i)K_k^{(p,N)}(i)]_{-1}^{j-1} + \frac{1}{\lambda_k^{(p)}} \sum_{i=-1}^{j-2} \Delta[\sigma(i+1)\omega^{(p,N)}(i+1)\nabla K_k^{(p,N)}(i+1)] \right\} \\
&\qquad = \frac{\overline{\Omega}^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)} \left\{ \Omega^{(p,N)}(j-1)K_k^{(p,N)}(j-1) + \frac{1}{\lambda_k^{(p)}}\sigma(j)\omega^{(p,N)}(j)\nabla K_k^{(p,N)}(j) \right\}.
\end{aligned}$$

On the other hand, in the same way,

$$\sum_{i=j}^{N}\mathbf{\Gamma}_{i,j}^{(p,N)}[\sqrt{\sigma(i)}\nabla K_k^{(p,N)}(i)]\omega^{(p,N)}(i)=\frac{\Omega^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)}\sum_{i=j}^{N}\overline{\Omega}^{(p,N)}(i-1)\nabla K_k^{(p,N)}(i)$$

$$=\frac{\Omega^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)}\sum_{i=j-1}^{N-1}\overline{\Omega}^{(p,N)}(i)\Delta K_k^{(p,N)}(i)$$

$$=\frac{\Omega^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)}\left\{[\overline{\Omega}^{(p,N)}(i)K_k^{(p,N)}(i)]_{j-1}^{N}-\sum_{i=j-1}^{N-1}[\Delta\overline{\Omega}^{(p,N)}(i)K_k^{(p,N)}(i+1)]\right\}$$

$$=\frac{\Omega^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)}\left\{[\overline{\Omega}^{(p,N)}(i)K_k^{(p,N)}(i)]_{j-1}^{N}+\sum_{i=j-1}^{N-1}[\omega^{(p,N)}(i+1)K_k^{(p,N)}(i+1)]\right\}$$

$$=\frac{\Omega^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)}\left\{[\overline{\Omega}^{(p,N)}(i)K_k^{(p,N)}(i)]_{j-1}^{N}-\frac{1}{\lambda_k^{(p)}}\sum_{i=j-1}^{N-1}\Delta[\sigma(i+1)\omega^{(p,N)}(i+1)\nabla K_k^{(p,N)}(i+1)]\right\}$$

$$=\frac{\Omega^{(p,N)}(j-1)}{\sqrt{\sigma(j)}\omega^{(p,N)}(j)}\left\{-\overline{\Omega}^{(p,N)}(j-1)K_k^{(p,N)}(j-1)+\frac{1}{\lambda_k^{(p)}}\sigma(j)\omega^{(p,N)}(j)\nabla K_k^{(p,N)}(j)\right\}.$$

By summing these two equalities, we obtain

$$\sum_{i=1}^{N}\mathbf{\Gamma}_{i,j}^{(p,N)}\sqrt{\sigma(i)}\nabla K_k^{(p,N)}(i)\omega^{(p,N)}(i)=[\Omega^{(p,N)}(j-1)+\overline{\Omega}^{(p,N)}(j-1)]\cdot\frac{1}{\lambda_k^{(p)}}[\sqrt{\sigma(j)}\nabla K_k^{(p,N)}(j)],$$

which is the desired result. □

Lemma 7.2. *One has, as $\theta\to 0$, for $1\le i\le N$*

$$\Omega^{(p+\theta,N)}(i-1)-\Omega^{(p,N)}(i-1)\sim -N\theta\omega^{(p,N-1)}(i-1), \tag{49}$$

$$\frac{\{\Omega^{(p+\theta,N)}(i-1)-\Omega^{(p,N)]}(i-1)\}^2}{\sigma(i)\omega^{(p,N)}(i)}\sim\frac{N\theta^2}{p}\omega^{(p,N-1)}(i-1), \tag{50}$$

$$\sum_{j=1}^{N}\frac{\{\Omega^{(p+\theta,N)}(j-1)-\Omega^{(p,N)]}(j-1)\}^2}{\sigma(j)\omega^{(p,N)}(j)}\sim\frac{N\theta^2}{p}. \tag{51}$$

Proof. From (6) we infer, for $0\le i\le N$,

$$\frac{\partial\Omega^{(p,N)}(i)}{\partial p}=-\frac{N!}{(N-i-1)!i!}p^iq^{N-1-i}=-N\omega^{(p,N-1)}(i),$$

from which (49) follows. Then (50) is straightforward in view of the equality

$$\sigma(i)\omega^{(p,N)}(i)=Np\omega^{(p,N-1)}(i-1).$$

Then (51) follows from the fact that $\sum_{i=0}^{N-1}\omega^{(p,N-1)}(i)=1$. □

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