Filomat 37:14 (2023), 4761–4770
https://doi.org/10.2298/FIL2314761B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Addressing impulsive fractional integro-differential equations with Caputo-Fabrizio via monotone iterative technique in Banach spaces

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Abstract. In this paper, we investigate the existence of solutions for initial value problem of the impulsive integro-differential for fractional differential equations involving a Caputo-Fabrizio fractional derivative of order $r \in (0, 1)$ in Banach spaces. Under some monotonicity conditions and the noncompactness measure condition of nonlinearity functions. We obtain the existence of extremal solutions between lower and upper solutions.

1. Introduction

Recently, fractional differential equations have attracted considerable interest in both mathematics and applications, since they have been proved to be valuable tools in modeling many physical phenomena (see [3, 9]). There has been significant development in fractional differential equations in recent years, see the monographs of Samko et al.[44], Kilbas et al.[32], Miller and Ross [37], Podlubny [43], the papers [1, 18, 20] and the references therein.

Heymans and Podlubny [31] have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals on the field of the viscoelasticity, and such initial conditions are more appropriate than physically interpretable initial conditions.

But now a days the researchers are studying a new type of fractional derivative which is called Caputo-Fabrizio fractional derivative. In 2015, Caputo and Fabrizio together introduced this derivative [17]. Latter on, Caputo-Fabrizio derivative was used by many researchers for modeling various problems in engineering sciences see [41]. Further, this type of derivative have many applications. Such as it is use an exponential decay kernel to a novel HIV/AIDS epidemic model that includes an anti-retrovirus treatment compartment [42], and also some researcher apply this new type of the fractional for the dynamical system with both chaotic and non-chaotic behaviors [13], hyper-chaotic behaviors, optimal control and synchronization [14], nonstandard finite differece scheme and non-identical synchronization of noval fractional chaotic system [15].

On the other hand, fractional impulsive differential equations have played an important role in the modeling of phenomena, chiefly in the description of dynamics to sudden changes as well as other phenomena

2020 *Mathematics Subject Classification*. 26A33, 47H08
Keywords. Fractional dierential equation, Caputo-Fabrizio integral of fractional order, Caputo-Fabrizio fractional derivatives, Lower and upper solutions
Received: 12 August 2022; Revised: 26 November 2022; Accepted: 28 December 2022
Communicated by Maria Alessandra Ragusa
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such as crops, diseases, and nonlinearity.
The theory of impulsive fractional differential equations is new and important branch of fractional differential equation theory, which has an extensive physical background and realistic mathematical model and hence has been emerging as an important area of investigation in recent years, see [12].

The monotone iterative technique based on lower and upper solutions is an effective and flexible mechanism see [7, 10]. It yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions.

In this paper, we use a monotone iterative technique in the presence of lower and upper solutions to discuss the existence of solutions for the initial value problem of the impulsive differential equations the type Caputo-Fabrizio derivative order, in a Banach space E

$$\begin{cases} {}^{CF}D_0^r u(t) = f(t, u(t), Gu(t)); \quad \text{a.e. } t \in J := [0, b], \ \ t \neq t_k, \quad k = 1, 2, \cdot\cdot, m, \\ \Delta u|_{t=t_k} = I_k(u_k(t_k)), \quad k = 1, 2, \cdot\cdot, m, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $b > 0$, ${}^{CF}D_0^r$ is the standard Caputo-Fabrizio fractional derivative of order $r \in (0, 1)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_k \in C(E, E)$, is an impulsive function, and $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$,$k = 1, 2, \cdots, m$, $u_0 \in E$, and $f : J \times E \to E$ map satisfying some assumptions. The operator G is given by

$$Gu(t) = \int_0^t K(t, s)u(s)ds, \tag{2}$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \{(t, s) : 0 \leq s \leq t \leq b\}$. Throughout this paper, we always assume that

$$K_0 = \sup_{t \in J} \int_0^t K(t, s)ds.$$

2. Preliminaries

Let $J := [0, b]$ and E be an ordered Banach space with the norm $\| \cdot \|_E$ and partial order $\leq$, whose positive cone $P = \{u \in E, u \geq \theta\}$ is normal with normal constant N. Let $C(J, E)$ be the spaces of E-valued continuous functions on I, respectively, endowed with the uniform norm topology

$$\|u\|_\infty = \sup\{\|u(t)\|, t \in J\}.$$

Set $L^1(J, E)$ denote the Banach space of functions, $u : J \to E$ which are Bochner integrable and normed by

$$\|f\|_{L^1} = \int_0^b \|f(t)\|dt.$$

Let $AC(J, E) = \{u : I \to E \mid u(t) \text{ is continuous at } t \neq t_k, \text{ and left continuous at } t = t_k, \text{ and } u(t_k^+) \text{ exists}, \ k = 1, 2, \cdots, m\}$. $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$ respectively.

Further, we present some results and properties from the fractional calculus.

Definition 2.1. *[17, 21] The Caputo-Fabrizio fractional integral of order $0 < r < 1$ for a function $h \in L^1(J)$ is defined by*

$${}^{CF}I^r h(\tau) = \frac{2(1-r)}{M(\alpha)(2-r)} h(\tau) + \frac{2r}{M(r)(2-r)} \int_0^\tau h(s)ds, \ \tau \geq 0, \tag{3}$$

where $M(r)$ is normalization constant depending on r.

Definition 2.2. *[17] The Caputo-Fabrizio fractional derivative for a function $h \in AC(J)$ of order $0 < r < 1$, is defined by for $\tau \in J$,*

$$^{CF}D^{r}h(\tau) = \frac{(2-r)M(r)}{2(1-r)} \int_0^{\tau} \exp\left(-\frac{r}{1-r}(\tau - s)\right) h'(s)ds. \tag{4}$$

Note that $(^{CF}D^{r})(h) = 0$ if only if h is a constant function.

Next, we recall some definitions and properties of measure of noncompactness, for more details, we refer the reader to [[5],[8],[33][45]].

Definition 2.3. *Let E be a Banach space and $(\mathcal{A}, \geq)$ a partially ordered set. A map $\beta : \mathcal{P}(E) \longrightarrow \mathcal{A}$ is called a measure of noncompactness (MNC) on E, if for every subset $\Omega \in \mathcal{P}(E)$, we have*

$$\beta(\overline{co}\Omega) = \beta(\Omega).$$

Notice that if D is dense in Ω, then $\overline{co}\Omega = \overline{co}D$ and hence $\beta(\Omega) = \beta(D)$.

Definition 2.4. *[3, 8] A measure of noncompactness β is called*

(a) *Monotone, if $\Omega_0, \Omega_1 \in \mathcal{P}(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$.*

(b) *Nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$ and $\Omega \in \mathcal{P}(E)$.*

(c) *Invariant with respect to the union with compact sets, if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset E$ and $\Omega \in \mathcal{P}(E)$.*

(d) *Real, if $\mathcal{A} = \overline{\mathbb{R}_+} = [0, \infty]$ and $\beta(\Omega) < \infty$ for every bounded Ω.*

(e) *Regular, if th condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.*

(f) *If $\{W_n\}_n^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset and $\lim\limits_{n\to+\infty} \beta(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact,*

(g) *Algebraically semiadditive, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(E)$.*

As example of an MNC, are may consider the Hausdorff measure

$$\chi(\Omega) = \inf\{\epsilon > 0, \text{ for which } \Omega \text{ has a finite } \epsilon\text{-net in } E\}.$$

For any $W \subset AC(J, E)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t u(s)ds : u \in W, \text{ for } t \in J = [0, T] \right\},$$

where $W(s) = \{u(s) \in E : u \in w\}$.

Lemma 2.5. *If $W \subset AC(J, E)$ is bounded and equicontinuous then $\overline{co}(W)$ is also bounded equicontinuous continuous on J*

Lemma 2.6. *[8] Let E be a Banach space, $\Omega \subset AC(J, E)$ be bounded and equicontinuous. Then $\beta(\Omega(t))$ is continuous on J, and*

$$\beta(\Omega) = \max_{t\in J} \beta(\Omega(t)) = \beta(\Omega(I)).$$

Lemma 2.7. *[33] Let E be a Banach space, $\Omega = \{u_n\} \subset AC(I, E)$ be bounded and countable set. Then $\beta(\Omega(t))$ is Lebesgue integral on J, and*

$$\beta\left(\left\{ \int_I u_n(t) | n \in \mathbb{N} \right\}\right) \leq 2 \int_I \beta(\Omega(t))dt.$$

Lemma 2.8. *Let E be a Banach space , $D \subset E$ be bounded. Then there exist a countable set $D_0 \subset D$, such that $\beta(D) \leq 2\beta(D_0)$.*

Lemma 2.9. *Let $u \in AC(J)$, and let $\lambda > 0$, the solution of the Cauchy problem ${}^{CF}D^r u(t) + \lambda u(t) = h(t)$ with initial boundary condition defined by $u(0) = u_0$, is given by*

$$u(t) = \frac{u_0}{1+\lambda(1-r)} \exp\left(\frac{-\lambda r t}{1+\lambda(1-r)}\right) + \int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right) h(s) ds \tag{5}$$

Proof. We first apply the Laplace transform, thus it follows that

$$\mathcal{L}({}^{CF}D^r u(t)) = -\lambda \mathcal{L}(u(t)) + H(s), \tag{6}$$

such that $H(s) = \mathcal{L}(h)(s)$, on other hand

$$\begin{aligned} \frac{s\mathcal{L}\{u(t)\}(s) - u_0}{s + r(1-s)} &= -\lambda \mathcal{L}(u(t)) \\ s\mathcal{L}\{u(t)\}(s) - u_0 &= -s\lambda \mathcal{L}(u(t)) - r(1-s)\lambda \mathcal{L}(u(t)), \end{aligned}$$

so

$$\begin{aligned} [s(1+\lambda(1-r)) + \lambda r]\, \mathcal{L}\{u(t)\}(s) &= u_0 \\ \left[s + \lambda r(\lambda(1-r))^{-1}\right] \mathcal{L}\{u(t)\}(s) &= u_0(1+\lambda(1-r))^{-1}. \end{aligned}$$

Hence

$$\mathcal{L}_p(u(t)) = \frac{u_0(1+\lambda(1-r))^{-1}}{s + \lambda r(\lambda(1-r))^{-1}}.$$

Then by (6) we have

$$\mathcal{L}_p = \frac{u_0(1+\lambda(1-r))^{-1}}{s + \lambda r(\lambda(1-r))^{-1}} + \frac{H(s)}{s + \lambda r(\lambda(1-r))^{-1}}$$

Applying the inverse of Laplace transform on previous equation, we obtain (5). □

3. Main Results

We consider the linear initial value problem (LIVPF) of the impulsive fractional differential equation

$$\begin{cases} {}^{CF}D_0^r u(t) + \lambda u(t) = h(t); \quad \text{a.e. } t \in J := [0,T], \;\; t \neq t_k, \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \cdot\cdot, m, \\ u(0) = x, \end{cases} \tag{7}$$

where $h \in L^1(J, E)$, $x \in E$ and $y_k \in E$, for $k = 1, 2, \cdots, m$.

Theorem 3.1. *For any $h \in L^1(I, E)$, $x \in E$ and $y_k \in E$, $k = 1, 2, \cdots, m$, the (LIVPF)(7) has a unique solution $u \in AC(J, E)$ given by*

$$u(t) = \frac{\exp\left(\frac{-\lambda r t}{1+\lambda(1-r)}\right)}{1+\lambda(1-r)} x + \int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right) h(s) ds + \sum_{t_k < t} \exp\left(\frac{-\lambda r(t-t_k)}{1+\lambda(1-r)}\right) y_k \tag{8}$$

Proof. Let $I_k = [t_{k-1}, t_k], k = 1, 2, \cdots, m+1$, where $t_0 = 0$ and $t_{m+1} = T$. If $u \in AC(J, E)$ is a solution of (LIVPF)(7), then the restriction of u in I_k satisfies the initial value problem of the linear fractional differential equation

$$\begin{cases} {}^{CF}D_0^r u(t) + \lambda u(t) = h(t); \quad t_{k-1} < t \leq t_k, \\ u(t_{k-1}^+) = u(t_{k-1}) + y_{k-1}. \end{cases} \tag{9}$$

Hence, in $(t_{k-1}, t_k]$, $u(t)$ can be expressed by

$$u(t) = \frac{\exp\left(\frac{-\lambda rt}{1+\lambda(1-r)}\right)}{1+\lambda(1-r)} u(t_{k-1}) + \exp\left(\frac{-\lambda r(t-t_{k-1})}{1+\lambda(1-r)}\right) y_{k-1} + \int_{t_{k-1}}^{t} \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right) h(s) ds.$$

Iterating successively in the above equality with $u(t_j)$ for $j = k-1, k-2, \cdots, 1$, we see that u satisfies (8). Inversely, we can verify directly that the function $u \in AC(J, E)$ defined by (8) is a solution of (LIVPF)(7). Therefore, the conclusion of Lemma 3.1 holds. □

Evidently, $AC(J, E)$ is also an ordered Banach space with the partial order $'' \leq ''$ reduced by the positive function cone $K_{AC} = \{u \in AC(J, E) \mid u(t) \geq 0, \ t \in I\}$, K_{AC} is also normal with the same normal constant N. For $v(\cdot), w(\cdot) \in AC(J, E)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in AC(J, E) \mid v \leq u \leq w\}$, and $[v(t), w(t)]$ to denote the order interval $\{x \in E \mid v(t) \leq x \leq w(t)\}$ in E. By (8) if $h \geq \theta$, $x \geq \theta$ and $y_k \geq 0$, $k = 1, 2, \cdots, m$, the solution of (LIVPF)(7), $u \geq 0$.

Definition 3.2. *If a function $v(\cdot) \in AC(J, E)$ satisfies that*

$$\begin{cases} {}^{CF}D_0^{\alpha} v(t) \leq f(t, v(t), Gv(t)), \quad t \in J, \\ \Delta v(t)|_{t=t_k} \leq I_k(v(t_k)), \quad k = 1, 2, \cdots, m \\ u(0) \leq u_0, \end{cases} \tag{10}$$

we call it a lower solution of the problem (1), *if all inequalities of* (10) *are inverse, we call it an upper solution of the problem* (1).

Theorem 3.3. *Assume that E is an ordered Banach space, its positive cone P is normal, $f \in C(J \times E, E)$ and $u_0 \in E$. If problem* (1) *has a lower solution $v_0(\cdot) \in AC(J, E)$ and upper solution $w_0(\cdot) \in AC(J, E)$ with $v_0 \leq w_0$, and the following conditions are satisfied*

(H_1) *there exists a constant $\lambda > 0$ such that*

$$f(t, u_2) - f(t, u_1) \geq -\lambda(u_2 - u_1),$$

for any $t \in I$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $Gv_0(t) \leq y_1 \leq y_2 \leq Gw_0(t)$.

(H_2) *$I_k(u)$ is increasing on order interval $[v_0(t), w_0(t)]$ for $t \in J$, $k = 1, 2, \cdots, m$,*

(H_3) *there exists a constant $L > 0$ such that*

$$\beta(\{f(t, u_n)\}) \leq L(\beta(\{u_n\} + \beta(\{y_n\})),$$

for any $t \in I$, and increasing or decreasing monotonic sequences $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{y_n\} \subset [Gv_0(t), Gw_0(t)]$.

(H_4) *let $v_n = Nv_{n-1}$, $w_n = Nw_{n-1}$, $n = 1, 2, \cdots$, such that the sequences $v_n(0)$ and $w_n(0)$ are convergent.*

Then the (IVPF)(1) *has minimal and maximal solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Proof. We define the mapping $A : AC(J, E) \to AC(J, E)$ by

$$Au(t) = \frac{\exp\left(\frac{-\lambda rt}{1+\lambda(1-r)}\right)}{1+\lambda(1-r)}u_0 + \int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right)(f(s, u(s), Gu(s)) + \lambda u(s))ds + \sum_{t_k<t} \exp\left(\frac{-\lambda r(t-t_k)}{1+\lambda(1-r)}\right) I_k(u(t_k)). \tag{11}$$

Clearly, operator A is continuous. By Lemma the solution of (IVPF)(1) is equivalent to fixed point of A. By assumptions (H_1) and (H_2), A is increasing in $[v_0, w_0]$, and maps any bounded set.
We show that $v_0 \leq Av_0$, $Aw_0 \leq w_0$. Let $h(t) = {}^{CF}D_0^\alpha v(t) + \lambda v_0(t)$, by the definition of lower solution, $h \in AC(J, E)$ and $h(t) \leq f(t, v_0(t)) + \lambda v_0(t)$ for $t \in I$. Because $v_0(t)$ is a solution of (LIVPF)(8) for $x = v_0(0)$ and $y_k = \Delta v_0|_{t=t_k}$, $k = 1, 2, \cdots, m$, by Lemma 3.1

$$\begin{aligned} v_0(t) &= \frac{\exp\left(\frac{-\lambda rt}{1+\lambda(1-r)}\right)}{1+\lambda(1-r)}v_0(0) + \int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right)h(s)ds + \sum_{t_k<t} \exp\left(\frac{-\lambda r(t-t_k)}{1+\lambda(1-r)}\right)\Delta v_0|_{t=t_k} \\ &\leq \frac{\exp\left(\frac{-\lambda rt}{1+\lambda(1-r)}\right)}{1+\lambda(1-r)}u_0 + \int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right)h(s)ds + \sum_{t_k<t} \exp\left(\frac{-\lambda r(t-t_k)}{1+\lambda(1-r)}\right)I_k v(t_k) \\ &\leq Av_0, \quad t \in J, \end{aligned}$$

namely $v_0 \leq Av_0$. Similarly, it can be shown that $Aw_0 \leq w_0$. Combining these facts and the increasing property of A in $[v_0, w_0]$ into itself, and $A : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator.
Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Av_{n-1}, \quad w_n = Aw_{n-1}, \quad n = 1, 2, \cdots \tag{12}$$

Then from the monotonicity of A, it follows that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdot\cdot \beta \leq w_n \leq \cdots \leq w_1 \leq w_0. \tag{13}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are uniformly convergent in J.
For convenience, let $B = \{v_n \mid n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$. Since $B = A(B_0)$, by (11) and the boundedness of B_0, we easily see that B is equicontinuous in every interval J'_k, where $J'_1 = [0, t_1]$ and $J'_k = (t_{k-1}, t_k]$, $k = 2, 3, \cdots, m$. From $B_0 = B \cup \{v_0\}$ it follows that $\beta(B_0(t)) = \beta(B(t))$ for $t \in J$. Let $\varphi(t) := \beta(B(t)) = \beta(B_0(t))$, $\quad t \in I$ by Lemma 2.6, $\varphi \in AC(I, \mathbb{R}^+)$.
For $t \in J$ there exists a J'_k such that $t \in J'_k$. By (2) and Lemma2.6, we have

$$\begin{aligned} \beta(G(B_0)(t)) &= \beta\left(\left\{\int_0^t K(t, s)v_{n-1}(s)ds \ \ |n \in \mathbb{N}\right\}\right) \\ &\leq \sum_{j=1}^{k-1} \beta\left(\left\{\int_{t_{j-1}}^{t_j} K(t, s)v_{n-1}(s)ds \ \ |n \in \mathbb{N}\right\}\right) \\ &+\beta\left(\left\{\int_{t_{k-1}}^{t} K(t, s)v_{n-1}(s)ds \ \ |n \in \mathbb{N}\right\}\right) \\ &\leq K_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \beta(B_0(s))ds + K_0 \int_{t_{k-1}}^{t} \beta(B_0(s))ds \\ &= K_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \varphi(s)ds + K_0 \int_{t_{k-1}}^{t} \varphi(s)ds \\ &= K_0 \int_0^t \varphi(s)ds, \end{aligned}$$

and therefore,

$$\int_0^t \beta(G(B_0)(s))ds \le bK_0 \int_0^t \varphi(s)ds. \tag{14}$$

Going from J'_1 to I'_{m+1} interval by interval we show that $\varphi(t) \equiv 0$ in J. For $t \in I'_1$, from (11), using Definition 2.4 and Lemma 2.7, assumption (H_3) and (14), we have

$$\begin{aligned}
\varphi(t) &= \beta(B(t)) = \beta(A(B_0)(t)) \\
&= \beta\left(\left\{\int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right)(f(s, v_{n-1}(s), Gv_{n-1}(s)) + \lambda v_{n-1}(s))ds\right\}\right) \\
&\le 2\int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right)\beta(\{f(s, v_{n-1}(s), Gv_{n-1}(s)) + \lambda v_{n-1}(s)\})ds \\
&\le 2\int_0^t (L(\beta(B_0(s)) + \beta(G(B_0)(s))) + \lambda\beta(B_0(s)))ds \\
&\le 2(L + \lambda + bK_0L)\int_0^t \varphi(s)ds.
\end{aligned}$$

Hence by the Gronwall's inequality, $\varphi(t) \equiv 0$ on J'_1. In particular, $\beta(B(t_1)) = \beta(B_0(t_1)) = \varphi(t_1) = 0$, this means that $B(t_1)$ and $B_0(t_1)$ are precompact in E. Thus $I_1(B_0(t_1))$ is precompact in E, and $\beta(I_1(B_0(t_1))) = 0$. Now, for $t \in J'_2$, by (11) and the above argument for $t \in I'_1$, we have

$$\begin{aligned}
\varphi(t) &= \beta(B(t)) = \beta(A(B_0)(t)) \\
&= \beta\left(\left\{\int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right)(f(s, v_{n-1}(s), Gv_{n-1}(s))) + \lambda v_{n-1}(s))ds\right\}\right) \\
&\quad + \beta(\{I_1(v_{n-1}(t_1))\}) \\
&\le 2\int_0^t \exp\left(\frac{-\lambda r(t-s)}{1+\lambda(1-r)}\right)\beta(\{f(s, v_{n-1}(s), Gv_{n-1}(s))) + \lambda v_{n-1}(s)\})ds \\
&\le 2(L + \lambda + bK_0L)\int_{t_1}^t \varphi(s)ds + \beta(I_1(B_0(s))) \\
&\le 2(L + \lambda + bK_0L)\int_{t_1}^t \varphi(s)ds.
\end{aligned}$$

Again by the Gronwall's inequality, $\varphi(t) \equiv 0$ in J'_2, from which we obtain that $\beta(B_0(t_2)) = 0$ and $\beta(I_2(B_0(t_2))) = 0$.

Continuing such a process interval by interval up to J'_{m+1}, we can prove that $\varphi(t) \equiv 0$ in every J'_k, $k = 1, 2, \cdots, m+1$. For any J_k if we modify the value of v_n at $t = t_{k-1}$ via $v_n(t_{k-1}) = v_n(t^+_{k-1})$, $n \in \mathbb{N}$, then $\{v_n\} \subset C(J_k, E)$ and it is equicontinuous. Since $\beta(\{v_n(t)\} \equiv 0$, $\{v_n(t)\}$ is precompact in E for every $t \in J_k$. By the Arzela-Ascoli theorem, $\{v_n(t)\}$ is precompact in $C(J_k, E)$. Hence, $\{v_n\}$ has a convergent subsequence in $C(J_k, E)$. Combining this with the monotonicity (13) we easily prove that $\{v_n\}$ itself is convergent in $C(J_k, E)$. In particular, $\{v_n(t)\}$ is uniformly convergent in J'_k. Consequently, $\{v_n(t)\}$ is uniformly convergent over the whole of J. Using a similar argument to that for $\{v_n(t)\}$, we can prove that $w_n(t)\}$ is also uniformly convergent in J. Hence, $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent in $AC(J, E)$. Set

$$\underline{u} = \lim_{n\to\infty} v_n, \quad \overline{u} = \lim_{n\to\infty}, \quad \text{in} \quad AC(J, E). \tag{15}$$

Letting $n \to \infty$ in (12) and (13), we see that $v_0 \le \underline{u} \le \overline{u} \le w_0$ and

$$\underline{u} = A\underline{u}, \quad \overline{u} = A\overline{u}. \tag{16}$$

By the monotonicity of A, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of A in $[v_0, w_0]$, and therefore, they are the minimal and maximal solutions of (IVPF)(1) in $[v_0, w_0]$, respectively. This completes the proof of Theorem 3.3. □

Now we discuss the uniqueness of the solution to (IVPF)(1) in $[v_0, w_0]$. if we replace the assumption (H_3) by the following assumption:

(H_5) there exist positive constants C_1 and C_2 such that

$$f(t, u_2, y_2) - f(t, u_1, y_1) \leq C_1(u_2 - u_1) + C_2(y_2 - y_1),$$
$$\forall t \in J, \quad \text{and} \quad v_0(t) \leq u_1 \leq u_2 \leq w_0(t), \quad Gv_0(t) \leq y_1 \leq y_2 \leq Gw_0(t),$$

we have the following unique existence result

Theorem 3.4. *Let E be an ordered Banach space, whose positive cone P is normal, $f \in C(J \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$. If the (IVPF)(1) has a lower solution $v_0 \in AC(J, E)$ and an upper solution $w_0 \in AC(J, E)$ with $v_0 \leq w_0$, such that conditions $(H_1) - (H_3)$ and (H_5) hold, then the (IVPF)(1) has a unique solution between v_0 and w_0 which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

Proof. We firstly prove that (H_1) and (H_5) imply (H_3). For $t \in J$, let $\{u_n\} \subset [v_0, w_0]$ be sequence. For $m, n \in \mathbb{N}$ with $m > n$, by (H_1) and (H_5),

$$\begin{aligned}\theta &\leq (f(t, u_m, y_m) - f(t, u_n, y_n)) + \lambda(u_m - u_n) \\ &\leq (C_1 + \lambda)(u_m - u_n) + C_2(y_m - y_n).\end{aligned}$$

By this and the normality of cone P, we have

$$\begin{aligned}\|f(t, u_m) - f(t, u_n)\| &\leq N\|(C_1 + \lambda)(u_1 - u_1) + C_2(y_m - y_n)\| + \lambda\|u_m - u_n\| \\ &\leq (\lambda + N\lambda + NC_1)\|u_m - u_n\| + NC_2\|y_m - y_n\|.\end{aligned}$$

From this inequality and definition of the measure of noncompactness, it follows that

$$\beta(\{f(t, u_n, y_n))\}) \leq (\leq L(\beta(\{u_n\}) + \beta(\{y_n\})), \tag{17}$$

where $L = \lambda + N\lambda + NC_1 + NC_2$. If $\{u_n\}$ are decreasing sequence, the above inequality is also valid. Hence (H_3) holds.

Therefore, by Theorem 3.3, the (IVPF)(1) has minimal solution $\underline{u}$ and maximal solution $\overline{u}$ in $[v_0, w_0]$. By the proof of Theorem3.3,(12),(13), (15), and (16) are valid. Going from J'_1 to J'_{m+1} interval by interval we show that $\underline{u}(t) \equiv \overline{u}(t)$ in every J'_k. For $t \in J'_1$, by (16) and (11) and assumption (H_5), we have

$$\begin{aligned}\theta &\leq \overline{u}(t) - \underline{u}(t) = A\overline{u}(t) - A\underline{u}(t) \\ &= \int_0^t \exp\left(\frac{-\lambda r(t-s)}{1 + \lambda(1-r)}\right)(f(s, \overline{u}(s), G\overline{u}(s)) - f(s, \underline{u}(s), G\underline{u}(s)) + \lambda(\overline{u}(t) - \underline{u}(t)))ds \\ &\leq \int_0^t \exp\left(\frac{-\lambda r(t-s)}{1 + \lambda(1-r)}\right)(\lambda + C_1)(\overline{u}(s) - \underline{u}(s)) + C_2(G\overline{u}(s) - G\underline{u}(s)))ds \\ &\leq (\lambda + C_1 + bC_2K_0)\int_0^t (\overline{u}(s) - \underline{u}(s))ds.\end{aligned}$$

From this and the normalityof cone P it follows that

$$\|\overline{u}(t) - \underline{u}(t)\| \leq N(\lambda + C_1 + bC_2K_0)\int_0^t \|\overline{u}(s) - \underline{u}(s)\|ds.$$

By this the Gronwall's inequality, we obtain that $\underline{u}(t) \equiv \overline{u}(t)$ in J_1'. For $t \in J_2'$, since $I_1(\overline{u}(t_1) = I_1(\underline{u}(t_1)$ using (11) and completely the same argument as above for $t \in J_1'$, we can prove that

$$\begin{aligned} \|\overline{u}(t) - \underline{u}(t)\| &\leq N(\lambda + C_1 + bC_2K_0)\int_0^t \|\overline{u}(s) - \underline{u}(s)\|ds \\ &= N(\lambda + C_1 + bC_2K_0)\int_{t_1}^t \|\overline{u}(s) - \underline{u}(s)\|ds. \end{aligned}$$

Again, by the Gronwall's inequality, we obtain that $\underline{u}(t) \equiv \overline{u}(t)$ in J_2'.
Continuing such a process interval by interval up to J_{m+1}', we see that $\underline{u}(t) \equiv \overline{u}(t)$ over the whole of J. Hence, $u^* := \underline{u} = \overline{u}$ is the unique solution of (IVPF)(1) in $[v_0, w_0]$, which can be obtained by the monotone iterative procedure(15) starting from v_0 or w_0. □

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