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Certain Lie algebraic structures on Riemannian manifolds with semi-symmetric non-metric connection

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Abstract. As a natural consequence of the Levi-Civita connection on a Riemannian manifold, there is a Lie algebra structure on a Riemannian manifold. Lie Algebras and Lie Groups are the mathematical structure of continuous symmetries in physics. In this paper, semi-symmetric non-metric connection is considered instead of Levi-Civita connection of Riemann manifold, and accordingly the existence of algebraic structures is investigated. First, it is shown that there is not always a Lie algebra structure on a Riemannian manifold with a semi-symmetric non-metric connection. Then, necessary and sufficient conditions for Lie admissible algebra, pre-Lie algebra and post Lie algebra on a Riemann manifold with semi-symmetric non-metric connection are obtained depending on geometric terms. In addition, the cases of the Riemannian manifold with such algebraic structures according to the semi-symmetric non-metric connection being Einstein manifold and being flat manifold have been also investigated.

1. Introduction

Pre-Lie algebras have been introduced independently by Vinberg [19] and Gerstenhaber [8]. These algebraic structures have been appeared in different research areas such as combinatorics, mathematical physics, differential geometry, Lie theory and numerical analysis; see [3], [6], [11].

Post-Lie algebras were introduced by Vallette in [18]. They were further analyzed a few years later by Munthe-Kaas and Ludervold [12] showed that such algebraic structures are suitable objects for representing vector fields and flows, parallel transport, analytical and numerical flow maps. For details on the importance of post-Lie algebras and their use in geometric integration, see reference [7].

It is well known that there is a unique Riemannian connection on a Riemannian manifold that is torsion free and compatible with the metric. The existence of connections other than Riemannian connections on a Riemannian manifold has always attracted the attention of researchers. The first studies on this subject is belong to Hayden [9] and Yano[20]. Hayden defined the semi-symmetric metric connection, and Yano [20] later studied these Riemannian manifolds in details. On the other hand, Agashe and Chafle[1] defined the semi-symmetric non-metric connection and they also studied the submanifolds of Riemannian manifolds with this type of connection [2]. Manifolds with this type of connection have been studied extensively by researchers [10, 13, 15, 16, 22]. For example, Özgür and Mihai [13] studied Chen inequalities

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for submanifolds of manifolds with semi-symmetric non-metric connections. Studies on manifolds with non-semi-symmetric non-metric connections and their submanifolds continue to be an important research area, see; [4, 5, 14, 23–25]

The main purpose of this paper is to investigate pre-Lie algebra, Lie admissible algebra and post-Lie algebra structures on the space of the vector fields of Riemannian manifolds with semi-symmetric non-metric connection according to this semi-symmetric non-metric connection. First of all, a bracket is defined with the help of semi-symmetric non-metric connection, and it is examined whether the space of vector fields has a Lie Algebra structure according to this new bracket, and a necessary and sufficient condition has been obtained for it to be Lie algebra. Then we investigate the existence conditions for pre-Lie algebra, Lie admissible algebra and post-Lie algebra on the space of vector fields of Riemannian manifolds with respect to semi-symmetric non-metric connection. The results show that the space of vector fields of Riemannian manifolds with semi-symmetric non-metric connection having Pre-Lie algebra, Lie admissible algebra and post lie algebra structures according to this semi-symmetric non-metric connection gives important information about curvature tensor field of the Levi-Civita connection of Riemannian manifold.

2. Preliminaries

In this section, certain notions related to the paper are reminded. Considering the readers unfamiliar with the subject, the basic notions of manifolds considered in this paper are also given.

Let $\chi(M)$ be the space of vector fields on a manifold M. A linear connection on a manifold is a map degree preserving map

$$\nabla: \ \chi(M)\times\chi(M)\to\chi(M); \ \ (X,Y)\mapsto\nabla_X Y,$$

which satisfies the following
1) Bi-linearity

$$\nabla_X(Y+Z)=\nabla_X Y+\nabla_X Z; \ \ \nabla_{X+Y}Z=\nabla_X Z+\nabla_Y Z,$$

2)$C^\infty(M)$-linearrity in the first argument

$$\nabla_{fX}Y=f\nabla_X Y,$$

3)The Leibniz rule

$$\nabla_X(fY)=X(f)Y+f\nabla_X Y,$$

for all vector fields $X,Y,Z\in\chi(M)$ and $f\in C^\infty(M)$. The torsion tensor of an affine connection $T_\nabla: \ \chi(M)\times\chi(M)\to\chi(M)$ is defined as

$$T_\nabla(X,Y):=\nabla_X Y-\nabla_Y X-[X,Y], \tag{1}$$

for any $X,Y\in\chi(M)$. An affine connection is said to be symmetric if the torsion vanishes. An affine connection on a Riemannian manifold (M,g) is said to be metric compatible if and only if

$$X\langle Y,Z\rangle_g=\langle\nabla_X Y,Z\rangle_g+\langle Y,\nabla_X Z\rangle_g\,,$$

for any $X,Y,Z\in\chi(M)$. The Riemannian curvature tensor of an affine connection

$$R_\nabla: \ \chi(M)\times\chi(M)\times\chi(M)\to\chi(M)$$

is defined as

$$R_\nabla(X,Y)Z=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}Z,$$

for all X,Y and $Z\in\chi(M)$. Directly from the definition it is clear that

$$R_\nabla(X,Y)Z=-R_\nabla(Y,X)Z, \tag{2}$$

for all X, Y and $Z \in \chi(M)$. It also satisfies the Bianchi identities

$$\circlearrowleft_{X,Y,Z}(T_\nabla(T_\nabla(X,Y),Z) + ((\nabla_X T_\nabla)(Y,Z) \quad = \quad \circlearrowleft_{X,Y,Z}(R_\nabla(X,Y)Z) \tag{3}$$

$$\circlearrowleft_{X,Y,Z}(\nabla_X R_\nabla)(Y,Z) \quad + \quad R_\nabla(T_\nabla(X,Y),Z) = 0. \tag{4}$$

$\circlearrowleft_{X,Y,Z}$ denotes cyclic summation with respect to X, Y and Z.

We will now recall the definitions of algebraic notions used in this paper. Since the algebraic notions presented in this paper are new, and especially their relation to geometric structures, an example will be given after each algebraic notion is presented. In this way the reader will see the geometric significance of these algebraic structures.

Definition 2.1. *[8, 19] A pre-Lie algebra over a field k is a k-vector space $\mathcal{A}$ with a binary bilinear product $\triangleright$ that satisfies the pre-Lie identity:*

$$[x,y,z] = a(x,y,z) - a(y,x,z) = 0, \tag{5}$$

for all x, y, $z \in \mathcal{A}$, where $a(x,y,z)$ is the associator of the product $\triangleright$ given by

$$a(x,y,z) = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z. \tag{6}$$

Example 2.2. *[12] Let $\chi(M)$ be the space of vector fields on a manifold M, equipped with a linear connection ∇. The covariant derivative $\nabla_X Y$ of Y in the direction of X defines an $\mathbb{R}$-linear, non-associative binary product $X \triangleright Y$ on $\mathcal{X}(M)$. From (2) and (1), we have*

$$R_\nabla(X,Y)Z = a(X,Y,Z) - a(Y,X,Z) + T_\nabla(X,Y) \triangleright Z \tag{7}$$

for vector fields X, Y and Z. Thus if $R_\nabla = 0$ and ∇ is symmetric, then we get $a(X,Y,Z) - a(Y,X,Z) = 0$. Thus $(\chi(M), \nabla)$ is a pre-lie algebra.

An algebra $(\mathcal{A}, \triangleright)$ is Lie-admissible if

$$\circlearrowleft_{x,y,z}[x,y,z] = 0 \tag{8}$$

where $\circlearrowleft_{x,y,z}$ denotes cyclic summation with respect to x, y and z.

Example 2.3. *[12] Let $\chi(M)$ be the space of vector fields on a manifold M, equipped with a linear connection ∇. If $T_\nabla = 0$, then from (3), we have $\circlearrowleft_{X,Y,Z}(R_\nabla(X,Y)Z) = 0$. Then from (7) we have*

$$\circlearrowleft_{X,Y,Z}(a(X,Y,Z) - a(Y,X,Z)) = 0.$$

Thus $(\chi(M), \triangleright)$ is a Lie-admissible algebra.

Definition 2.4. *[12] A post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \triangleright)$ consists of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ and a binary product $\triangleright : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ such that, for all elements $x, y, z \in \mathfrak{g}$ the following relations hold*

$$x \triangleright [y,z] = [x \triangleright y, z] + [y, x \triangleright z], \tag{9}$$

$$[x,y] \triangleright z = \mathrm{a}_\triangleright(x,y,z) - \mathrm{a}_\triangleright(y,x,z), \tag{10}$$

where the associator $\mathrm{a}_\triangleright(x,y,z) := x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z$.

Example 2.5. *[7] Let $\chi(M)$ be the space of vector fields on a manifold M, equipped with a linear connection ∇. If the connection is flat and has constant torsion, i.e., $R = 0 = \nabla T$, we have that $(\mathcal{X}(M), -T(\cdot,\cdot), \triangleright)$ defines a post-Lie algebra. Indeed, the first Bianchi identity shows that $-T(\cdot,\cdot)$ obeys the Jacobi identity; as T is skew-symmetric it therefore defines a Lie bracket. Moreover, flatness is equivalent to* (10) *as can be seen by inserting* (1) *into the statement $R = 0$, whilst* (9) *follows from the definition of the covariant differential of T:*

$$0 = \nabla T(Y,Z;X) = X \triangleright T(Y,Z) - T(Y, X \triangleright Z) - T(X \triangleright Y, Z).$$

Finally in this section, we will recall the notion of semi-symmetric non-metric connection on a Riemannian manifold from [1]. Let N be a Riemannian manifold with Riemannian metric g and the Levi-Civita connection ∇. Let $\overset{*}{\nabla}$ be a linear connection defined by

$$\overset{*}{\nabla}_X Y = \nabla_X Y + \eta(Y)X \tag{11}$$

for vector fields X and Y on N, where η is a 1− form on N. If we consider Lie-Bracket $[,]$ corresponding to the Levi-Civita connection ∇, then we have

$$\overset{*}{T}(X,Y) = \overset{*}{\nabla}_X Y - \overset{*}{\nabla}_Y X - [X,Y] = \eta(Y)X - \eta(X)Y \tag{12}$$

and

$$\begin{aligned} \overset{*}{R}(X,Y)Z &= \overset{*}{\nabla}_X \overset{*}{\nabla}_Y Z - \overset{*}{\nabla}_Y \overset{*}{\nabla}_X Z - \overset{*}{\nabla}_{[X,Y]} Z \\ &= R_\nabla(X,Y)Z + [(\nabla_X \eta)Z - \eta(X)\eta(Z)]Y \\ &- [(\nabla_Y \eta)Z - \eta(Y)\eta(Z)]X \end{aligned} \tag{13}$$

3. A new Lie algebra on Riemannian manifolds with symmetric non-metric connection

In this section, a bracket is defined with the help of $\overset{*}{\nabla}$ and conditions are investigated for this bracket to provide Jacobi identity. In this way, a new Lie algebra is constructed on the Riemannian manifold (M, g, ∇). Let (M, g, ∇) be a Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$. Then we define a new bracket as

$$[X,Y]_* = \overset{*}{\nabla}_X Y - \overset{*}{\nabla}_Y X \tag{14}$$

for $X, Y \in \chi(M)$. Using (11) we get

$$[X,Y]_* = [X,Y] + \eta(Y)X - \eta(X)Y. \tag{15}$$

It is well known that Lie bracket corresponding to torsion free connection ∇ on a manifold makes $(\chi(M), [,])$ a Lie algebra. So the natural question arises as follows. Does the bracket $[,]_*$ make $(\chi(M), [,]_*)$ a Lie algebra. In this section we are going to show that $(\chi(M), [,]_*)$ is not a Lie algebra in general. Therefore we are going to obtain an appropriate condition for $(\chi(M), [,]_*)$ to be a Lie algebra. More precisely, we have the following result.

Theorem 3.1. *Let (M, g, ∇) be a Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$. Then $(\chi(M), [,]_*)$ is a Lie algebra if and only if*

$$\circlearrowleft_{X,Y,Z} 2d\eta(X,Y)Z = \circlearrowleft_{X,Y,Z} \eta(Z)[X,Y]$$

for $X, Y, Z \in \chi(M)$.

Proof. It is easy to see that $[,]_*$ is bilinear and skew-symmetric. Therefore we need to show that $[,]_*$ satisfies the Jacobi identity $[X,[Y,Z]_*]_* + [Y,[Z,X]_*]_* + [Z,[X,Y]_*]_* = 0$ for $X, Y, Z \in \chi(M)$. From (15) and bilinearity we have

$$\begin{aligned} [X,[Y,Z]_*]_* &= [X,[Y,Z] + \eta(Z)Y - \eta(Y)Z]_* \\ &= [X,[Y,Z]]_* + [X,\eta(Z)Y]_* - [X,\eta(Y)Z]_* \\ &= [X,[Y,Z]] + \eta([Y,Z])X - \eta(X)[Y,Z] \\ &+ [X,\eta(Z)Y] + \eta(Z)\eta(Y)X - \eta(X)\eta(Z)Y \\ &- [X,\eta(Y)Z] - \eta(Y)\eta(Z)X + \eta(X)\eta(Y)Z. \end{aligned}$$

Using the identity $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for differential functions f and g, and vector fields X, Y in the above equation, we get

$$\begin{aligned} [X,[Y,Z]_*]_* &= [X,[Y,Z]] + \eta([Y,Z])X - \eta(X)[Y,Z] \\ &+ \eta(Z)[X,Y] + X(\eta(Z))Y - \eta(X)\eta(Z)Y \\ &- \eta(Y)[X,Z] - X(\eta(Y))Z + \eta(X)\eta(Y)Z. \end{aligned} \tag{16}$$

For the second term, we have

$$\begin{aligned} [Y,[Z,X]_*]_* &= [Y,[Z,X]] + \eta([Z,X])Y - \eta(Y)[Z,X] \\ &+ \eta(X)[Y,Z] + Y(\eta(X))Z - \eta(Y)\eta(X)Z \\ &- \eta(Z)[Y,X] - Y(\eta(Z))X + \eta(Y)\eta(Z)X. \end{aligned} \tag{17}$$

Likewise for the third term, we derive

$$\begin{aligned} [Z,[X,Y]_*]_* &= [Z,[X,Y]] + \eta([X,Y])Z - \eta(Z)[X,Y] \\ &+ \eta(Y)[Z,X] + Z(\eta(Y))X - \eta(Z)\eta(Y)X \\ &- \eta(X)[X,Y] - Z(\eta(X))Y + \eta(Z)\eta(X)Y. \end{aligned} \tag{18}$$

Combining (16), (17), (18) and applying the Jacobi identity for [,], we arrive at

$$\begin{aligned} \circlearrowleft_{X,Y,Z}[X,[Y,Z]_*]_* &= (X\eta(Z) - Z\eta(X) - \eta([X,Z]))Y \\ &+ (Y\eta(X) - X\eta(Y) - \eta([Y,X]))Z \\ &+ (Z\eta(Y) - Y\eta(Z) - \eta([Z,Y]))X \\ &+ \eta(X)[Y,Z] + \eta(Z)[X,Y] + \eta(Y)[Z,X]. \end{aligned} \tag{19}$$

Recall that the exterior diferential operator d for a $r-$ form ω is

$$\begin{aligned} d\omega(X_1,...,X_{r+1}) &= \frac{1}{r+1}\{\sum_{i=1}^{r+1}(-1)^{i+1}X_i(\omega(X_1,...,\hat{X}_i,...,X_{r+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X_{r+1})\} \end{aligned} \tag{20}$$

for vector fields $X_1, ..., X_{r+1}$ on M, where the symbols covered by ˆ are omitted. Thus proof is complete due to (20) and (19). □

4. Pre-Lie and Lie-admissible algebras on Riemannian manifolds with semi-symmetric non-metric connection

In this section, we are going to investigate the necessary and sufficient conditions for the space of vector fields of Riemannian manifolds with semi-symmetric non-metric connection to be pre-lie algebra and Lie admissible algebra. Let (M, g, ∇) be a Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$. Then the covariant derivative $\overset{*}{\nabla}_X Y$ of Y in the direction of X defines an $\mathbb{R}$-linear, non-associative binary product $X \overset{*}{\triangleright} Y$ on $\chi(M)$.We have the following result.

Theorem 4.1. *Let (M, g, ∇) be a Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$. Then $(\chi(M), \overset{*}{\triangleright})$ is a pre-Lie algebra on M if and only if*

$$R_\nabla(X,Y)Z = \circlearrowleft^{sgn}_{X,Y}(\nabla_X\eta)(Z)Y - \eta(Y)\nabla_X Z \tag{21}$$

for $X, Y, Z \in \chi(M)$, where $\circlearrowleft^{sgn}_{X,Y} f(X,Z,Y)$ is defined as $\circlearrowleft^{sgn}_{X,Y} f(X,Z,Y) = f(X,Z,Y) - f(Y,Z,X)$.

Proof. From (5) and (6) we have

$$\begin{aligned}[X,Y,Z] &= a(X,Y,Z) - a(Y,X,Z) \\ &= X \overset{*}{\triangleright} (Y \overset{*}{\triangleright} Z) - (X \overset{*}{\triangleright} Y) \overset{*}{\triangleright} Z \\ &- Y \overset{*}{\triangleright} (X \overset{*}{\triangleright} Z) + (Y \overset{*}{\triangleright} X) \overset{*}{\triangleright} Z.\end{aligned}$$

Using (11) and (2), after a long computation we obtain

$$\begin{aligned}[X,Y,Z] &= R_{\nabla}(X,Y)Z - (\nabla_Y \eta)(Z)X + (\nabla_X \eta)(Z)Y \\ &- \eta(Y)\nabla_X Z + \eta(X)\nabla_Y Z\end{aligned} \tag{22}$$

which gives the assertion. □

We recall that the Ricci tensor field of a Riemannian manifold is given by

$$Ric(X,Y) = \sum_{i=1}^{n} g(R(e_i,X)Y,e_i) \tag{23}$$

for X and Y on M, where $e_1,...,e_n$ are local orthonormal vector fields of M. If the Ricci tensor S is of the form

$$Ric(X,Y) = \lambda g(X,Y) \tag{24}$$

where λ is a constant, then M is called an Einstein manifold [21].

Theorem 4.2. *Let (M,g,∇) be a connected Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$ such that $(\chi(M),\overset{*}{\triangleright})$ is a pre-Lie algebra on M. If*

$$trace(\nabla_X \eta)(.) = trace\eta(.)\nabla_X(.)$$

for arbitrary vector field X on M, then M is an Einstein manifold.

Proof. For a local orthonormal frame field $\{e_i\}_{i=1}^n$ on M, geodesic at $p \in M$. At p, from (22) we have

$$\begin{aligned}Ric(X,Y) &= \sum_{i=1}^{n} g(R(e_i,X)Y,e_i) \\ &= -\sum_{i=1}^{n} g(R(e_i,X)e_i,Y) \\ &= \sum_{i=1}^{n} (\nabla_{e_i}\eta)(e_i)g(X,Y) - (\nabla_X\eta)(e_i)g(e_i,Y) \\ &+ \eta(e_i)g(\nabla_X e_i,Y)\end{aligned}$$

for X and Y on M. Thus, if $trace(\nabla_X\eta)(.) = trace\eta(.)\nabla_X(.)$ then we get

$$Ric(X,Y) = \sum_{i=1}^{n} (\nabla_{e_i}\eta)(e_i)g(X,Y).$$

Since $\sum_{i=1}^{n}(\nabla_{e_i}\eta)(e_i)$ is a function on M, proof follows from [21, page:38,Theorem 3.3]. □

From Theorem 4.1, we have the following result.

Theorem 4.3. *Let (M, g, ∇) be a Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$. Then $(\chi(M), \overset{*}{\triangleright})$ is a Lie admissible algebra on M if and only if*

$$\circlearrowleft_{X,Z,Y} 2d\eta(X, Z)Y + \eta(Y)[Z, X] = 0 \tag{25}$$

for $X, Y, Z \in \chi(M)$.

Proof. From (8) and (22), we get

$$\begin{aligned}\circlearrowleft_{X,Y,Z}[X, Y, Z] &= \circlearrowleft_{X,Y,Z}(R_\nabla(X, Y)Z + [(\nabla_X\eta)Z - (\nabla_Z\eta)X]Y \\ &+ \eta(Y)[Z, X])\end{aligned}$$

Since (M, g, ∇) is a Riemannian manifold R_∇ satisfies the Jacobi identity. Thus we derive

$$\circlearrowleft_{X,Y,Z}[X, Y, Z] = \circlearrowleft_{X,Y,Z}([(\nabla_X\eta)Z - (\nabla_Z\eta)X]Y + \eta(Y)[Z, X]). \tag{26}$$

We now recall that for a torsion free connection ∇ we have the following relation between the exterior differentation of an $r-$ form ω and covariant derivative of an $r-$ form ω as

$$d\omega(X_1, ..., X_{r+1}) = \frac{1}{r+1}\sum_{i=1}^{r+1}(-1)^{r+1}(\nabla_{X_i}\omega)(X_1, ..., \hat{X}_i, ..., X_{r+1})$$

for vector fields $X_1, ..., X_{r+1}$ on M. Thus proof follows from $2d\eta(X, Y) = (\nabla_X\eta)Y - (\nabla_Y\eta)X$. □

(22) gives the following geometric characterization.

Theorem 4.4. *Let (M, g, ∇) be a Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$. Then any two conditions below imply the third;*

(1) $(\chi(M), \overset{*}{\triangleright})$ is a pre-lie algebra on M,
(2) (M, g, ∇) is flat,
(3) $\circlearrowleft^{sgn}_{X,Y}(\nabla_X\eta)(Z)Y - \eta(Y)\nabla_X Z = 0$.

for $X, Y, Z \in \chi(M)$.

5. Post-Lie algebras on Riemannian manifolds with semi-symmetric non-metric connection

In this section, we are going to investigate the necessary and sufficient conditions for the space of vector fields of Riemannian manifolds with semi-symmetric non-metric connection to be Post-Lie algebra. We have the following result.

Theorem 5.1. *Let (M, g, ∇) be a Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$. Then $(\chi(M), \overset{*}{\triangleright}, [,])$ is a post-Lie algebra on M if and only if the following conditions are satisfied*

$$\nabla_{[X,Y]}Z + \eta(Z)[X, Y] = R_\nabla(X, Y)Z + \circlearrowleft^{sgn}_{X,Y}(\nabla_X\eta)(Z)Y - \eta(Y)\nabla_X Z \tag{27}$$

and

$$\circlearrowleft^{sgn}_{Y,Z}(R_\nabla(X, Y)Z + (\nabla_Z\eta)(Y)X - (Y \triangleright X) \triangleright Z + \eta(Z)[X, Y]) = 0 \tag{28}$$

for $X, Y, Z \in \chi(M)$.

Proof. From (22), (11) and (10), it follows that $[X,Y]\overset{*}{\triangleright}Z=[X,Y,Z]$ if and only if

$$\begin{aligned}\nabla_{[X,Y]}Z+\eta(Z)[X,Y] &= R_\nabla(X,Y)Z-(\nabla_Y\eta)(Z)X+(\nabla_X\eta)(Z)Y\\ &- \eta(Y)\nabla_XZ+\eta(X)\nabla_YZ \end{aligned}\tag{29}$$

for $X,Y,Z\in\chi(M)$. From the definition of the covariant differential of T with respect to $\overset{*}{\nabla}$ we have

$$(\overset{*}{\nabla}_X\overset{*}{T})(Y,Z)=\overset{*}{\nabla}_X\overset{*}{T}(Y,Z)-\overset{*}{T}(\overset{*}{\nabla}_XY,Z)-\overset{*}{T}(Y,\overset{*}{\nabla}_XZ).$$

Then using (11) and curvature tensor field $\overset{*}{R}$ with respect to $\overset{*}{\nabla}$, we get

$$\begin{aligned}(\overset{*}{\nabla}_X\overset{*}{T})(Y,Z) &= \overset{*}{R}(X,Y)Z+\overset{*}{R}(Z,X)Y+(Z\triangleright X)\triangleright Y\\ &\quad -(Y\triangleright X)\triangleright Z+\eta(Y)[Z,X]+\eta(Z)[X,Y]\\ &\quad -(X\overset{*}{\triangleright}[Y,Z]-[X\overset{*}{\triangleright}Y,Z]-[Y,X\overset{*}{\triangleright}Z]).\end{aligned}\tag{30}$$

On the other hand, by using (11) and (12) we derive

$$\begin{aligned}(\overset{*}{\nabla}_X\overset{*}{T})(Y,Z) &= (\nabla_X\eta)(Z)Y-(\nabla_X\eta)(Y)Z\\ &\quad +\eta(Y)\eta(X)Z-\eta(X)\eta(Z)Y\end{aligned}\tag{31}$$

for $X,Y,Z\in\chi(M)$. Then (30) and (32) imply that

$$\begin{aligned}(\nabla_X\eta)(Z)Y &- (\nabla_X\eta)(Y)Z+\eta(Y)\eta(X)Z-\eta(X)\eta(Z)Y=\overset{*}{R}(X,Y)Z\\ \overset{*}{R}(Z,X)Y &+ (Z\triangleright X)\triangleright Y-(Y\triangleright X)\triangleright Z+\eta(Y)[Z,X]+\eta(Z)[X,Y]\\ &- (X\overset{*}{\triangleright}[Y,Z]-[X\overset{*}{\triangleright}Y,Z]-[Y,X\overset{*}{\triangleright}Z]).\end{aligned}\tag{32}$$

Inserting (13) in (32), we obtain

$$\begin{aligned}X\overset{*}{\triangleright}[Y,Z] &- [X\overset{*}{\triangleright}Y,Z]-[Y,X\overset{*}{\triangleright}Z]=R_\nabla(X,Y)Z+(\nabla_Z\eta)(Y)X\\ &- (Y\triangleright X)\triangleright Z+\eta(Z)[X,Y]-\{R_\nabla(X,Z)Y+(\nabla_Y\eta)(Z)X\\ &- (Z\triangleright X)\triangleright Y+\eta(Y)[Z,X]\}.\end{aligned}\tag{33}$$

Then proof follows from (29) and (33). □

We also have the following geometric result about flatness of (M,g,∇).

Theorem 5.2. *Let (M,g,∇) be a Riemannian manifold with semi-symmetric non-metric connection $\overset{*}{\nabla}$. Then any two conditions below imply the third;*

(1) The connection $\overset{*}{\triangleright}$ satisfies (9) on $(\chi(M),[,])$,
(2) (M,g,∇) is flat,
(3) $2d\eta(Z,Y)=\circlearrowleft^{sgn}_{Z,Y}((Z\triangleright X)\triangleright Y+\eta(Z)[X,Y])$.

for $X,Y,Z\in\chi(M)$.

Proof. Since ∇ is torsion free, R_∇ satisfies the Jacobi identity, thus we have

$$R_\nabla(X,Y)Z+R_\nabla(X,Z)Y=R_\nabla(Z,Y)X.\tag{34}$$

Then putting (34) and $2d\eta(Z,Y)=(\nabla_Z\eta)(Y)-(\nabla_Y\eta)Z$ in (33) we arrive at

$$\begin{aligned}X\overset{*}{\triangleright}[Y,Z] &- [X\overset{*}{\triangleright}Y,Z]-[Y,X\overset{*}{\triangleright}Z]=R_\nabla(Z,Y)X+2d\eta(Z,Y)X\\ &+ (Z\triangleright X)\triangleright Y+\eta(Z)[X,Y]-(Y\triangleright X)\triangleright Z-\eta(Y)[X,Z]\}\end{aligned}$$

which gives assertion. □

Concluding Remarks

In this paper, the existence of algebraic structures other than Lie algebra in the case of a semi-symmetric and non-metric connection on a Riemannian manifold is investigated. There are studies on a Riemannian manifold by considering many connections other than the Levi-Civita connection. Tripathi [17] has also introduced a new connection that includes 17 different connections. All these connections can be seen in Tripathi's paper. Thus we can conclude that the problem considered in this study may lead to many new research problems.

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