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Some approximation properties of the parametric generalization of Bleimann-Butzer-Hahn operators

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Abstract. The present paper deals with a new generalization of Bleimann-Butzer-Hahn operators that depends on a real non-negative parameter α and is therefore called the α-Bleimann-Butzer-Hahn operators. We examined the uniform convergence of the newly defined operators with the help of the Korovkin type approximation theorem. The rate of convergence is investigated by means of the modulus of continuity and by Lipschitz type maximal functions. A Voronovskaya type theorem is also obtained and lastly graphical examples are given in order to illustrate the convergence of the operators to the given functions.

1. Introduction

It was 1980 that Bleimann, Butzer and Hahn introduced a Bernstein type linear positive operators on $C[0,\infty)$, the space of continuous functions on the interval $[0,\infty)$.

The Bleimann-Butzer-Hahn operators- shortly written as BBH- were defined by

$$(L_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} f\left(\frac{k}{n-k+1}\right), \quad x \geq 0,\ n \in \mathbb{N} \tag{1}$$

and investigated in several different papers. In [7], assuming $f \in C_B[0,\infty)$ (the space of continuous and bounded real valued functions defined on $[0,\infty)$) the authors proved the pointwise and uniform convergence of the operators on $[0,\infty)$ and on any compact subset of $[0,\infty)$, respectively. In [9], the monotonicity properties of the BBH operators were examined. In 1999, Gadjiev and Çakar [13] presented a Korovkin type theorem by using the test functions $\left(\frac{x}{x+1}\right)^{\nu}$, $\nu = 0,1,2$. They first defined a special class of functions -denoted by H_{ϖ} - on some subspace of bounded continuous functions and then proved the uniform convergence of the sequence of linear positive operators to the functions in H_{ϖ}. They applied this result to the BBH operators and investigated the uniform convergence result for these operators.

In 1999, Abel and Ivan [1] obtained some results on the BBH operators by using their connection to the well known Bernstein operators. Altın et al. [4] introduced a bivariate case of the BBH operators and studied their approximation properties. For some relatively new generalizations of BBH operators we recommend the papers ([5], [10], [11], [12], [21]).

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Very recently, a new kind of generalization of some well known operators has appeared, which depends on a nonnegative real parameter α. Unsuprisingly, Bernstein operators were the first to be examined, and it is followed by Baskakov and Meyer König Zeller operators. Chen et. al. [8] constructed a new family of Bernstein operators as

$$B_{n,\alpha}(f;x)=\sum_{i=0}^{n}p_{n,i}^{(\alpha)}(x)f\left(\frac{i}{n}\right)$$

where

$$p_{n,i}^{(\alpha)}(x)=\left\{(1-\alpha)x\binom{n-2}{i}+(1-\alpha)(1-x)\binom{n-2}{i-2}+\alpha(1-x)x\binom{n}{i}\right\}x^{i-1}(1-x)^{n-i-1}$$

for $x\in[0,1], n\geq 2$.

These operators, henceforth, are known as α-Bernstein operators in the literature. In their paper, the authors investigated some basic properties of the newly constructed operators; they proved uniform convergence of the operators and also estimated the error in terms of modulus of continuity. Following this study, there appeared severel studies on α-Bernstein operators and among them the most recent one belong to Sofyalıoğlu et.al. [20]. Inspired by α-Bernstein operators, in 2019, Aral and Erbay [6] proposed a parametric generalization of Baskakov operators and called them α-Baskakov operators. These operators are defined as, for every $f\in C_B[0,\infty)$

$$L_{n,\alpha}(f;x)=\sum_{k=0}^{\infty}p_{n,k}^{(\alpha)}(x)f\left(\frac{k}{n}\right)$$

where $x\in[0,\infty), n\geq 1$ and

$$p_{n,k}^{(\alpha)}(x)=\left\{\alpha\frac{x}{1+x}\binom{n+k-1}{k}-(1-\alpha)(1+x)\binom{n+k-3}{k-2}+(1-\alpha)x\binom{n+k-1}{k}\right\}\frac{x^{k-1}}{(1+x)^{n+k-1}},$$

with $\binom{n-3}{-2}=\binom{n-2}{-1}=0$. They examined the approximation properties of the operators in detail. Using the same basis function $p_{n,k}^{(\alpha)}$, in 2022, Kajlaa et. al [15] introduced α-Baskakov-Jain operators and studied their weighted approximation properties.

Very recently M. Sofyalıoğlu et.al. [19] constructed a parametric generalization of Meyer-König-Zeller operators as

$$(M_r^{(\alpha)}h)(x)=\sum_{y=0}^{\infty}B_{r,y}^{(\alpha)}(x)h\left(\frac{y}{r+y+1}\right)$$

where $r\in\mathbb{N}$, $0\leq\alpha<1$, $x\in[0,1)$ and

$$B_{r,y}^{(\alpha)}(x)=\left\{\alpha x(1-x)\binom{r+y}{y}-(1-\alpha)\binom{r+y-2}{y-2}+(1-\alpha)x\binom{r+y}{y}\right\}x^{y-1}(1-x)^{r}$$

with $\binom{r-2}{-2}=\binom{r-1}{-1}=0$. The Korovkin type theorem and the rate of convergence of the operators are investigated and graphical analysis is demonstrated in this study. For more information on the approximation of the operators and estimations for the rate of the convergence of the operators we refer the papers ([2], [3], [14], [17], [18]).

The above mentioned studies motivated us to construct a new generalization of BBH operators depending on a real nonnegative parameter α. The paper is arranged in the following way: In Section 2, α-Bleimann-Butzer-Hahn operators are constructed. In Section 3, approximation properties of these operators are examined. Section 4 is devoted to the Voronovskaya theorem and in the last section graphical illustrations are given.

2. Construction of the operators

For every $f \in C_B[0,\infty)$, we define the parametric form of BBH operators as follows:

$$(B_n^{(\alpha)} f)(x) = \sum_{k=0}^{n} \ell_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n-k+1}\right) \tag{2}$$

where $x \geq 0, n \in \mathbb{N}$ and $\ell_{n,k}^{(\alpha)}(x)$, is given by

$$\ell_{n,k}^{(\alpha)}(x) = \frac{x^{k-1}}{(1+x)^{n-1}} \left\{ \alpha \frac{x}{1+x}\binom{n}{k} + (1-\alpha)\binom{n-2}{k-2} + (1-\alpha)x\binom{n-2}{k} \right\} \tag{3}$$

with

$$\binom{n-2}{-2} = \binom{n-2}{-1} = 0.$$

For simplicity, we denote the parametric form of BBH operators by α-BBH and one can check that, for $\alpha = 1$,

$$\begin{aligned}
(B_n^{(1)} f)(x) &= \sum_{k=0}^{n} \ell_{n,k}^{(1)}(x) f\left(\frac{k}{n-k+1}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} f\left(\frac{k}{n-k+1}\right) \\
&= (L_n f)(x)
\end{aligned}$$

That is, for $\alpha = 1$, α-BBH operators turn into to the classical BBH operators.

Theorem 2.1. *The α-BBH operators given in (2) can also be written in the form*

$$(B_n^{(\alpha)} f)(x) = (1-\alpha)\sum_{k=0}^{n-1} g_k \binom{n-1}{k} \frac{x^k}{(1+x)^{n-1}} + \alpha \sum_{k=0}^{n} h_k \binom{n}{k} \frac{x^k}{(1+x)^n} \tag{4}$$

where

$$h_k = f\left(\frac{k}{n-k+1}\right) \tag{5}$$

and

$$g_k = \left(1 - \frac{k}{n-1}\right) f\left(\frac{k}{n-k+1}\right) + \frac{k}{n-1} f\left(\frac{k+1}{n-k}\right). \tag{6}$$

Proof. From (2) and (3) we can rewrite the α-BBH operators as

$$(B_n^{(\alpha)} f)(x) = (1-\alpha)(p_1 + p_2) + \alpha \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} f\left(\frac{k}{n-k+1}\right) \tag{7}$$

where

$$p_1 = \sum_{k=0}^{n} \binom{n-2}{k} \frac{x^k}{(1+x)^{n-1}} f\left(\frac{k}{n-k+1}\right) \quad \text{and} \quad p_2 = \sum_{k=0}^{n} \binom{n-2}{k-2} \frac{x^{k-1}}{(1+x)^{n-1}} f\left(\frac{k}{n-k+1}\right).$$

Note that in p_1, the term for $k = n$ and in p_2, the term for $k = 0$ are both zero. So we can rewrite p_1 and p_2 as

$$p_1 = \sum_{k=0}^{n-1} \binom{n-2}{k} \frac{x^k}{(1+x)^{n-1}} f\left(\frac{k}{n-k+1}\right); \qquad \binom{n-2}{n-1} = 0$$

and

$$p_2 = \sum_{k=1}^{n} \binom{n-2}{k-2} \frac{x^{k-1}}{(1+x)^{n-1}} f\left(\frac{k}{n-k+1}\right); \qquad \binom{n-2}{-1} = 0.$$

Arranging the indices of p_2, we get

$$p_2 = \sum_{k=0}^{n-1} \binom{n-2}{k-1} \frac{x^k}{(1+x)^{n-1}} f\left(\frac{k+1}{n-k}\right).$$

Remember that

$$\binom{n-2}{k} = \left(1 - \frac{k}{n-1}\right)\binom{n-1}{k} \quad \text{and} \quad \binom{n-2}{k-1} = \frac{k}{n-1}\binom{n-1}{k}.$$

Using the above identities, we can write

$$p_1 + p_2 = \sum_{k=0}^{n-1} \left[\left(1 - \frac{k}{n-1}\right) f\left(\frac{k}{n-k+1}\right) + \frac{k}{n-1} f\left(\frac{k+1}{n-k}\right)\right] \binom{n-1}{k} \frac{x^k}{(1+x)^{n-1}}$$

which, substituting into the equation (7) gives the desired representation of α-BBH operators. □

Lemma 2.2. *For $n \in \mathbb{N}$, the following identities hold for the operators α-BBH.*

$$B_n^{(\alpha)}(1;x) = 1 \tag{8}$$

$$B_n^{(\alpha)}\left(\frac{t}{1+t};x\right) = \frac{n}{n+1}\frac{x}{1+x} \tag{9}$$

$$B_n^{(\alpha)}\left(\frac{t^2}{(1+t)^2};x\right) = \frac{(n-2)(n+1)+2\alpha}{(n+1)^2}\frac{x^2}{(1+x)^2} + \frac{n+2-2\alpha}{(n+1)^2}\frac{x}{1+x} \tag{10}$$

$$\begin{aligned} B_n^{(\alpha)}\left(\frac{t^3}{(1+t)^3};x\right) &= \frac{(n-2)[(n+2)(n-3)+6\alpha]}{(n+1)^3}\frac{x^3}{(1+x)^3} + \frac{3(n-2)(n+3)-6\alpha(n-3)}{(n+1)^3}\frac{x^2}{(1+x)^2} \\ &\quad + \frac{n+6-6\alpha}{(n+1)^3}\frac{x}{1+x} \end{aligned} \tag{11}$$

$$\begin{aligned} B_n^{(\alpha)}\left(\frac{t^4}{(1+t)^4};x\right) &= \frac{(n-2)(n-3)[(n+3)(n-4)+12\alpha]}{(n+1)^4}\frac{x^4}{(1+x)^4} + \frac{6(n-2)\left(n^2+(1-2\alpha)n-12(1-\alpha)\right)}{(n+1)^4}\frac{x^3}{(1+x)^3} \\ &\quad + \frac{7n^2+(29-36\alpha)n-86(1-\alpha)}{(n+1)^4}\frac{x^2}{(1+x)^2} + \frac{n+14(1-\alpha)}{(n+1)^4}\frac{x}{1+x} \end{aligned} \tag{12}$$

Proof. The proof of the above lemma is not difficult, yet it involves some complicated calculations. We only give the proof for the first three identity.
For $f(x) = 1$, we have $g_k = h_k = 1$ from the equations (5) and (6). Thus it follows

$$\begin{aligned} (B_n^{(\alpha)}1)(x) &= (1-\alpha)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{x^k}{(1+x)^{n-1}} + \alpha\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{(1+x)^n} \\ &= (1-\alpha)L_{n-1}(1;x) + \alpha L_n(1;x) \\ &= 1-\alpha+\alpha = 1. \end{aligned}$$

For $f(x) = \frac{x}{1+x}$, we have

$$\begin{aligned} g_k &= \left(1 - \frac{k}{n-1}\right) f\left(\frac{k}{n-k+1}\right) + \frac{k}{n-1} f\left(\frac{k+1}{n-k}\right) \\ &= \left(1 - \frac{k}{n-1}\right) \frac{k}{n+1} + \frac{k}{n-1} \frac{k+1}{n+1}. \end{aligned}$$

Substitution of the above relation into the equation (4), we obtain

$$\begin{aligned} B_n^{(\alpha)}(\frac{t}{1+t};x) &= (1-\alpha) \sum_{k=0}^{n-1} \left(\frac{n.k}{n^2-1}\right) \binom{n-1}{k} \frac{x^k}{(1+x)^{n-1}} + \alpha \sum_{k=0}^{n} \frac{k}{n+1} \binom{n}{k} \frac{x^k}{(1+x)^n} \\ &= (1-\alpha) \sum_{k=0}^{n-2} \left(\frac{n}{n+1}\right) \binom{n-2}{k} \frac{x^{k+1}}{(1+x)^{n-1}} + \alpha \sum_{k=0}^{n} \frac{k}{n+1} \binom{n}{k} \frac{x^k}{(1+x)^n} \\ &= (1-\alpha) \frac{n}{n+1} \frac{x}{1+x} + \alpha \frac{n}{n+1} \frac{x}{1+x} \end{aligned}$$

from which we get the result. Lastly for $f(x) = \dfrac{x^2}{(1+x)^2}$, we have

$$\begin{aligned} g_k &= \left(1 - \frac{k}{n-1}\right) f\left(\frac{k}{n-k+1}\right) + \frac{k}{n-1} f\left(\frac{k+1}{n-k}\right) \\ &= \left(1 - \frac{k}{n-1}\right) \frac{k^2}{(n+1)^2} + \frac{k}{n-1} \frac{(k+1)^2}{(n+1)^2}. \end{aligned}$$

Hence we have

$$B_n^{(\alpha)}(\frac{t^2}{(1+t)^2};x) = (1-\alpha) \sum_{k=0}^{n-1} \frac{(n+1)\,k^2 + k}{(n-1)\,(n+1)^2} \binom{n-1}{k} \frac{x^k}{(1+x)^{n-1}} + \alpha \sum_{k=0}^{n} \frac{k^2}{(n+1)^2} \binom{n}{k} \frac{x^k}{(1+x)^n}.$$

Making the necessary computations yields,

$$B_n^{(\alpha)}(\frac{t^2}{(1+t)^2};x) = (1-\alpha) \left\{ \frac{n-2}{n+1} \frac{x^2}{(1+x)^2} + \left(\frac{1}{n+1} + \frac{1}{(n+1)^2}\right) \frac{x}{1+x} \right\} + \alpha \left\{ \frac{n\,(n-1)}{(n+1)^2} \frac{x^2}{(1+x)^2} + \frac{n}{(n+1)^2} \frac{x}{1+x} \right\}.$$

Arranging the terms we obtain the desired equality. □

Remark 2.3. One can see that, $B_n^{(\alpha)}(e_i;x) = (L_n)(e_i;x)$ for $i = 0, 1$ where $e_i = \left(\frac{t}{1+t}\right)^i$.

The immediate consequence of the above lemma is the following corollary.

Corollary 2.4. *The identities given below hold for the operator $B_n^{(\alpha)} f$.*

$$\mu_{n,1}^{\alpha}(x) := B_n^{(\alpha)}\left(\frac{t}{1+t} - \frac{x}{1+x};x\right) = \frac{-1}{n+1} \frac{x}{1+x} \tag{13}$$

$$\mu_{n,2}^{\alpha}(x) := B_n^{(\alpha)}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right) = \frac{-n-1+2\alpha}{(n+1)^2} \frac{x^2}{(1+x)^2} + \frac{n+2-2\alpha}{(n+1)^2} \frac{x}{1+x} \tag{14}$$

$$\begin{aligned}
\mu_{n,4}^{\alpha}(x) &= B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^4;x\right)\\
&= \left(\frac{(n-2)(n-3)\left[(n-3)(n-4)+12\alpha\right]}{(n+1)^4}-\frac{4(n-2)\left[(n-2)(n+3)+6\alpha\right]}{(n+1)^3}+\frac{6(n-2)(n+1)+2\alpha}{(n+1)^2}\right.\\
&\left.-\frac{4n}{n+1}+1\right)\frac{x^4}{(1+x)^4}\\
&+\left(\frac{6(n-2)\left(n^2+(1-2\alpha)n-12(1-\alpha)\right)}{(n+1)^4}-\frac{4(n-2)(n-3)-2\alpha(n-3)}{(n+1)^3}+\frac{6(n+2-2\alpha)}{(n+1)^2}\right)\frac{x^3}{(1+x)^3}\\
&+\left(\frac{7n^2+(29+36\alpha)n-86(1-\alpha)}{(n+1)^4}-\frac{4(n+6-6\alpha)}{(n+1)^3}\right)\frac{x^2}{(1+x)^2}+\frac{n+14(1-\alpha)}{(n+1)^4}\frac{x}{1+x}.
\end{aligned}$$

3. Approximation properties of the operators

In this section, we will give a Korovkin type theorem and obtain some convergence results of the α-BBH operators. For this reason, we recall the function ω, a general function of the type of modulus of continuity that satisfies the following conditions:

(a) ω is a continuous, nonnegative, increasing function on $[0,\infty)$,

(b) $\omega(\delta_1+\delta_2)\leq\omega(\delta_1)+\omega(\delta_2)$,

(c) $\lim_{\delta\to 0}\omega(\delta)=0$.

Now let us recall the space H_ω : H_ω indicates the space of all real-valued functions defined on semi axis $[0,\infty)$ and satisfy the condition

$$\left|f(x)-f(y)\right|\leq\omega\left(\left|\frac{x}{1+x}-\frac{y}{1+y}\right|\right) \tag{15}$$

for any $x,y\in[0,\infty)$. Recall that $H_\omega\subset C_B[0,\infty)$, where $C_B[0,\infty)$ is the space of functions f which is continuous and bounded on $[0,\infty)$ endowed with the norm

$$\left\|f\right\|_{C_B}=\sup_{x\geq 0}\left|f(x)\right|.$$

Gadjiev and Çakar [13] were the first to establish the following Korovkin type theorem for the linear positive operators acting from H_ω to $C_B[0,\infty)$.

Theorem 3.1. ([13]) *If A_n is the sequence of linear positive operators acting from H_ω to $C_B[0,\infty)$ and satisfy the three conditions*

$$\lim_{n\to\infty}\left\|A_n\left(\left(\frac{t}{1+t}\right)^\tau;x\right)-\left(\frac{x}{1+x}\right)^\tau\right\|_{C_B}=0,\quad \textit{for}\ \ \tau=0,1,2, \tag{16}$$

then for any function $f\in H_\omega$

$$\lim_{n\to\infty}\left\|A_n(f)-f\right\|_{C_B}=0.$$

Theorem 3.2. *Let $B_n^{(\alpha)}f$ be the operator defined in (2). For any $f\in H_\omega$, we have*

$$\lim_{n\to\infty}\left\|B_n^{(\alpha)}(f)-f\right\|_{C_B}=0.$$

Proof. We have to show that three conditions given in Theorem (3.1) are satisfied. The first condition of (16), corresponding to $\tau = 0$, is fullfilled for $B_n^{(\alpha)} f$ from (8) in Lemma (2.2).
For $\tau = 1$, we have

$$\begin{aligned}\left\| B_n^{(\alpha)}\left(\frac{t}{1+t};x\right) - \frac{x}{1+x} \right\|_{C_B} &= \left|\frac{n}{n+1} - 1\right| \sup_{x\geq 0} \frac{x}{1+x} \\ &\leq \frac{1}{n+1}\end{aligned}$$

which implies the second condition of (16) is fullfilled for $B_n^{(\alpha)} f$. Lastly for $\tau = 2$, we address the identity given in (10). We obtain

$$\begin{aligned}\left\| B_n^{(\alpha)}\left(\left(\frac{t}{1+t}\right)^2;x\right) - \left(\frac{x}{1+x}\right)^2 \right\|_{C_B} &= \left|\frac{-3n-3+2\alpha}{(n+1)^2}\right| \sup_{x\geq 0} \frac{x^2}{(1+x)^2} + \left|\frac{n+2-2\alpha}{(n+1)^2}\right| \sup_{x\geq 0} \frac{x}{1+x} \\ &\leq \frac{3n+3+2\alpha}{(n+1)^2} + \frac{n+2+2\alpha}{(n+1)^2} \\ &= \frac{4n+5+4\alpha}{(n+1)^2}\end{aligned}$$

which implies that the third condition of (16) is fulfilled for $B_n^{(\alpha)} f$. Hence the proof of the theorem is completed. □

Theorem 3.3. *Let $B_n^{(\alpha)} f$ be an operator defined in (2). For each $x \geq 0$ and for any $f \in H_\omega$, the following inequality holds:*

$$\left|B_n^{(\alpha)}(f;x) - f(x)\right| \leq 2\omega\left(\sqrt{\nu_n(x)}\right)$$

where

$$\nu_n(x) = \frac{2\alpha - n - 1}{(n+1)^2}\frac{x^2}{(1+x)^2} + \frac{n+2-2\alpha}{(n+1)^2}\frac{x}{1+x}. \tag{17}$$

Proof. Since $L_n^{(\alpha)}(1;x) = 1$, we have

$$\left|B_n^{(\alpha)}(f;x) - f(x)\right| \leq B_n^{(\alpha)}\left(\left|f(t) - f(x)\right|;x\right).$$

On the other hand, from the properties of the modulus of continuity function ω, we have

$$\begin{aligned}\left|f(t) - f(x)\right| &\leq \omega\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|\right) \\ &\leq \left(1 + \frac{\left|\frac{t}{1+t} - \frac{x}{1+x}\right|}{\delta}\right)\omega(\delta).\end{aligned}$$

Applying the operator $B_n^{(\alpha)}(f)$ to the above inequality we obtain

$$\left|B_n^{(\alpha)}(f;x) - f(x)\right| \leq \left(1 + \frac{1}{\delta}B_n^{(\alpha)}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|;x\right)\right)\omega(\delta).$$

The Cauchy-Schwarz inequality leads to the following inequality,

$$\left|B_n^{(\alpha)}(f;x) - f(x)\right| \leq \left\{1 + \frac{1}{\delta}\left[B_n^{(\alpha)}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right)\right]^{1/2}\right\}\omega(\delta)$$

from which, letting $\nu_n(x) = \frac{-n-1+2\alpha}{(n+1)^2}\frac{x^2}{(1+x)^2} + \frac{n+2-2\alpha}{(n+1)^2}\frac{x}{1+x}$, and choosing $\delta = (\nu_n(x))^{1/2}$, we get the desired result. □

Now, we will investigate the rate of convergence of the operators with the aid of of Lipschitz-type maximal functions [16]. The space of general Lipschitz type maximal functions space on $U \subset R_+$ is denoted by $\tilde{W}_{\gamma,U}$ and defined as,

$$\tilde{W}_{\gamma,U} = \left\{ f : \sup (1+x)^{\gamma} \tilde{f_{\gamma}}(x,y) \leq M \frac{1}{(1+y)^{\gamma}}, x \geq 0, y \in U \right\}$$

where f is a bounded and continuous function on R_+ , $M > 0$ is a constant, $0 < \gamma \leq 1$ and $\tilde{f_{\gamma}}(x)$ is the function given by:

$$\tilde{f_{\gamma}}(x,t) = \frac{\left|f(t) - f(x)\right|}{|x-t|^{\gamma}}.$$

Let also $d(x, U)$ be the distance between x and U, that is

$$d(x,U) = \inf \left\{ \left|x - y\right| : y \in U \right\}.$$

Theorem 3.4. *Let $B_n^{(\alpha)} f$ be the operator defined in (2). Then for any $f \in \tilde{W}_{\gamma,U}$*

$$\left|B_n^{(\alpha)}(f;x) - f(x)\right| \leq M\left(\nu_n^{\gamma/2}(x) + 2\,(d(x,U))^{\gamma}\right) \tag{18}$$

where $\nu_n(x)$ is defined in (17).

Proof. If $\bar{U}$ denotes the closure of the set U, then there exists an $x_0 \in \bar{U}$ such that $|x - x_0| = d(x, U)$, where $x \in R_+$. We now write the inequality

$$\left|f - f(x)\right| \leq \left|f - f(x_0)\right| + \left|f(x_0) - f(x)\right|.$$

Since $B_n^{(\alpha)} f$ is a linear positive operator and $f \in \tilde{W}_{\gamma,U}$, applying $B_n^{(\alpha)}$ to the above inequality, we get

$$\begin{aligned}\left|B_n^{(\alpha)}(f;x) - f(x)\right| &\leq B_n^{(\alpha)}\left(\left|f - f(x_0)\right|;x\right) + \left|f(x_0) - f(x)\right| \\ &\leq MB_n^{(\alpha)}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\gamma};x\right) + M\frac{|x-x_0|^{\gamma}}{(1+x_0)^{\gamma}(1+x)^{\gamma}}. \end{aligned} \tag{19}$$

With the help of the basic inequality $(a + b)^{\gamma} \leq a^{\gamma} + b^{\gamma}$ for $a \geq 0$ and $b \geq 0$, for $0 < \gamma \leq 1$ and $t \in R_+$, we can write

$$\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\gamma} \leq \left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\gamma} + \left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\gamma}.$$

Hence we obtain

$$B_n^{(\alpha)}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\gamma};x\right) \leq B_n^{(\alpha)}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\gamma};x\right) + \frac{|x-x_0|^{\gamma}}{(1+x_0)^{\gamma}(1+x)^{\gamma}}.$$

Applying Hölder's inequality with $p = 2/\gamma$ and $q = 2/\left(2-\gamma\right)$ to the first term on the right hand side of the above inequality, we get

$$B_n^{(\alpha)}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\gamma};x\right) \leq B_n^{(\alpha)}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right)^{\gamma/2} + \frac{|x-x_0|^{\gamma}}{(1+x_0)^{\gamma}(1+x)^{\gamma}}$$

where we have used the equality $B_n^{(\alpha)}(1;x) = 1$. Thus substituting the last inequality into (19), we finally obtain (18). □

4. Voronovskaya theorem

This part is devoted to the Voronovskaya-type asymptotic formula for the operators $B_n^{(\alpha)}$.

Theorem 4.1. *Suppose that* $f'\left(\frac{x}{1+x}\right)$ *and* $f''\left(\frac{x}{1+x}\right)$ *exist for* $x \geq 0$. *We have,*

$$\lim_{n\to\infty}(n+1)\left(B_n^{(\alpha)}\left(f\left(\frac{t}{1+t}\right);x\right)-f\left(\frac{x}{1+x}\right)\right)=\frac{-x}{1+x}f^!\left(\frac{x}{1+x}\right)+\frac{1}{2}\left[-\left(\frac{x}{1+x}\right)^2+\frac{x}{1+x}\right]f''\left(\frac{x}{1+x}\right).$$

Proof. Using Taylor's Theorem we can write

$$f(\eta)=f(t)+(\eta-t)f'(t)+\frac{(\eta-t)^2}{2}f''(t)+(\eta-t)^2\varphi(\eta-t)$$

where $\left|\varphi(h)\right| \leq H$ for all h and $\lim_{h\to 0}\varphi(h)=0$. Taking $\eta=\frac{k}{n+1}$ and $t=\frac{x}{1+x}$ in Taylor's formula, we obtain

$$f\left(\frac{k}{n+1}\right)=f\left(\frac{x}{1+x}\right)+\left(\frac{k}{n+1}-\frac{x}{1+x}\right)f'\left(\frac{x}{1+x}\right)+\frac{\left(\frac{k}{n+1}-\frac{x}{1+x}\right)^2}{2}f''\left(\frac{x}{1+x}\right)+\left(\frac{k}{n+1}-\frac{x}{1+x}\right)^2\varphi\left(\frac{k}{n+1}-\frac{x}{1+x}\right).$$

Applying the operators $B_n^{(\alpha)}$ to the equality above, one gets

$$\begin{aligned}B_n^{(\alpha)}\left(f\left(\frac{t}{1+t}\right);x\right)-f\left(\frac{x}{1+x}\right)&=f'\left(\frac{x}{1+x}\right)B_n^{(\alpha)}\left(\frac{t}{1+t}-\frac{x}{1+x};x\right)+\frac{1}{2}f''\left(\frac{x}{1+x}\right)B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2;x\right)\\&\quad+B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2\varphi\left(\frac{k}{n+1}-\frac{x}{1+x}\right);x\right).\end{aligned}$$

From Corollary (2.4), we can write

$$\begin{aligned}B_n^{(\alpha)}\left(f\left(\frac{t}{1+t}\right);x\right)-f\left(\frac{x}{1+x}\right)&=\frac{-1}{n+1}\frac{x}{1+x}f'\left(\frac{x}{1+x}\right)\\&\quad+\frac{1}{2}\left[\frac{-n-1+2\alpha}{(n+1)^2}\frac{x^2}{(1+x)^2}+\frac{n+2-2\alpha}{(n+1)^2}\frac{x}{1+x}\right]f''\left(\frac{x}{1+x}\right)\\&\quad+B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2\varphi\left(\frac{k}{n+1}-\frac{x}{1+x}\right);x\right).\end{aligned}\tag{20}$$

Multiplying both sides of 20 by $(n+1)$ and then taking the limit as $n\to\infty$, we get

$$\begin{aligned}\lim_{n\to\infty}(n+1)\left(B_n^{(\alpha)}\left(f\left(\frac{t}{1+t}\right);x\right)-f\left(\frac{x}{1+x}\right)\right)&=\frac{-x}{1+x}f'\left(\frac{x}{1+x}\right)+\frac{1}{2}\left[-\left(\frac{x}{1+x}\right)^2+\frac{x}{1+x}\right]f''\left(\frac{x}{1+x}\right)\\&\quad+\lim_{n\to\infty}(n+1)B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2\varphi\left(\frac{t}{1+t}-\frac{x}{1+x}\right);x\right).\end{aligned}\tag{21}$$

We will apply Cauchy-Schwarz inequality for the last term on the right hand side of the above inequality, from which we get

$$\begin{aligned}(n+1)B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2\varphi\left(\frac{t}{1+t}-\frac{x}{1+x}\right);x\right)&\leq\left[(n+1)^2B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^4;x\right)\right]^{1/2}\\&\quad\times\left[B_n^{(\alpha)}\left(\varphi^2\left(\frac{t}{1+t}-\frac{x}{1+x}\right);x\right)\right]^{1/2}.\end{aligned}\tag{22}$$

After some simple-yet complicated-calculations, we get,

$$B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^4;x\right)=\frac{1}{(x+1)^4}\frac{1}{(n+1)^4}$$
$$\times\left[x^4+(11n+50(1-\alpha))x^3+\left(3n^2+4(1-3\alpha)n-68(1-\alpha)\right)x^2+(n+14(1-\alpha))\,x\right]$$

with the aid of Corollary (2.4). Now it is obvious that,

$$\lim_{n\to\infty}(n+1)^2\,B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^4;x\right)=\frac{3x^2}{(x+1)^4}. \tag{23}$$

We also have

$$\lim_{n\to\infty}B_n^{(\alpha)}\left(\varphi^2(\frac{t}{1+t}-\frac{x}{1+x});x\right)=0. \tag{24}$$

Hence taking the positivity of the operator $B_n^{(\alpha)}$ into account and using the equations (23) and (24), from (22), we can write

$$\lim_{n\to\infty}(n+1)\,B_n^{(\alpha)}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2\varphi(\frac{t}{1+t}-\frac{x}{1+x});x\right)=0. \tag{25}$$

Substitution of (25) into (21) gives the desired result of the theorem. □

5. Graphical illustrations

In this part, by using Maple software, we present some graphical examples related to our operator. In Figure 1, we approximate the function $f(x)=x(x+\frac{1}{2})$ on the interval $[0,2]$ with the newly defined operators $B_n^{(\alpha)}$. In Fig.1a, we fixed the value of the parameter α as $\alpha=0.5$ and choose different n values, $n=10,20,50$. In Fig.1b we choose n, as $n=50$ and observe the graphs for different choices of α values.
In Figure 2, we take the function $f(x)=e^{\frac{-x}{5}}$ on the interval $[0,3]$. In Fig.2a, similarly as in Fig.1a, we fixed the value of the parameter α as $\alpha=0.5$ and choose different n values, $n=10,20,50$. In Fig.2b we choose n, as $n=5$ and investigate the graphs for different choices of α values.
In both figures we observe that, as the value of n gets larger the graphs of the operators gets closer to the graphs of the given functions, implying that we get better convergence for larger n values. We also observe that in Figure 1, for fixed $n=10$, the best approximation is carried out when $\alpha=0.9$, while in Figure 2 it is achieved when $\alpha=0.1$ with $n=5$ is fixed.

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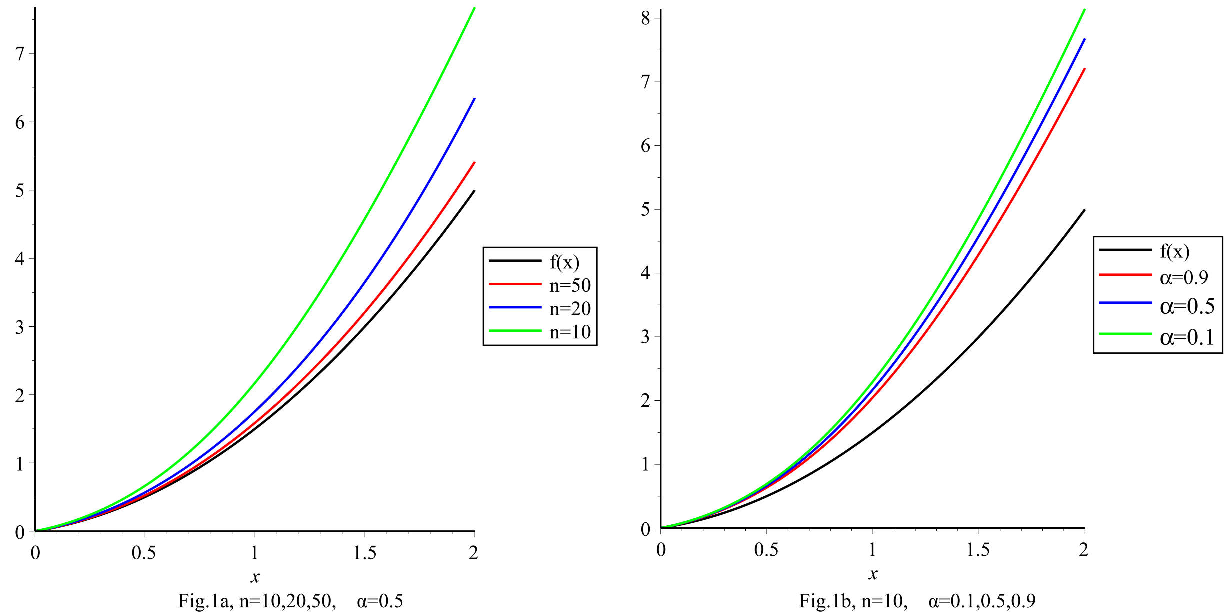

Figure 1: Approximation of $f(x) = x(x + \frac{1}{2})$ by $\alpha - BBH$ Operators

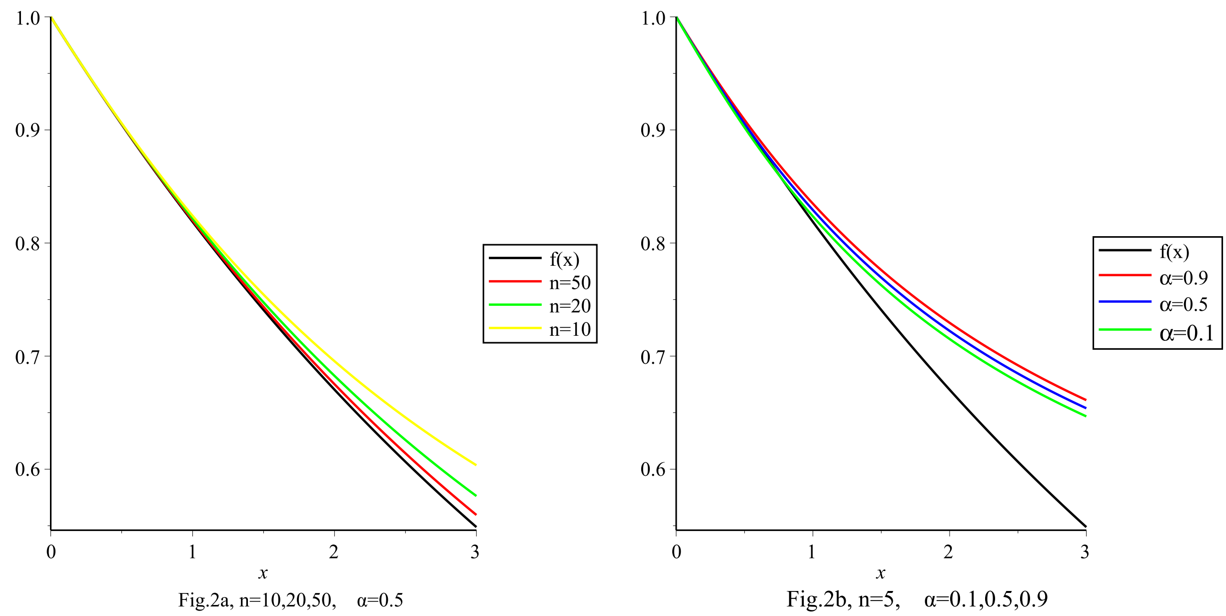

Figure 2: Approximation of $f(x) = \exp \frac{-x}{5}$ by $\alpha - BBH$ Operators

References

[1] U. Abel, M. Ivan, Some identities for the operator of Bleimann, Butzer and Hahn involving divided differences, Calcolo 36 (1999) 143–160.

[2] U. Abel, V. Gupta, An estimate of the rate of convergence of a Bézier variant of the Baskakov-Kantorovich operators for bounded variation functions, Demonstr. Math. 36 (2003) 123–136.

[3] H. Aktuğlu, S.Y. Zaheriani, Approximation of Functions by Generalized Parametric Blending-Type Bernstein Operators, Iranian J. Sci. Techn., Trans. A: Sci. 44 (2020) 1495–1504.

[4] A. Altın, O. Doğru, M.A. Özarslan, Korovkin type approximation properties of bivariate Bleimann, Butzer and Hahn operators. In: Proc. 8th WSEAS Internat. Conf. Appl. Math., Tenerife, Spain, December 16-18 (2005) 234–238.

[5] A. Aral, O. Doğru, Bleimann, Butzer and Hahn operators based on the q-integers, J. Inequal. Appl. 2007 (2007) Article ID 79410 12 pages.

[6] A. Aral, H. Erbay, Parametric generalization of Baskakov operators, Math. Commun. 24 (2019) 119–131.

[7] G. Bleimann, P.L. Butzer, L. Hahn, A Bernstein type operator approximating continuous functions on the semi axes, Proc. Netherl. Sci. 83 (1980) (Indag. Math. 42)255–262.

[8] X. Chen, J. Tan, Z. Liu, J. Xie, Approximation of fuMathematicalnctions by a new family of generalized Bernstein operators, J. Math. Anal. Appl. 450 (2017) 244–261.

[9] B. Della Vecchia, Some properties of a rational operator of Bernstein type, Progress Approx. Theory, Academic Press, New York (1991) 177–185.

[10] O. Doğru, On Bleimann, Butzer and Hahn type generalization of Balázs operators, Studia Univ. Babes-Bolyai Math. 47 (2002) 37–45.

[11] O. Doğru, V. Gupta, Monotonicity and the asymptotic estimate of Bleimann Butzer and Hahn operators based on q-integers, Georgian Math. J. 12 (2005) 415–422.

[12] S. Ersan, O. Doğru, Statistical approximation properties of q-Bleimann, Butzer and Hahn operators, Math. Comput. Modelling 49 (2009) 1595–1606.

[13] A.D. Gadjiev, Ö. Çakar, On uniform approximation by Bleimann-Butzer and Hahn operators on all positive semi-axis, Trans. AS Azerbaijan. Ser. Phys. Techn. Math. Sci. 19 (1999) 21–26 .

[14] V. Gupta, M.Th Rassias. Moments of linear positive operators and approximation. Switzerland: Springer International Publishing, 2019.

[15] A. Kajlaa, S.A. Mohiuddine, A. Alotaibic, Approximation by α-Baskakov-Jain type operators, Filomat 36 (2022) 1733–1741.

[16] B.Lenze, Bernstein-Baskakov-Kantorovich operators and Lipschitz-type maximal functions, Approx. Theory Kecskemét, Hungary, Colloq. Math. Soc. Janos Bolyai 58 (1990) 469–496.

[17] M. Nasiruzzaman, N. Rao, S. Wazir, R. Kumar, Approximation on parametric extension of Baskakov-Durrmeyer operators on weighted spaces, J. Inequal. Appl. 2019 (2019) 103.

[18] M. Nasiruzzaman, N. Rao, M. Kumar, R. Kumar, Approximation on bivariate parametric-extension of Baskakov-Durrmeyer-operators Filomat 35 (2021) 2783–2800.

[19] M. Sofyalıoğlu, K. Kanat, B. Çekim, Parametric generalization of the Meyer-König-Zeller operators, Chaos, Solitons & Fractals 152 (2021) 111417.

[20] M. Sofyalıoğlu, K. Kanat, B. Çekim, Parametric generalization of the modified Bernstein operators, Filomat 36 (2022) 1699–1709.

[21] D. Söylemez-Özden, Some properties of the generalized Bleimann, Butzer and Hahn operators, Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat. 64 (2015) 55–62.