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On local density and local weak density of the hyperspace of sets with finitely many components

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Abstract. This paper is devoted to the investigation of cardinal invariants such as the local density, the local weak density and the relation between the tightness of the space $C_n(X)$ of closed sets with finitely many components and the density of a topological space itself. Moreover, it is shown that the functor $C_n\colon Comp \to Comp$ preserves the local density and the local weak density of compact spaces. As a result, criteria for locally separability and locally weakly separability of compact spaces are obtained.

1. Introduction

The cardinal invariants are considered as topological invariants with values in the class of all cardinal numbers and are used to describe various topological properties of spaces. For example, the weight, the character, the density, the Lindelöf number, the Souslin number and the tightness of a topological space are some classical cardinal invariants. Many researches have been devoted to the investigation of cardinal invariants (see for example, [1, 2, 12, 13, 16]) and their important role in Topology verifies that this study should be continued. Thus, in recent years, related investigations enriched this topic (see for example, [3, 4, 8–10, 20]). Hyperspaces and Hattori spaces [15] and their cardinal invariants attracted also a particular interest.

Moreover, in recent researches an interest in the theory of cardinal invariants and their behavior under the influence of various covariant functors is increasing fast. In [5–7] the authors investigated several cardinal invariants under the influence of some weakly normal and normal functors, as well as, some types of hyperspace. In the investigations [11] and [18] the concept of hyperspace of nonempty closed sets consisting of finitely many of components is introduced. In particular, in [7] the functor $C_n\colon Comp \to Comp$ is introduced, as well as, some cardinal and categorical properties of this functor are investigated.

The current paper is devoted to the investigation of cardinal invariants such as the local density and the local weak density. Also, the relation between the tightness of the space $C_n(X)$ of closed sets with finitely many components and the density of a topological space itself is studied. Moreover, it is shown that the functor $C_n\colon Comp \to Comp$ preserves the local density and the local weak density of compact spaces. As a consequence, criteria for locally separability and locally weakly separability of compact spaces are obtained.

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More precisely, the paper is organized as follows. In Section 2, we recall basic notions and notations that will be used in the rest of the study. In Section 3, we study basic facts and results for the weak density and the local weak density. In Section 4, we study cardinal invariants for the space $C_n(X)$ of closed sets with finitely many components and finally, in Section 5, we present some open problems for further investigation on this topic.

Throughout the paper all spaces are assumed to be completely regular, τ means an infinite cardinal number and by ω we denote the countable cardinal number. Also, by *Comp* we denote the category of compact spaces and their continuous mappings.

2. Preliminary notes

Let X be a topological T_1-space. The set of all non-empty closed subsets of a topological space X is denoted by $\exp X$. The family of all sets of the form

$$O\langle U_1, ..., U_n\rangle = \left\{ F : \; F \in \exp(X), F \subset \bigcup_{i=1}^{n} U_i, F \cap U_i \neq \emptyset, \; i = 1, 2, ..., n \right\},$$

where $U_1, ..., U_n$ are open subsets of X, generates a base of the topology on the set $\exp(X)$. This topology is called the *Vietoris topology*. The set $\exp(X)$ with the Vietoris topology is called *exponential space* or *the hyperspace of a space* X [14].

By $C_n(X)$ we denote the set of all closed subsets of X consisting of no more than n (n is natural) components. This space contains the hyperspace $\exp_n(X)$ of closed sets consisting of no more than n elements and the hyperspace of closed connected sets $\exp^c(X)$.

Put

$$\exp_n(X) = \{F \in \exp(X) : |F| \leq n\},$$

$$\exp_\omega(X) = \bigcup \left\{\exp_n(X) : n = 1, 2, ...\right\},$$

$$\exp^c(X) = \{F \in \exp(X) : \text{ F is connected in } X\}$$

and

$$C_\omega(X) = \bigcup \{C_n(X) : n = 1, 2, ...\}.$$

It is clear that $\exp^c(X) \subset C_n(X) \subset \exp(X)$ and $\exp_n(X) \subset C_n(X)$ for any topological space X. On the sets $C_n(X)$ and $C_\omega(X)$ the topology induced from the hyperspace $\exp(X)$ is considered. Note that $\exp_n(X) = C_n(X)$ for a discrete space X. Moreover, it is clear that we have $\exp^c(X) = C_1(X)$.

Let $X, Y \in Comp$ and let $f : X \to Y$ be a continuous map between compact spaces X and Y. For any set $F \in C_n(X)$ put $C_n(f)(F) = f(F)$. Then $C_n(f) : C_n(X) \to C_n(Y)$ is a continuous map. Thus, the structure C_n forms a covariant functor in the category *Comp* of compacta.

Definition 2.1. ([14]) A covariant functor $F : Comp \to Comp$ acting in the category of compacta is called normal, if it

1) preserves the weight;
2) preserves singletons and empty set;
3) is monomorphic (preserves embeddings);
4) is epimorphic (preserves surjections);
5) preserves intersections of closed subsets;
6) preserves inverse images;
7) is continuous with respect to inverse limits.

The following example shows that the functor $C_n : Comp \to Comp$ is not normal.

Example 2.2. ([7]) Consider the sets $X = [-3, -1] \cup [1, 3]$ and $Y = [-2, 2]$. In these sets we consider the natural topology induced from the real line R. We construct a map $f : X \to Y$ as follows:

$$f(x) = \begin{cases} x+1, & \text{when } x \in [-3, -1] \\ x-1, & \text{when } x \in [1, 3]. \end{cases}$$

It is clear that the mapping $f : X \to Y$ is continuous and "onto". The set $F = [-1, 1] \subset Y$ consists of a single component, and therefore $F \in C_1(Y)$. But none of elements from $C_1(X)$ is transformed by $C_1(f) : C_1(X) \to C_1(Y)$ onto F. Hence, the map $C_1(f) : C_1(X) \to C_1(Y)$ is not surjective. Therefore, we have shown that the functor $C_n : Comp \to Comp$ does not preserve epimorphisms.

In [7] the authors obtain the following result.

Theorem 2.3. ([7]) *The functor $C_n : Comp \to Comp$ satisfies all the conditions of normality except of preserving epimorphisms.*

3. Weak density and local weak density

In this section we give basic results and facts for the weak density and the local weak density of topological spaces.

Definition 3.1. The *density* of a topological space X, denoted by $d(X)$, is defined as follows:

$$d(X) = min\{|A| \colon A \text{ is a dense subset of } X\}.$$

A topological space is separable, if $d(X) \leq \omega$. Moreover, a collection $\lambda = \{E_\alpha \colon \alpha \in A\}$ of nonempty subsets of a topological space X is said to be a *π-network* of the space X if for an arbitrary open subset $U \subset X$ there exists $E_\alpha \in \lambda$ such that $E_\alpha \subset U$. A π-network consisting of only open sets is called a *π-base*.

It is said that a family γ of subsets of a topological space has the *finite intersection property* if every finitely many elements of γ has nonempty intersection. A family which has this property is said to be a *centered system*.

Definition 3.2. The weak density of a topological space X, denoted by $wd(X)$, is the smallest cardinal number $\tau \geq \aleph_0$ such that there is a π-base in X coinciding with τ centered systems of open sets, i.e. there is a π-base $B = \bigcup\{B_\alpha : \alpha \in A\}$, where B_α is a centered system of open sets for each $\alpha \in A$ and $|A| = \tau$.

If $wd(X) = \aleph_0$, then we say that a topological space X is weakly separable.

Proposition 3.3. *The weak density of a topological space X is τ if and only if there exists a π-network coinciding with the union of τ centered systems.*

Proof. If the weak density of a topological space X is τ, then according to Definition 3.2, there is a π-base in X coinciding with τ centered systems of open sets, i.e. there is a π-base $B = \bigcup\{B_\alpha : \alpha \in A\}$, where B_α is a centered system of open sets for each $\alpha \in A$ and $|A| = \tau$. Since every π-base is also a π-network, we have the "necessity" of the proposition.

Now suppose that $\gamma = \cup\{\gamma_\alpha \colon \alpha \in A, |A| = \tau\}$ is a π-network for X and each γ_α is centered. We shall prove that $wd(X) = \tau$. Put

$$\sigma_\alpha = \{U \subset X \colon U \text{ is an open and contains some } E \in \gamma_\alpha\}.$$

Then clearly, the family σ_α is centered for every $\alpha \in A$. We show that the system

$$\sigma = \cup\{\sigma_\alpha \colon \alpha \in A, |A| = \tau\}$$

is a π-base for X. Let W be an arbitrary nonempty open subset of X. There exist $\alpha \in A$ and $E \in \gamma_\alpha$ such that $E \subset W$, since the system γ is a π-network of X. Then clearly, $W \in \sigma_\alpha$. Hence, σ coincides with τ centered systems of open sets, which means that $wd(X) = \tau$. Proposition 3.3 is proved. □

Proposition 3.4. *Let $d(X) = \tau \geq \omega$. Then $wd(X) \leq \tau$.*

Proof. Let $d(X) = \tau$, i.e. there exists a subset $M = \{a_\alpha\colon \alpha \in A, |A| = \tau\}$ in X such that $[M] = X$, where $[M]$ denotes the closure of M in X. Denote by σ_α the system of all open subsets of X containing a_α, i.e.

$$\sigma_\alpha = \{U_s^\alpha\colon a_\alpha \in U_s^\alpha \text{ and } U_s^\alpha \text{ is open in } X \text{ for every } \alpha\}.$$

Consider the system

$$\sigma = \cup\{\sigma_\alpha\colon \alpha \in A\}.$$

Now we show that the system σ is a π-base for X. Indeed, let G be an arbitrary nonempty open subset of X. Since the set M is dense in X, there exists a point $a_\alpha \in M \cap G$. The set G is open, therefore, there exists a neighborhood $U_s^\alpha \in \sigma_\alpha \subset \sigma$ such that $U_s^\alpha \subset G$. This means that the system σ is a π-base of the space X.

Further, we have to prove that σ_α is a centered system for each $\alpha \in A$. Take arbitrary elements $U_{s_1}^\alpha, U_{s_2}^\alpha, \ldots, U_{s_k}^\alpha$ of the family σ_α. In that case, we have

$$a_\alpha \in \cap\{U_{s_i}^\alpha\colon i = 1, 2, \ldots, k\} \neq \emptyset.$$

Thus, for each $\alpha \in A$ the system σ_α is centered. We have proved that $wd(X) \leq \tau$. Proposition 3.4 is proved. □

Definition 3.5. The *Souslin number* of a topological space X, denoted by $c(X)$, is defined as follows:

$$c(X) = \sup\{|\gamma| : \gamma \text{ is disjoint family of open subsets of } X\}.$$

The following statement establishes the relation between the weak density and the Souslin number of a topological space X.

Proposition 3.6. *For any topological space X the following inequalities hold:*

$$c(X) \leq wd(X) \leq d(X).$$

Proof. We show that $c(X) \leq wd(X)$. Let $wd(X) = \tau \geq \omega$, i.e. there exists a π-base

$$B = \cup\{B_\alpha\colon \alpha \in A, |A| = \tau\}$$

such that each $B_\alpha = \cup\{U_{s_k}^\alpha : a \in A\}$ is centered system of open sets. Now suppose that the Souslin number of the space X is greater than τ, i.e. $c(X) = \tau' > \tau$. In this case, there exists a system

$$\gamma = \{G_\beta\colon \beta \in B, |B| = \tau' > \tau\}$$

of nonempty open sets such that $G_\beta \cap G_{\beta'} = \emptyset$ for every pair of indexes $\beta \neq \beta'$.

For each open set $G_\beta \in \gamma$ there exists a set $U_s^\alpha \in B_\alpha$ such that $U_s^\alpha \subset G_\beta$, since the system B is a π-base for X. From the fact that the system B_α is centered, we obtain that distinct sets G_β can contain sets U_s^α from only distinct systems B_α. This is a contradiction, since the system B is a π-base. Therefore, $c(X) \leq \tau$.

Moreover, by Proposition 3.4 we have $wd(X) \leq d(X)$, and this completely proves our statement. Proposition 3.6 is proved. □

Theorem 3.7. *For every compact space X the following equality holds:*

$$wd(X) = d(X).$$

Proof. By Proposition 3.4 we have $wd(X) \leq d(X)$. Thus, it is sufficient to show the inequality $d(X) \leq wd(X)$. Suppose $wd(X) = \tau$ and

$$B = \bigcup\{B_\alpha\colon \alpha \in A, |A| = \tau\}$$

is a π-base for X. Consider the system

$$\mu_\alpha = \{[U] \colon U \in B_\alpha\}.$$

Since μ_α is a centered system of closed subsets of the compact space X, this system has nonempty intersection. Let $x_\alpha \in \bigcap\{[U] \colon U \in B_\alpha\}$. Put

$$X_0 = \{x_\alpha \colon \alpha \in A, |A| = \tau\}.$$

We show that X_0 is dense in X. Let V be a nonempty open subset of X. There exists a nonempty open subset W such that $[W] \subset V$ by completely regularity of the compact space X. Since B is a π-base for X, there exists $\alpha \in A$ and $U_s^\alpha \in B_\alpha$ such that $U_s^\alpha \subset W$. In this case, we have $[U_s^\alpha] \subset [W] \subset V$. Clearly, $x_\alpha \in [U_s^\alpha]$, a fortiori, $x_\alpha \in V$. Theorem 3.7 is proved. □

Problem 3.8. *Find a non-compact space X such that $wd(X) \neq d(X)$.*

From Theorem 3.7 we can get the following result.

Corollary 3.9. *A compact X is separable if and only if it is weakly separable.*

Proposition 3.10. *If Y is a dense subset of a space X, then $wd(Y) = wd(X)$.*

Proof. Let us first show that $wd(Y) \leq wd(X)$. Suppose $wd(X) = \tau \geq \omega$ and

$$B = \bigcup\{B_\alpha \colon \alpha \in A, |A| = \tau\}$$

is a π-base for X. Put

$$B'_\alpha = \{V \cap Y \colon V \in B_\alpha\}$$

and

$$B' = \bigcup_{\alpha\in A} B'_\alpha.$$

We firstly show that B' is a π-base for Y. Let $G' \subset Y$ be an arbitrary nonempty open subset of Y. There exists an open subset $G \subset X$ such that $G \cap X = G'$. Since the system B is a π-base for X, there exists $U_\alpha \in B_\alpha$ such that $U_\alpha \subset G$. In this case, we have $U_\alpha \cap Y \subset G \cap Y = G'$. Thus, the system B' is a π-base.

Now let us show that the system B' is centered for each $\alpha \in A$. Get an arbitrary $\alpha \in A$ and sets $U_{s_1}^\alpha, U_{s_2}^\alpha, \ldots, U_{s_k}^\alpha$ from the system B'_α. We have $\bigcap_{i=1}^k U_{s_i}^\alpha \neq \emptyset$. Indeed, for each $i = 1, 2, \ldots, k$ there exists an open set $V_{s_i}^\alpha \in B_\alpha$ such that $V_{s_i}^\alpha \cap Y = U_{s_i}^\alpha$. Since the system B_α is centered, we have $\bigcap_{i=1}^k V_{s_i}^\alpha \neq \emptyset$. Then

$$\bigcap_{i=1}^k U_{s_i}^\alpha = \bigcap_{i=1}^k \left(Y \cap V_{s_i}^\alpha\right) = \left(\bigcap_{i=1}^k V_{s_i}^\alpha\right) \cap Y \neq \emptyset.$$

Thus the system B' is a π-base and each B'_α is centered. This means that $wd(Y) \leq \tau$.

Now we show the inequality $wd(X) \leq wd(Y)$. Let $wd(Y) = \tau$. There exists a π-base

$$\Gamma = \cup\{\Gamma_\alpha \colon \alpha \in A\}$$

for Y such that $|A| = \tau$ and Γ_α is centered for every $\alpha \in A$. Denote by B_α the system of all open subsets $U \subset X$ for which there exists $V \in \Gamma_\alpha$ such that $V \subset U$. Clearly, each system B_α is centered. We shall prove that the system

$$B = \bigcup_{\alpha\in A} B_\alpha$$

is a π-base for X. Indeed, let W be an arbitrary open subset of X. Since Y is dense in X, $W' = W \cap Y$ is a nonempty open subset of Y. Then there exists $V_s^\alpha \in \Gamma_\alpha$ such that $V_s^\alpha \subset W' \subset W$. On the other side, by the construction of B_α, we have $W \in B_\alpha$. This means that the system B is a π-base for X, which can be represented as the union of τ-many centered systems. Therefore, we have $wd(X) \leq \tau$. Proposition 3.10 is proved. □

Corollary 3.11. *Let X be an arbitrary space and bX its arbitrary compact extension. Then the following equality holds:*

$$wd(X) = d(bX).$$

Proof. Let X be an arbitrary space. Since X is dense in its compact extension bX, by Proposition 3.10 we have $wd(X) = wd(bX)$. Now by Theorem 3.7 we obtain $wd(bX) = d(bX)$, since bX is compact. Therefore, $wd(X) = d(bX)$. Corollary 3.11 is proved. □

4. Cardinal invariants for $C_n(X)$

In this section we give results regarding cardinal properties of the space $C_n(X)$ such as the tightness, the local density and the local weak density.

Theorem 4.1. ([7]) *For every infinite T_1-space X the following equalities hold:*
1) $d(X) = d(C_n(X))$;
2) $wd(X) = wd(C_n(X))$.

Definition 4.2. The *tightness* of a topological space X, denoted by $t(X)$, is the smallest infinite cardinal number τ such that the following condition is satisfied: if $x \in X$, $A \subset X$ and $x \in [A]$, then there exists a set $B \subset A$ for which $|B| \leq \tau$ and $x \in [B]$.

Proposition 4.3. *Let X be an infinite topological space such that $X \in C_\omega(X)$, i.e. X has finitely many of components. Then we have*

$$d(X) \leq t(C_\omega(X)).$$

Proof. Let $t(C_\omega(X)) = \tau$. We have to show that $d(X) \leq \tau$. Suppose the opposite, i.e. $d(X) > \tau$ and M is a dense subset of X. By the assumption, $|M| > \tau$.

Now let γ be the family of all finite subsets of M. Clearly, $|\gamma| = |M|$ and $X \in [\gamma]_{\exp X}$. Consider an arbitrary neighborhood $O\langle V_1, ..., V_n\rangle$ of X in $\exp X$. Choose a point x_i from each intersection $V_i \cap M$, $i = 1, \ldots, n$. In that case, the set $F = \{x_1, ..., x_n\}$ belongs to $C_\omega(X)$, besides, $F \in O\langle V_1, ..., V_n\rangle$ at the same time.

Since $t(C_\omega(X)) \leq \tau$ there exists a subfamily $\mu \subset \gamma$ with $|\mu| \leq \tau$ such that $X \in [\mu]_{\exp X}$. But, in that case, the set $Y = \cup\mu$ is dense in X. Indeed, get an arbitrary open subset $V \subset X$ and consider the neighborhood $O\langle V, X\rangle$ of X in $\exp X$. Since $X \in [\mu]_{\exp X}$, there exists a closed subset $E \in \mu \cap O\langle V, X\rangle$. Therefore, we have $E \cap V \neq \emptyset$. On the other side, $E \subset \cup\mu = Y$ which implies that $Y \cap V \neq \emptyset$, i.e. Y is dense in X and $|Y| \leq \tau$. The last contradicts the assumption $d(X) > \tau$. Proposition 4.3 is proved. □

Proposition 3.4, Theorem 4.1 and Proposition 4.3 directly imply the following results.

Corollary 4.4. *For an infinite space X the following inequalities hold:*

$$wd(C_n(X)) \leq d(C_n(X)) \leq t(C_n(X)).$$

Corollary 4.5. *Let X be a topological space such that $X \in C_n(X)$. If $t(C_n(X)) \leq \omega$, then both X and $C_n(X)$ are separable.*

Definition 4.6. The local density at a point $x \in X$, denoted by $ld(x)$, is τ if τ is the smallest cardinal number such that x has a neighborhood of density τ in X.

The *local density* of a topological space X, denoted by $ld(X)$, is defined as the supremum of all numbers $ld(x)$ for $x \in X$. A topological space X is locally separable, if $ld(X) \leq \omega$.

Remark 4.7. Note that if $ld(X) \leq \tau$ and $K \subset X$ is a compact subset, then there exists a neighborhood OK of K such that $d(OK) \leq \tau$.

Indeed, the set

$$\lambda = \{Ox \colon x \in K \text{ and } d(Ox) \le \tau\}$$

is an open cover of K, consisting of sets of density not greater than τ. Since K is compact, there exist $Ox_1, Ox_2, ..., Ox_m$ in λ such that $K \subset \cup_{i=1}^{m} O_i x$. Let $OK = \cup_{i=1}^{m} O_i x$. Then clearly, $d(OK) \le \tau$.

Theorem 4.8. *For every infinite compact space X we have*

$$ld(X) = ld(C_n(X)) = ld(C_\omega(X)),$$

where n is an arbitrary natural number.

Proof. First we show the inequality $ld(C_\omega(X)) \le ld(X)$. Let $ld(X) = \tau$. We get an arbitrary element $F \in C_\omega(X)$ and show that $ld(F) \le \tau$ in $C_\omega(X)$. Suppose

$$F = F_1 \cup F_2 \cup ... \cup F_n,$$

where F_i is a component of F for $i = 1, 2, ..., n$. Since each component F_i is compact, by Remark 4.7 for every $i = 1, 2, ..., n$ there exists a neighborhood OF_i of F_i such that $d(OF_i) \le \tau$.

Consider a dense subset M_i of OF_i with $|M_i| \le \tau$ for each $i = 1, 2, ..., n$ and put $M = \bigcup\limits_{i=1}^{n} M_i$. Then M is dense in $\bigcup\limits_{i=1}^{n} OF_i$ and $|M| \le \tau$.

Put

$$\mu = \{F \in \exp_n X : F \subset \bigcup_{i=1}^{n} M_i\}.$$

It is clear that $\left|\mu\right| \le \tau$. We shall show that μ is dense in $O\langle OF_1, OF_2, ..., OF_n\rangle$. Let $O\langle V_1, V_2, ..., V_k\rangle$ $(k \le n)$ be an arbitrary nonempty open set of $O\langle OF_1, OF_2, ..., OF_n\rangle$. By Theorem 1 [19] we have $\bigcup\limits_{i=1}^{k} V_i \subset \bigcup\limits_{j=1}^{n} OF_j$, and consequently, $V_i \subset \bigcup\limits_{j=1}^{n} OF_j$ for every $i = 1, 2, ..., k$. Each V_i intersects M, since the set M is dense in $\bigcup\limits_{i=1}^{n} OF_i$. Choosing a point $y_i \in V_i \cap M$ for each $i = 1, 2, ..., k$, put $E = \{y_1, y_2, ..., y_k\}$. Then $E = \{y_1,\ y_2, ..., y_k\} \in \mu$ and, on the other hand, $E \in O\langle V_1,\ V_2, ..., V_k\rangle$. Thus, the set μ is dense in $O\langle OF_1, OF_2, ..., OF_n\rangle$ and $|M| \le \tau$. This proves the inequality $ld(F) \le \tau$ in $C_\omega(X)$.

Now we shall show $ld\,(C_\omega(X)) \ge ld\,(X)$. Let $ld\,(C_\omega(X)) = \tau$. We have to prove that $ld\,(X) \le \tau$. Consider an arbitrary point $x \in X$. Clearly, $\{x\} \in C_1(X) \subset C_\omega(X)$. Then there exists a neighborhood $O\langle U_x\rangle$ in $C_n(X)$ such that $d\,(O\langle U_x\rangle), \le \tau$, where U_x is an open neighborhood of the point x in X. Assume that $S = \{F_\alpha : \alpha \in A\}$ is a dense set in $O\langle U_x\rangle$ such that $|S| \le \tau$. Choose an arbitrary point $x_\alpha \in F_\alpha$ from each set F_α. Put

$$B = \{x_\alpha : x_\alpha \in F_\alpha, F_\alpha \in S\}.$$

Obviously, $|B| \le \tau$ and B is dense in U_x. Indeed, if $G \subset U_x$ is any nonempty open subset of U_x, then $O\langle G\rangle$ is an open subset of $O\langle U_x\rangle$. Since S is dense in $O\langle U_x\rangle$, there exists an element $F_\alpha \in S$ such that $F_\alpha \in O\langle G\rangle$. It is easy to see that $F_\alpha \subset G$. According to the choice of the points of B, we have $x_\alpha \in F_\alpha \subset G$. Thus, B is dense in U_x. Since the point $x \in X$ has been chosen arbitrarily, we see that $ld\,(X) \le \tau$. This proves the inequality $ld\,(C_\omega(X)) \ge ld\,(X)$.

From the above proven inequalities we obtain $ld\,(X) = ld\,(C_\omega(X))$. With a completely similar way, one may prove the equality $ld(X) = ld(C_n(X))$. Theorem 4.8 is proved. □

From Theorem 4.8 we directly obtain the following results.

Corollary 4.9. *For every infinite compact space X the following conditions are equivalent:*

1) X is locally separable;

2) $C_n(X)$ is locally separable;

3) $C_\omega(X)$ is locally separable.

Corollary 4.10. *The functor $C_n\colon Comp \to Comp$ preserves the local density of infinite compact spaces.*

Lemma 4.11. *Let X be an infinite topological space and $U_1, U_2, ..., U_n$ be its open subsets such that $wd(U_i) \le \tau$, $i = 1, 2, ..., n$, where τ is some infinite cardinal number. Then $wd\left(\bigcup_{i=1}^{n} U_i\right) \le \tau$.*

Proof. Assume that the system $\gamma_i = \bigcup_{\alpha\in A} \gamma_\alpha^{(i)}$, where $|A| \le \tau$, is a π-base coinciding with τ centered systems γ_α^i in U_i for $i = 1, 2, \ldots, n$. Then the system

$$\gamma = \bigcup_{i=1}^{n} \gamma_i$$

is a π-base. Indeed, suppose that V is any nonempty open subset of the space $\bigcup_{i=1}^{n} U_i$. Then there exists $i \in \{1, 2, .., n\}$ such that $V \cap U_i \ne \varnothing$ and this intersection is open in the subspace U_i. Since γ_i is a π-base in U_i, there exists an element G from $\gamma_i \subset \gamma$ such that $G \subset V \cap U_i$ and thus, $G \subset V$. Therefore γ is a π-base in $\bigcup_{i=1}^{n} U_i$. Moreover, the system γ can be represented as the union of τ centered systems of open sets. This implies that $wd\left(\bigcup_{i=1}^{n} U_i\right) \le \tau$. Lemma 4.11 is proved. □

Now, for an element $O = O\langle U_1, U_2, ..., U_n\rangle$ of the base of $\exp X$ put $S(O) = \{U_1, U_2, ..., U_n\}$, where $U_1, U_2, ..., U_n$ are open sets in X.

Lemma 4.12. *Let*

$$\Delta = \left\{O_\beta = O\left\langle U_1^\beta, U_2^\beta, ..., U_{n(i)}^\beta\right\rangle :\ \beta \in B\right\}$$

be a centered system of open subsets of $C_\omega(X)$, where U_i^β are open sets in X for $\beta \in B$ and $i = 1, \ldots, n(i)$. Then the family

$$\mu = \left\{W_\beta = \bigcup S(O_\beta) : O_\beta \in \Delta,\ \beta \in B\right\}$$

is a centered system of open sets in X.

Proof. Assume the opposite, i.e. there exists a finite sequence $W_{\beta_1}, W_{\beta_2}, ..., W_{\beta_k}$ of elements from μ with empty intersection. Since the system Δ has the finite intersection property in $C_\omega(X)$, we have

$$\bigcap_{j=1}^{k} O\left\langle U_1^{\beta_j}, U_2^{\beta_j}, ..., U_{n(j)}^{\beta_j}\right\rangle \ne \varnothing.$$

Then there exists $F \in C_n(X)$ such that $F \subset \bigcup_{i=1}^{n(j)} U_i^{\beta_j}$ for each $j = 1, 2, ..., k$. This implies that

$$F \subset \bigcap_{j=1}^{k}\left(\bigcup_{i=1}^{n(j)} U_i^{\beta_j}\right) = \bigcap_{j=1}^{k} W_{\beta_j}.$$

This contradiction proves that the system μ has the finite intersection property. Lemma 4.12 is proved. □

Definition 4.13. The *local weak density at a point* $x \in X$, denoted by $lwd(x)$, is τ if τ is the smallest cardinal number such that x has a neighborhood of weak density τ in X.

The *local weak density* of a topological space X, denoted by $lwd(X)$, is defined as the supremum of all numbers $lwd(x)$ for $x \in X$. A topological space X is locally weakly separable, if $lwd(X) \le \omega$.

With the similar way as in Remark 4.7 one can prove the following result.

Remark 4.14. If $lwd(X) \leq \tau$ and $K \subset X$ is a compact subset, then there exists a neighborhood OK of K such that $wd(OK) \leq \tau$.

Theorem 4.15. *For every infinite compact space X we have*

$$lwd(X) = lwd(C_n(X)) = lwd(C_\omega(X)),$$

where n is an arbitrary natural number.

Proof. We prove the equality $lwd(X) = lwd(C_\omega(X))$. First suppose that $lwd\,(X) = \tau \geq \aleph_0$. We have to show that $lwd\,(C_\omega(X)) \leq \tau$. Take an arbitrary element $F \in C_\omega(X)$. Assume that

$$F = F_1 \cup F_2 \cup \ldots \cup F_n,$$

where $F_1, F_2, \ldots, F_n$ are components of the set F in X. Since $lwd\,(X) = \tau \geq \aleph_0$, by Remark 4.14 there exist neighborhoods $OF_1, OF_2, \ldots, OF_n$ of the sets $F_1, F_2, \ldots, F_n$, respectively, such that $wd(OF_i) \leq \tau$ for each $i = 1, 2, ..., n$. Then by Lemma 4.12 we have $wd(\bigcup\limits_{i=1}^{n} OF_i) \leq \tau$. In that case, we have the inequality

$$wd\,(O\,\langle OF_1, OF_2, \ldots, OF_n\rangle) \leq \tau.$$

Indeed, since $wd(\bigcup\limits_{i=1}^{n} OF_i) \leq \tau$, there exists a π-base $\mu = \bigcup\limits_{\alpha\in A} \mu_\alpha$ for $\bigcup\limits_{i=1}^{n} OF_i$, coinciding with τ centered systems μ_α, i.e. $|A| \leq \tau$ and for each $\alpha \in A$ the system μ_α is centered. Put

$$\Sigma = \{B \subset A\colon B \text{ is finite}\},$$

$$M = \{\gamma \subset \mu\colon \gamma \text{ is finite}\}$$

and

$$O(M) = \{O\,\langle W_1, W_2, \ldots, W_k\rangle : \{W_1, W_2, \ldots, W_k\} \in M\}.$$

Now we prove that $O(M)$ is a π-base in $O\,\langle OF_1, OF_2, \ldots, OF_n\rangle$ and can be represented as the union of τ centered systems. Take an arbitrary open subset $O\,\langle U_1, U_2, ..., U_k\rangle$ of $O\,\langle OF_1, OF_2, \ldots, OF_n\rangle$. Clearly, $U_j \subset \bigcup\limits_{i=1}^{n} OF_i$ for $j = 1, 2, ..., k$. Since μ is a π-base in $\bigcup\limits_{i=1}^{n} OF_i$, there exists an element G_j from μ such that $G_j \subset U_j$ for each $j = 1, 2, ..., k$. Then it is clear that

$$O\,\langle G_1, G_2, ..., G_k\rangle \subset O\,\langle U_1, U_2, ..., U_k\rangle$$

and

$$O\,\langle G_1, G_2, ..., G_k\rangle \in O(M).$$

Therefore $O(M)$ is a π-base in $O\,\langle OF_1, OF_2, \ldots, OF_n\rangle$.

Now let us show that $O(M)$ can be represented as the union of τ centered systems of open sets in $O\,\langle OF_1, OF_2, \ldots, OF_n\rangle$. For each $\delta \in \Sigma$ put

$$O_\delta(M) = \{O\,\langle W_{\alpha_1}, W_{\alpha_2}, ..., W_{\alpha_m}\rangle \in O(M) : \{\alpha_1, \alpha_2, ..., \alpha_m\} = \delta\}.$$

Then this system is centered for every $\delta \in \Sigma$ and, clearly

$$O(M) = \bigcup_{\delta\in\Sigma} O_\delta(M).$$

Indeed, let us take an arbitrary finite sequence of elements of $O_\delta(M)$:

$$O\left\langle W^{(1)}_{\alpha_1}, W^{(1)}_{\alpha_2}, \ldots, W^{(1)}_{\alpha_m}\right\rangle, O\left\langle W^{(2)}_{\alpha_1}, W^{(2)}_{\alpha_2}, \ldots, W^{(2)}_{\alpha_m}\right\rangle, \ldots, O\left\langle W^{(r)}_{\alpha_1}, W^{(r)}_{\alpha_2}, \ldots, W^{(r)}_{\alpha_m}\right\rangle,$$

where r is some natural number. Since every system μ_α is centered, we have $\bigcap\limits_{j=1}^{r} W_{\alpha_i}^{(j)} \neq \varnothing$ for $i = 1, 2, ..., m$. Choose a point y_i from the intersection for each $i = 1, 2, ..., m$ and form the set $E = \{y_1, y_2, ..., y_m\}$. For each $j = 1, 2, ..., r$ we have $E \subset \bigcup\limits_{i=1}^{m} W_{\alpha_i}^{(j)}$ and $E \cap W_{\alpha_i}^{(j)} \neq \varnothing$, $i = 1, 2, ..., m$. This implies that

$$E \in \bigcap_{j=1}^{r} O\left\langle W_{\alpha_1}^{(j)}, W_{\alpha_2}^{(j)}, ..., W_{\alpha_m}^{(j)} \right\rangle.$$

We have shown that any finite sequence of elements of $O_\delta(\mathcal{M})$ has nonempty intersection. Therefore, $O_\delta(M)$ is centered for each $\delta \in \Sigma$, and consequently, we obtain

$$wd(O\langle OF_1, OF_2, \ldots, OF_n\rangle) \leq \tau.$$

The inequality $lwd\,(C_\omega(X)) \leq \tau$ is proved.

Now assume that $lwd\,(C_\omega(X)) = \tau \geq \aleph_0$. We shall show that $lwd\,(X) \leq \tau$. Take an arbitrary point $x \in X$. Then $\{x\} \in C_1(X) \subset C_\omega(X)$. From the relation $lwd\,(C_\omega(X)) = \tau$ it follows that there exists a neighborhood $O\langle U_x\rangle$ of the point $\{x\}$ such that $wd\,(O\langle U_x\rangle) \leq \tau$, where U_x is an open set in X. Let us now prove that $wd\,(U_x) \leq \tau$.

From $wd\,(O\langle U_x\rangle) \leq \tau$ it follows that $O\langle U_x\rangle$ has a π-base $\mathrm{O} = \bigcup\limits_{\alpha\in A} \mathrm{O}_\alpha$, where the system

$$O_\alpha = \{O\left\langle U_1^\beta, U_2^\beta, ..., U_n^\beta\right\rangle : \beta \in B_\alpha\}$$

is centered for each $\alpha \in A$ and $|A| \leq \tau$. For each $\alpha \in A$ consider the system

$$\mu_\alpha = \{W_\beta = \bigcup_{i=1}^{n} U_i^\beta : \beta \in B_\alpha\}$$

of open sets in U_x. Then by Lemma 4.12 the system μ_α is centered for each $\alpha \in A$.

Now let us show that the system $\mu = \bigcup\limits_{\alpha\in A} \mu_\alpha$ is a π-base in U_x. Let $G \subset U_x$ be any nonempty open subset of U_x. Then $O\langle G\rangle$ is a nonempty open set in $C_n(X)$ and $O\langle G\rangle \subset O\langle U_x\rangle$. Since the system O is a π-base in $C_\omega(X)$, there exists $O\left\langle U_1^\beta, U_2^\beta, ..., U_n^\beta\right\rangle \in \mathrm{O}$ such that $O\left\langle U_1^\beta, U_2^\beta, ..., U_n^\beta\right\rangle \subset O\langle G\rangle$. This implies that

$$W_\beta = \bigcup_{i=1}^{n} U_i^\beta \subset G.$$

The set W_β is contained to μ. Therefore, μ is a π-base in U_x. We constructed the π-base coinciding with τ centered systems in U_x. Therefore, $wd\,(U_x) \leq \tau$ and since the point $x \in X$ has been chosen arbitrarily, the inequality $lwd\,(X) \leq \tau$ is proved. Therefore, $lwd\,(X) = lwd\,(C_\omega(X))$.

With a similar way we can prove the equality $lwd\,(X) = lwd\,(C_n(X))$. Theorem 4.15 is proved. □

From Theorem 4.15 we directly obtain the following results.

Corollary 4.16. *For every infinite compact space X the following conditions are equivalent:*
1) X is locally weakly separable;
2) $C_n(X)$ is locally weakly separable;
3) $C_\omega(X)$ is locally weakly separable.

Corollary 4.17. *The functor C_n: Comp → Comp preserves the local weak density of infinite compact spaces.*

5. Future investigation

We complete our study, presenting open problems which are related to the topic of this paper. These problems combine cardinal invariants with the so-called universality problem. We recall that the universality problem for topological spaces is a question, which determines whether there are universal elements in a given class of spaces. Actually, this problem can be posed also for any topological property where the class of spaces considered is the totality of all spaces having the property.

Definition 5.1. A topological space T is said to be *universal* in a class $\mathbb{P}$ of spaces if the following conditions are satisfied:
(1) $T \in \mathbb{P}$ and
(2) for every $X \in \mathbb{P}$ there exists an embedding of X into T.

In order to succeed answers to the universality problem in various classes of topological spaces, the notion of saturated classes of spaces was introduced. The precise definition of the saturated class of spaces is given in [17]. Among many important results that have been proved for the universality problem, we state that in any saturated class of spaces, there exist universal elements (see [17]).

Problem 5.2. *Let τ be an infinite cardinal and* ci *be one of the cardinal invariants: density, weak density, Souslin number, local density, local weak density and tightness. Is the class of all completely regular topological spaces X with* $\mathrm{ci}(X) \leqslant \tau$ *saturated?*

Problem 5.3. *Let τ be an infinite cardinal and* ci *be one of cardinal invariants: density, weak density, Souslin number, local density, local weak density and tightness. Does there exist a universal space in the class of all completely regular topological spaces X with* $\mathrm{ci}(X) \leqslant \tau$*?*

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