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Weakly S-Noetherian modules

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Abstract. Let R be a commutative ring, S a multiplicative subset of R and M an R-module. We say that M satisfies weakly S-stationary on ascending chains of submodules (w-ACC$_S$ on submodules or weakly S-Noetherian) if for every ascending chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of M, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n M_n \subseteq M_k$ for some $s_n \in S$. In this paper, we investigate modules (respectively, rings) with w-ACC$_S$ on submodules (respectively, ideals). We prove that if R satisfies w-ACC$_S$ on ideals, then R is a Goldie ring. Also, we prove that a semilocal commutative ring with w-ACC$_S$ on ideals have a finite number of minimal prime ideals. This extended a classical well known result of Noetherian rings.

1. Introduction

In 1988, Hamann, Houston and Johnson ([4]) in their works on polynomial rings over integral domains, introduced the notion of almost principal ideals. They called an ideal I of $D[X]$ (where D is an integral domain) *almost principal* if there exist a $s \in D\backslash\{0\}$ and a $f \in I$ of positive degree with $sI \subseteq fD[X]$ and they called the polynomial ring $D[X]$ an *almost PID* if each ideal of $D[X]$ that extends to a proper ideal of $K[X]$ is almost principal (K the quotient field of D). Then Anderson, Kwak and Zafrullah defined agreeable domains. An integral domain D is called *agreeable* if for each fractional ideal F of $D[X]$ with $F \subseteq K[X]$ where K is the quotient field of D, there exists a $s \in D\backslash\{0\}$ with $sF \subseteq D[X]$. They also called an ideal I of $K[X]$ is *almost finitely generated* if there is a finite set of polynomials $\{f_1, f_2, \ldots, f_n\}$ contained in I and an element $s \in D \setminus \{0\}$ such that $sI \subseteq (f_1, f_2, \ldots, f_n)$, [2].

Later, Anderson and Dumitrescu generalized the concept of almost principal and almost finitely generated ideals to modules over commutative rings. Let R be a commutative ring and $S \subseteq R$ be a multiplicative set and M be an R-module. Following [1], we say that M is *S-finite* (resp., *S-principal*) if $sM \subseteq F$ for some $s \in S$ and some finitely generated (resp., principal) submodule F of M. Also, M is called *S-Noetherian* (resp., *S-PIR*) if each submodule of M is a S-finite (resp., S-principal) module.

In 2016, Ahmed and Sana ([5]) tried to characterize the concept of S-Noetherian modules via a suitable chain condition and a special kind of maximality. An increasing sequence $(N_n)_{n\in\mathbb{N}}$ of submodules of M is called *S-stationary* if there exists a positive integer k and $s \in S$ such that for each $n \geq k$, $sN_n \subseteq N_k$ and a submodule N_i is called *S-maximal* if for every $j \in \mathbb{N}$, $sN_j \subseteq N_i$, for some $i \in \mathbb{N}$. They showed that, if every nonempty set of ideals of R has a S-maximal element, then R is S-Noetherian and the later that,

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every increasing sequence of ideals of R is S-stationary. In 2017, Bilgin, Reyes and Tekir ([3]) characterize S-Noetherian modules over noncommutative rings. They proved that M is S-Noetherian if and only if every increasing sequence of submodules of M is S-stationary if and only if every nonempty set of submodules of M has a S-maximal element if and only if every nonempty S-saturated set of submodules of M has a maximal element.

In this paper, we study weakly S-Noetherian modules, dualizing the former notion of weakly S-Artinian modules introduced by Khani-Nasab and Hamed in [6]. We say that M satisfies *weakly S-stationary* on ascending chains of submodule (w-ACC$_S$ on submodules for short) if for every ascending chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of M, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n M_n \subseteq M_k$ for some $s_n \in S$. Let $\mathcal{F}$ be a set of submodules of M. We say that $N \in \mathcal{F}$ is *weakly S-maximal* if for every $L \in \mathcal{F}$ and $N \subseteq L$, there exists $s \in S$ such that $sL \subseteq N$. We compare Noetherian modules with modules which have w-ACC$_S$ on submodules. For example, we show that there exists a module with w-ACC$_S$ on finitely generated submodules which does not satisfies w-ACC$_S$ on submodules. In section 3, we consider the case where $S \subseteq R$ is a regular multiplicative set. We show that a module M which satisfies weakly S-stationary on submodules (ACC$_S$ for short) where S is regular multiplicative set is a hopfian module. Moreover, if R satisfies w-ACC$_S$ on ideals where S is regular, then R is a Goldie ring. Also, we show that the converse is not true in general. Finally, we prove that a semilocal commutative ring with w-ACC$_S$ on ideals where S is regular, have a finite number of minimal prime ideals and the regularity of S is necessary.

2. Weakly S-stationary and weakly S-maximal

Let R be a commutative ring, $S \subseteq R$ a multiplicative set and M an R-module. According to [5], an increasing sequence $(N_n)_{n\in\mathbb{N}}$ of submodules of M is called S-stationary if there exist a positive integer $k \in \mathbb{N}$ and $s \in S$ such that for all $n \geq k$, $sN_n \subseteq N_k$. We say that M satisfies ACC$_S$ on submodules if for every ascending chain of submodules of M is S-stationary. In this section we relaxes this property by introducing the notion of weakly S-stationary sequence of submodules. We study various properties of modules in which every ascending chain of submodules is weakly S-stationary.

Definition 2.1. *Let R be a commutative ring, $S \subseteq R$ a multiplicative set and M an R-module. We say that M satisfies* weakly S-stationary *on ascending chains of submodules (w-ACC$_S$ on submodules for short) if for every ascending chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of M, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n M_n \subseteq M_k$ for some $s_n \in S$.*

Examples 2.2.
1. Modules with ACC$_S$ on submodules satisfies w-ACC$_S$ on submodules. In Example 2.8, we prove that the reverse of this implication is not true in general.
2. Every S-Noetherian modules satisfies w-ACC$_S$ on submodules (follows from [5, Remark 2.3] and the fact that every module with ACC$_S$ on submodules satisfies w-ACC$_S$ on submodules).
3. Let p be a prime number, $S = \{1\} \cup (p\mathbb{Z}\backslash\{0\})$ and $M = \mathbb{Z}_{p^\infty}$ (as a $\mathbb{Z}$-module). Then M satisfies w-ACC$_S$ on submodules. Note that M does not satisfy ACC$_S$ on submodules, since for every $s \in S$ and every finitely generated submodule F of M $s(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty} \nsubseteq F$.
4. Every semisimple module satisfies w-ACC$_S$.

Definition 2.3. *Let R be a commutative ring, $S \subseteq R$ a multiplicative set and M an R-module.*

1. *Let $\mathcal{F}$ be a set of submodules of M. We say that $N \in \mathcal{F}$ is weakly S-maximal if for every $L \in \mathcal{F}$ and $N \subseteq L$, there exists $s \in S$ such that $sL \subseteq N$.*
2. *A submodule N of M is said to be weakly S-maximal if it is weakly S-maximal in the set of all proper submodules of M.*

Proposition 2.4. *Let R be a commutative ring, $S \subseteq R$ a multiplicative set and M an R-module. Then the following assertions are equivalent.*

1. *M satisfies w-ACC$_S$ on submodules.*

2. *Every nonempty set of submodules of M has a weakly S-maximal element.*

Proof. (1) ⇒ (2) Let $\mathcal{F}$ be a nonempty set of submodules of M such that for every submodule $N \in \mathcal{F}$, N is not weakly S-maximal. Let $N_1 \in \mathcal{F}$. Then N_1 is not weakly S-maximal and so there exists $N_2 \in \mathcal{F}$ such that $N_1 \subseteq N_2$ and for every $s \in S$, $sN_2 \nsubseteq N_1$. $N_2 \in \mathcal{F}$ is not weakly S-maximal, hence there exists $N_3 \in \mathcal{F}$ such that $N_2 \subseteq N_3$ and for every $s \in S$, $sN_3 \nsubseteq N_2$. By continuing this way, we obtain a chain of submodules $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ which is not weakly S-stationary. This shows that M does not satisfy w-ACC$_S$ on submodules.

(2) ⇒ (1) Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be a chain of submodules in M. Set

$$\mathcal{F} = \{N_i, \quad i = 1, 2, \ldots\}$$

By (2), $\mathcal{F}$ has a weakly S-maximal element like N_k where $k \in \mathbb{N}$. Clearly for every $n \geq k$, there exists $s_n \in S$ such that $s_n N_n \subseteq N_k$. □

Our next result gives equivalent conditions for an R-module M to be S-Noetherian, where S is a finite multiplicative subset of R. First let us recall the following notion. Let $\mathfrak{F}$ be a family of submodules of M. An element $N \in \mathfrak{F}$ is said to be *S-maximal* if there exists a $s \in S$ such that for each $L \in \mathfrak{F}$, if $N \subseteq L$, then $sL \subseteq N$ ([5]).

Proposition 2.5. *Let R be a commutative ring, $S \subseteq R$ a finite multiplicative set and M an R-module. Then the following assertions are equivalent.*

1. *M is a S-Noetherian module.*
2. *M satisfies ACC$_S$ on submodules.*
3. *M satisfies w-ACC$_S$ on submodules.*
4. *Every nonempty set of submodules of M has a weakly S-maximal element*
5. *Every nonempty set of submodules of M has a S-maximal. element.*

Proof. (1) ⇒ (2) Follows from Example 2.2(1).

(2) ⇒ (3) Obvious.

(3) ⇒ (4) Follows from Proposition 2.4.

(4) ⇒ (5) Follows from the fact that the weakly S-maximal and the S-maximal properties are the same when S is finite.

(5) ⇒ (1) Let $S = \{s_1, s_2, s_3, \ldots, s_n\}$ and N be a submodule of M. Set $s := s_1 s_2 \cdots s_n$. We show that N is S-finite. Suppose that $\mathcal{F}$ is the set of all finitely generated submodules of M included in N. Clearly, $\mathcal{F}$ is a nonempty set. By (5) there exists $F \in \mathcal{F}$ such that F is S-maximal. Let $x \in N$. Set $L = F + Rx$. Then $L \in \mathcal{F}$ and $F \subseteq L$. Since F is S-maximal, there exists $s_{i_0} \in S$ such that $s_{i_0} L \subseteq F$. Thus

$$(s_1 s_2 \cdots s_n)L \subseteq s_{i_0} L \subseteq F.$$

This implies that $sN \subseteq F$, and hence M is a S-Noetherian module. □

Corollary 2.6. *Let R be a commutative ring and S a finite regular multiplicative subset of R. Then R is Noetherian if and only if R satisfies w-ACC$_S$ on ideals. Indeed, by [5, Example 3.2], $S \subseteq \mathrm{U}(R)$; so R satisfies w-ACC$_S$ on ideals if and only if R satisfies ACC$_S$ on ideals if and only if R satisfies ACC on ideals if and only if R is a Noetherian ring.*

We know that M is a Noetherian module if and only if every ascending chain of finitely generated submodules stops. Next we construct an example of a module with w-ACC$_S$ on finitely generated submodules which does not satisfies w-ACC$_S$ on submodules. First we need the following Remark.

Remark 2.7. *Let R be a commutative ring, $S \subseteq R$ a multiplicative set and M an R-module. Assume that for every ascending chain $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ of submodules of M and for each $n \in \mathbb{N}$, there exists $s_n \in S$ such that $s_n L_n = 0$, then M satisfies w-ACC$_S$ on submodules.*

Example 2.8. *Consider $M = \bigoplus_{p\in P} \mathbb{Z}_p$ as a $\mathbb{Z}$-module where P is the set of all prime integers. Let $S = \mathbb{Z} \setminus \{0\}$. First we show that M satisfy w-ACC$_S$ on finitely generated submodules. Let L be a finitely generated submodule of M. Then there exists $p_1, p_2, \ldots, p_n \in P$ such that $L \hookrightarrow \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_n}$ and hence L is finite. By Remark 2.7, every finite module satisfies w-ACC$_S$ on submodules. This shows that M satisfies w-ACC$_S$ on submodules.*

Next we introduce a chain of submodules of M which does not satisfy the w-ACC$_S$ on submodules. Let $p_1 \leq p_2 \leq p_3 \leq \cdots$ be all prime numbers. Suppose that for every p we replace $\iota_p(\mathbb{Z}_p)$ by $\mathbb{Z}_p$ where $\iota_p : \mathbb{Z}_p \longmapsto M$. Set $L = \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots$ and $K = \mathbb{Z}_{p_2} \oplus \mathbb{Z}_{p_4} \oplus \cdots$. Since $I = \{p_1, p_3, p_5, \ldots\}$ is infinite, there exist infinite subsets I_1 and I_2 of I such that $I = I_1 \cup I_2$. Also, I_1 is infinite. So there exist infinite subsets I_3 and I_4 of I_1 such that $I_1 = I_3 \cup I_4$. Continuing in this way, we get a sequence $I_1, I_3, I_5, \ldots$ such that $I_n = I_{n+2} \cup I_{n+3}$. Define $L_i = \bigoplus_{p\in I_i} \mathbb{Z}_p$ for every $i \in \mathbb{Z}$ we have the following chain

$$K \oplus L_2 \subsetneq K \oplus L_2 \oplus L_4 \subsetneq K \oplus L_2 \oplus L_4 \oplus L_6 \subsetneq \cdots$$

Suppose that there exists $k \in \mathbb{N}$ such that for every $n \geq k$

$$s_n(K \oplus L_2 \oplus L_4 \oplus \cdots \oplus L_{2n+2}) \subseteq K \oplus L_2 \oplus L_4 \oplus \cdots \oplus L_{2n}$$

for some $s_n \in S$. Thus

$$s_n(L_{2n+2}) \subseteq K \oplus L_2 \oplus L_4 \oplus \ldots \oplus L_{2n}$$

Hence $s_n(L_{2n+2}) = 0$. I_{2n+2} is an infinite set of prime numbers. Let $t_1, t_2, \ldots$ be all distinct elements of I_{2n+2}. Then $L_{2n+2} = \bigoplus_{p\in\{t_1,t_2,\ldots\}} \mathbb{Z}_p$. Since $s_n L_{2n+2} = 0$, for every $i \in \mathbb{N}$, $t_i | s_n$, a contradiction. Thus M does not satisfy w-ACC$_S$ on submodules.

Next proposition investigates w-ACC$_S$ on ideals for direct product of rings.

Proposition 2.9. *Let $S_1, S_2, \cdots, S_n$ be multiplicative subsets of rings $R_1, R_2, \cdots, R_n$, respectively. Set $R = \prod_{i=1}^n R_i$ and $S = \prod_{i=1}^n S_i$. Then the following conditions are equivalent.*

1. *R satisfies w-ACC$_S$ on ideals*
2. *For each $i \in \{1, ..., n\}$, R_i satisfies w-ACC$_{S_i}$ on ideals*

Proof. (1) ⇒ (2) Obvious.

(2) ⇒ (1) Suppose that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending chain of ideals in R. Then for every $i \in \mathbb{N}$, $I_i = L_{i1} \times L_{i2} \times \cdots \times L_{in}$ where L_{ij} is an ideal of R_j, for all $j \in \{1, 2, \ldots, n\}$. Since every R_j satisfies w-ACC$_{S_i}$ on ideals, we can find $k \in \mathbb{N}$ such that for each $n \geq k$ and $j \in \{1, 2, \ldots, n\}$ there exists $s_{nj} \in S_j$ such that $s_{nj} L_{nj} \subseteq L_{kj}$. Therefore, for every $n \geq k$, $s_n = (s_{n1}, s_{n2}, \ldots, s_{nn}) \in \prod_{i=1}^n S_i$ and we have $s_n I_n \subseteq I_k$. This shows that R has w-ACC$_S$ on ideals where $S = \prod_{i=1}^n S_i$. □

Unlike finite product of rings, an infinite product of rings not necessarily has w-ACC$_S$ on ideals.

Example 2.10. *Let $R = \prod_{i\in I} R_i$ and $S = \{1_R\}$ be a multiplicative subset of R where index set of I is infinite. Since I is infinite, there exist infinite subsets I_1 and I_2 of I such that $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. Set $J = \bigoplus_{i\in I_1} R_i$ and $K = \bigoplus_{i\in I_2} R_i$. So $J \subsetneq J \oplus K$ and continuing in this way, we can form an ascending chain of ideals of R. Thus R does not satisfy w-ACC$_S$ on ideals.*

Proposition 2.11. *Let M be an R-module, N a proper submodule of M and S a multiplicative subset of R. Then the following assertions are equivalent.*

1. *M satisfies w-ACC$_S$ on submodules.*
2. *N and M/N both satisfy w-ACC$_S$ on submodules.*

Proof. (1) ⇒ (2) Assume that M has w-ACC$_S$ on submodules. It is immediate that N satisfies w-ACC$_S$ on submodules. Let $L_1/N \subseteq L_2/N \subseteq L_3/N \subseteq \cdots$ be a chain of submodules in M/N. Since $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ is a chain in M and M satisfies w-ACC$_S$ on submodules, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, there exists $s_n \in S$ with $s_n L_n \subseteq L_k$. This implies that for every $n \geq k$, $s_n(L_n/N) \subseteq L_k/N$. Hence M/N satisfies w-ACC$_S$ on submodules.

(2) ⇒ (1) Let $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ be a chain in M. By assumption, there exists a positive integer k such that for each $n \geq k$, there exists $s_n \in S$ with $s_n(L_n + N)/N \subseteq (L_k + N)/N$ and there exists $s'_n \in S$ such that $s'_n(N \cap L_n) \subseteq N \cap L_k$. We prove that for each $n \geq k$, $s'_n s_n(L_n) \subseteq L_k$. Since $L_n \subseteq L_n + N$, $s_n(L_n) \subseteq s_n(L_n + N) \subseteq L_k + N$. Let $x \in L_n$. Then $s_n x \in L_k + N$ and there exist $l \in L_k$ and $y \in N$ such that $s_n x - l = y$. Thus $s_n x - l \in N \cap L_n$, and so $s'_n(s_n x - l) \in N \cap L_k$. Therefore $s' s_n x \in L_k$, as desire. □

Corollary 2.12. *Let R be a ring and S be a multiplicative subset of R. Then R satisfies w-ACC$_S$ on ideals if and only if for each $n \in \mathbb{N}^*$, R^n satisfies w-ACC$_S$ on submodules.*

Proof. Assume that R satisfies w-ACC$_S$ on ideals. We will show this via induction. Let $P(n)$ be the property that R^n satisfies w-ACC$_S$ on submodules. For $n = 1$, R satisfies w-ACC$_S$ on ideals if and only if for each R as an R-module satisfies w-ACC$_S$ on submodules. Suppose that the property holds for $1 \leq n$. Let's prove $P(n + 1)$. The module R^n is isomorphic to the submodule $N = R^n \times \{0\}$. Hence, by the induction hypothesis and Proposition 2.11, N satisfies w-ACC$_S$. Clearly $R^{n+1}/N \simeq R$. Thus by Proposition 2.11, R^{n+1} satisfies w-ACC$_S$ on submodules. The other implication is obvious. □

Theorem 2.13. *Let R be a commutative ring, S a multiplicative subset of R and M a finitely generated R-module. If R satisfies w-ACC$_S$ on ideals, then M satisfies w-ACC$_S$ on submodules.*

Proof. Since M is a finitely generated R-module, there exist $n \in \mathbb{N}^*$ and a surjective module homomorphism $f : R^n \longrightarrow M$, such that $R^n/Ker(f) \simeq M$. By Corollary 2.12, R^n satisfies w-ACC$_S$ on submodules; so by Proposition 2.11, $R^n/Ker(f)$ satisfies w-ACC$_S$. Therefore M satisfies w-ACC$_S$ on submodules. □

Corollary 2.14. *Let R be a commutative ring, $S \subseteq R$ a multiplicative set and M a S-finite R-module. If R satisfies w-ACC$_S$ on ideals, then M satisfies w-ACC$_S$ on submodules.*

Proof. Since M is S-finite, there exist $s \in S$ and a finitely generated submodule F of M such that $sM \subseteq F$. Suppose that $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ is a chain of submodules in M. By Theorem 2.13, F satisfies w-ACC$_S$ on submodules. Since for each n, sN_n is a submodule of F, the chain $sN_1 \subseteq sN_2 \subseteq sN_3 \subseteq \cdots$ is a chain in F; so there exists $k \in \mathbb{N}$ such that, for each $n \geq k$ there exists $t_n \in S$ with $t_n(sN_n) \subseteq sN_k \subseteq N_k$. For each $n \geq k$, let $s_n := st_n \in S$. Thus for each $n \geq k$, $s_n N_n \subseteq N_k$. This shows that M satisfies w-ACC$_S$ on submodules. □

3. Weakly S-stationary when S is a regular multiplicative set

In this section we prove a relation between modules satisfying the w-ACC$_S$ property and some classical well known modules (hopfian modules, Goldie rings, ...) where S is a regular multiplicative set. We start this section by the following definition.

Definition 3.1. *For an R-module M and $s \in R$, we say that s is a nonzero divisor for M, if for each $m \in M$, $sm = 0$ implies that $m = 0$. A regular multiplicative set S over M is a set in which for every $s \in S$, s is nonzero divisor for M.*

Example 3.2. Let R be a valuation ring and let S be a multiplicative set of regular elements of R. Set $K = \bigcap_{s \in S} Rs$. Then $K \trianglelefteq R$. Consider the ring $\overline{R} := R/K$ and $\overline{S} := \{s + K \mid s \in S\} \subseteq \overline{R}$.

1. $\overline{S}$ is closed under multiplication.
2. $1_{\overline{R}} = 1 + K \in \overline{S}$.
3. $0_{\overline{R}} \notin \overline{S}$ if and only $K \neq R$.

If $K \neq R$, then $\overline{S}$ is a multiplicative regular set in $\overline{R}$. In this case, $\overline{R}$ satisfies w-ACC$_S$ on ideals.

Proof. (1). Clear.

(2). Clear.

(3). If $K = R$, then $S \subseteq K$ and hence $\overline{S} = \{0 + K\}$. Conversely, if $K \neq R$, then $1 \notin K$. Thus, there exists $s_0 \in S$ such that $1 \notin s_0R$. Suppose to the contrary, $0 + K \in \overline{S}$. There exists $s_1 \in S$ such that $0 + K = s_1 + K$. Hence $s_1 \in K \subseteq s_0s_1R$; so there exists $r \in R$ such that $s_1 = rs_0s_1$, which implies that $1 = rs_0$ since S is regular. Therefore $s_0R = R$, a contradiction.

We want to prove that if $K \neq R$, then $\overline{S}$ is regular. Let $(s + K)(r + K) = 0_{\overline{R}}$ where $s \in S$ and $r \in R$. Let $s' \in K$. Then $sr \in ss'R$. There exists $x \in R$ such that $sr = ss'x$. Since S is regular, $r = s'x \in s'R$. Thus $r \in K$, as desire.

Now, we show that $\overline{R}$ satisfies ACC$_{\overline{S}}$ on ideals. Let I/K be a nonzero ideal in $\overline{R}$. Then $K \subset I \trianglelefteq R$ and $I \nsubseteq K$. Hence, there exists $s_0 \in S$ such that $I \nsubseteq Rs_0$. Since R is a valuation ring, $s_0R \subseteq I$ and $s_0I \subseteq s_0R \subseteq I$. It follows that

$$(s_0 + K)I/K = (s_0I + K)/K \subseteq (s_0R + K)/K = (s_0 + K)\overline{R} \subseteq I/K.$$

Thus $\overline{R}$ is a $\overline{S}$-Noetherian ring, and hence satisfies w-ACC$_{\overline{S}}$ on ideals. □

An R-module M is said to be *hopfian* if any surjective endomorphism of M is an isomorphism. We know that Noetherian modules are hopfian. Our next result relaxes the Noetherian property by the w-ACC$_S$ notion.

Proposition 3.3. *Let R be a commutative ring, M an R-module and $S \subseteq R$ is a regular multiplicative set over M. If M satisfies w-ACC$_S$ on submodules, then M is hopfian.*

Proof. Let $\phi : M \to M$ be a surjective homomorphism. Consider the following chain

$$\mathrm{Ker}(\phi) \subseteq \mathrm{Ker}(\phi^2) \subseteq \mathrm{Ker}(\phi^3) \subseteq \cdots$$

Since M satisfies w-ACC$_S$ on submodules, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n\mathrm{Ker}(\phi^{n+1}) \subseteq \mathrm{Ker}(\phi^n)$ for some $s_n \in S$. Let $m \in \mathrm{Ker}(\phi)$. Since ϕ is surjective, there exists $m' \in M$ such that $m = \phi^n(m')$. Then $\phi(m) = \phi(\phi^n(m'))$ implies that $0 = \phi^{n+1}(m')$ and thus $m' \in \mathrm{Ker}(\phi^{n+1})$. Multiplying s_n, we have $s_nm' \in s_n\mathrm{Ker}(\phi^{n+1}) \subseteq \mathrm{Ker}(\phi^n)$. Thus $s_nm' \in \mathrm{Ker}(\phi^n)$, and so $s_n\phi^n(m') = \phi^n(s_nm') = 0$. Since S is regular on M, $m = \phi^n(m') = 0$. Hence ϕ is an isomorphism. □

Lemma 3.4. *Let R be a commutative ring, M an R-module and $S \subseteq R$ a regular multiplicative set over M. Assume that R satisfies w-ACC$_S$ on ideals. Then R satisfies ACC on annihilators of subsets of M.*

Proof. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending sequence in R such that for every $j \in \mathbb{N}$, $I_j = \mathrm{ann_R(A_i)}$ for some $A_i \subseteq M$. Since R satisfies w-ACC$_S$ on ideals, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_nI_n \subseteq I_k$ for some $s_n \in S$. Let $n \geq k$ and $a \in I_n$, $s_na \in I_k$. So $s_naA_k = 0$. By regularity of S on M we have $aA_k = 0$. It follows that $a \in I_k$. Therefore, $I_n \subseteq I_k \subseteq I_n$, and hence $I_n = I_k$. Thus R satisfies ACC on annihilators of subsets of M. □

Remark 3.5. *Let R be a commutative ring, M an R-module and $S \subseteq R$ a regular multiplicative set over M. Assume that R satisfies w-ACC$_S$ on ideals. Then by the previous Lemma 3.4, the set $\mathrm{X} = \{\mathrm{ann_R(A)} \mid \mathrm{A} \subseteq \mathrm{M} \setminus \{0\}\}$ has a maximal element.*

Let R be a commutative ring and M an R-module. We denoted by $\mathrm{Z}(M)$ the set $\mathrm{Z}(M) = \{r \in R \mid xr = 0, \text{for some nonzero } x \in M\} = \bigcup_{0 \neq x \in M} \mathrm{ann_R(x)}$.

Theorem 3.6. *Let R be a commutative ring, M an R-module and $S \subseteq R$ a regular multiplicative set over M. Let $X = \{\mathrm{ann_R(x)} \mid \mathrm{x} \in \mathrm{M}\setminus\{0\}\}$. Assume that R and M both satisfy w-ACC$_S$ on submodules. Then*

1. *X has only a finite number of maximal elements.*
2. $\mathrm{Z}(M)$ *is a union of a finite number of associated primes of M.*

Proof. (1). Assume $\{\mathrm{ann_R(x_i)}\}_{\mathrm{i\in\mathbb{N}}}$ is a set of (distinct) maximal elements of X. Consider the chain $x_1R \subseteq x_1R + x_2R \subseteq \cdots$ in M. Since M satisfies w-ACC$_S$ on submodules, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n(\sum_{j=1}^{n} x_jR) \subseteq \sum_{j=1}^{k} x_jR$ for some $s_n \in S$. This implies that $s_n(\sum_{j=1}^{k+1} x_jR) \subseteq \sum_{j=1}^{k} x_jR$; so $s_nx_{k+1} \in \sum_{j=1}^{k} x_jR$. Thus, there exist $r_1, r_2, \ldots, r_k \in R$ such that $s_nx_{k+1} = r_1x_1 + \cdots + r_kx_k$. For $i \in \mathbb{N}$, set $P_i = \mathrm{ann_R(x_i)}$. Then $P_1P_2 \cdots P_k(r_1x_1 + \cdots + r_kx_k) = 0$; so $P_1P_2 \cdots P_ks_nx_{k+1} = 0$. Since S is regular, $P_1P_2 \cdots P_kx_{k+1} = 0$, and hence $P_1P_2 \ldots P_k \subseteq P_{k+1} = \mathrm{ann_R(x_{k+1})}$. It is easy to see that each maximal element of X is a prime ideal in R and so P_{k+1}. Thus there exists $j < k + 1$ such that $P_j \supseteq P_{k+1}$. Since $P_{k+1} \in X$, maximality of P_j implies that $P_j = P_{k+1}$, a contradiction.

(2). By the first assertion, X has only a finite number of maximal elements, say $\mathrm{ann_R(x_1)}, ..., \mathrm{ann_R(x_n)}$, where $x_1, \ldots, x_n \in M$. We show that $\mathrm{Z}(M) = \bigcup_{j=1}^{n} \mathrm{ann_R(x_j)}$. Clearly, $\bigcup_{j=1}^{n} \mathrm{ann_R(x_j)} \subseteq \mathrm{Z(M)}$. Conversely, let $a \in \mathrm{Z}(M)$. Then there exists $x \in M \setminus \{0\}$ such that $ax = 0$. Consider $Y = \{\mathrm{ann_R(y)} \mid 0 \neq \mathrm{y} \in \mathrm{M}, \mathrm{ann_R(x)} \subseteq \mathrm{ann_R(y)}\}$. Then $\mathrm{ann_R(x)} \in \mathrm{Y}$, and so $Y \neq \emptyset$. By Lemma 3.4, R satisfies ACC on annihilators of subsets of M; so Y has a maximal element, say $\mathrm{ann_R(y)}$. But $\mathrm{ann_R(y)}$ is a maximal element of X. So there exists $i \in \{1, \ldots, n\}$ such that $\mathrm{ann_R(y)} = \mathrm{ann_R(x_i)}$. Hence $a \in \mathrm{ann_R(x)} \subseteq \mathrm{ann_R(y)} = \mathrm{ann_R(x_i)} \subseteq \bigcup_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{ann_R(x_j)}$. Therefore, $\mathrm{Z}(M) = \bigcup_{j=1}^{n} \mathrm{ann_R(x_j)}$. It is not hard to see that P_j is an associated prime of M. □

Example 3.7. A commutative ring R with w-ACC$_S$ on ideals where S is a multiplicative non regular set of R may not have ACC on annihilators.

Let F be a field and $R = F[x_1, x_2, \ldots]/\langle x_ix_j; i \neq j\rangle$. Suppose that $S = \{\overline{x_1^i} | i \geq 0\}$. Then S is a multiplicative set of R. Since the chain $\langle \overline{x_1} \rangle \subseteq \langle \overline{x_1}, \overline{x_2} \rangle \subseteq \cdots$ is not stationary, R is not Noetherian. It is enough to show that R is S-Noetherian. First define the following mapping; $\theta : R \to F[\overline{x_1}]$, with $f \mapsto f_1(\overline{x_1})$, where $f = f_1(\overline{x_1}) + \overline{x_2}f_2(\overline{x_2}) + \cdots + \overline{x_n}f_n(\overline{x_n})$.

Clearly θ is a surjective homomorphism and $\ker(\theta) = \{\sum_{i=2}^{n} \overline{x_i}f_i(\overline{x_i}) \mid n \in \mathbb{N}\}$. Let I be an ideal of R. Then $\overline{x_1}I$ is an ideal of R too. Thus $\theta(\overline{x_1}I)$ is an ideal of $F[\overline{x_1}]$. Therefore, $\theta(\overline{x_1}I)$ is principal. Since $\overline{x_1}I \cap \ker(\theta) = 0$, $\overline{x_1}I$ is principal. Hence R is S-Noetherian. Thus R satisfies w-ACC$_S$ on ideals. Now, we introduce a chain of annihilators in R which is not stationary:

$$\mathrm{ann_R(x_1, x_2, \ldots)} \subsetneq \mathrm{ann_R(x_2, x_3, \ldots)} \subsetneq \mathrm{ann_R(x_3, x_4, \ldots)} \subsetneq \cdots$$

So the regularity of S is necessary in Lemma 3.4.

Recall that a module M is called a *uniform* module if the intersection of any two nonzero submodules is nonzero. A submodule N of M is said to be an *essential* submodule of M if for every submodule H of M, $H \cap N = \{0\}$ implies that $H = \{0\}$. The *uniform dimension* of a module M, denoted u.dim(M), is defined to be n if there exists a finite set of uniform submodules U_i such that $\bigoplus_{i=1}^{n} U_i$ is an essential submodule of M. A ring R is said to be a *Goldie* ring if it has finite uniform dimension as a module over itself, and satisfies the ascending chain condition on annihilators of subsets of R. With aid of following lemma, we show that a ring with w-ACC$_S$ on ideals where $S \subseteq R$ is a regular multiplicative set, is a Goldie ring.

Lemma 3.8. *Let R be a commutative ring and M an R-module which satisfies w-ACC$_S$ on submodules, where $S \subseteq R$ is a regular multiplicative set over M. Then M has finite uniform dimension.*

Proof. Suppose to the contrary that M has not finite uniform dimension. Then there exists a family of independent nonzero submodules such as $\{N_1, N_2, N_3, \ldots\}$. Consider the following chain of submodules of M:

$$N_1 \subseteq N_1 \oplus N_2 \subseteq N_1 \oplus N_2 \oplus N_3 \subseteq \cdots$$

Since M satisfies w-ACC$_S$ on submodules, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n(\bigoplus_{i=1}^{n} N_i) \subseteq \bigoplus_{i=1}^{k} N_i$ for some $s_n \in S$. In particular, $s_n N_{k+1} = 0$. Since S is regular over M, we must have $N_{k+1} = 0$, a contradiction. So M has finite uniform dimension. □

Theorem 3.9. *Let R be a commutative ring and $S \subseteq R$ a regular multiplicative set. If R satisfies w-ACC$_S$ on ideals, then R is Goldie.*

Proof. Follows directly from Lemma 3.4 and Lemma 3.8. □

Following example shows that the converse of Theorem 3.9 is not true in general.

Example 3.10. *Let $R = \mathbb{Z}[x_1, x_2, \ldots]$ and $S = \{x_1^i \mid i \geq 0\}$. Clearly S is a regular multiplicative set of R. Also, R is a Goldie ring. The following chain shows that R does not satisfies w-ACC$_S$ on ideals:*

$$\langle x_2 \rangle \subsetneq \langle x_2, x_3 \rangle \subsetneq \langle x_2, x_3, x_4 \rangle \subsetneq \cdots$$

So the converse of Theorem 3.9 does not hold.

In the next result, we show that a commutative semilocal ring with w-ACC$_S$ on ideals have a finite number of minimal prime ideals. First, we need the following Lemma.

Lemma 3.11. *Let R be a commutative ring and $S, T \subseteq R$ be two multiplicative sets of R. If R satisfies w-ACC$_S$ on ideals, then $T^{-1}R$ satisfies w-ACC$_S$ on ideals.*

Proof. Suppose that $A_1 \subseteq A_2 \subseteq \cdots$ be an ascending sequence of ideals of $T^{-1}R$. Then for each $n \in \mathbb{N}^*$, $A_n = T^{-1}B_n$, for some ideal B_n of R. For each $n \in \mathbb{N}^*$, set $I_n := \sum_{i=1}^{n} B_i$. Then $(I_n)_n$ is an ascending sequence of ideals of R. Since R satisfies w-ACC$_S$ on ideals, there exists $k \in \mathbb{N}^*$ such that for each $n \geq k$, $s_n I_n \subseteq I_k$ for some $s_n \in S$. This implies that for each $n \geq k$, $s_n(T^{-1}I_n) \subseteq T^{-1}I_k$.

Now, for each $n \in \mathbb{N}^*$,

$$\begin{aligned} T^{-1}I_n &= T^{-1}(B_1 + \cdots + B_n) \\ &= T^{-1}B_1 + \cdots + T^{-1}B_n \\ &= A_1 + \cdots + A_n \\ &= A_n. \end{aligned}$$

Thus for each $n \geq k$, $s_n A_n \subseteq A_k$, which implies that the sequence $(A_n)_n$ is weakly S-stationary. Hence $T^{-1}R$ satisfies w-ACC$_S$ on ideals. □

Theorem 3.12. *Let R be a commutative semilocal ring and S a regular multiplicative subset of R. If R satisfies w-ACC$_S$ on ideals, then R contains only a finite number of minimal primes.*

Proof. A commutative semilocal ring has only a finite number of maximal ideals. Since every minimal prime ideal of R is contained in a maximal ideal, it is enough to show that every maximal ideal of R contains only a finite number of minimal primes. But for every maximal ideal M of R, the minimal prime ideals of R which are contained in M correspond to the minimal prime ideals of the ring $T^{-1}R$ for $T = R \backslash M$. Thus it suffices to consider the case when R is a local ring. It is clear that for every ideal I in R, R/I has w-ACC$_{\overline{S}}$ on ideals, where $\overline{S} = \{s + I \mid s \in S\}$. So considering the quotient of R modulo its prime radical, we may assume that R is semiprime. Now, by [8, Theorem 11.43], R has only a finite number of minimal primes if and only if R has finite uniform dimension. From Proposition 3.8, we obtain that R has finite uniform dimension. Hence R contains only a finite number of minimal prime ideals. □

Following example shows that a ring with w-ACC_S on ideals where S is a non-regular multiplicative set of R may have infinitely many minimal prime ideals.

Example 3.13. Consider R as in Example 3.7. R has w-ACC_S on ideals and S is a non regular multiplicative subset of R. Then $M := \langle x_1, x_2, x_3, \ldots\rangle / \langle x_j \mid i \neq j \in \mathbb{N}\rangle$ is a maximal ideal of R. It is easy to show that for every $k \in \mathbb{N}$, $\langle x_j \mid j \in \mathbb{N} \setminus \{k\}\rangle / \langle x_i x_j \mid i \neq j \in \mathbb{N}\rangle$ is a minimal prime ideal. Thus the localization R_M has infinitely many minimal prime ideals. This shows that the regularity of S is necessary in Theorem 3.12.

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