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A new type of exponential operator

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Abstract. In the present research, we investigate a novel type of exponential operator. This operator is developed using $p(x) = x^{4/3}$. Here, we establish the direct estimate, quantitative variants of the Voronovskaja theorem, same quantification for functions having exponential growth and some other convergence estimates for the newly defined exponential-type operator. Later in the end, we analyze graphically the convergence of the new operator for the exponential function e^{-4x}.

1. Introduction

Ismail and May [10] first studied the exponential operators and established the method of construction of some new exponential type approximation operators four decades ago. Since then no other new exponential-type operator is constructed by researchers. We are developing here a new type of exponential operator associated with $p(x) = x^{4/3}$. For the establishment of same, consider the following form of exponential operator:

$$(M_n g)(x) = \int_0^\infty K_n(x,t)\, g(t)\, dt,$$

where the kernel $K_n(x,t)$ satisfies the following partial differential equation:

$$\frac{\partial}{\partial x} K_n(x,t) = \frac{nt}{p(x)}\left(1 - \frac{x}{t}\right) K_n(x,t). \tag{1}$$

By simple calculations, the solution to (1) is provided by

$$K_n(x,t) = exp\left(\frac{-3nt}{x^{1/3}} - \frac{3nx^{2/3}}{2}\right) F(n,t),$$

where $x \in (0,\infty)$.

In order to have normalization, we must have

$$\int_0^\infty exp\left(\frac{-3nt}{x^{1/3}}\right) F(n,t)\, dt = exp\left(\frac{3nx^{2/3}}{2}\right). \tag{2}$$

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Obviously, by simple computations, we have

$$F(n,t) = \delta(t) + \sum_{m=1}^{\infty} \frac{t^{2m-1}(3n)^{3m}}{2^m\, m!\,\Gamma(2m)},$$

where $\delta(t)$ is the Dirac delta function.

Thus our new operator connected with $x^{4/3}$ for $x \in (0,\infty)$ takes the following form:

$$(M_n g)(x) = e^{-\frac{3n}{2}x^{2/3}}\Big[g(0) + \sum_{m=1}^{\infty} \frac{(3n)^{3m}}{2^m\, m!\,\Gamma(2m)} \int_0^{\infty} e^{\frac{-3nt}{x^{1/3}}}\, t^{2m-1}\, g(t)\, dt\Big] \tag{3}$$

which, as revealed by its construction, is of the exponential-type.

Remark 1.1. *Alternatively, eq. (2) can be obtained using the fascinating approach devised by Ismail and May [10] in the following form:*
Consider

$$q(x) = \int_1^x \frac{dt}{t^{4/3}}.$$

Following [10, eq. (3.2)], we get

$$exp\left(n\int_1^x \frac{t}{t^{4/3}}dt\right) = \int_{-\infty}^{\infty} exp\left[3nt\left(1 - \frac{1}{x^{1/3}}\right)\right] C(n,t)dt$$

implying

$$exp\left(\frac{3nx^{2/3}}{2}\right) = \int_{-\infty}^{\infty} exp\left(-\frac{3nt}{x^{1/3}}\right) F(n,t)dt,$$

where

$$F(n,t) = \begin{cases} \delta(t) + \sum_{m=1}^{\infty} \frac{t^{2m-1}(3n)^{3m}}{2^m\, m!\,\Gamma(2m)}, & t > 0 \\ 0, & t < 0 \end{cases}$$

and we get the operator (3).

Remark 1.2. *For $x \in (0,\infty)$ and a real parameter λ, we have*

$$(M_n e^{\lambda t})(x) = exp\left(\frac{3\lambda n(6n - \lambda x^{1/3})x}{2(3n - \lambda x^{1/3})^2}\right). \tag{4}$$

Also, if $(M_n e_r)(x); e_r(t) = t^r$, r=0,1,2,... ,then coefficient of $\frac{\lambda^r}{r!}$ in the expansion of (4) will provide us the r-th order moment.

Obviously for each $r \geq 1$, $s \in \mathbf{R}$ and $x > 0$,

$$\lim_{n\to\infty} (M_{rn} e^{isnt})\left(\frac{x}{n}\right) = e^{isx} = Id(e^{ist}; x)$$

and

$$\lim_{n\to\infty} (M_r e^{ist/n})(nx) = e^{isx} = Id(e^{ist}; x).$$

Following [1, Th. 1.1] and references therein, for $g \in \hat{C}_B(0,\infty)$ (the space of all bounded functions on positive real axis which are continuous), it can be seen that

$$\lim_{n\to\infty} (M_{rn} g(nt))\left(\frac{x}{n}\right) = g(x),$$

and

$$\lim_{n\to\infty} \left(M_r g\left(\frac{t}{n}\right)\right)(nx) = g(x).$$

2. Estimation of Moments

Lemma 2.1. *Using the representation (4), we conclude that for certain constants $a_i, i = 0, 1, 2, ..., a_i \neq 0$*

$$\begin{aligned}\left(M_n \sum_{i\geq 0} a_i e_i\right)(x) &= a_0 + a_1 x + a_2\left(x^2 + \frac{x^{4/3}}{n}\right) + a_3\left(x^3 + \frac{3x^{7/3}}{n} + \frac{4x^{5/3}}{3n^2}\right)\\ &+a_4\left(x^4 + \frac{6x^{10/3}}{n} + \frac{25x^{8/3}}{3n^2} + \frac{20x^2}{9n^3}\right) + a_5\left(x^5 + \frac{10x^{13/3}}{n} + \frac{85x^{11/3}}{3n^2} + \frac{220x^3}{9n^3} + \frac{40x^{7/3}}{9n^4}\right)\\ &+a_6\left(x^6 + \frac{15x^{16/3}}{n} + \frac{215x^{14/3}}{3n^2} + \frac{385x^4}{3n^3} + \frac{700x^{10/3}}{9n^4} + \frac{280x^{8/3}}{27n^5}\right) + ...\end{aligned}$$

Proof follows using simple calculations on (4).

Lemma 2.2. *If $\mu_{n,r}(x) = (M_n(e_1 - e_0 x)^r)(x)$, $r \in \mathbf{N}_0$(the set of all whole numbers), then we have*

$$\mu_{n,r}(x) = \left[\frac{\partial^r}{\partial\lambda^r} exp\left(-\lambda x + \frac{3\lambda n(6n - \lambda x^{1/3})x}{2(3n - \lambda x^{1/3})^2}\right)\right]_{\lambda=0}.$$

By basic computations, for some non-zero constants b_j, $j = 0, 1, 2, ...$, the central moments satisfy

$$\begin{aligned}\sum_{j\geq 0} b_j \mu_{n,j}(x) &= b_0 + b_2\frac{x^{4/3}}{n} + b_3\frac{4x^{5/3}}{3n^2} + b_4\left(\frac{3x^{8/3}}{n^2} + \frac{20x^2}{9n^3}\right)\\ &+b_5\left(\frac{40x^3}{3n^3} + \frac{40x^{7/3}}{9n^4}\right) + b_6\left(\frac{15x^4}{n^3} + \frac{460x^{10/3}}{9n^4} + \frac{280x^{8/3}}{27n^5}\right) + ...\end{aligned}$$

Lemma 2.3. *Using representation (4), we have the following:*

$$\begin{aligned}\left(M_n e_1 e^{\lambda t}\right)(x) &= \left[\frac{27n^3 x}{(3n - \lambda x^{1/3})^3}\right] \cdot exp\left(\frac{3\lambda n(6n - \lambda x^{1/3})x}{2(3n - \lambda x^{1/3})^2}\right)\\ \left(M_n e_2 e^{\lambda t}\right)(x) &= \left[\frac{81n^3(9n^2 - 6\lambda n x^{1/3} + \lambda^2 x^{2/3} + 9n^3 x^{2/3})x^{4/3}}{(3n - \lambda x^{1/3})^6}\right] \cdot exp\left(\frac{3\lambda n(6n - \lambda x^{1/3})x}{2(3n - \lambda x^{1/3})^2}\right).\end{aligned}$$

The proof follows by differentiating (4) successively with respect to λ.

Lemma 2.4. *For $n \in \mathbf{N}$, $\lambda > 0$, $x \in (0, \infty)$ and $3n \geq 2\lambda x^{1/3}$, we have*

$$\left(M_n(e_1 - e_0 x)^2 e^{\lambda t}\right)(x) \leq \Psi(\lambda, x)\mu_{n,2}(x),$$

where

$$\Psi(\lambda, x) = \left(\frac{2}{3}\right)^6 \cdot \left(729 + 81\lambda^2(x^{2/3} + 9x^{4/3}) + 135\lambda^4 x^2 + \lambda^6 x^{8/3}\right) \cdot e^{4\lambda x}.$$

Proof. Using Lemma 2.3 and the linearity feature of the operator M_n, we obtain

$$\begin{aligned}\left(M_n(e_1 - xe_0)^2 e^{\lambda t}\right)(x) &= \mu_{n,2}(x) \cdot \frac{1}{(3n - \lambda x^{1/3})^6} \cdot exp\left(\frac{3\lambda n(6n - \lambda x^{1/3})x}{2(3n - \lambda x^{1/3})^2}\right)\\ &\cdot\Big[729n^6 - 486\lambda n^5 x^{1/3} + 81\lambda^2 n^4 x^{2/3} + 729\lambda^2 n^5 x^{4/3}\\ &-486\lambda^3 n^4 x^{5/3} + 135\lambda^4 n^3 x^2 - 18\lambda^5 n^2 x^{7/3} + \lambda^6 n x^{8/3}\Big].\end{aligned}$$

For $3n \geq 2\lambda x^{1/3}$, we have the following:

$$\frac{1}{(3n-\lambda x^{1/3})^6} \leq \left(\frac{2}{3n}\right)^6,$$

and

$$\frac{3\lambda n x(6n-\lambda x^{1/3})}{2(3n-\lambda x^{1/3})^2} \leq 4\lambda x - \frac{2\lambda^2 x^{4/3}}{3n}.$$

Thus we get

$$\left(M_n(e_1 - xe_0)^2 e^{\lambda t}\right)(x) \leq \Psi(\lambda, x)\mu_{n,2}(x),$$

as required. □

3. Direct estimate

Consider the K-functional:

$$K_2(g,\varrho) = \inf\{\|g-f\|_\infty + \varrho\|f''\|_\infty : f,\ f',\ f'' \in \hat{C}_B(0,\infty)\},$$

$\|g_1\|_\infty = \sup_{x\in(0,\infty)} |g_1(x)|$, $\varrho > 0$.

Theorem 3.1. *If $g \in \hat{C}_B(0,\infty)$, then we have*

$$\left|(M_n g)(x) - g(x)\right| \leq C\omega_2\left(g, x^{2/3}n^{-1/2}\right),$$

where C is an absolute constant and $\omega_2(g,.)$ is the moduli of continuity of order two.

Proof. Let $h \in \hat{C}_B(0,\infty)$ be such that $h', h'' \in \hat{C}_B(0,\infty)$ and $x, t > 0$, then by Taylor's formula, we have

$$h(t) - h(x) = h'(x)(t-x) + \int_x^t (t-v)h''(v)dv.$$

In view of Lemma 2.1 and Lemma 2.2, we have

$$|(M_n h)(x) - h(x)| = \left|\left(M_n \int_x^t (t-v)h''(v)dv\right)(x)\right| \leq \frac{x^{4/3}}{2n}\|h''\|_\infty. \tag{5}$$

Also, we have

$$\left|(M_n g)(x)\right| \leq \|g\|_\infty. \tag{6}$$

Therefore, using (5) and (6), we get

$$\begin{aligned}\left|(M_n g)(x) - g(x)\right| &\leq \left|(M_n(g-h))(x) - (g-h)(x)\right| + |(M_n h)(x) - h(x)| \\ &\leq 2\left(\|g-h\|_\infty + \frac{x^{4/3}}{4n}\|h''\|_\infty\right).\end{aligned}$$

Considering the infimum on the right side over all $h \in \hat{C}_B^2(0,\infty)$ and using $K_2(g,\varrho) \leq C\omega_2(g, \sqrt{\varrho})$, $\varrho > 0$ due to [4], we get the required result. □

4. Quantitative variants of Voronovskaja theorem

Gadjiev-Aral in [6] studied the relevance of defining weighted modulus of continuity and explained the difficulties in estimating the rate of approximation of high-growth-rate functions at infinity using modulus ω. Thus, the weighted modulus of continuity, whose definition incorporates function's growth at infinity, can be utilized for this purpose. Motivated by the idea, Păltănea [11] proposed the following modulus of continuity with weights:

$$\grave{\omega}_{\Theta}(g,\varrho) = \sup_{u,v\geq 0}\left\{|g(u)-g(v)| : |u-v|\leq \varrho\Theta\left(\frac{u+v}{2}\right)\right\},\ \varrho\geq 0$$

where $\Theta(u)=\frac{u^{1/2}}{u^m+1}$ is the weight function, $u\geq 0$ and $m=2,3,4,...$

Now, consider the same weighted modulus $\grave{\omega}_{\Theta}(g,.)$ on the interval $(0,\infty)$ and denote $G_{\Theta}(0,\infty)$ as the subspace containing all the real valued functions on the interval $(0,\infty)$ such that $\lim_{\varrho\to 0}\grave{\omega}_{\Theta}(g,\varrho)=0$. Further let W is the subspace of $C(0,\infty)$ such that $C_r(0,\infty)\subset W$, where

$$C_r(0,\infty)=\{g\in C(0,\infty) : |g(u)|\leq C(1+u^r),\ \forall u>0,\ C>0\},\quad r\in\mathbf{N}.$$

Also, denote $C^2(0,\infty)$ as the space of two times continuously differentiable functions on the interval $(0,\infty)$. Following [11, Th. 2], the subspace $G_{\Theta}(0,\infty)$ contains those functions g for which $g\circ e_2$ is uniformly continuous on $(0,1]$ and $g\circ e_s$, $s=\frac{2}{1+2m}$, is uniformly continuous on $[1,\infty)$. Based on the weighted modulus $\grave{\omega}_{\Theta}(g,\varrho)$, Gupta-Tachev in [7] produced Voronovskaja kind asymptotic formula. The quantitative estimate for our exponential-type operator M_n assumes the following form:

Theorem 4.1. *For $g\in W\cap C^2(0,\infty)$, $g''\in G_{\Theta}(0,\infty)$, $r=max\{m+3,2m,6\}$ with $x\in(0,\infty)$, we have*

$$\begin{aligned}&\left|(M_ng)(x)-g(x)-\frac{x^{4/3}}{2n}g''(x)\right|\\ \leq\ &\left[\frac{x^{4/3}}{2n}+\sqrt{\frac{1}{2}\left(M_n\left(1+\left(x+\frac{|t-x|}{2}\right)^m\right)^2\right)(x)}\right]\\ &\grave{\omega}_{\Theta}\left(g'',\sqrt{\frac{15x^3}{n^3}+\frac{460x^{7/3}}{9n^4}+\frac{280x^{5/3}}{27n^5}}\right).\end{aligned}$$

Theorem 4.2. *For $g\in W\cap C^2(0,\infty)$, $g''\in G_{\Theta}(0,\infty)$, $r=max\{m+3,4\}$ and $x\in(0,\infty)$, we have*

$$\begin{aligned}&\left|n\left[(M_ng)(x)-g(x)-\frac{x^{4/3}}{2n}g''(x)\right]\right|\\ \leq\ &\frac{x^{4/3}}{2}\left[1+\sqrt{\frac{2}{x}}b_{n,m}(x)\right]\grave{\omega}_{\Theta}\left(g'',\sqrt{\frac{3x^{4/3}}{n}+\frac{20x^{2/3}}{9n^2}}\right),\end{aligned}$$

where

$$b_{n,m}(x)=1+\frac{1}{(M_n|t-x|^3)(x)}\sum_{i=0}^{m}\binom{m}{i}x^{m-i}\frac{(M_n|t-x|^{i+3})(x)}{2^i}.$$

Using Lemma 2.2 and the techniques incorporated in [8, pp.91], the proofs of the above two theorems follow.

5. Voronovskaja's quantification for functions featuring exponential growth

For $g \in U^*(0,\infty)$ and $\lambda > 0$, consider

$$\omega_1(g,\varrho,\lambda) = \sup_{u>0, h\leq\varrho} |g(u) - g(u+h)|e^{-\lambda u}, \qquad \varrho > 0,$$

as the modulus of continuity of first order defined by Ditzian [5] and $U^*(0,\infty) := \{g \in C(0,\infty) : \|g\|_\lambda = \sup_{u>0} |g(u)e^{-\lambda u}| < \infty\}$.

Further, for $\alpha \in (0,1]$, consider the following Lipschitz space:

$$Lip_\alpha(\lambda) = \{g \in U^*(0,\infty) : \omega_1(g,\varrho,\lambda) \leq C\varrho^\alpha, \ \forall \varrho < 1\}.$$

Theorem 5.1. *Let $g \in U^*(0,\infty) \cap C^2(0,\infty)$, $g'' \in Lip_\alpha(\lambda)$, $\alpha \in (0,1]$, then for $3n \geq 2\lambda x^{1/3}$ with $x \in (0,\infty)$, we have*

$$\begin{aligned} &\left| (M_n g)(x) - g(x) - \frac{x^{4/3}}{2n} g''(x) \right| \\ \leq\ & \omega_1\left(g'', \sqrt{\frac{3x^{4/3}}{n} + \frac{20x^{2/3}}{9n^2}}, \lambda\right) \cdot \left[e^{2\lambda x} + \frac{\Psi(\lambda,x)}{2} + \frac{\sqrt{\Psi(2\lambda,x)}}{2}\right] \cdot \frac{x^{4/3}}{n}. \end{aligned}$$

Proof. According to Taylor's expansion, δ exists between x and t such that

$$g(t) = \sum_{k=0}^{2} \frac{(t-x)^k}{k!} g^{(k)}(x) + \Xi(t,x)(e_1 - xe_0)^2, \tag{7}$$

where

$$\Xi(t,x) := \frac{g''(\delta) - g''(x)}{2}.$$

On applying the operator M_n to (7), we get

$$\left| (M_n g)(x) - g(x) - \frac{x^{4/3}}{2n} g''(x) \right| \leq \left(M_n |\Xi(t,x)|(e_1 - xe_0)^2\right)(x). \tag{8}$$

Following [8, pp. 101], we get

$$\begin{aligned} \left(M_n |\Xi(t,x)|(e_1 - xe_0)^2\right)(x) \leq\ & \frac{\omega_1(g'',h,\lambda)}{2} \cdot \left[\left(M_n e^{\lambda t}|e_1 - xe_0|^2\right)(x) \right. \\ & \left. + \frac{1}{h}\left(M_n e^{\lambda t}|e_1 - xe_0|^3\right)(x) + e^{2\lambda x}\left(\frac{x^{4/3}}{n} + \frac{1}{h}\left(M_n |e_1 - xe_0|^3\right)(x)\right)\right]. \end{aligned} \tag{9}$$

Using Cauchy-Schwarz inequality and Lemma 2.4 for $3n \geq 2\lambda x^{1/3}$, we have the following:

$$\left(M_n |e_1 - xe_0|^3 e^{\lambda t}\right)(x) \leq \sqrt{\frac{3x^4}{n^3} + \frac{20x^{10/3}}{9n^4}} \sqrt{\Psi(2\lambda,x)}, \tag{10}$$

and

$$\left(M_n |e_1 - xe_0|^3\right)(x) \leq \sqrt{\frac{3x^4}{n^3} + \frac{20x^{10/3}}{9n^4}}. \tag{11}$$

Substituting $h = \sqrt{\frac{3x^{4/3}}{n} + \frac{20x^{2/3}}{9n^2}}$ in (9) and combining (8), (10), (11) and Lemma 2.4, we get the desired result. □

6. Convergence estimates

Since the publication of Korovkin's theorem on the convergence of positive linear operators in 1953, there have been numerous contributions that refine the original classical Korovkin's theory. A Korovkin-type theorem for exponential functions has been produced in [3] and generalized by Altomare-Campiti in [2]. Holhoş [9] added to the work of [2] and obtained an estimation for the rate of convergence of the operators satisfying the general Korovkin-type result for exponential functions.

Denote the space of all continuous real-valued functions $g(u)$ having finite limit for sufficiently large element u by $\hat{C}^*(0,\infty)$. Further, let us consider the following moduli:

$$\breve{\omega}(g,\varrho) = \sup_{u,v\in(0,\infty)} \{|g(u)-g(v)| : |e^{-u}-e^{-v}| \leq \varrho\}, \qquad \varrho \geq 0.$$

Theorem 6.1. *([9]) For the functions g, belonging to the class $\hat{C}^*(0,\infty)$ and for an operator Λ_n, whose domain and range belongs to the class $\hat{C}^*(0,\infty)$. If it satisfies the conditions*

$$\|(\Lambda_n e^{-sv})(x) - e^{-su}\|_{(0,\infty)} = \lambda_{s,n};\ \ s=0,1,2$$

and $\lambda_{s,n} \to 0$ as $n\to\infty$, then we obtain

$$\|\Lambda_n g - g\|_{(0,\infty)} \leq \lambda_{0,n}\cdot\|g\|_{(0,\infty)} + (2+\lambda_{0,n})\cdot\breve{\omega}(g, \sqrt{\lambda_{0,n}+2\lambda_{1,n}+\lambda_{2,n}}).$$

Now, we present the application of Theorem 6.1 for the operator (3).

Theorem 6.2. *For $g \in \hat{C}^*(0,\infty)$, we have*

$$\|M_n g - g\|_{(0,\infty)} \leq 2\breve{\omega}(g, \sqrt{2\lambda_{1,n}+\lambda_{2,n}}),$$

where $\lambda_{1,n}, \lambda_{2,n} \to 0$ as $n\to\infty$.

Proof. As the constant function is preserved by the operator M_n therefore, we have $\lambda_{0,n} = 0$. Next

$$(M_n e^{-t})(x) = exp\left(\frac{-3nx(6n+x^{1/3})}{2(3n+x^{1/3})^2}\right).$$

Consider

$$g_n(x) = exp\left(\frac{-3nx(6n+x^{1/3})}{2(3n+x^{1/3})^2}\right) - e^{-x}.$$

Since $g_n(0) = \lim_{x\to+\infty} g_n(x) = 0$, a point $\varsigma_n > 0$ exists, such that

$$\|g_n\|_{(0,\infty)} = g_n(\varsigma_n) := \lambda_{1,n}.$$

Also $g_n'(\varsigma_n) = 0$ implies

$$exp\left(\frac{-3n\varsigma_n(6n+\varsigma_n^{1/3})}{2(3n+\varsigma_n^{1/3})^2}\right)\left[\frac{-3n(6n+\varsigma_n^{1/3})}{2(3n+\varsigma_n^{1/3})^2} - \frac{n\varsigma_n^{1/3}}{2(3n+\varsigma_n^{1/3})^2} + \frac{n\varsigma_n^{1/3}(6n+\varsigma_n^{1/3})}{(3n+\varsigma_n^{1/3})^3}\right]$$
$$= -e^{-\varsigma_n}.$$

Thus, we have

$$\lambda_{1,n} = exp\left(\frac{-3n\varsigma_n(6n+\varsigma_n^{1/3})}{2(3n+\varsigma_n^{1/3})^2}\right)\cdot\frac{18n^2\varsigma_n^{1/3}+8n\varsigma_n^{2/3}+\varsigma_n}{(3n+\varsigma_n^{1/3})^3} \to 0$$

as $n \to \infty$.
Finally

$$(M_n e^{-2t})(x) = exp\left(\frac{-3nx\,(6n + 2x^{1/3})}{(3n + 2x^{1/3})^2}\right).$$

Let

$$h_n(x) = exp\left(\frac{-3nx\,(6n + 2x^{1/3})}{(3n + 2x^{1/3})^2}\right) - e^{-2x}.$$

Since $h_n(0) = \lim_{x\to+\infty} h_n(x) = 0$, a point $\tau_n > 0$ exists, such that

$$\|h_n\|_{(0,\infty)} = h_n(\tau_n) := \lambda_{2,n}.$$

Also $h_n'(\tau_n) = 0$ implies

$$\begin{aligned} & exp\left(\frac{-3n\tau_n\,(6n + 2\tau_n^{1/3})}{(3n + 2\tau_n^{1/3})^2}\right)\left[\frac{-3n\,(6n + 2\tau_n^{1/3})}{(3n + 2\tau_n^{1/3})^2} - \frac{2n\tau_n^{1/3}}{(3n + 2\tau_n^{1/3})^2} + \frac{4n\tau_n^{1/3}\,(6n + 2\tau_n^{1/3})}{(3n + 2\tau_n^{1/3})^3}\right] \\ = \;& -2e^{-2\tau_n}. \end{aligned}$$

Thus, we have

$$\lambda_{2,n} = exp\left(\frac{-6n\tau_n\,(3n + \tau_n^{1/3})}{(3n + 2\tau_n^{1/3})^2}\right) \cdot \frac{4\left(9n^2\tau_n^{1/3} + 8n\tau_n^{2/3} + 2\tau_n\right)}{(3n + 2\tau_n^{1/3})^3} \to 0$$

as $n \to \infty$.

The proof is completed. □

Theorem 6.3. *If g, g'' belongs to $\hat{C}^*(0,\infty)$, then for $x \in (0,\infty)$, the following inequality exists:*

$$\begin{aligned} & \left| n\left[(M_n g)(x) - g(x)\right] - \frac{x^{4/3}}{2} g''(x) \right| \\ \leq \;& \frac{1}{2}\breve{\omega}\left(g'', \frac{1}{\sqrt{n}}\right)\left[x^{4/3} + \left(\frac{20x^2}{9n} + 3x^{8/3}\right)^{1/2}\left[n^2\left(M_n\left(\frac{1}{e^x} - \frac{1}{e^t}\right)^4\right)(x)\right]^{1/2}\right]. \end{aligned}$$

Proof. Application of the operator M_n to (7) and using Lemma 2.2, we have the following:

$$\left| n\left[(M_n g)(x) - g(x)\right] - \frac{x^{4/3}}{2} g''(x) \right| \leq n\left(M_n\,|\Xi(t,x)|\,(e_1 - e_0 x)^2\right)(x).$$

Using the property of $\breve{\omega}(g, \varrho)$ given by

$$|g(t) - g(x)| \leq \left(1 + \frac{1}{\varrho^2}\left(\frac{1}{e^x} - \frac{1}{e^t}\right)^2\right)\breve{\omega}(g, \varrho),$$

we obtain

$$|\Xi(t,x)| \leq \frac{1}{2}\left(1 + \frac{1}{\varrho^2}\left(\frac{1}{e^x} - \frac{1}{e^t}\right)^2\right)\breve{\omega}\left(g'', \varrho\right).$$

Hence, applying Cauchy-Schwarz inequality and choosing $\varrho = \frac{1}{\sqrt{n}}$, we get

$$\begin{aligned} & n\left(M_n\,|\Xi(t,x)|\,(e_1 - xe_0)^2\right)(x) \\ \leq \;& \frac{1}{2}\breve{\omega}\left(g'', n^{-1/2}\right)\left[n\mu_{n,2}(x) + \sqrt{n^2\left(M_n\,(e^{-x} - e^{-t})^4\right)(x)}\sqrt{n^2\mu_{n,4}(x)}\right]. \end{aligned}$$

Finally, we obtain the required outcome by using Lemma 2.2. □

Remark 6.4. *The convergence of the operator M_n in the preceding theorem takes place for n large enough. Further, for $A = -1, -2, -3, -4$ in (4), we get*

$$
\begin{aligned}
& \lim_{n\to\infty} n^2 \left(M_n \left(e^{-x} - e^{-t}\right)^4\right)(x) \\
= \; & \lim_{n\to\infty} n^2 \Big[e^{-4x} - 4e^{-3x}(M_n e^{-t})(x) + 6e^{-2x}(M_n e^{-2t})(x) \\
& -4e^{-x}(M_n e^{-3t})(x) + (M_n e^{-4t})(x) \Big] \\
= \; & \lim_{n\to\infty} n^2 \Big[e^{-4x} - 4e^{-3x} exp\left(\frac{-3nx(6n + x^{1/3})}{2(3n + x^{1/3})^2}\right) + 6e^{-2x} exp\left(\frac{-3nx(6n + 2x^{1/3})}{(3n + 2x^{1/3})^2}\right) \\
& -4e^{-x} exp\left(\frac{-3nx(2n + x^{1/3})}{2(n + x^{1/3})^2}\right) + exp\left(\frac{-6nx(6n + 4x^{1/3})}{(3n + 4x^{1/3})^2}\right) \Big] \\
= \; & 3e^{-4x} x^{8/3}.
\end{aligned}
$$

7. Graphical representation

In the following graphs, we analyze the convergence of the operator M_n for the exponential function $g(x) = e^{-4x}$.

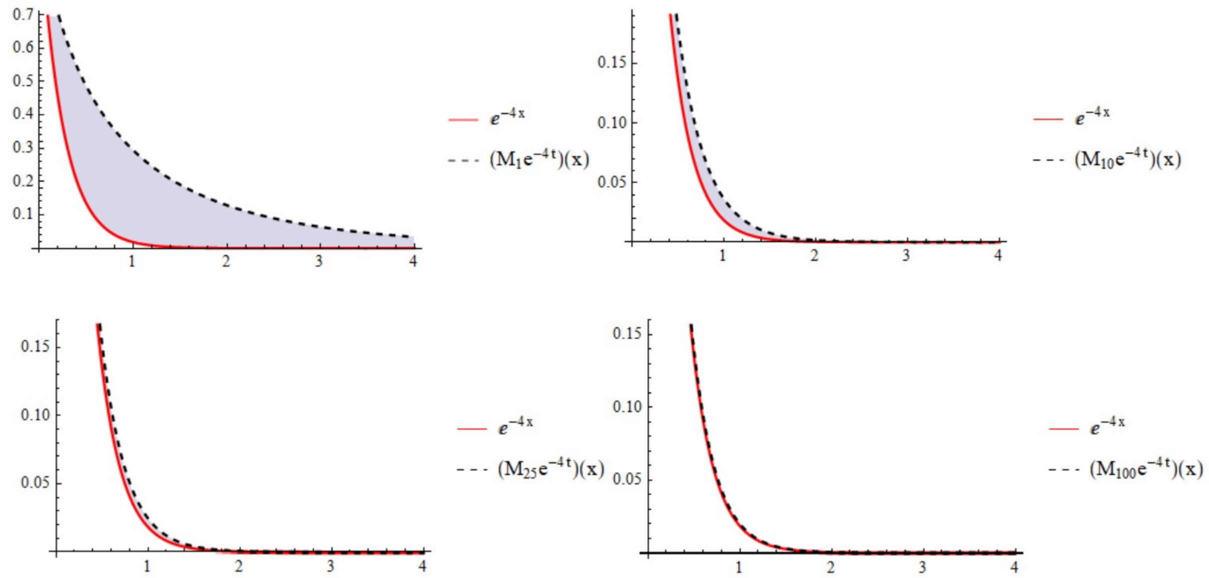

Figure 1: Convergence of the operator $(M_n g)(x)$ for $g(x) = e^{-4x}$

We can see that the shaded area in the graphs of Fig. 1 indicates the gap between given function $g(x) = e^{-4x}$ (represented by red) and approximation of $g(x)$ through the operator M_n (represented by black dotted curve) for various values of n. Finally, we can conclude that as n increases, the gap between them shrinks and the operator converges more rapidly to the function.

Data Availability Statement

The authors declare that the manuscript has no associated data.

Conflict of interest

The authors declare that they have no conflict of interest.

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