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The w-core inverse of a product and its applications

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Abstract. Let R be a unital $*$-ring. This paper establishes necessary and sufficient conditions for the existence of the w-core inverse of a regular element by units in R. Then, the existence criterion of the w-core inverse of the product of three elements are derived. As applications, the existence criterion and expression of the w-core inverse of 2×2 matrices over a ring are given.

1. Introduction

Due to the importance of the existence theorems in linear algebra involving factorizations, lots of scholars considered necessary and sufficient conditions for the existence of generalized inverses of the product of matrices or elements in rings. For instance, given $A \in R_{m\times n}$, the set of all $m \times n$ matrices over a ring R, Gouveia and Puystjens [14] characterized the existence of the group inverse and the Moore-Penrose inverse of PAQ in terms of the corresponding generalized inverse of A provided that there exist $P', Q' \in R_{m\times n}$ such that $P'PA = A = AQQ'$, where $P, Q \in R_{n\times m}$. Precisely, they proved that if $A \in R_{n\times n}$ has group inverse $A^\#$, then PAQ has a group inverse if and only if $AA^\#QPA + 1 - AA^\#$ is invertible. An analogous result for Moore-Penrose inverses was also given in [14]. At the beginning of the first decade of 21st century, Patrício [22] determined the Moore-Penrose inverse of PAQ by invertible matrices, where P, Q are invertible and A is regular. Shortly thereafter, the Moore-Penrose inverse of such a factorization PAQ was further investigated by Patrício [21] in a general case of $P'PA = A = AQQ'$. Recently, Gao and Chen [13] established existence criteria and formulae for the pseudo core inverse (a.k.a. core-EP inverse) of a product PAQ. For the case of a product of three elements in a ring R, given any $p', p, a, q, q' \in R$ such that $p'pa = a = aqq'$, the author in [35] investigated $\{1, 3\}$-inverses and $\{1, 4\}$-inverses of a factorization paq by units in a ring. Later, a more general case for the existence of (Mary's) inverse along a product paq was considered in [32]. Then, the core invertibility of paq in a ring was considered by Ke et al. [17].

Recently, the author et al. [33] in a $*$-semigroup introduced the w-core inverse, extending the core inverse [1], the Moore-Penrose inverse [23] and the core-EP inverse [24].

Inspired by the aforementioned work, it is of interest to consider the existence criterion of w-core inverses of the product of three elements in a ring.

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The paper is organized as follows. In Section 2, we give some characterizations of the w-core inverse of a regular element by units. In particular, we in Proposition 2.6 prove that a is w-core invertible if and only if $(aw)^m + 1 - aa^- \in R^{-1}$ and $a\{1,3\} \neq \emptyset$ for any positive integer m. In Section 3, it is shown that paq is w-core invertible if and only if a is qwp-core invertible if and only if pa is qw-core invertible, under certain conditions. Also, we in Theorems 3.11 and 3.13 characterize both w-core invertibility and dual v-core invertibility of a product by ideals and units, respectively. As an application, the existence criterion and representation of the w-core inverse of 2×2 matrices over a ring R are given in Section 4.

For the convenience of reader, let us now recall several basic notions of generalized inverses in a ring.

Let R be an associate ring. An element $a \in R$ is (von Neumann) regular if there exists an $x \in R$ such that $a = axa$. Such an x is called an inner inverse of a, and is denoted by a^-. By $a\{1\}$ we denote the set of all inner inverses of a. An element $a \in R$ is called group invertible (see, e.g., [2]) if there exists some $x \in R$ such that $axa = a, xax = x, ax = xa$. Such an x is called a group inverse of a. It is unique if it exists, and is denoted by $a^\#$. We denote by $R^\#$ the set of all group invertible elements in R.

Let $a, d \in R$. An element a is called left invertible along d [29] if there exists some $b \in R$ such that $bad = d$ and $b \in Rd$. Such an element b is called a left inverse of a along d, and is denoted by $a_l^{\|d}$. It is known that a is left invertible along d if and only if $d \in Rdad$. Dually, an element a is called right invertible along d [29] if there exists some $c \in R$ such that $dac = d$ and $c \in dR$. Such an element c is called a right inverse of a along d, and is denoted by $a_r^{\|d}$. Also, it is known that a is right invertible along d if and only if $d \in dadR$. An element a is invertible along d [19] if it is both left and right invertible along d, or equivalently if there exists an $x \in R$ such that $dax = d = xad$ and $x \in dR \cap Rd$. Such an x is called an inverse of a along d. It is unique if it exists, and is denoted by $a^{\|d}$. Symbols $R_l^{\|d}$, $R_r^{\|d}$ and $R^{\|d}$ will stand for the sets of all left invertible, right invertible and invertible elements along d in R, respectively.

Mary [19, Theorem 10] illustrated that $a \in R^\#$ if and only if $a^{\|a}$ exists. In this case, $a^\# = a^{\|a}$. One also knows from [20, Corollary 3.4] that $a \in R^\#$ if and only if $1^{\|a}$ exists. Moreover, $1^{\|a} = aa^\#$. More results on the inverse along an element can be seen in [3, 4, 10, 11, 19, 20, 29, 32].

Throughout this paper, R denotes a unital $*$-ring, which is a ring with unity 1 and an involution $*$ satisfying $(a^*)^* = a, (ab)^* = b^*a^*$, and $(a + b)^* = a^* + b^*$ for any $a, b \in R$.

Follow [23], an element $a \in R$ is Moore-Penrose invertible if there exists an $x \in R$ satisfying the following four equations:

$$(1)\ axa = a,\ (2)\ xax = x,\ (3)\ (ax)^* = ax,\ (4)\ (xa)^* = xa.$$

Such an x is called a Moore-Penrose inverse of a. It is unique if it exists, and is denoted by $a^\dagger$. If $a, x \in R$ satisfy (1) $axa = a$ and (3) $(ax)^* = ax$, then x is called a $\{1, 3\}$-inverse of a, and is denoted by $a^{(1,3)}$. If a and x satisfy (1) $axa = a$ and (4) $(xa)^* = xa$, then x is called a $\{1, 4\}$-inverse of a, and is denoted by $a^{(1,4)}$. We use $a\{1, 3\}$ and $a\{1, 4\}$ to represent the sets of all $\{1, 3\}$-inverses and $\{1, 4\}$-inverses of a, respectively. It is worth pointing out that the products $aa^{(1,3)}$ and $a^{(1,4)}a$ are invariant, although a could have different $\{1, 3\}$-inverses and $\{1, 4\}$-inverses. We denote by $R^\dagger$, $R^{(1,3)}$ and $R^{(1,4)}$ the sets of all Moore-Penrose invertible, $\{1, 3\}$-invertible and $\{1, 4\}$-invertible elements in R, respectively. It is well known that $a \in R^\dagger$ if and only if $a \in R^{(1,3)} \cap R^{(1,4)}$, in which case, $a^\dagger = a^{(1,4)}aa^{(1,3)}$.

Given any $a, w, v \in R$, an element a is called w-core invertible [33] if there exists some $x \in R$ such that $xawa = a$, $awx^2 = x$, $awx = (awx)^*$. Such an x is called a w-core inverse of a. It is unique if it exists, and is denoted by $a_w^{\text{#}}$. It is proved in [33] that a is w-core invertible if and only if w is invertible along a and a is {1,3}-invertible. Moreover, $a_w^{\text{#}} = w^{\|a}a^{(1,3)}$. The dual v-core inverse of a, if exists, is defined as the unique $a_{v,\text{#}}$ such that $avaa_{v,\text{#}} = a$, $a_{v,\text{#}}^2va = a_{v,\text{#}}$, $a_{v,\text{#}}va = (a_{v,\text{#}}va)^*$. Analogously, it is shown in [33] that a is dual v-core invertible if and only if v is invertible along a and a is {1,4}-invertible. Moreover, $a_{v,\text{#}} = a^{(1,4)}v^{\|a}$. The set of all w-core (resp., dual v-core) invertible elements in R is denoted by $R_w^{\text{#}}$ (resp., $R_{v,\text{#}}$). An element a is called core invertible if it is 1-core invertible or a-core invertible. The core inverse of a is unique if it exists, and is denoted by $a^{\text{#}}$. The standard notion of the core inverse of an element can be referred to [28]. More results of w-core inverses can be referred to [33, 34].

2. Characterizations of the w-core inverse

In this section, existence criteria for the w-core inverse of a regular element are derived by units in a ring.

Recall that an element $a \in R$ is left invertible if there exists some $x \in R$ such that $xa = 1$, and a is right invertible if $ay = 1$ for some $y \in R$. An element is invertible if it is both left and right invertible. As usual, by R_l^{-1}, R_r^{-1} and R^{-1} we denote the sets of all left invertible, right invertible and invertible elements in R, respectively.

Lemma 2.1. [18, Exercise 1.6] *Let $a, b \in R$. Then*
(i) $1 + ab \in R_l^{-1}$ *if and only if* $1 + ba \in R_l^{-1}$. *Moreover, if* $x(1 + ab) = 1$ *for some* $x \in R$, *then* $(1 - bxa)(1 + ba) = 1$.
(ii) $1 + ab \in R_r^{-1}$ *if and only if* $1 + ba \in R_r^{-1}$. *Moreover, if* $(1 + ab)y = 1$ *for some* $y \in R$, *then* $(1 + ba)(1 - bya) = 1$.

It follows from Lemma 2.1 that $1+ab \in R^{-1}$ if and only if $1+ba \in R^{-1}$. Moreover, $(1+ba)^{-1} = 1-b(1+ab)^{-1}a$, which is known as the Jacobson's Lemma.

Lemma 2.2. [35, Lemma 2.2] *Let $a \in R$. Then*
(i) $a^{(1,3)}$ *exists if and only if* $a \in Ra^*a$. *If* $xa^*a = a$ *for some* $x \in R$, *then* x^* *is a* $\{1, 3\}$*-inverse of* a.
(ii) $a^{(1,4)}$ *exists if and only if* $a \in aa^*R$. *If* $aa^*y = a$ *for some* $y \in R$, *then* y^* *is a* $\{1, 4\}$*-inverse of* a.

Lemma 2.3. [20, Theorem 2.2] *Let $a, w \in R$. Then $w \in R^{\|a}$ if and only if $a \in awaR \cap Rawa$. In this case, $w^{\|a} = ax = ya$, where $x, y \in R$ satisfy $a = awax = yawa$.*

Lemma 2.4. [10, Corollary 2.9] *Let $a, w \in R$ with a regular and let $m \geq 1$ be an integer. Then the following conditions are equivalent*:
(i) w *is invertible along* a.
(ii) $u = (aw)^m + 1 - aa^- \in R^{-1}$.
(iii) $u' = (wa)^m + 1 - a^-a \in R^{-1}$.
In this case, $w^{\|a} = u^{-1}(aw)^{m-1}a = a(wa)^{m-1}(u')^{-1}$.

Lemma 2.5. [33] *Let $a, w \in R$. Then*
(i) $a \in R_w^{\#}$ *if and only if* $w \in R^{\|a}$ *and* $a\{1, 3\} \neq \emptyset$. *In this case,* $a_w^{\#} = w^{\|a}a^{(1,3)}$.
(ii) $a \in R_{v,\#}$ *if and only if* $v \in R^{\|a}$ *and* $a\{1, 4\} \neq \emptyset$. *In this case,* $a_{v,\#} = a^{(1,4)}v^{\|a}$.

As a consequence of Lemmas 2.4 and 2.5, we have the following result.

Proposition 2.6. *Let $a, w \in R$ with a regular and let $m \geq 1$ be an integer. Then the following conditions are equivalent*:
(i) $a \in R_w^{\#}$.
(ii) $u = (aw)^m + 1 - aa^- \in R^{-1}$ *and* $a\{1, 3\} \neq \emptyset$.
(iii) $u' = (wa)^m + 1 - a^-a \in R^{-1}$ *and* $a\{1, 3\} \neq \emptyset$.
In this case, $a_w^{\#} = u^{-1}(aw)^{m-1}aa^{(1,3)} = a(wa)^{m-1}(u')^{-1}a^{(1,3)}$.

Theorem 2.7. *Let $a, w \in R$ with a regular and let $m \geq 1$ be an integer. Then the following conditions are equivalent*:
(i) $a \in R_w^{\#}$.
(ii) $u = (aw)^m + 1 - aa^- \in R^{-1}$ *and* $v = ((aw)^*)^m + 1 - aa^- \in R^{-1}$ *for some* $a^- \in a\{1\}$.
(iii) $u = (aw)^m + 1 - aa^- \in R^{-1}$ *and* $v = ((aw)^*)^m + 1 - aa^- \in R_l^{-1}$ *for some* $a^- \in a\{1\}$.
(iv) $s = ((aw)^m)^*(aw)^m + 1 - aa^- \in R^{-1}$ *and* $v = ((aw)^*)^m + 1 - aa^- \in R^{-1}$ *for some* $a^- \in a\{1\}$.
In this case, $a_w^{\#} = u^{-1}(aw)^{2m-1}t^*$, *where* $t \in R$ *is a left inverse of* v.

Proof. (i) $\Rightarrow$ (ii) As $a \in R_w^{\#}$, then $a \in R^{(1,3)}$ and $(aw)^m + 1 - aa^- \in R^{-1}$ by Proposition 2.6. Take $a^{(1,3)} \in a\{1\}$, it follows that $(aw)^m + 1 - aa^{(1,3)} \in R^{-1}$ and hence $((aw)^m + 1 - aa^{(1,3)})^* = ((aw)^m)^* + 1 - aa^{(1,3)} \in R^{-1}$.
(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i) Lemma 2.4 ensures that $w \in R^{\|a}$ since $(aw)^m + 1 - aa^- \in R^{-1}$. It next suffices to show that $a \in R^{(1,3)}$ by Lemma 2.5. As $v = ((aw)^*)^m + 1 - aa^- \in R_l^{-1}$, then there exists some $t \in R$ such that $tv = 1$, and consequently $a = tva = t((aw)^*)^m a = t((aw)^*)^{m-1} w^* a^* a \in Ra^*a$, which implies $a \in R^{(1,3)}$ and $w(aw)^{m-1}t^* \in a\{1,3\}$ by Lemma 2.2, as required.

(ii) $\Leftrightarrow$ (iv) According to (iii) $\Rightarrow$ (i), $a^{(1,3)}$ exists. Take $a^{(1,3)} \in a\{1\}$, we have at once $((aw)^m)^*(aw)^m + 1 - aa^{(1,3)} = (((aw)^m)^* + 1 - aa^{(1,3)})((aw)^m + 1 - aa^{(1,3)}) \in R^{-1}$. Since $((aw)^*)^m + 1 - aa^- \in R^{-1}$, it follows that $((aw)^m)^*(aw)^m + 1 - aa^- \in R^{-1}$ if and only if $(aw)^m + 1 - aa^- \in R^{-1}$ for some $a^- \in a\{1\}$.

Finally, we give the formula of $a_w^{\#}$. It follows from the implication (iii) $\Rightarrow$ (i) that $w(aw)^{m-1}t^* \in a\{1,3\}$. Also, by Lemma 2.4, $w^{\|a} = u^{-1}(aw)^{m-1}a$. Thus, $a_w^{\#} = w^{\|a}a^{(1,3)} = u^{-1}(aw)^{2m-1}t^*$. $\square$

One knows from Theorem 2.7 that the existence of $a_w^{\#}$ can be characterized by the invertibilities of $u = (aw)^m + 1 - aa^-$ and $v = ((aw)^*)^m + 1 - aa^-$, where $m \geq 1$ is an integer. It is natural to consider whether it can be described by one-sided invertibilities of $u = (aw)^m + 1 - aa^-$ and $v = ((aw)^*)^m + 1 - aa^-$. The following result shows that the assumption is accurate when the index of m-th power term in $v = ((aw)^*)^m + 1 - aa^-$ is no less than two.

Proposition 2.8. *Let $a, w \in R$ with a regular and let $m \geq 1$, $n \geq 2$ be two integers. Then the following conditions are equivalent*:

(i) $a \in R_w^{\#}$.

(ii) $u = (aw)^m + 1 - aa^- \in R_l^{-1}$ *and* $v = ((aw)^*)^n + 1 - aa^- \in R_l^{-1}$ *for some* $a^- \in a\{1\}$.

In this case, $a_w^{\#} = (aw)^{n-1}t^*$, *where* $t \in R$ *is a left inverse of* v.

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 2.7.

(ii) $\Rightarrow$ (i) Since $(aw)^m + 1 - aa^- \in R_l^{-1}$ and $((aw)^*)^n + 1 - aa^- \in R_l^{-1}$, there exist $s, t \in R$ such that $s((aw)^m + 1 - aa^-) = 1 = t(((aw)^*)^n + 1 - aa^-)$. Thus, by Theorem 2.7 (iii) $\Rightarrow$ (i), we could get $a \in R^{(1,3)}$ and $w(aw)^{n-1}t^* \in a\{1,3\}$. So, $a = aa^{(1,3)}a = aw(aw)^{n-1}t^*a \in awaR$. Also, $a = s((aw)^m + 1 - aa^-)a = s(aw)^m a \in Rawa \cap awaR$, which shows $w \in R^{\|a}$ and $w^{\|a} = s(aw)^{m-1}a$ by Lemma 2.3. As a consequence, $a \in R_w^{\#}$ and $a_w^{\#} = w^{\|a}a^{(1,3)} = s(aw)^{m-1}aw(aw)^{n-1}t^* = (aw)^{n-1}t^*$. $\square$

Remark 2.9. We here remind the reader that, in Proposition 2.8, (ii) does not imply (i) in general, for the case of $n = 1$. Precisely, for any integer $m \geq 1$, if $u = (aw)^m + 1 - aa^- \in R_l^{-1}$ and $v = (aw)^* + 1 - aa^- \in R_l^{-1}$ for some $a^- \in a\{1\}$, then a may not be w-core invertible. For instance, let R be an infinite complex matrix ring whose rows and columns are both finite and let the involution $*$ be the transpose. Suppose
$$A = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ 1 & & & & & 1 \end{bmatrix} \in R.$$
Take
$$A^- = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & 1 & \\ -1 & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix} \in A\{1\}.$$
By a direct check,
$$V = (AW)^* + 1 - AA^- = \begin{bmatrix} 1 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix} \in$$

R_l^{-1} and $V_l^{-1} = \begin{bmatrix} 0 & & & & & 1 \\ 1 & 0 & & & & -1 \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{bmatrix}$. In addition, we have $AWA = A^2 = \begin{bmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ 1 & 0 & 0 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & 0 & 0 & \\ & & & & 1 & 0 & 0 \end{bmatrix}$,

it is easy to get $A \in RAWA$, that is $W \in R_l^{\|A}$. [10, Corollary 2.3] ensures $U = (AW)^m + 1 - AA^- \in R_l^{-1}$. However, $A \notin AWAR$, so that $W \notin R^{\|A}$ and hence $A \notin R_W^{\#}$.

The following result shows that if $v = ((aw)^*)^m + 1 - aa^- \in R^{-1}$ for any $a^- \in a\{1\}$, then a is w-core invertible.

Proposition 2.10. *Let $a, w \in R$ with a regular and let $m \geq 1$ be an integer. If $v = ((aw)^*)^m + 1 - aa^- \in R^{-1}$ for any $a^- \in a\{1\}$, then $a \in R_w^{\#}$.*

Proof. Given $v \in R^{-1}$, then $va = ((aw)^*)^m a$ and $a = v^{-1}((aw)^*)^{m-1} w^* a^* a$. Hence, by Lemma 2.2, $a \in R^{(1,3)}$, whence, $v = ((aw)^*)^m + 1 - aa^{(1,3)} \in R^{-1}$, so that $v^* = (aw)^m + 1 - aa^{(1,3)} \in R^{-1}$. It follows from Lemma 2.4 that $w \in R^{\|a}$. So, $a \in R_w^{\#}$. □

Remark 2.11. (i) In Proposition 2.10, if $(aw)^* + 1 - aa^- \in R^{-1}$ for some $a^- \in a\{1\}$, then a may not be w-core invertible. Such as let $R = M_2(\mathbb{C})$ be the ring of all 2×2 complex matrices and suppose that the involution $*$ is the conjugate transpose. Let $A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, $W = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in R$. Take $A^- = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \in A\{1\}$. Hence, $(AW)^* + I - AA^- = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \in R^{-1}$. However, $AWA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so that $W \notin R^{\|A}$ and $A \notin R_W^{\#}$.

(ii) The converse statement of Proposition 2.10 does not necessarily hold. More precisely, if $a \in R_w^{\#}$, then $((aw)^*)^m + 1 - aa^-$ may not be invertible for any $a^- \in a\{1\}$. For example, let R be the ring as one that in (i). Given $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$, $W = \begin{bmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{bmatrix} \in R$, then $AWA = -3A$, and hence $A_W^{\#}$ exists. Taking $A^- = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ 0 & 0 \end{bmatrix}$, then $(AW)^* + I - AA^- = \frac{1}{5}\begin{bmatrix} 6 & 4 \\ -24 & -16 \end{bmatrix}$ is not invertible.

We can see from Theorem 2.7, Propositions 2.8 and 2.10 above that the invertibilities of $u = (aw)^m + 1 - aa^-$ and $v = ((aw)^*)^m + 1 - aa^-$ have a strongly connection with the existence of $a_w^{\#}$.

It is known that $a \in R_w^{\#}$ if and only if $w \in R^{\|a}$ and $a \in R^{(1,3)}$. Given any $a \in R_w^{\#}$, then $wa_w^{\#} \in a\{1,3\}$ and $a^* = a^* awa_w^{\#}$ as $awa_w^{\#} a = a$ and $awa_w^{\#} = (awa_w^{\#})^*$. Pre-multiplying $a^* = a^* awa_w^{\#}$ by w^* gives $(aw)^* = (aw)^* awa_w^{\#}$. It follows from Lemma 2.4 that $aw + 1 - awa_w^{\#} \in R^{-1}$ since $w \in R^{\|a}$. Then, $(aw + 1 - awa_w^{\#})^* = 1 + (aw)^* - awa_w^{\#} = 1 + (aw)^* awa_w^{\#} - awa_w^{\#} \in R^{-1}$. By Jacobson's Lemma, $awa_w^{\#}(aw)^* + 1 - awa_w^{\#} \in R^{-1}$. Note also that $((aw)^*)^m + 1 - awa_w^{\#} \in R^{-1}$ since $a \in R_w^{\#}$. So, we have $(((aw)^*)^m + 1 - awa_w^{\#})(awa_w^{\#}(aw)^* + 1 - awa_w^{\#}) = ((aw)^*)^{m+1} + 1 - awa_w^{\#} \in R^{-1}$.

Since $wa_w^{\#} \in a\{1,3\} \subseteq a\{1\}$, we claim that, for any integer $m \geq 1$, $((aw)^*)^{m+1} + 1 - aa^- \in R^{-1}$ if and only if $((aw)^*)^m + 1 - aa^- \in R^{-1}$ for some $a^- \in a\{1\}$.

It is of interest to ask whether $((aw)^*)^{m+1} + 1 - aa^- \in R^{-1}$ is equivalent to $((aw)^*)^m + 1 - aa^- \in R^{-1}$ or not, for any $a^- \in a\{1\}$. The following theorem gives a positive answer.

Theorem 2.12. *Let $a, w \in R$ with $a \in R_w^{\#}$ and let $m \geq 1$ be an integer. For any $a^- \in a\{1\}$, the following conditions are equivalent:*

(i) $((aw)^*)^m + 1 - aa^- \in R^{-1}$.

(ii) $((aw)^*)^{m+1} + 1 - aa^- \in R^{-1}$.

(iii) $((aw)^*)^m (aw)^m + 1 - aa^- \in R^{-1}$.

Proof. (i) ⇔ (ii) Note that the equality $(((aw)^*)^m + 1 - aa^-)(awa_w^{\#}(aw)^* + 1 - awa_w^{\#}) = ((aw)^*)^{m+1} + 1 - aa^-$. Then $((aw)^*)^m + 1 - aa^- \in R^{-1}$ if and only if $((aw)^*)^{m+1} + 1 - aa^- \in R^{-1}$ since $awa_w^{\#}(aw)^* + 1 - awa_w^{\#} \in R^{-1}$.

(i) ⇔ (iii) Note that $a \in R_w^{\#}$. Then $(aw)^m + 1 - awa_w^{\#} \in R^{-1}$ in terms of Lemma 2.4. As $(((aw)^*)^m + 1 - aa^-)((aw)^m + 1 - awa_w^{\#}) = ((aw)^*)^m (aw)^m + 1 - aa^-$, then $((aw)^*)^m (aw)^m + 1 - aa^- \in R^{-1}$ if and only if $((aw)^*)^m + 1 - aa^- \in R^{-1}$. □

3. The w-core inverse of the product of three elements

Given any $p, p', a, q, q' \in R$ such that $p'pa = a = aqq'$, Patrício [21] proved that paq is Moore-Penrose invertible if and only if pa is $\{1,3\}$-invertible and aq is $\{1,4\}$-invertible. In this case, $(paq)^\dagger = (aq)^{(1,4)}a(pa)^{(1,3)}$. A more general characterization for the inverse along an element was given in [31], i.e., paq is invertible along d if and only if aq is left invertible along dp and pa is right invertible along qd. In this case, $(paq)^{\|d} = (aq)_l^{\|dp}a(pa)_r^{\|qd}$.

Inspired by the aforementioned results, we next give several results for w-core inverses and dual v-core inverses of the product of three elements, for which we firstly present the following lemma.

Lemma 3.1. *Let $p, a, q, w \in R$. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:

(i) $w \in R^{\|paq}$.

(ii) $qw \in R^{\|pa}$.

(iii) $wp \in R^{\|aq}$.

In this case, $w^{\|paq} = (qw)^{\|pa}q = p(wp)^{\|aq}$.

Proof. (i) $\Rightarrow$ (ii) As $w \in R^{\|paq}$, then, by Lemma 2.3, $paq \in Rpaqwpaq \cap paqwpaqR$, and consequently $pa \in Rpaqwpa \cap paqwpaR$, so that $qw \in R^{\|pa}$.

(ii) $\Rightarrow$ (i) Given $qw \in R^{\|pa}$, then $paqw(qw)^{\|pa} = pa = (qw)^{\|pa}qwpa$ and $(qw)^{\|pa} \in paR \cap Rpa$. It follows from $(qw)^{\|pa} \in paR \cap Rpa$ that $(qw)^{\|pa} = pax = ypa$ for suitable $x, y \in R$. Let $z = (qw)^{\|pa}q$. Then z is the inverse of w along paq. Indeed, we have

(1) $paqwz = paqw(qw)^{\|pa}q = paq = (qw)^{\|pa}qwpaq = zwpaq$.

(2) $z = (qw)^{\|pa}q = paxq = paqq'xq \in paqR$ by $a = aqq'$.

(3) $z = (qw)^{\|pa}q = ypaq \in Rpaq$.

Therefore, $w \in R^{\|paq}$ and $w^{\|paq} = (qw)^{\|pa}q$.

(i) $\Leftrightarrow$ (iii) is similar to the proof of (i) $\Leftrightarrow$ (ii). □

Theorem 3.2. *Let $p, a, q, w \in R$. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:

(i) $paq \in R_w^{\#}$.

(ii) $pa \in R_{qw}^{\#}$.

If $p' = p^*$, then (i) *or* (ii) *is equivalent to*

(iii) $aq \in R_{wp}^{\#}$.

In this case, $(paq)_w^{\#} = (pa)_{qw}^{\#} = p(aq)_{wp}^{\#}p^*$.

Proof. (i) $\Leftrightarrow$ (ii) By Lemma 3.1, we know that $w \in R^{\|paq}$ if and only if $qw \in R^{\|pa}$. According to Lemma 2.5, it next suffices to show that $paq \in R^{(1,3)}$ if and only if $pa \in R^{(1,3)}$. Suppose $paq \in R^{(1,3)}$. Then we have $paq \in R(paq)^*paq$ by Lemma 2.2, which together with $a = aqq'$ implies $pa \in R(pa)^*pa$, i.e., $pa \in R^{(1,3)}$. Conversely, given $pa \in R^{(1,3)}$, we next show that $x = q'(pa)^{(1,3)} \in (paq)\{1,3\}$ for any $(pa)^{(1,3)} \in (pa)\{1,3\}$. Indeed, $paqxpaq = paqq'(pa)^{(1,3)}paq = pa(pa)^{(1,3)}paq = paq$ and $paqx = paqq'(pa)^{(1,3)} = pa(pa)^{(1,3)} = (paqx)^*$.

(i) $\Leftrightarrow$ (iii) Lemma 3.1 ensures that $w \in R^{\|paq}$ if and only if $wp \in R^{\|aq}$. To show (i) $\Leftrightarrow$ (iii), it suffices to prove that $paq \in R^{(1,3)}$ if and only if $aq \in R^{(1,3)}$. Suppose $paq \in R^{(1,3)}$. Then $paq \in R(paq)^*paq$, which combines with $p^*pa = a$ to guarantee $aq \in R(aq)^*aq$, i.e., $aq \in R^{(1,3)}$. Conversely, given $aq \in R^{(1,3)}$, we next show that $y = (aq)^{(1,3)}p^* \in (paq)\{1,3\}$ for any $(aq)^{(1,3)} \in (aq)\{1,3\}$. Indeed, $paqypaq = paq(aq)^{(1,3)}p^*paq = paq(aq)^{(1,3)}aq = paq$ and $paqy = paq(aq)^{(1,3)}p^* = (paqy)^*$, as required.

We now give the representation of $(paq)_w^{\#}$. It follows from Lemma 3.1 that $w^{\|paq} = (qw)^{\|pa}q$. By (i) $\Leftrightarrow$ (ii), we know that $q'(pa)^{(1,3)} \in (paq)\{1,3\}$ for any $(pa)^{(1,3)} \in (pa)\{1,3\}$. Hence, $(paq)_w^{\#} = w^{\|paq}(paq)^{(1,3)} = (qw)^{\|pa}qq'(pa)^{(1,3)}$. Since $(qw)^{\|pa} \in Rpa$, it follows that $(qw)^{\|pa}qq' = (qw)^{\|pa}$ and hence $(paq)_w^{\#} = (qw)^{\|pa}qq'(pa)^{(1,3)} = (qw)^{\|pa}(pa)^{(1,3)} = (pa)_{qw}^{\#}$. Similarly, $(paq)_w^{\#} = p(aq)_{wp}^{\#}p^*$. □

Dually, a similar result for the dual v-core inverse of a product of three elements can be given, whose proof is analogous to one that of Theorem 3.2.

Theorem 3.3. *Let $p, a, q, v \in R$. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:

(i) $paq \in R_{v,\circledast}$.

(ii) $aq \in R_{vp,\circledast}$.

If $q' = q^*$, then (i) *or* (ii) *is equivalent to*

(iii) $pa \in R_{qv,\circledast}$.

In this case, $(paq)_{v,\circledast} = (aq)_{vp,\circledast} = q^*(pa)_{qv,\circledast}q$.

Lemma 3.4. *Let $p, a, q, w \in R$. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:

(i) $w \in R^{\|paq}$.

(ii) $qwp \in R^{\|a}$.

In this case, $w^{\|paq} = p(qwp)^{\|a}q$.

Proof. (i) ⇒ (ii) As $w \in R^{\|paq}$, then $paq \in Rpaqwpaq \cap paqwpaqR$ by Lemma 2.3, which implies $a \in Raqwpa \cap aqwpaR$ by $p'pa = a = aqq'$, so that $qwp \in R^{\|a}$.

(ii) ⇒ (i) Since $qwp \in R^{\|a}$, we have $aqwp(qwp)^{\|a} = a = (qwp)^{\|a}qwpa$ and $(qwp)^{\|a} \in aR \cap Ra$. From $(qwp)^{\|a} \in aR \cap Ra$, then there exist $x, y \in R$ such that $(qwp)^{\|a} = ax = ya$. Let $z = p(qwp)^{\|a}q$. We next show that z is the inverse of w along paq.

(1) $paqwz = paqwp(qwp)^{\|a}q = paq = p(qwp)^{\|a}qwpaq = zwpaq$.

(2) $z = p(qwp)^{\|a}q = paxq = paqq'xq \in paqR$ by $a = aqq'$.

(3) $z = p(qwp)^{\|a}q = pyaq = pyp'paq \in Rpaq$ by $p'pa = a$.

Therefore, $w \in R^{\|paq}$ and $w^{\|paq} = p(qwp)^{\|a}q$. □

Let $p', p, a, q, q' \in R$ such that $p'pa = a = aqq'$. Lemma 3.4 guarantees that w is invertible along paq if and only if qwp is invertible along a. One may ask whether the qwp-core invertibility of a is equivalent to the w-core invertibility of paq, under the same condition $p'pa = a = aqq'$. In fact, the answer to this question is negative. See the following example.

Example 3.5. Let R be the ring of all 2×2 complex matrices and suppose the involution $*$ is the transpose. Let $a = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, w = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, p = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}, q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, p' = \begin{bmatrix} 1 & 0 \\ 0 & -2i \end{bmatrix}, q' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R$. Then $p'pa = a = aqq'$. By a direct calculation, we get $a \in R^{\circledast}_{qwp}$ and $a^{\circledast}_{qwp} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. However, paq is not w-core invertible. In fact, since $(paq)^*paq = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we have $paq \notin R(paq)^*paq$, which implies that paq is not $\{1, 3\}$-invertible from Lemma 2.2. Therefore, paq is not w-core invertible by Theorem 2.5.

Given $p, a, w, q \in R$, a natural question is under what conditions $paq \in R^{\circledast}_w$ if and only if $a \in R^{\circledast}_{qwp}$.

Theorem 3.6. *Let $p, a, q, w \in R$. If there exists $q' \in R$ such that $p^*pa = a = aqq'$, then the following conditions are equivalent*:

(i) $paq \in R^{\circledast}_w$.

(ii) $a \in R^{\circledast}_{qwp}$.

In this case, $(paq)^{\circledast}_w = pa^{\circledast}_{qwp}p^*$.

Proof. Lemma 3.4 ensures that $w \in R^{\|paq}$ if and only if $qwp \in R^{\|a}$. To prove the equivalence of (i) and (ii), it next suffices to prove that $paq \in R^{(1,3)}$ if and only if $a \in R^{(1,3)}$. Suppose $paq \in R^{(1,3)}$. Then we have $paq \in R(paq)^*paq$, which combines with $p^*pa = a = aqq'$ to imply $a \in Ra^*a$. So, $a \in R^{(1,3)}$.

Conversely, given $a \in R^{(1,3)}$, write $z := q'a^{(1,3)}p^*$ for any $a^{(1,3)} \in a\{1, 3\}$, then $z \in (paq)\{1, 3\}$. Indeed, $paqzpaq = paqq'a^{(1,3)}p^*paq = paq$ and $paqz = paqq'a^{(1,3)}p^* = paa^{(1,3)}p^* = (paqz)^*$. Moreover, $(paq)^{\circledast}_w = w^{\|paq}(paq)^{(1,3)} = p(qwp)^{\|a}qq'a^{(1,3)}p^*$. Note that $(qwp)^{\|a} \in Ra$. Then $(qwp)^{\|a}$ can be written as the form sa for suitable $s \in R$. Consequently, $(paq)^{\circledast}_w = p(qwp)^{\|a}qq'a^{(1,3)}p^* = psaqq'a^{(1,3)}p^* = p(qwp)^{\|a}a^{(1,3)}p^* = pa^{\circledast}_{qwp}p^*$. □

Dually, substituting $p'pa = a = aqq^*$ for the condition $p'pa = a = aqq'$ in Lemma 3.4, then we can get the following result for the dual v-core inverse of paq. The proof is dual to one that of Theorem 3.6.

Theorem 3.7. *Let $p, a, q, v \in R$. If there exists $p' \in R$ such that $p'pa = a = aqq^*$, then the following conditions are equivalent*:
(i) $paq \in R_{v,\oplus}$.
(ii) $a \in R_{qvp,\oplus}$.
In this case, $(paq)_{v,\oplus} = q^* a_{qvp,\oplus} q$.

By Theorems 3.2, 3.3 and 3.7, we have the following corollary.

Corollary 3.8. *Let $p, a, q \in R$ with $p^*pa = a = aqq^*$. Then the following conditions are equivalent*:
(i) $paq \in R^{(1,3)}$.
(ii) $a \in R^{(1,3)}$.
(iii) $aq \in R^{(1,3)}$.
(iv) $pa \in R^{(1,3)}$.

Let $a, w, v \in R$. As is well known, $a \in R_w^{\oplus} \cap R_{v,\oplus}$ if and only if $w, v \in R^{\|a}$ and $a \in R^\dagger$. Moreover, $a_w^{\oplus} = w^{\|a} a^\dagger$ and $a_{v,\oplus} = a^\dagger v^{\|a}$.

We next characterize the existence criteria of both w-core inverse and dual v-core inverse of a product of three elements by ideals and units.

Lemma 3.9. [25, Theorem 3.12] *Let $a \in R$. Then a is Moore-Penrose invertible if and only if $a \in aa^*aR$ if and only if $a \in Raa^*a$. Moreover, $a^\dagger = (ax)^* = (ya)^*$, where $a = aa^*ax = yaa^*a$ for some $x, y \in R$.*

Lemma 3.10. [32, Theorem 2.3] *Let $p, a, q, w \in R$ with a regular. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:
(i) $w \in R^{\|paq}$.
(ii) $s = aqwpaa^- + 1 - aa^- \in R^{-1}$.
(iii) $t = a^-aqwpa + 1 - a^-a \in R^{-1}$.
(iv) $s' = aqwp + 1 - aa^- \in R^{-1}$.
(v) $t' = qwpa + 1 - a^-a \in R^{-1}$.
In this case, $w^{\|paq} = ps^{-1}aq = p(s')^{-1}aq = pat^{-1}q = pa(t')^{-1}q$.

Generally, Lemma 3.10 above also holds for the case of one-sided invertibility, namely, $w \in R_l^{\|paq}$ (resp., $w \in R_r^{\|paq}$) if and only if $s = aqwpaa^- + 1 - aa^- \in R_l^{-1}$ (resp., R_r^{-1}) if and only if $t = a^-aqwpa + 1 - a^-a \in R_l^{-1}$ (resp., R_r^{-1}).

Recall that a ring R is Dedekind-finite if $xy = 1 \Rightarrow yx = 1$ for any $x, y \in R$. We remark the reader that all left (resp., right) invertible elements are always right (resp., left) invertible in a Dedekind-finite ring. Generally, if a is left (resp., right) invertible along d, then a is right (resp., left) invertible along d in a Dedekind-finite ring.

Theorem 3.11. *Let R be a Dedekind-finite ring and $p, a, q, w, v \in R$. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:
(i) $paq \in R_w^{\oplus} \cap R_{v,\oplus}$.
(ii) $Raq(paq)^*pa = Ra = Raqwpa$ *and* $aqvpaR = aR$.
(iii) $aqwpaR = aR = aq(paq)^*paR$ *and* $Raqvpa = Ra$.

Proof. (i) $\Rightarrow$ (ii) Given $paq \in R_w^{\oplus} \cap R_{v,\oplus}$, then $w, v \in R^{\|paq}$ and $paq \in R^\dagger$ by Lemma 2.5. So, $paq = paq(paq)^\dagger paq$, which combines with $p'pa = a = aqq'$ to imply $a = aq(paq)^\dagger pa$, i.e., a is regular. From Lemma 3.10, we know that $w, v \in R^{\|paq}$ imply that $s = aqwpaa^- + 1 - aa^- \in R^{-1}$ and $t = a^-aqvpa + 1 - a^-a \in R^{-1}$. We have at once $sa = aqwpa$ and $at = aqvpa$. Thus, $a = s^{-1}aqwpa$ and $a = aqvpat^{-1}$, so that $Ra = Raqwpa$ and $aR = aqvpaR$. Also, it follows from Lemma 3.9 that $paq \in Rpaq(paq)^*paq$ since $paq \in R^\dagger$, which combines with $p'pa = a = aqq'$ to guarantee $a \in Raq(paq)^*pa$, so that $Ra = Raq(paq)^*pa$.

(ii) $\Rightarrow$ (i) Post-multiplying $Raq(paq)^*pa = Ra$ by q gives $Raq(paq)^*paq = Raq$, which combines with $p'pa = a$ to guarantee that $Rpaq(paq)^*paq = Raq(paq)^*paq = Raq = Rpaq$. So, $paq \in R^\dagger$ by Lemma 3.9. From the implication of (i) $\Rightarrow$ (ii), we know that a is regular. To prove $paq \in R_w^{\circledast} \cap R_{v,\circledast}$, it next suffices to show $w, v \in R^{\parallel paq}$. As $Ra = Raqwpa$, then there is suitable $y \in R$ such that $a = yaqwpa$. By a direct calculation, we have $(yaa^- + 1 - aa^-)(aqwpaa^- + 1 - aa^-) = 1$, which shows $s = aqwpaa^- + 1 - aa^- \in R_l^{-1}$, i.e., $w \in R_l^{\parallel paq}$. Note the fact that $w \in R_l^{\parallel paq}$ implies $w \in R_r^{\parallel paq}$ in a Dedekind-finite ring R. Thus, $w \in R^{\parallel paq}$. Similarly, we can get $v \in R^{\parallel paq}$ by $aqvpaR = aR$.

(i) $\Leftrightarrow$ (iii) is analogous to the proof of (i) $\Leftrightarrow$ (ii). □

In Theorem 3.11 above, set $v = w$, then we can get the following characterization of the w-core inverse and the dual w-core inverse by ideals in a general ring.

Theorem 3.12. *Let $p, a, q, w \in R$. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:

(i) $paq \in R_w^{\circledast} \cap R_{w,\circledast}$.

(ii) $Raq(paq)^*pa = Ra = Raqwpa$ *and* $aqwpaR = aR$.

(iii) $aqwpaR = aR = aq(paq)^*paR$ *and* $Raqwpa = Ra$.

It next allows us to give the existence criterion for the w-core and the dual v-core inverses of paq by units in a Dedekind-finite ring.

Theorem 3.13. *Let R be a Dedekind-finite ring and $p, a, q, w, v \in R$ with a regular. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:

(i) $paq \in R_w^{\circledast} \cap R_{v,\circledast}$.

(ii) $\tilde{u} = aq(paq)^*paqwpaqvp + 1 - aa^- \in R^{-1}$.

(iii) $\tilde{v} = aqwpaqvpaq(paq)^*p + 1 - aa^- \in R^{-1}$.

(iv) $\tilde{s} = q(paq)^*paqwpaqvpa + 1 - a^-a \in R^{-1}$.

(v) $\tilde{t} = qwpaqvpaq(paq)^*pa + 1 - a^-a \in R^{-1}$.

In this case, $(paq)_w^{\circledast} = paqvpaq(paq)^*pa\tilde{t}^{-1}q(paqwpaqvpa\tilde{s}^{-1}q)^*$, $(paq)_{v,\circledast} = (paqwpaqvpa\tilde{s}^{-1}q)^*p\tilde{u}^{-1}aq(paq)^*paqwpaq$.

Proof. (ii) $\Leftrightarrow$ (iv) and (iii) $\Leftrightarrow$ (v) follow from Jacobson's Lemma.

(i) $\Rightarrow$ (ii) As $paq \in R_w^{\circledast} \cap R_{v,\circledast}$, then $w, v \in R^{\parallel paq}$ and $paq \in R^{\dagger}$. Since $w, v \in R^{\parallel paq}$, one can get $aqwpaa^- + 1 - aa^- \in R^{-1}$ and $aqvp + 1 - aa^- \in R^{-1}$ by Lemma 3.10. Thus, we have $(aqwpaa^- + 1 - aa^-)(aqvp + 1 - aa^-) = aqwpaqvp + 1 - aa^- \in R^{-1}$. Also, $paq \in R^{\dagger}$ ensures that $aq(paq)^*paa^- + 1 - aa^- \in R^{-1}$ by [35, Theorem 2.7]. Consequently, $\tilde{u} = aq(paq)^*paqwpaqvp + 1 - aa^- = (aq(paq)^*paa^- + 1 - aa^-)(aqwpaqvp + 1 - aa^-) \in R^{-1}$.

(ii) $\Rightarrow$ (i) To prove $paq \in R_w^{\circledast} \cap R_{v,\circledast}$, it next suffices to prove $w, v \in R^{\parallel paq}$ and $paq \in R^{\dagger}$. Given $\tilde{u} = aq(paq)^*paqwpaqvp + 1 - aa^- \in R^{-1}$ for some $a^- \in a\{1\}$, then we have $\tilde{u}a = aq(paq)^*paqwpaqvpa$ and whence $a = \tilde{u}^{-1}aq(paq)^*paqwpaqvpa$. Multiplying the equation $a = \tilde{u}^{-1}aq(paq)^*paqw\, paqvpa$ by p on the left side and by q on the right side give $paq = p\tilde{u}^{-1}aq(paq)^*\, paqwpaqvpaq \in Rpaqvpaq$, which implies $v \in R_l^{\parallel paq}$. So, $v \in R^{\parallel paq}$ in a Dedekind-finite ring. According to Lemma 3.10, one gets that $aqvp + 1 - aa^- \in R^{-1}$. Similarly, from the equivalence of (ii) and (iv), we have $paq = paq(paq)^*paqwpaqvpa\tilde{s}^{-1}q \in paq(paq)^*paqR$, which shows that $paq \in R^{\dagger}$ and $(paq)^{\dagger} = (paqwpaqvpa\tilde{s}^{-1}q)^*$ by Lemma 3.9. This in turn gives $aq(paq)^*paa^- + 1 - aa^- \in R^{-1}$ by [35, Theorem 2.7].

Note that $(aq(paq)^*paa^- + 1 - aa^-)(aqwpaqvp + 1 - aa^-) = aq(paq)^*paqwpa\, qvp + 1 - aa^- \in R^{-1}$. Then $aqwpaqvp + 1 - aa^- \in R^{-1}$. Also, since $(aqwpaa^- + 1 - aa^-)(aqvp + 1 - aa^-) = aqwpaqvp + 1 - aa^- \in R^{-1}$, we have $aqwpaa^- + 1 - aa^- \in R^{-1}$, which guarantees that $w \in R^{\parallel paq}$ by Lemma 3.10. Consequently, $paq \in R_w^{\circledast} \cap R_{v,\circledast}$.

(i) $\Leftrightarrow$ (iii) can be proved by a similar way of (i) $\Leftrightarrow$ (ii).

We next give the formulae of $(paq)_w^{\circledast}$ and $(paq)_{v,\circledast}$, respectively. Since $\tilde{t} = qwpaqvpaq(paq)^*pa + 1 - a^-a \in R^{-1}$, we have $a\tilde{t} = aqwpaqvpaq(paq)^*pa$ and $a = aqwpaqvpaq(paq)^*pa\tilde{t}^{-1}$. So, $paq = paqwpaqvpaq(paq)^*pa\tilde{t}^{-1}q$. As $w \in R^{\parallel paq}$, then $w^{\parallel paq} = paqvpaq(paq)^*pa\tilde{t}^{-1}q$. Thus, $(paq)_w^{\circledast} = w^{\parallel paq}(paq)^{\dagger} = paqvpaq(paq)^*pa\tilde{t}^{-1}q(paqwpaqvpa\tilde{s}^{-1}q)^*$.

Similarly, $(paq)_{v,\circledast} = (paq)^{\dagger}v^{\parallel paq} = (paqwpaqvpa\tilde{s}^{-1}q)^*p\tilde{u}^{-1}aq(paq)^*paqw\, paq$. □

In Theorem 3.13 above, set $v = w$, then we can get the following result of the w-core inverse and the dual w-core inverse of paq by units in a general case, whose proof is left to the reader.

Theorem 3.14. *Let $p, a, q, w \in R$ with a regular. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:

(i) $paq \in R_w^{\circledast} \cap R_{w,\circledast}$.

(ii) $\tilde{u} = aq(paq)^*paqwp + 1 - aa^- \in R^{-1}$.

(iii) $\tilde{v} = aqwpaq(paq)^*p + 1 - aa^- \in R^{-1}$.

(iv) $\tilde{s} = q(paq)^*paqwpa + 1 - a^-a \in R^{-1}$.

(v) $\tilde{t} = qwpaq(paq)^*pa + 1 - a^-a \in R^{-1}$.

(vi) $\tilde{u}' = aq(paq)^*paqwp - 1 + aa^- \in R^{-1}$.

(vii) $\tilde{v}' = aqwpaq(paq)^*p - 1 + aa^- \in R^{-1}$.

(viii) $\tilde{s}' = q(paq)^*paqwpa - 1 + a^-a \in R^{-1}$.

(ix) $\tilde{t}' = qwpaq(paq)^*pa - 1 + a^-a \in R^{-1}$.

In this case, $(paq)_w^{\circledast} = p\tilde{u}^{-1}aq(paq)^* = p(\tilde{u}')^{-1}aq(paq)^*$ *and* $(paq)_{w,\circledast} = (paq)^*pa\tilde{v}^{-1}q = (paq)^*pa(\tilde{v}')^{-1}q$.

Suppose $p = q = 1$ in Theorem 3.14 above. Then we can obtain the following result, where equivalences (i)-(v) were presented in [33, Theorem 3.9].

Corollary 3.15. *Let $a, w \in R$ with a regular. Then the following conditions are equivalent*:

(i) $a \in R_w^{\circledast} \cap R_{w,\circledast}$.

(ii) $\tilde{u} = aa^*aw + 1 - aa^- \in R^{-1}$.

(iii) $\tilde{v} = waa^*a + 1 - a^-a \in R^{-1}$.

(iv) $\tilde{s} = a^*awa + 1 - a^-a \in R^{-1}$.

(v) $\tilde{t} = awaa^* + 1 - aa^- \in R^{-1}$.

(vi) $\tilde{u}' = aa^*aw - 1 + aa^- \in R^{-1}$.

(vii) $\tilde{v}' = waa^*a - 1 + a^-a \in R^{-1}$.

(viii) $\tilde{s}' = a^*awa - 1 + a^-a \in R^{-1}$.

(ix) $\tilde{t}' = awaa^* - 1 + aa^- \in R^{-1}$.

In this case, $a_w^{\circledast} = \tilde{u}^{-1}aa^* = (\tilde{u}')^{-1}aa^*$ *and* $a_{w,\circledast} = a^*a\tilde{v}^{-1} = a^*a(\tilde{v}')^{-1}$.

For $p, a, q, w \in R$ with $a \in R^{(1,4)}$ (or $a \in R^{(1,3)}$), we close this section with the existence of the w-core inverse of paq. An auxiliary lemma is presented.

Lemma 3.16. [7] *Let $p, a, q \in R$ with $p'pa = a = aqq'$ for some $p', q' \in R$.*

(i) *If $a\{1,4\} \neq \emptyset$, then $(paq)\{1,3\} \neq \emptyset$ if and only if $\alpha = (pa)^*pa + 1 - a^{(1,4)}a \in R^{-1}$. In this case, $q'a^{(1,4)}a\alpha^{-1}(pa)^* \in (paq)\{1,3\}$.*

(ii) *If $a\{1,3\} \neq \emptyset$, then $(paq)\{1,4\} \neq \emptyset$ if and only if $\beta = aq(aq)^* + 1 - aa^{(1,3)} \in R^{-1}$. In this case, $(aq)^*\beta^{-1}aa^{(1,3)}p' \in (paq)\{1,4\}$.*

Theorem 3.17. *Let $p, a, q, w \in R$ such that $a\{1,4\} \neq \emptyset$. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$, then the following conditions are equivalent*:

(i) $paq \in R_w^{\circledast}$.

(ii) $\hat{s} = aqwpa(pa)^*p + 1 - aa^{(1,4)} \in R^{-1}$.

(iii) $\hat{t} = qwpa(pa)^*pa + 1 - a^{(1,4)}a \in R^{-1}$.

In this case, $(paq)_w^{\circledast} = pa(pa)^*pa\hat{t}^{-1}qq'(p\hat{s}^{-1}aqwpa)^*$.

Proof. (ii) ⇔ (iii) follows from Jacobson's Lemma.

(i) ⇒ (ii) As $paq \in R_w^{\circledast}$, then $w \in R^{\|paq}$ and $paq \in R^{(1,3)}$. Lemma 3.16 ensures that $paq \in R^{(1,3)}$ if and only if $(pa)^*pa + 1 - a^{(1,4)}a \in R^{-1}$, so that $a(pa)^*p + 1 - aa^{(1,4)} \in R^{-1}$ by Jacobson's Lemma. Also, we know that $w \in R^{\|paq}$ implies $aqwpaa^{(1,4)} + 1 - aa^{(1,4)} \in R^{-1}$ by Lemma 3.10. Consequently, $\hat{s} = aqwpa(pa)^*p + 1 - aa^{(1,4)} = (aqwpaa^{(1,4)} + 1 - aa^{(1,4)})(a(pa)^*p + 1 - aa^{(1,4)}) \in R^{-1}$.

(ii) ⇒ (i) Given $\hat{s} = aqwpa(pa)^*p + 1 - aa^{(1,4)} \in R^{-1}$, then $\hat{s}a = aqwpa(pa)^*pa$ and whence $a = \hat{s}^{-1}aqwpa(pa)^*pa$, which combines with $a = aqq'$ to ensure that $paq = p\hat{s}^{-1}aqwpa(paqq')^*paq = p\hat{s}^{-1}aqwpa(q')^*(paq)^*paq$. This gives $paq \in R^{(1,3)}$ and $q'(p\hat{s}^{-1}aqwpa)^* \in paq\{1,3\}$ from Lemma 2.2. We have at once $a(pa)^*p + 1 - aa^{(1,4)} \in R^{-1}$

by Lemma 3.16. Note that $\hat{s} = aqwpa(pa)^*p + 1 - aa^{(1,4)} = (aqwpaa^{(1,4)} + 1 - aa^{(1,4)})(a(pa)^*p + 1 - aa^{(1,4)}) \in R^{-1}$. Then $aqwpaa^{(1,4)} + 1 - aa^{(1,4)} \in R^{-1}$, and consequently $w \in R^{\|paq}$ by Lemma 3.10.

We next give the formula of $(paq)^{\circledast}_w$. Since $\hat{t} = qwpa(pa)^*pa + 1 - a^{(1,4)}a \in R^{-1}$, we have $a\hat{t} = aqwpa(pa)^*pa$ and $a = aqwpa(pa)^*pa\hat{t}^{-1}$. So, $paq = paqwpaqq'(pa)^*pa\hat{t}^{-1}q \in paqwpaqR$, which impiles $w \in R^{\|paq}$ and $w^{\|paq} = pa(pa)^*pa\hat{t}^{-1}q$ by Lemma 2.3. Therefore, $(paq)^{\circledast}_w = w^{\|paq}(paq)^{(1,3)} = pa(pa)^*pa\hat{t}^{-1}qq'(p\hat{s}^{-1}aqwpa)^*$. □

Another characterization for the existence of the w-core inverse of paq is given, provided that $a\{1,3\} \neq \varnothing$.

Theorem 3.18. *Let $p, a, q, w \in R$ such that $a\{1,3\} \neq \varnothing$. If there exist $p', q' \in R$ such that $p'pa = a = aqq'$ and $p(1 - aa^{(1,3)}) = 1 - aa^{(1,3)}$, then the following conditions are equivalent*:

(i) $paq \in R^{\circledast}_w$.

(ii) $\check{s} = aqwpaa^{(1,3)}p^*p + 1 - aa^{(1,3)} \in R^{-1}$.

(iii) $\check{t} = qwpaa^{(1,3)}p^*pa + 1 - a^{(1,3)}a \in R^{-1}$.

In this case, $(paq)^{\circledast}_w = paa^{(1,3)}p^*pa\check{t}^{-1}qq'a^{(1,3)}(p\check{s}^{-1}aqwp)^*$.

Proof. (ii) ⇔ (iii) follows immediately from Jacobson's Lemma. It is sufficient to prove (i) ⇔ (ii).

(i) ⇒ (ii) As $paq \in R^{\circledast}_w$, then $w \in R^{\|paq}$ and $(paq)(1,3) \neq \varnothing$. According to Theorem 3.2 (i) ⇔ (ii) and [35, Theorem 2.10], one gets that $(paq)\{1,3\} \neq \varnothing$ if and only if $(pa)\{1,3\} \neq \varnothing$ if and only if $h = p^*paa^{(1,3)} + 1 - aa^{(1,3)} \in R^{-1}$. By Jacobson's Lemma, we have $h' = aa^{(1,3)}p^*p + 1 - aa^{(1,3)} \in R^{-1}$. Also, $w \in R^{\|paq}$ gives $s = aqwpaa^{(1,3)} + 1 - aa^{(1,3)} \in R^{-1}$ by Lemma 3.10. Hence, $\check{s} = aqwpaa^{(1,3)}p^*p + 1 - aa^{(1,3)} = (aqwpaa^{(1,3)} + 1 - aa^{(1,3)})(aa^{(1,3)}p^*p + 1 - aa^{(1,3)}) \in R^{-1}$.

(ii) ⇒ (i) It follows from $\check{s} = aqwpaa^{(1,3)}p^*p + 1 - aa^{(1,3)} \in R^{-1}$ that $\check{s}a = aqwpaa^{(1,3)}p^*pa$, which implies $a = \check{s}^{-1}aqwpaa^{(1,3)}p^*pa$. Multiplying the equation $a = \check{s}^{-1}aqwpaa^{(1,3)}p^*pa$ by p on the left side and by q on the right side give $paq = p\check{s}^{-1}aqwpaa^{(1,3)}p^*paq = p\check{s}^{-1}aqwp(paa^{(1,3)})^*paq = p\check{s}^{-1}aqwp(paqq'a^{(1,3)})^*paq = p\check{s}^{-1}aqwp(q'a^{(1,3)})^*(paq)^*paq \in R(paq)^*paq$. So, $paq \in R^{(1,3)}$ and $q'a^{(1,3)}(p\check{s}^{-1}aqwp)^* \in paq\{1,3\}$ by Lemma 2.2.

To prove $paq \in R^{\circledast}_w$, it suffices to prove $w \in R^{\|paq}$. It is known that $(paq)\{1,3\} \neq \varnothing$ if and only if $(pa)\{1,3\} \neq \varnothing$ if and only if $h' = aa^{(1,3)}p^*p + 1 - aa^{(1,3)} \in R^{-1}$. Note that $\check{s} = aqwpaa^{(1,3)}p^*p + 1 - aa^{(1,3)} = (aqwpaa^{(1,3)} + 1 - aa^{(1,3)})(aa^{(1,3)}p^*p + 1 - aa^{(1,3)}) \in R^{-1}$. Then $s = aqwpaa^{(1,3)} + 1 - aa^{(1,3)} \in R^{-1}$, i.e., $w \in R^{\|paq}$, as required.

We finally give the expression of $(paq)^{\circledast}_w$. As $\check{t} = qwpaa^{(1,3)}p^*pa + 1 - a^{(1,3)}a \in R^{-1}$, then $a\check{t} = aqwpaa^{(1,3)}p^*pa$ and $a = aqwpaa^{(1,3)}p^*pa\check{t}^{-1}$, which together with $a = aqq'$ give $paq = paqwpaqq'a^{(1,3)}p^*pa\check{t}^{-1}q \in paqwpaqR$. Hence, $w^{\|paq} = paa^{(1,3)}p^*pa\check{t}^{-1}q$ by Lemma 2.3.

Therefore, $(paq)^{\circledast}_w = w^{\|paq}(paq)^{(1,3)} = paa^{(1,3)}p^*pa\check{t}^{-1}qq'a^{(1,3)}(p\check{s}^{-1}aqwp)^*$. □

4. Applications to 2 × 2 matrices over a ring

Given any $W = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $A = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ with d_4 invertible. In this section, we mainly derive the existence criterion and formula of the W-core inverse of a regular 2×2 matrix A over a ring.

For any $A = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ with d_4 invertible, we have the following decomposition

$$A = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & d_3d_4^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s & 0 \\ 0 & d_4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ d_4^{-1}d_2 & 1 \end{bmatrix} := PMQ, \tag{1}$$

where $s = d_1 - d_3d_4^{-1}d_2$ is the Schur complement of d_4 in the matrix A. It is well known that A is regular if and only if M is regular. Similarly, if d_1 is invertible, then we get a similar decomposition of (1), and $d_4 - d_2d_1^{-1}d_3$ is called the Schur complement of d_1 in the matrix A.

Lemma 4.1. [32, Corollary 3.5] *Let $W = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $A = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ with d_4 invertible. If $a^{\|s}$ exists, then $W^{\|A}$ exists if and only if $\xi = \beta - \alpha a^{\|s}(ad_3d_4^{-1} + c)$ is invertible.*

In this case, $W^{\|A} = \begin{bmatrix} x_1 d_2 + x_2 s & x_1 d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1} d_4 \end{bmatrix}$*, where*

$$\begin{aligned} s &= d_1 - d_3 d_4^{-1} d_2, \\ u &= sa + 1 - ss^+, \\ \alpha &= d_2 a + d_4 b, \\ \beta &= \alpha d_3 d_4^{-1} + d_2 c + d_4 d, \\ \xi &= \beta - \alpha a^{\|s}(a d_3 d_4^{-1} + c), \\ x_1 &= [(1 - a^{\|s} a) d_3 d_4^{-1} - a^{\|s} c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1 \alpha u^{-1}. \end{aligned}$$

For any $A = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$, where d_1 is invertible, Castro-González et al. in [5, Theorem 4.1] characterized $\{1,3\}$-inverses of A.

A similar characterization for {1,3}-inverses of such form A can be given, provided that d_4 is invertible. We denote by I the identity matrix in $R_{2\times 2}$.

Lemma 4.2. *Let* $A = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ *with* d_4 *invertible and* $s = d_1 - d_3 d_4^{-1} d_2$*. Assume* $s\{1,3\} \neq \emptyset$ *and let* $s^{(1,3)} \in s\{1,3\}$*. Then* $A\{1,3\} \neq \emptyset$ *if and only if* $\eta = 1 + (d_3 d_4^{-1})^* e d_3 d_4^{-1} \in R^{-1}$*. In this case, a* $\{1,3\}$*-inverse of* A *is given by*

$$A^{(1,3)} = \begin{bmatrix} s^{(1,3)}(1 - d_3 d_4^{-1} \eta^{-1} (d_3 d_4^{-1})^* e) & -s^{(1,3)} d_3 d_4^{-1} \eta^{-1} \\ l\eta^{-1}(d_3 d_4^{-1})^* e - d_4^{-1} d_2 s^{(1,3)} & l\eta^{-1} \end{bmatrix},$$

where $e = 1 - ss^{(1,3)}$, $\eta = 1 + (d_3 d_4^{-1})^* e d_3 d_4^{-1}$ *and* $l = (1 + d_4^{-1} d_2 s^{(1,3)} d_3) d_4^{-1}$*.*

Proof. Let P, M, Q as in (1). Then a $\{1,3\}$-inverse of M can be written as the form

$$M^{(1,3)} = \begin{bmatrix} s^{(1,3)} & 0 \\ 0 & d_4^{-1} \end{bmatrix}.$$

Then

$$E = I - MM^{(1,3)} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix},$$

where $e = 1 - ss^{(1,3)}$. Hence, $PE = E$. Set

$$F = E(I - P^{-1}) = \begin{bmatrix} 0 & e d_3 d_4^{-1} \\ 0 & 0 \end{bmatrix}, \textit{then}$$

$$I + F^* F = \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}, \textit{where } \eta = 1 + (d_3 d_4^{-1})^* e d_3 d_4^{-1}.$$

It follows from [5, Theorem 3.1] that PM is $\{1,3\}$-invertible if and only if $\eta \in R^{-1}$. Moreover, a $\{1,3\}$-inverse of PM is of the form $(PM)^{(1,3)} = M^{(1,3)} P^{-1} (I + F^* F)^{-1} (I + F^*)$. Therefore,

$$\begin{aligned} (PM)^{(1,3)} &= \begin{bmatrix} s^{(1,3)} & 0 \\ 0 & d_4^{-1} \end{bmatrix} \begin{bmatrix} 1 & -d_3 d_4^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \eta^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (d_3 d_4^{-1})^* e & 1 \end{bmatrix} \\ &= \begin{bmatrix} s^{(1,3)}(1 - d_3 d_4^{-1} \eta^{-1} (d_3 d_4^{-1})^* e) & -s^{(1,3)} d_3 d_4^{-1} \eta^{-1} \\ d_4^{-1} \eta^{-1} (d_3 d_4^{-1})^* e & d_4^{-1} \eta^{-1} \end{bmatrix}. \end{aligned}$$

As Q is invertible, then, by Theorem 3.2 (i) $\Leftrightarrow$ (ii), $PMQ\{1,3\} \neq \emptyset$ if and only if $PM\{1,3\} \neq \emptyset$, in which case, $A^{(1,3)} = (PMQ)^{(1,3)} = Q^{-1}(PM)^{(1,3)}$.

As a consequence,

$$A^{(1,3)} = \begin{bmatrix} s^{(1,3)}(1 - d_3d_4^{-1}\eta^{-1}(d_3d_4^{-1})^*e) & -s^{(1,3)}d_3d_4^{-1}\eta^{-1} \\ l\eta^{-1}(d_3d_4^{-1})^*e - d_4^{-1}d_2s^{(1,3)} & l\eta^{-1} \end{bmatrix}.$$

□

Applying Lemmas 4.1 and 4.2, we get the following characterization and expression for the W-core inverse of a 2×2 matrix A.

Theorem 4.3. *Let $W = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $A = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ with d_4 invertible and $s = d_1 - d_3d_4^{-1}d_2$. If $s_a^{\#}$ exists, then $A_W^{\#}$ exists if and only if $\xi = \beta - \alpha a^{\|s}(ad_3d_4^{-1} + c) \in R^{-1}$ and $\eta = 1 + (d_3d_4^{-1})^*ed_3d_4^{-1} \in R^{-1}$.*

In this case,

$$A_W^{\#} = \begin{bmatrix} x_1\eta^{-1}(d_3d_4^{-1})^*e + x_2sas_a^{\#}t & x_1\eta^{-1} - x_2sas_a^{\#}d_3d_4^{-1}\eta^{-1} \\ \xi^{-1}\eta^{-1}(d_3d_4^{-1})^*e - \xi^{-1}\alpha s_a^{\#}t & \xi^{-1}\eta^{-1} + \xi^{-1}\alpha s_a^{\#}d_3d_4^{-1}\eta^{-1} \end{bmatrix},$$

where

$$\begin{aligned} s &= d_1 - d_3d_4^{-1}d_2, \\ u &= sa + 1 - ss^+, \\ \alpha &= d_2a + d_4b, \\ \beta &= \alpha d_3d_4^{-1} + d_2c + d_4d, \\ \xi &= \beta - \alpha a^{\|s}(ad_3d_4^{-1} + c), \\ x_1 &= [(1 - a^{\|s}a)d_3d_4^{-1} - a^{\|s}c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1\alpha u^{-1}, \\ t &= 1 - d_3d_4^{-1}\eta^{-1}(d_3d_4^{-1})^*e, \\ e &= 1 - ss^{(1,3)}, \\ \eta &= 1 + (d_3d_4^{-1})^*ed_3d_4^{-1}, \\ l &= (1 + d_4^{-1}d_2s^{(1,3)}d_3)d_4^{-1}. \end{aligned}$$

Proof. Since $s_a^{\#}$ exists, it follows that $a^{\|s}$ and $s^{(1,3)}$ both exist. Lemma 4.1 ensures that $W^{\|A}$ exists if and only if $\xi = \beta - \alpha a^{\|s}(ad_3d_4^{-1} + c) \in R^{-1}$. By the existence of $s^{(1,3)}$ and Lemma 4.2, it follows that $A\{1,3\} \neq \emptyset$ if and only if $\eta = 1 + (d_3d_4^{-1})^*ed_3d_4^{-1} \in R^{-1}$. Moreover, a $\{1,3\}$-inverse of A can be written as the form

$$A^{(1,3)} = \begin{bmatrix} s^{(1,3)}(1 - d_3d_4^{-1}\eta^{-1}(d_3d_4^{-1})^*e) & -s^{(1,3)}d_3d_4^{-1}\eta^{-1} \\ l\eta^{-1}(d_3d_4^{-1})^*e - d_4^{-1}d_2s^{(1,3)} & l\eta^{-1} \end{bmatrix},$$

where $e = 1 - ss^{(1,3)}$, $\eta = 1 + (d_3d_4^{-1})^*ed_3d_4^{-1}$ and $l = (1 + d_4^{-1}d_2s^{(1,3)}d_3)d_4^{-1}$.

By Lemma 2.5, one knows that $A_W^{\#}$ exists if and only if $\xi = \beta - \alpha a^{\|s}(ad_3d_4^{-1} + c) \in R^{-1}$ and $\eta = 1 + (d_3d_4^{-1})^*ed_3d_4^{-1} \in R^{-1}$. Hence, we have

$$\begin{aligned} & d_4l\eta^{-1} - d_2s^{(1,3)}d_3d_4^{-1}\eta^{-1} \\ &= d_4(1 + d_4^{-1}d_2s^{(1,3)}d_3)d_4^{-1}\eta^{-1} - d_2s^{(1,3)}d_3d_4^{-1}\eta^{-1} \\ &= \eta^{-1}, \end{aligned}$$

and $d_4l\eta^{-1}(d_3d_4^{-1})^*e - d_2s^{(1,3)}d_3d_4^{-1}\eta^{-1}(d_3d_4^{-1})^*e = \eta^{-1}(d_3d_4^{-1})^*e$.

Note that $as_a^{\circledast}$ is a $\{1,3\}$-inverse of s. Then $ss^{(1,3)} = sas_a^{\circledast}$. Thus,

$$\begin{aligned} A_W^{\circledast} &= W^{\|A}A^{(1,3)} \\ &= \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1}d_4 \end{bmatrix} \begin{bmatrix} s^{(1,3)}(1 - d_3d_4^{-1}\eta^{-1}(d_3d_4^{-1})^*e & -s^{(1,3)}d_3d_4^{-1}\eta^{-1} \\ l\eta^{-1}(d_3d_4^{-1})^*e - d_4^{-1}d_2s^{(1,3)} & l\eta^{-1} \end{bmatrix} \\ &= \begin{bmatrix} x_1\eta^{-1}(d_3d_4^{-1})^*e + x_2sas_a^{\circledast}t & x_1\eta^{-1} - x_2sas_a^{\circledast}d_3d_4^{-1}\eta^{-1} \\ \xi^{-1}\eta^{-1}(d_3d_4^{-1})^*e - \xi^{-1}\alpha s_a^{\circledast}t & \xi^{-1}\eta^{-1} + \xi^{-1}\alpha s_a^{\circledast}d_3d_4^{-1}\eta^{-1} \end{bmatrix}. \end{aligned}$$

□

It was shown [33] in a $*$-ring that a is core invertible if and only if a is a-core invertible or 1-core invertible. Moreover, $a^{\circledast} = a_1^{\circledast} = a^{\#}aa^{(1,3)} = aa_a^{\circledast}$.

Set $W = I$ in Theorem 4.3, the existence criterion and the formula of the core inverse of A are given as follows.

Corollary 4.4. *Let* $W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ *with* d_4 *invertible. If* $s^{\circledast}$ *exists, then* $A^{\circledast}$ *exists if and only if* $\xi = \beta - d_2s^{\circledast}sd_3d_4^{-1} \in R^{-1}$ *and* $\eta = 1 + (d_3d_4^{-1})^*ed_3d_4^{-1} \in R^{-1}$. *In this case,*

$$A^{\circledast} = \begin{bmatrix} x_1\eta^{-1}(d_3d_4^{-1})^*e + x_2ss^{\circledast}t & x_1\eta^{-1} - x_2ss^{\circledast}d_3d_4^{-1}\eta^{-1} \\ \xi^{-1}\eta^{-1}(d_3d_4^{-1})^*e - \xi^{-1}\alpha s^{\circledast}t & \xi^{-1}\eta^{-1} + \xi^{-1}\alpha s^{\circledast}d_3d_4^{-1}\eta^{-1} \end{bmatrix},$$

where

$$\begin{aligned} s &= d_1 - d_3d_4^{-1}d_2, \\ u &= s + 1 - ss^+, \\ \beta &= d_2d_3d_4^{-1} + d_4, \\ \xi &= \beta - d_2s^{\circledast}sd_3d_4^{-1}, \\ x_1 &= (1 - s^{\circledast}s)d_3d_4^{-1}\xi^{-1}, \\ x_2 &= u^{-1} - x_1d_2u^{-1}, \\ t &= 1 - d_3d_4^{-1}\eta^{-1}(d_3d_4^{-1})^*e, \\ e &= 1 - ss^{(1,3)}, \\ \eta &= 1 + (d_3d_4^{-1})^*ed_3d_4^{-1}, \\ l &= (1 + d_4^{-1}d_2s^{(1,3)}d_3)d_4^{-1}. \end{aligned}$$

References

[1] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58 (2010) 681-697.
[2] A. Ben-Israel, T.N.E. Greville, Generalized Inverses : Theory and Applications, (2nd edition), Springer, New York 2003.
[3] J. Benítez, E. Boasso, The inverse along an element in rings, Electron. J. Linear Algebra 31 (2016) 572-592.
[4] J. Benítez, E. Boasso, The inverse along an element in rings with an involution, Banach algebras and C*-algebras, Linear Multilinear Algebra 65 (2017) 284-299.
[5] N. Castro-González, J.L. Chen, L. Wang, Further results on generalized inverses in rings with involution, Electron. J. Linear Algebra 30 (2015) 118-134.
[6] N. Castro-González, J. Robles, J.Y. Vélez-Cerrada, The group inverse of 2×2 matrices over a ring, Linear Algebra Appl. 438 (2013) 3600-3609.
[7] J.L. Chen, A note on generalized inverses of a product, Northeast Math. 12 (1996) 431-440.
[8] X.F. Chen, J.L. Chen, Right core inverses of a product and a companion matrix, Linear Multilinear Algebra 69 (2021) 2245-2263.
[9] J.L. Chen, H.H. Zhu, P. Patrício, Y.L. Zhang, Characterizations and representations of core and dual core inverses, Canad. Math. Bull. 60 (2017) 269-282.
[10] J.L. Chen, H.L. Zou, H.H. Zhu, P. Patrício, The one-sided inverse along an element in semigroups and rings, Mediterr. J. Math. 14 (2017) 208-225.
[11] D.S. Cvetković-Ilić, Y.M. Wei, Algebraic Properties of Generalized Inverses, Series: Developments in Mathematics, Vol. 52, Springer, 2017.
[12] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506-514.

[13] Y.F. Gao, J.L. Chen, The pseudo core inverse of a lower triangular matrix, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 113 (2019) 423-434.
[14] M.C. Gouveia, R. Puystjens, About the group inverse and Moore-Penrose inverse of a product, Linear Algebra Appl. 150 (1991) 361-369.
[15] R.E. Hartwig, Block generalized inverses, Arch. Rational Mech. Anal. 61 (1976) 197-251.
[16] R.E. Hartwig, P. Patrício, When does the Moore-Penrose inverse flip, Oper. Matrices. 6 (2012) 181-192.
[17] Y.Y. Ke, L. Wang, J.L. Chen, The core inverse of a product and 2×2 matrices, Bull Malays Math Sci Soc. 42 (2019) 51-66.
[18] T.Y. Lam, A first course in noncommutative rings, Graduate Text in Mathematics, vol. 131, 2nd edn. Springer, Berlin (2001).
[19] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 434 (2011) 1836-1844.
[20] X. Mary, P. Patrício, Generalized inverses modulo $\mathcal{H}$ in semigroups and rings, Linear Multilinear Algebra 61 (2013) 886-891.
[21] P. Patrício, The Moore-Penrose inverse of a factorization, Linear Algebra Appl. 370 (2003) 227-235.
[22] P. Patrício, The Moore-Penrose inverse of von Neumann regular matrices over a ring, Linear Algebra Appl. 332-334 (2001) 469-483.
[23] R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 51 (1955) 406-413.
[24] K.M. Prasad, K.S. Mohana, Core-EP inverse, Linear Multilinear Algebra 62 (2014) 792-802.
[25] R. Puystjens, R.E. Hartwig, The group inverse of a companion matrix, Linear Multilinear Algebra 43 (1997) 137-150.
[26] R. Puystjens, D.W. Robinson, Symmetric morphisms and the existence of Moore-Penrose inverses, Linear Algebra Appl. 131 (1990) 51-69.
[27] D.S. Rakić, N.C. Dinčić, D.S. Djordjević, Core inverse and core partial order of Hilbert space operators, Appl. Math. Comput. 244 (2014) 283-302.
[28] D.S. Rakić, N.C. Dinčić, D.S. Djordjević, Group, Moore-Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014) 115-133.
[29] H.H. Zhu, J.L. Chen, Further results on the inverse along an element in semigroups and rings, Linear Multilinear Algebra 64 (2016) 393-403.
[30] H.H. Zhu, J.L. Chen, X.X. Zhang, P. Patrício, The Moore-Penrose inverse of 2×2 matrices over a certain $*$-regular ring, Appl. Math. Comput. 246 (2014) 263-267.
[31] H.H. Zhu, P. Patrício, A note on clean elements and inverses along an element, Filomat 32 (2018) 4285-4290.
[32] H.H. Zhu, P. Patrício, J.L. Chen, The inverse along a product and its applications, Linear Multilinear Algebra 64 (2016) 834-841.
[33] H.H. Zhu, L.Y. Wu, J.L. Chen, A new class of generalized inverses in semigroups and rings with involution, Comm. Algebra (2022), DOI: https://doi.org/10.1080/00927872.2022.2150771.
[34] H.H. Zhu, L.Y. Wu, D. Mosić, One-sided w-core inverses in rings with an involution, Linear Multilinear Algebra (2022), https://doi.org/10.1080/03081087.2022.2035308.
[35] H.H. Zhu, X.X. Zhang, J.L. Chen, Generalized inverses of a factorization in a ring with involution, Linear Algebra Appl. 472 (2015) 142-1506.