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Some new and general versions of q-Hermite-Hadamard-Mercer inequalities

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Abstract. In this paper, we establish two new different and general variants of q-Hermite–Hadamard-Mercer inequalities by using the newly defined q-integrals. The main edge of these inequalities is that they can be converted into some existing and new inequalities for different choices of $q \in (0,1)$ and $\lambda \in (0,1]$. Finally, we study some mathematical examples to assure the validity of newly established inequalities.

1. Introduction

The Hermite-Hadamard inequality, named after Charles Hermite and Jacques Hadamard and commonly known as Hadamard's inequality, says that if a function $f : [a,b] \to \mathbb{R}$ is convex, the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

If f is a concave mapping, the above inequality holds in the opposite direction. The inequality (1) can be proved using the Jensen inequality. There has been much research done in the direction of Hermite-Hadamard for different kinds of convexities. For example, in [1, 2], the authors established some inequalities linked with midpoint and trapezoid formulas of numerical integration for convex functions.

In 2003, Mercer [3] proved another version of Jensen inequality, which is called Jensen-Mercer inequality and stated as:

Theorem 1.1. *For a convex mapping $f : [a,b] \to \mathbb{R}$, the following inequality holds for each $x_j \in [a,b]$:*

$$f\left(a+b-\sum_{j=i}^{n} u_j x_j\right) \leq f(a)+f(b)-\sum_{j=1}^{n} u_j f\left(x_j\right),$$

where $u_j \in [0,1]$ and $\sum\limits_{j=1}^{n} u_j = 1$.

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After that, in 2013, Kian et al. [4] used this new Jensen inequality and established the following new versions of Hermite-Hadamard inequality:

Theorem 1.2. *For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold for all $x,y\in[a,b]$ and $x<y$:*

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq f(a)+f(b)-\frac{1}{y-x}\int_x^y f(u)\,du \\ &\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned} \tag{2}$$

and

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(u)\,du \\ &\leq \frac{f(a+b-x)+f(a+b-y)}{2} \\ &\leq f(a)+f(b)-\frac{f(x)+f(y)}{2}. \end{aligned} \tag{3}$$

Remark 1.3. *It is easy to see that the inequality (3) becomes the traditional Hermite-Hadamard inequality (1) for convex functions by setting $a=x$, $b=y$.*

For more recent inequalities related to (2) and (3), one can consult [5–10].

On the other hand, quantum calculus is a very important branch of calculus and it has a wide range of applications in the fields of mathematics and physics. Because of the numerous applications of quantum calculus (shortly, q-calculus) without limit calculus, many researchers began working on it and applying its concepts in areas such as differential equations, integral equalitions, mathematical modeling, and integral equations.

In [11–14], the authors used the q-integrals (defined in Section 2) to prove four different versions of q-Hermite-Hadamard inequalities and some estimates. The q-Hermite-Hadamard inequalities are described as:

Theorem 1.4. *[11, 12] For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold for $q\in(0,1)$:*

$$f\left(\frac{qa+b}{[2]_q}\right) \leq \frac{1}{b-a}\mathcal{I}_{a+}^{q}f(b)\leq\frac{qf(a)+f(b)}{[2]_q}, \tag{4}$$

$$f\left(\frac{a+qb}{[2]_q}\right) \leq \frac{1}{b-a}\mathcal{I}_{b-}^{q}f(a)\leq\frac{f(a)+qf(b)}{[2]_q}, \tag{5}$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\,{}^{T}\mathcal{I}_{a+}^{q}f(b)\leq\frac{f(a)+f(b)}{2}, \tag{6}$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\,{}^{T}\mathcal{I}_{b-}^{q}f(a)\leq\frac{f(a)+f(b)}{2}. \tag{7}$$

Remark 1.5. *It is very easy to observe that by adding (4) and (5), we have the following q-Hermite-Hadamard inequality (see, [12]):*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2(b-a)}\left[\mathcal{I}_{a+}^{q}f(b)+\mathcal{I}_{b-}^{q}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{8}$$

Remark 1.6. *We also observe that by adding (6) and (7), we get the following inequality:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2(b-a)}\left[{}^{T}\mathcal{I}_{a+}^{q}f(b)+\,{}^{T}\mathcal{I}_{b-}^{q}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{9}$$

Recently, Ali et al. [15] and Sitthiwirattham et al. [16] used new techniques to prove the following two different and new versions of Hermite-Hadamard type inequalities:

Theorem 1.7. *[15, 16] For a convex mapping $f : [a,b] \to \mathbb{R}$, the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\mathcal{I}^{q}_{\frac{a+b}{2}-}f(a) + \mathcal{I}^{q}_{\frac{a+b}{2}+}f(b)\right] \leq \frac{f(a)+f(b)}{2}, \tag{10}$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\mathcal{I}^{q}_{a+}f\left(\frac{a+b}{2}\right) + \mathcal{I}^{q}_{b-}f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}. \tag{11}$$

Remark 1.8. *By setting the limit as $q \to 1^-$ in (4)-(11), we recapture the traditional Hermite-Hadamard inequality (1).*

There has been much research done in the direction of q-integral inequalities for different kinds of convexities. For example, in [17], some new midpoint and trapezoidal type inequalities for q-integrals and q-differentiable convex functions were established. The authors of [18–22] used q-integral and established Simpson's type inequalities for q-differentiable convex and general convex functions. For more recent inequalities in q-calculus, one can consult [23–28].

Inspired by these ongoing studies, we consider Jensen-Mercer inequality and establish q-Hermite-Hadamard inequalities using ${}_aT_q$ and ${}^bT_q$ integrals. The main edge of these inequalities is that they can be converted into some existing and new inequalities for different choices of $q \in (0,1)$ and $\lambda \in (0,1]$.

2. Preliminaries of q-Calculus and Some Inequalities

We shall recall some basics of quantum calculus in this section. For the sake of brevity, let $q \in (0,1)$ and we use the following notation (see, [29]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}.$$

Definition 2.1. *[28] The left or q_a-derivative of $f : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is expressed as:*

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \ x \neq a. \tag{12}$$

Definition 2.2. *[12] The right or q^b-derivative of $f : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is expressed as:*

$${}^bD_q f(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}, \ x \neq b.$$

Definition 2.3. *[28] The left or q_a-integral of $f : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is defined as:*

$$\begin{aligned}\mathcal{I}^{q}_{a+}f(x) &= \int_a^x f(t)\ {}_ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &= (b-a)\int_0^1 f(tx + (1-t)a)\ {}_0d_qt.\end{aligned}$$

Definition 2.4. *[11] The right or q^b-integral of $f : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is defined as:*

$$\begin{aligned}\mathcal{I}^{q}_{b-}f(x) &= \int_x^b f(t)\ {}^bd_qt = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b) \\ &= (b-a)\int_0^1 f(tx + (1-t)b)\ {}_0d_qt.\end{aligned}$$

Definition 2.5. *[13] The ${}_aT_q$-integral of $f:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is defined as:*

$$\begin{aligned}{}^{T}\mathcal{I}^{q}_{a+}f(x) &= \int_a^x f(t)\ {}_ad^T_qt \\ &= \frac{(1-q)(x-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)a)-f(x)\right] \\ &= (b-a)\int_0^1 f(tx+(1-t)a)\ {}_0d^T_qt.\end{aligned}$$

Definition 2.6. *[14] The ${}^bT_q$-integral of $f:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is defined as:*

$$\begin{aligned}{}^{T}\mathcal{I}^{q}_{b-}f(x) &= \int_x^b f(t)\ {}^bd^T_qt \\ &= \frac{(1-q)(b-x)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)b)-f(x)\right] \\ &= (b-a)\int_0^1 f(tx+(1-t)b)\ {}_0d^T_qt.\end{aligned}$$

3. q-Hermite–Hadamard-Mercer Inequalities

In this section, we prove two new and different versions of general q-Hermite-Hadamard-Mercer type inequalities.

Theorem 3.1. *For a convex function $f:[a,b]\to\mathbb{R}$, the following inequalities hold:*

$$\begin{aligned}&f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & f(a)+f(b)-\frac{1}{2\lambda(y-x)}\left[{}^{T}\mathcal{I}^{q}_{x+}f(\lambda y+(1-\lambda)x)+{}^{T}\mathcal{I}^{q}_{y-}f(\lambda x+(1-\lambda)y)\right] \\ \leq\ & f(a)+f(b)-f\left(\frac{x+y}{2}\right)\end{aligned} \tag{13}$$

and

$$\begin{aligned}&f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2\lambda(y-x)}\left[\begin{array}{c}{}^{T}\mathcal{I}^{q}_{(a+b-y)+}f(\lambda(a+b-x)+(1-\lambda)(a+b-y)) \\ +\,{}^{T}\mathcal{I}_{(a+b-x)-}f(\lambda(a+b-y)+(1-\lambda)(a+b-x))\end{array}\right] \\ \leq\ & \frac{f(a+b-x)+f(a+b-y)}{2} \\ \leq\ & f(a)+f(b)-\frac{f(x)+f(y)}{2}\end{aligned} \tag{14}$$

for all $x,y\in[a,b]$ and $x\leq y$.

Proof. From the Jensen-Mercer inequality, we have

$$f\left(a+b-\frac{u+v}{2}\right)\leq f(a)+f(b)-\frac{1}{2}\left[f(u)+f(v)\right]. \tag{15}$$

By setting $u = t(\lambda x + (1-\lambda) y) + (1-t) y$ and $v = t(\lambda y + (1-\lambda) x) + (1-t) x$, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & f(a)+f(b)-\frac{1}{2}\left[f(t(\lambda x+(1-\lambda) y)+(1-t) y)+f(t(\lambda y+(1-\lambda) x)+(1-t) x)\right]. \end{aligned} \tag{16}$$

T_q-Integrating (16) with respect to t over $[0,1]$, also from Definitions 2.5 and 2.6, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & f(a)+f(b)-\frac{1}{2}\left[\int_0^1 (f(t(\lambda x+(1-\lambda) y)+(1-t) y)+f(t(\lambda y+(1-\lambda) x)+(1-t) x))\, d_q^T t\right] \\ =\ & f(a)+f(b)-\frac{1}{2\lambda(y-x)}\left[{}^T\mathcal{I}_{x+}^q f(\lambda y+(1-\lambda) x)+{}^T\mathcal{I}_{y-}^q f(\lambda x+(1-\lambda) y)\right]. \end{aligned}$$

Thus, the first inequality in (13) is proved. To prove the second inequality, we use the inequalities (6) and (7), we have

$$\frac{1}{2\lambda(y-x)}\left[{}^T\mathcal{I}_{x+}^q f(\lambda y+(1-\lambda) x)+{}^T\mathcal{I}_{y-}^q f(\lambda x+(1-\lambda) y)\right] \geq f\left(\frac{x+y}{2}\right),$$

which implies that

$$\begin{aligned} & f(a)+f(b)-\frac{1}{2\lambda(y-x)}\left[{}^T\mathcal{I}_{x+}^q f(\lambda y+(1-\lambda) x)+{}^T\mathcal{I}_{y-}^q f(\lambda x+(1-\lambda) y)\right] \\ \leq\ & f(a)+f(b)-f\left(\frac{x+y}{2}\right). \end{aligned}$$

Thus, the proof of inequality (13) is completed.

Now we prove the inequality (14), from the convexity, we have

$$\begin{aligned} f\left(a+b-\frac{u+v}{2}\right) &= f\left(\frac{a+b-u+a+b-v}{2}\right) \\ &\leq \frac{1}{2}\left[f(a+b-u)+f(a+b-v)\right]. \end{aligned}$$

By setting $u = t(\lambda(a+b-y)+(1-\lambda)(a+b-x))+(1-t)(a+b-x)$ and $v = t(\lambda(a+b-x)+(1-\lambda)(a+b-y))+(1-t)(a+b-y)$, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2}\left[f(t(\lambda(a+b-y)+(1-\lambda)(a+b-x))+(1-t)(a+b-x))\right. \\ & \left.+f(t(\lambda(a+b-x)+(1-\lambda)(a+b-y))+(1-t)(a+b-y))\right]. \end{aligned} \tag{17}$$

T_q-Integrating (17) with respect to t over $[0,1]$, also from Definitions 2.5 and 2.6, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2\lambda(y-x)}\left[\begin{array}{c} {}^T\mathcal{I}_{(a+b-y)+}^q f(\lambda(a+b-x)+(1-\lambda)(a+b-y)) \\ \\ +\,{}^T\mathcal{I}_{(a+b-x)-} f(\lambda(a+b-y)+(1-\lambda)(a+b-x)) \end{array}\right]. \end{aligned}$$

Thus, the first inequality in (14) is proved. We again use convexity to prove the second inequality in (14) as follows:

$$\begin{aligned} & f(t(\lambda(a+b-y)+(1-\lambda)(a+b-x))+(1-t)(a+b-x)) \\ \leq\; & t[\lambda f(a+b-y)+(1-\lambda) f(a+b-x)]+(1-t) f(a+b-x) \end{aligned} \tag{18}$$

and

$$\begin{aligned} & f(t(\lambda(a+b-x)+(1-\lambda)(a+b-y))+(1-t)(a+b-y)) \\ \leq\; & t[\lambda f(a+b-x)+(1-\lambda) f(a+b-y)]+(1-t) f(a+b-y). \end{aligned} \tag{19}$$

By applying convexity after adding (18) and (19), we have

$$\begin{aligned} & f(t(\lambda(a+b-y)+(1-\lambda)(a+b-x))+(1-t)(a+b-x)) \\ & +f(t(\lambda(a+b-x)+(1-\lambda)(a+b-y))+(1-t)(a+b-y)) \\ \leq\; & f(a+b-x)+f(a+b-y) \\ \leq\; & 2[f(a)+f(b)]-[f(x)+f(y)]. \end{aligned} \tag{20}$$

Thus, we obtain the required inequality by T_q-Integrating (20) with respect to t over $[0,1]$ and from Definitions 2.5 and 2.6. □

Remark 3.2. *In Theorem 3.1, if we set $\lambda = 1$, then we recapture the following inequalities*

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) & \leq f(a)+f(b)-\frac{1}{2(y-x)}\left[{}^{T}\mathcal{I}_{x+}^{q} f(y)+{}^{T}\mathcal{I}_{y-}^{q} f(x)\right] \\ & \leq f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned}$$

and

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\; & \frac{1}{2(y-x)}\left[{}^{T}\mathcal{I}_{(a+b-y)+}^{q} f(a+b-x)+{}^{T}\mathcal{I}_{(a+b-x)-} f(a+b-y)\right] \\ \leq\; & \frac{f(a+b-x)+f(a+b-y)}{2} \\ \leq\; & f(a)+f(b)-\frac{f(x)+f(y)}{2}. \end{aligned}$$

These inequalities are established by Gulshan et al. in [30].

Remark 3.3. *In Theorem 3.1, if we set $\lambda = 1$, $x = a$ and $y = b$, then we recapture the inequality (9).*

Remark 3.4. *In Theorem 3.1, if we set $\lambda = 1$ and the limit as $q \to 1^-$, then we recapture the inequalities (2) and (3).*

Corollary 3.5. *In Theorem 3.1, if we set $\lambda = \frac{1}{2}$, then we obtain the following new Hermite-Hadamard-Mercer type inequalities:*

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) & \leq f(a)+f(b)-\frac{1}{(y-x)}\left[{}^{T}\mathcal{I}_{x+}^{q} f\left(\frac{x+y}{2}\right)+{}^{T}\mathcal{I}_{y-}^{q} f\left(\frac{x+y}{2}\right)\right] \\ & \leq f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned}$$

and

$$
\begin{aligned}
& f\left(a+b-\frac{x+y}{2}\right) \\
\leq\; & \frac{1}{(y-x)}\left[{}^{T}\mathcal{I}^{q}_{(a+b-y)+}f\left(a+b-\frac{x+y}{2}\right)+{}^{T}\mathcal{I}_{(a+b-x)-}f\left(a+b-\frac{x+y}{2}\right)\right] \\
\leq\; & \frac{f(a+b-x)+f(a+b-y)}{2} \\
\leq\; & f(a)+f(b)-\frac{f(x)+f(y)}{2}.
\end{aligned}
$$

Corollary 3.6. *In Theorem 3.1, if we set $\lambda=\frac{1}{3}$, then we obtain the following new Hermite-Hadamard-Mercer type inequalities:*

$$
\begin{aligned}
f\left(a+b-\frac{x+y}{2}\right) &\leq f(a)+f(b)-\frac{3}{2(y-x)}\left[{}^{T}\mathcal{I}^{q}_{x+}f\left(\frac{2x+y}{3}\right)+{}^{T}\mathcal{I}^{q}_{y-}f\left(\frac{x+2y}{3}\right)\right] \\
&\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& f\left(a+b-\frac{x+y}{2}\right) \\
\leq\; & \frac{3}{2(y-x)}\left[{}^{T}\mathcal{I}^{q}_{(a+b-y)+}f\left(a+b-\frac{x+2y}{3}\right)+{}^{T}\mathcal{I}_{(a+b-x)-}f\left(a+b-\frac{2x+y}{3}\right)\right] \\
\leq\; & \frac{f(a+b-x)+f(a+b-y)}{2} \\
\leq\; & f(a)+f(b)-\frac{f(x)+f(y)}{2}.
\end{aligned}
$$

Corollary 3.7. *In Theorem 3.1, if we set $\lambda=\frac{1}{4}$, then we obtain the following new Hermite-Hadamard-Mercer type inequalities:*

$$
\begin{aligned}
f\left(a+b-\frac{x+y}{2}\right) &\leq f(a)+f(b)-\frac{2}{y-x}\left[{}^{T}\mathcal{I}^{q}_{x+}f\left(\frac{3x+y}{4}\right)+{}^{T}\mathcal{I}^{q}_{y-}f\left(\frac{x+3y}{4}\right)\right] \\
&\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& f\left(a+b-\frac{x+y}{2}\right) \\
\leq\; & \frac{2}{y-x}\left[{}^{T}\mathcal{I}^{q}_{(a+b-y)+}f\left(a+b-\frac{x+3y}{4}\right)+{}^{T}\mathcal{I}_{(a+b-x)-}f\left(a+b-\frac{3x+y}{4}\right)\right] \\
\leq\; & \frac{f(a+b-x)+f(a+b-y)}{2} \\
\leq\; & f(a)+f(b)-\frac{f(x)+f(y)}{2}.
\end{aligned}
$$

Remark 3.8. *It is worth mentioning here that we can obtain infinite new Hermite–Hadamard-Mercer type inequalities from Theorem 3.1 for different choices of $\lambda\in(0,1]$.*

Theorem 3.9. *For a convex function $f:[a,b]\rightarrow\mathbb{R}$, the following inequalities hold:*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & f(a)+f(b)-\frac{1}{2\lambda(y-x)}\left[{}^{T}\mathcal{I}^{q}_{(\lambda y+(1-\lambda)x)-}f(x)+{}^{T}\mathcal{I}^{q}_{(\lambda x+(1-\lambda)y)+}f(y)\right] \\ \leq\ & f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned} \tag{21}$$

and

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2\lambda(y-x)}\left[\begin{array}{c} {}^{T}\mathcal{I}^{q}_{(\lambda(a+b-x)+(1-\lambda)(a+b-y))-}f(a+b-y) \\ +{}^{T}\mathcal{I}^{q}_{(\lambda(a+b-y)+(1-\lambda)(a+b-x))+}f(a+b-x) \end{array}\right] \\ \leq\ & \frac{f(a+b-x)+f(a+b-y)}{2} \\ \leq\ & f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{22}$$

for all $x,y\in[a,b]$ and $x\leq y$.

Proof. By the Jensen-Mercer inequality, we have

$$f\left(a+b-\frac{u+v}{2}\right)\leq f(a)+f(b)-\frac{1}{2}\left[f(u)+f(v)\right].$$

By setting $u=tx+(1-t)(\lambda y+(1-\lambda)x)$ and $v=ty+(1-t)(\lambda x+(1-\lambda)y)$, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & f(a)+f(b)-\frac{1}{2}\left[f(tx+(1-t)(\lambda y+(1-\lambda)x))+f(ty+(1-t)(\lambda x+(1-\lambda)y))\right]. \end{aligned} \tag{23}$$

By applying Definitions 2.5 and 2.6 after T_q-integrating (23) over $[0,1]$ with respect to t, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & f(a)+f(b)-\frac{1}{2}\left[\int_0^1\left(f(tx+(1-t)(\lambda y+(1-\lambda)x))+f(ty+(1-t)(\lambda x+(1-\lambda)y))\right)\, d_q^T t\right] \\ =\ & f(a)+f(b)-\frac{1}{2\lambda(y-x)}\left[{}^{T}\mathcal{I}^{q}_{(\lambda y+(1-\lambda)x)-}f(x)+{}^{T}\mathcal{I}^{q}_{(\lambda x+(1-\lambda)y)+}f(y)\right]. \end{aligned}$$

Hence, we get the first inequality in (13). We use inequality (7) to prove the second inequality of (21) as follows:

$$\frac{1}{2\lambda(y-x)}\left[{}^{T}\mathcal{I}^{q}_{(\lambda y+(1-\lambda)x)-}f(x)+{}^{T}\mathcal{I}^{q}_{(\lambda x+(1-\lambda)y)+}f(y)\right]\geq f\left(\frac{x+y}{2}\right),$$

which implies that

$$\begin{aligned} & f(a)+f(b)-\frac{1}{2\lambda(y-x)}\left[{}^{T}\mathcal{I}^{q}_{(\lambda y+(1-\lambda)x)-}f(x)+{}^{T}\mathcal{I}^{q}_{(\lambda x+(1-\lambda)y)+}f(y)\right] \\ \leq\ & f(a)+f(b)-f\left(\frac{x+y}{2}\right). \end{aligned}$$

Hence, the inequality (21) proof is now completed.

We apply convexity to prove (22) and we have

$$\begin{aligned} f\left(a+b-\frac{u+v}{2}\right) &= f\left(\frac{a+b-u+a+b-v}{2}\right) \\ &\leq \frac{1}{2}\left[f(a+b-u)+f(a+b-v)\right]. \end{aligned}$$

By setting $u = t(a+b-y) + (1-t)(\lambda(a+b-x) + (1-\lambda)(a+b-y))$ and $v = t(a+b-x) + (1-t)(\lambda(a+b-y) + (1-\lambda)(a+b-x))$, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2}\left[f(t(a+b-y)+(1-t)(\lambda(a+b-x)+(1-\lambda)(a+b-y)))\right. \\ & \left.+f(t(a+b-x)+(1-t)(\lambda(a+b-y)+(1-\lambda)(a+b-x)))\right]. \end{aligned} \tag{24}$$

By applying Definitions 2.5 and 2.6 after T_q-integrating (24) over $[0,1]$ with respect to t, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2\lambda(y-x)}\left[\begin{array}{c} {}^{T}\mathcal{I}^{q}_{(\lambda(a+b-x)+(1-\lambda)(a+b-y))-}f(a+b-y) \\ +\ {}^{T}\mathcal{I}^{q}_{(\lambda(a+b-y)+(1-\lambda)(a+b-x))+}f(a+b-x) \end{array}\right]. \end{aligned}$$

Hence, we get the first inequality in (21). We again use convexity to prove the second inequality in (21) as follows:

$$\begin{aligned} & f(t(a+b-y)+(1-t)(\lambda(a+b-x)+(1-\lambda)(a+b-y))) \\ \leq\ & tf(a+b-y)+(1-t)\left[\lambda f(a+b-x)+(1-\lambda)f(a+b-y)\right] \end{aligned} \tag{25}$$

and

$$\begin{aligned} & f(t(a+b-x)+(1-t)(\lambda(a+b-y)+(1-\lambda)(a+b-x))) \\ \leq\ & tf(a+b-x)+(1-t)\left[\lambda f(a+b-y)+(1-\lambda)f(a+b-x)\right]. \end{aligned} \tag{26}$$

By applying Jensen-Mercer inequality after adding (25) and (26), we have

$$\begin{aligned} & f(t(a+b-y)+(1-t)(\lambda(a+b-x)+(1-\lambda)(a+b-y))) \\ & +f(t(a+b-x)+(1-t)(\lambda(a+b-y)+(1-\lambda)(a+b-x))) \\ \leq\ & f(a+b-x)+f(a+b-y) \\ \leq\ & 2\left[f(a)+f(b)\right]-\left[f(x)+f(y)\right]. \end{aligned}$$

Thus, we obtain the required inequality by applying Definitions 2.5 and 2.6 after T_q-integrating (20) over $[0,1]$ with respect to t. □

Corollary 3.10. *In Theorem 3.9, if we set $\lambda = 1$, then we recapture the following inequalities*

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq f(a)+f(b)-\frac{1}{2(y-x)}\left[{}^{T}\mathcal{I}^{q}_{y-}f(x)+\ {}^{T}\mathcal{I}^{q}_{x+}f(y)\right] \\ &\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned}$$

and

$$
\begin{aligned}
& f\left(a+b-\frac{x+y}{2}\right) \\
\leq\ & \frac{1}{2(y-x)}\left[{}^{T}\mathcal{I}^{q}_{(a+b-x)-}f(a+b-y)+{}^{T}\mathcal{I}^{q}_{(a+b-y)+}f(a+b-x)\right] \\
\leq\ & \frac{f(a+b-x)+f(a+b-y)}{2} \\
\leq\ & f(a)+f(b)-\frac{f(x)+f(y)}{2}.
\end{aligned}
$$

Remark 3.11. *In Theorem 3.9, if we set $\lambda = 1$, $x = a$ and $y = b$, then we recapture the inequality (9).*

Remark 3.12. *In Theorem 3.9, if we set $\lambda = 1$ and the limit as $q \to 1^-$, then we recapture the inequalities (2) and (3).*

Corollary 3.13. *In Theorem 3.9, if we set $\lambda = \frac{1}{2}$, then we obtain the following new Hermite-Hadamard-Mercer type inequalities:*

$$
\begin{aligned}
f\left(a+b-\frac{x+y}{2}\right) &\leq f(a)+f(b)-\frac{1}{y-x}\left[{}^{T}\mathcal{I}^{q}_{\left(\frac{x+y}{2}\right)-}f(x)+{}^{T}\mathcal{I}^{q}_{\left(\frac{x+y}{2}\right)+}f(y)\right] \\
&\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& f\left(a+b-\frac{x+y}{2}\right) \\
\leq\ & \frac{1}{y-x}\left[{}^{T}\mathcal{I}^{q}_{\left(a+b-\frac{x+y}{2}\right)-}f(a+b-y)+{}^{T}\mathcal{I}^{q}_{\left(a+b-\frac{x+y}{2}\right)+}f(a+b-x)\right] \\
\leq\ & \frac{f(a+b-x)+f(a+b-y)}{2} \\
\leq\ & f(a)+f(b)-\frac{f(x)+f(y)}{2}.
\end{aligned}
$$

Corollary 3.14. *In Theorem 3.9, if we set $\lambda = \frac{1}{3}$, then we obtain the following new Hermite-Hadamard-Mercer type inequalities:*

$$
\begin{aligned}
f\left(a+b-\frac{x+y}{2}\right) &\leq f(a)+f(b)-\frac{3}{2(y-x)}\left[{}^{T}\mathcal{I}^{q}_{\left(\frac{2x+y}{3}\right)-}f(x)+{}^{T}\mathcal{I}^{q}_{\left(\frac{x+2y}{3}\right)+}f(y)\right] \\
&\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& f\left(a+b-\frac{x+y}{2}\right) \\
\leq\ & \frac{3}{2(y-x)}\left[{}^{T}\mathcal{I}^{q}_{\left(a+b-\frac{x+2y}{3}\right)-}f(a+b-y)+{}^{T}\mathcal{I}^{q}_{\left(a+b-\frac{2x+y}{3}\right)+}f(a+b-x)\right] \\
\leq\ & \frac{f(a+b-x)+f(a+b-y)}{2} \\
\leq\ & f(a)+f(b)-\frac{f(x)+f(y)}{2}.
\end{aligned}
$$

Corollary 3.15. *In Theorem 3.9, if we set $\lambda = \frac{1}{4}$, then we obtain the following new Hermite-Hadamard-Mercer type inequalities:*

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\le f(a)+f(b)-\frac{2}{y-x}\left[{}^{T}\mathcal{I}^{q}_{\left(\frac{3x+y}{4}\right)-}f(x)+{}^{T}\mathcal{I}^{q}_{\left(\frac{x+3y}{4}\right)+}f(y)\right] \\ &\le f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned}$$

and

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \le\ & \frac{2}{y-x}\left[{}^{T}\mathcal{I}^{q}_{\left(a+b-\frac{x+3y}{4}\right)-}f(a+b-y)+{}^{T}\mathcal{I}^{q}_{\left(a+b-\frac{3x+y}{4}\right)+}f(a+b-x)\right] \\ \le\ & \frac{f(a+b-x)+f(a+b-y)}{2} \\ \le\ & f(a)+f(b)-\frac{f(x)+f(y)}{2}. \end{aligned}$$

Remark 3.16. *It is worth to mention here that we can obtain infinite new Hermite–Hadamard-Mercer type inequalities from Theorem 3.9 for different choices of $\lambda \in (0,1]$.*

4. Examples

In this section, we give some mathematical examples to show the validation of established inequalities in last section for different values of λ.

Example 4.1. *Let us consider the convex function $f:[1,2]\to\mathbb{R}$ defined by $f(t)=t^2$ and let $x=\frac{4}{3}$ and $y=\frac{5}{3}$. Under these assumptions, we have*

$$f\left(a+b-\frac{x+y}{2}\right)=\frac{9}{4}$$

and

$$f(a)+f(b)-f\left(\frac{x+y}{2}\right)=\frac{11}{4}.$$

By Definition 2.5, we get

$$\begin{aligned} {}^{T}\mathcal{I}^{q}_{x+}f(\lambda y+(1-\lambda)x) &= \int_{\frac{4}{3}}^{\frac{4}{3}+\frac{\lambda}{3}} t^2 \ {}_{\frac{4}{3}}d^{T}_{q}t \\ &= \frac{(1-q)\lambda}{6q}\left[(1+q)\sum_{n=0}^{\infty}q^n\left(q^n\left(\frac{4}{3}+\frac{\lambda}{3}\right)+(1-q^n)\frac{4}{3}\right)^2-\left(\frac{4}{3}+\frac{\lambda}{3}\right)^2\right] \\ &= \frac{(1-q)\lambda}{54q}\left[(1+q)\sum_{n=0}^{\infty}q^n\left(\lambda^2q^{2n}+8\lambda q^n+16\right)-\left(\lambda^2+8\lambda+16\right)\right] \\ &= \frac{(1-q)\lambda}{54q}\left[(1+q)\left(\frac{\lambda^2}{1-q^3}+\frac{8\lambda}{1-q^2}+\frac{16}{1-q}\right)-\left(\lambda^2+8\lambda+16\right)\right] \\ &= \frac{\lambda}{54q}\left(\frac{\lambda^2(1+q)}{1+q+q^2}+8\lambda q+32q-\lambda^2(1-q)\right) \end{aligned}$$

and similarly by Definition 2.6, we have

$$\begin{aligned} {}^{T}\mathcal{I}^{q}_{y-}f(\lambda x+(1-\lambda)y) &= \int_{\frac{5}{3}-\frac{\lambda}{3}}^{\frac{5}{3}} t^{2}\ {}^{\frac{5}{3}}d_{q}^{T}t \\ &= \frac{\lambda}{54q}\left(\frac{\lambda^{2}(1+q)}{1+q+q^{2}}-10\lambda q+50q-\lambda^{2}(1-q)\right). \end{aligned}$$

Thus, the middle term of the inequality (13) reduces to

$$\begin{aligned} & f(a)+f(b)-\frac{1}{2\lambda(y-x)}\left[{}^{T}\mathcal{I}^{q}_{x+}f(\lambda y+(1-\lambda)x)+\ {}^{T}\mathcal{I}^{q}_{y-}f(\lambda x+(1-\lambda)y)\right] \\ =\ & 5-\frac{\lambda}{18q}\left(\frac{\lambda^{2}(1+q)}{1+q+q^{2}}+(41-\lambda)q-\lambda^{2}(1-q)\right). \end{aligned}$$

By the inequality (13), we have

$$\frac{9}{4}\leq 5-\frac{\lambda}{18q}\left(\frac{\lambda^{2}(1+q)}{1+q+q^{2}}+(41-\lambda)q-\lambda^{2}(1-q)\right)\leq\frac{9}{4}. \tag{27}$$

One can see the validity of the inequality (27) for some values of λ in Figure 1.

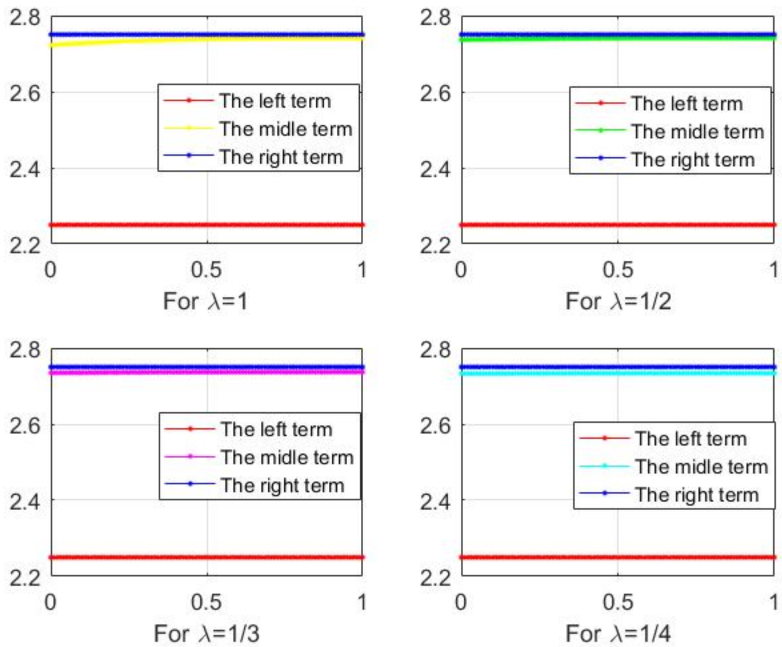

Figure 1: An example to the inequality (13)

Example 4.2. *Let us consider the convex function $f:[0,1]\to\mathbb{R}$ defined by $f(t)=t^{2}$ and let $x=\frac{1}{3}$ and $y=\frac{2}{3}$. Under these assumptions, we have*

$$f\left(a+b-\frac{x+y}{2}\right)=\frac{1}{4}$$

and

$$\frac{f(a+b-x)+f(a+b-y)}{2}=\frac{5}{18}.$$

By Definition 2.5, we get

$$\begin{aligned}
& {}^{T}\mathcal{I}^{q}_{(a+b-y)+}f(\lambda(a+b-x)+(1-\lambda)(a+b-y)) \\
= & \int_{\frac{1}{3}}^{\frac{1}{3}+\frac{\lambda}{3}} t^2 \,{}_{\frac{1}{3}}d_q^T t \\
= & \frac{(1-q)\lambda}{6q}\left[(1+q)\sum_{n=0}^{\infty}q^n\left(q^n\left(\frac{1}{3}+\frac{\lambda}{3}\right)+(1-q^n)\frac{1}{3}\right)^2-\left(\frac{1}{3}+\frac{\lambda}{3}\right)^2\right] \\
= & \frac{(1-q)\lambda}{54q}\left[(1+q)\sum_{n=0}^{\infty}q^n\left(\lambda^2q^{2n}+2\lambda q^n+1\right)-\left(\lambda^2+2\lambda+1\right)\right] \\
= & \frac{(1-q)\lambda}{54q}\left[(1+q)\left(\frac{\lambda^2}{1-q^3}+\frac{2\lambda}{1-q^2}+\frac{1}{1-q}\right)-\left(\lambda^2+2\lambda+1\right)\right] \\
= & \frac{\lambda}{54q}\left(\frac{\lambda^2(1+q)}{1+q+q^2}+2\lambda q+2q-\lambda^2(1-q)\right).
\end{aligned}$$

Similarly, by Definition 2.6, we have

$$\begin{aligned}
& {}^{T}\mathcal{I}_{(a+b-x)-}f(\lambda(a+b-y)+(1-\lambda)(a+b-x)) \\
= & \int_{\frac{2}{3}-\frac{\lambda}{3}}^{\frac{2}{3}} t^2 \,{}^{\frac{2}{3}}d_q^T t \\
= & \frac{\lambda}{54q}\left(\frac{\lambda^2(1+q)}{1+q+q^2}-4\lambda q+8q-\lambda^2(1-q)\right).
\end{aligned}$$

Thus the middle term of the inequality (14) reduces to

$$\begin{aligned}
& \frac{1}{2\lambda(y-x)}\left[\begin{array}{c} {}^{T}\mathcal{I}^{q}_{(a+b-y)+}f(\lambda(a+b-x)+(1-\lambda)(a+b-y)) \\ \\ +\,{}^{T}\mathcal{I}_{(a+b-x)-}f(\lambda(a+b-y)+(1-\lambda)(a+b-x)) \end{array}\right] \\
= & \frac{\lambda}{18q}\left(\frac{\lambda^2(1+q)}{1+q+q^2}+(5-\lambda)q-\lambda^2(1-q)\right).
\end{aligned}$$

By the inequality (14), we have the inequality

$$\frac{9}{4}\le 5-\frac{\lambda}{18q}\left(\frac{\lambda^2(1+q)}{1+q+q^2}+(41-\lambda)q-\lambda^2(1-q)\right)\le\frac{9}{4}. \tag{28}$$

One can see the validity of the inequality (28) for some values of λ in Figure 2.

Example 4.3. *Let us consider the convex function f defined in Example 4.2 with $x=\frac{1}{4}$ and $y=\frac{1}{2}$. Then, we have*

$$f\left(a+b-\frac{x+y}{2}\right)=\frac{25}{64}$$

and

$$f(a)+f(b)-f\left(\frac{x+y}{2}\right)=\frac{55}{64}.$$

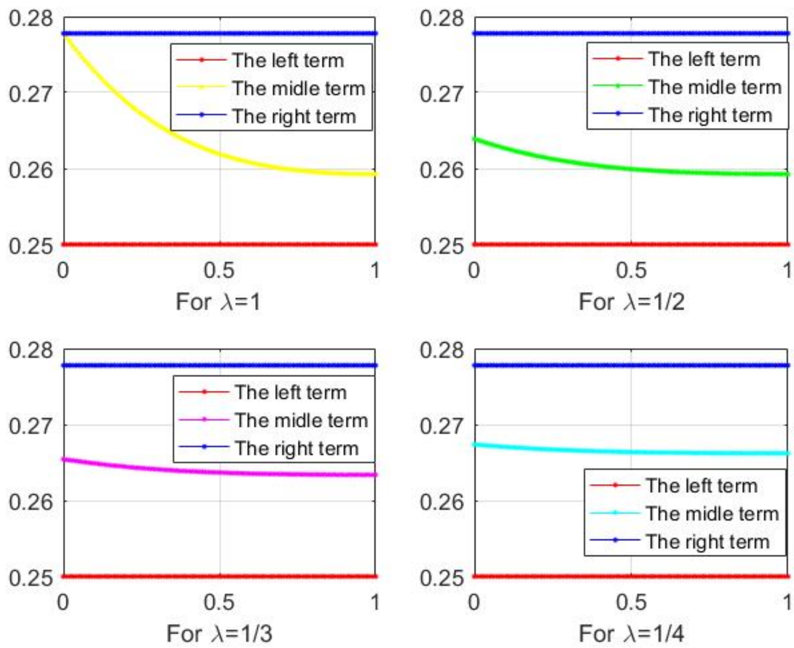

Figure 2: An example to the inequality (14)

By Definition 2.6, we get

$$
\begin{aligned}
& {}^{T}\mathcal{I}^{q}_{(\lambda y+(1-\lambda)x)-}f(x) \\
= & \int_{\frac{1}{4}}^{\frac{1}{4}+\frac{\lambda}{4}} t^{2}\ {}_{\frac{1}{4}+\frac{\lambda}{4}}d_{q}^{T}t \\
= & \frac{(1-q)\lambda}{8q}\left[(1+q)\sum_{n=0}^{\infty}q^{n}\left(q^{n}\frac{1}{4}+(1-q^{n})\left(\frac{1}{4}+\frac{\lambda}{4}\right)\right)^{2}-\left(\frac{1}{4}\right)^{2}\right] \\
= & \frac{(1-q)\lambda}{128q}\left[(1+q)\sum_{n=0}^{\infty}q^{n}\left(\lambda^{2}q^{2n}-2\lambda(1+\lambda)q^{n}+(1+\lambda)^{2}\right)-1\right] \\
= & \frac{(1-q)\lambda}{128q}\left[(1+q)\left(\frac{\lambda^{2}}{1-q^{3}}-\frac{2\lambda(1+\lambda)}{1-q^{2}}+\frac{(1+\lambda)^{2}}{1-q}\right)-\left(\lambda^{2}+2\lambda+1\right)\right] \\
= & \frac{\lambda}{128q}\left(\frac{\lambda^{2}(1+q)}{1+q+q^{2}}-2\lambda(1+\lambda)+(1+\lambda)^{2}(1+q)-1+q\right).
\end{aligned}
$$

Similarly, by Definition 2.5, we have

$$
\begin{aligned}
& {}^{T}\mathcal{I}^{q}_{(\lambda x+(1-\lambda)y)+}f(y) \\
= & \int_{\frac{1}{2}-\frac{\lambda}{4}}^{\frac{1}{2}} t^{2}\ {}_{\frac{1}{2}-\frac{\lambda}{4}}d_{q}^{T}t \\
= & \frac{\lambda}{128q}\left(\frac{\lambda^{2}(1+q)}{1+q+q^{2}}+2\lambda(2-\lambda)+(2-\lambda)^{2}(1+q)-4(1-q)\right).
\end{aligned}
$$

Thus the middle term of the inequality (21) reduces to

$$f(a)+f(b)-\frac{1}{2\lambda(y-x)}\left[{}^{T}\mathcal{I}^{q}_{(\lambda y+(1-\lambda)x)-}f(x)+{}^{T}\mathcal{I}^{q}_{(\lambda x+(1-\lambda)y)+}f(y)\right]$$
$$1-\frac{\lambda}{64q}\left(\frac{2\lambda^{2}(1+q)}{1+q+q^{2}}+2\lambda(1-2\lambda)+\left((2-\lambda)^{2}+(1+\lambda)^{2}\right)(1+q)-5(1-q)\right).$$

By the inequality (21), we have the inequality

$$\frac{25}{64}\leq 1-\frac{\lambda}{64q}\left(\frac{2\lambda^{2}(1+q)}{1+q+q^{2}}+2\lambda(1-2\lambda)+\left((2-\lambda)^{2}+(1+\lambda)^{2}\right)(1+q)-5(1-q)\right)\leq\frac{55}{64}. \tag{29}$$

One can see the validity of the inequality (29) for some values of λ in Figure 3.

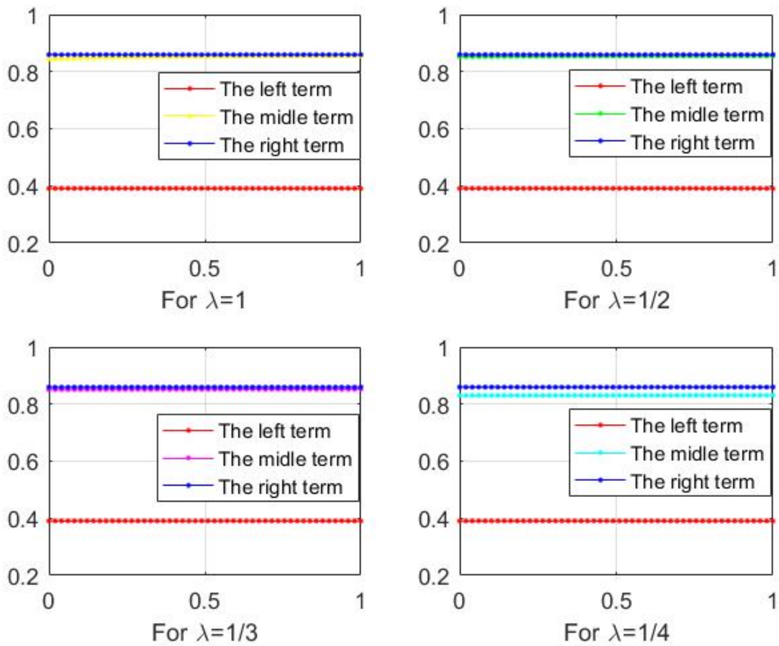

Figure 3: An example to the inequality (21)

Example 4.4. *Consider the convex function f defined in Example 4.1 with $x=\frac{1}{4}$ and $y=\frac{1}{2}$. Then, we have*

$$f\left(a+b-\frac{x+y}{2}\right)=\frac{169}{64}$$

and

$$\frac{f(a+b-x)+f(a+b-y)}{2}=\frac{85}{32}.$$

By Definition 2.6, we get

$$\begin{aligned}
& {}^{T}\mathcal{I}^{q}_{\left(\lambda(a+b-x)+(1-\lambda)\left(a+b-y\right)\right)-}f\left(a+b-y\right) \\
= & \int_{\frac{3}{2}}^{\frac{3}{2}+\frac{\lambda}{4}} t^{2} \ {}_{\frac{3}{2}+\frac{\lambda}{4}}d_{q}^{T}t \\
= & \frac{(1-q)\lambda}{8q}\left[(1+q)\sum_{n=0}^{\infty}q^{n}\left(q^{n}\frac{3}{2}+(1-q^{n})\left(\frac{3}{2}+\frac{\lambda}{4}\right)\right)^{2}-\left(\frac{3}{2}\right)^{2}\right] \\
= & \frac{(1-q)\lambda}{128q}\left[(1+q)\sum_{n=0}^{\infty}q^{n}\left(\lambda^{2}q^{2n}-2\lambda(6+\lambda)q^{n}+(6+\lambda)^{2}\right)-36\right] \\
= & \frac{(1-q)\lambda}{128q}\left[(1+q)\left(\frac{\lambda^{2}}{1-q^{3}}-\frac{2\lambda(6+\lambda)}{1-q^{2}}+\frac{(6+\lambda)^{2}}{1-q}\right)-36\right] \\
= & \frac{\lambda}{128q}\left(\frac{\lambda^{2}(1+q)}{1+q+q^{2}}-2\lambda(6+\lambda)+(6+\lambda)^{2}(1+q)-36(1-q)\right).
\end{aligned}$$

Similarly, by Definition 2.5, we have

$$\begin{aligned}
& {}^{T}\mathcal{I}^{q}_{\left(\lambda\left(a+b-y\right)+(1-\lambda)(a+b-x)\right)+}f\left(a+b-x\right) \\
= & \int_{\frac{7}{4}-\frac{\lambda}{4}}^{\frac{7}{4}} t^{2} \ {}_{\frac{7}{4}-\frac{\lambda}{4}}d_{q}^{T}t \\
= & \frac{\lambda}{128q}\left(\frac{\lambda^{2}(1+q)}{1+q+q^{2}}+2\lambda(7-\lambda)+(7-\lambda)^{2}(1+q)-79(1-q)\right).
\end{aligned}$$

Thus the middle term of the inequality (21) reduces to

$$\begin{aligned}
& \frac{1}{2\lambda(y-x)}\left[\begin{array}{c} {}^{T}\mathcal{I}^{q}_{\left(\lambda(a+b-x)+(1-\lambda)\left(a+b-y\right)\right)-}f\left(a+b-y\right) \\ +\,{}^{T}\mathcal{I}^{q}_{\left(\lambda\left(a+b-y\right)+(1-\lambda)(a+b-x)\right)+}f\left(a+b-x\right) \end{array}\right] \\
& \frac{\lambda}{64q}\left(\frac{2\lambda^{2}(1+q)}{1+q+q^{2}}+2\lambda(1-2\lambda)+\left((7-\lambda)^{2}+(6+\lambda)^{2}\right)(1+q)-85(1-q)\right).
\end{aligned}$$

By the inequality (21), we have the inequality

$$\frac{169}{64}\leq\frac{\lambda}{64q}\left(\frac{2\lambda^{2}(1+q)}{1+q+q^{2}}+2\lambda(1-2\lambda)+\left((7-\lambda)^{2}+(6+\lambda)^{2}\right)(1+q)-85(1-q)\right)\leq\frac{85}{32}. \quad (30)$$

One can see the validity of the inequality (30) for some values of λ in Figure 4.

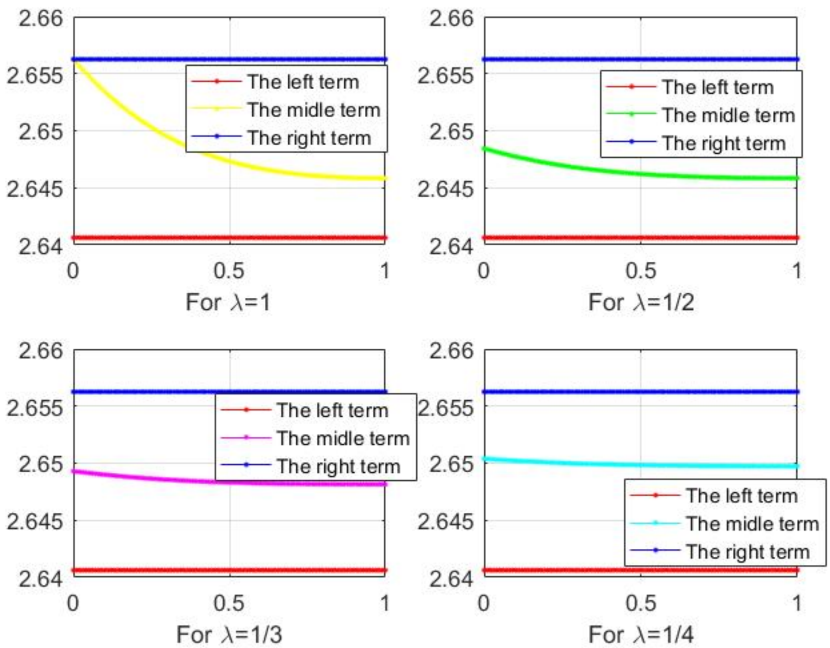

Figure 4: An example to the inequality (22)

5. Conclusion

In this paper, we established several new variants of q-Hermite–Hadamard-Mercer inequalities in the framework of q-calculus. We proved that the inequalities established here are the generalizations of existing inequalities inside the literature. We also studied some mathematical examples to assure that the newly established inequalities are true. It is interesting and new problem that the upcoming researchers can obtain similar inequalities for different kinds of convexity using the different q-integrals.

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