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On generalizations of post quantum midpoint and trapezoid type inequalities for (α, m)-convex functions

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Abstract. The aim of current study is to establish two crucial $(p,q)^b$-integral identities for midpoint and trapezoid type inequalities. Utilizing these identities, we developed some new variant of midpoint and trapezoid type integral inequalities of differential (α, m)-convex functions using right post quantum integral approach. Moreover, we have presented the application of derived results related to special means of positive real numbers.

1. Introduction

The notion of convexity is key to many branches of applied mathematics. Modern analysis devised the applications of convexity in various disciplines of engineering and mathematics. Convex functions are useful in the study of optimization theory and integral inequalities of convex functions has become an emerging area of research for last few decades. The concept of convexity has been generalized to a great extent and different types of convex functions like quasi-convex [1], (α, m) convex [2], h-convex [3], p-convex [4], *exp*-convex [5], *log*-convex [6], harmonically convex [7], E-convex [8], s-convex [9] etc. have been developed and analyzed thoroughly. A function $F : I \to \mathbb{R}$ is said to be convex if

$$F(\xi\varkappa + (1-\xi)\gamma) \leq \xi F(\varkappa) + (1-\xi)F(\gamma)$$

where $\varkappa, \gamma \in I$ and $\xi \in [0,1]$.

Mihesan introduced the class of (α, m)-convex functions and stated as:

Definition 1.1. *[10] A function $F : I \to \mathbb{R}$ is called (α, m)-convex, if the inequality*

$$F(\xi\varkappa + m(1-\xi)\gamma) \leq \xi^{\alpha} F(\varkappa) + m(1-\xi^{\alpha}) F(\gamma)$$

holds for all $\varkappa, \gamma \in I$, $\xi \in [0,1]$, $\alpha \in [0,1]$ and $m \in [0,1]$.

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It is notable that F is convex if and only if it satisfies the Hermite-Hadamard's inequality, stated below:

$$F\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} F(\varkappa)d\varkappa \leq \frac{F(\sigma)+F(\rho)}{2}$$

where $F : I \to \mathbb{R}$ is a convex function and $\sigma, \rho \in I$ with $\sigma < \rho$. The idea of convexity is substantially unified with quantum and post quantum calculus to develop generalized versions of integral inequalities [11]. Quantum and post quantum variants of Hermite Hadamard's inequalities, Simpson's type inequalities, Midpoint type inequalities, Trapezoid type inequalities, Newton's type inequalities etc. have been focused in recent research publications.

On the other hand, the work of Leonhard Euler (1707-1783) on newton's infinite series initiated the field of quantum calculus in early eighteenth century. Quantum calculus is also known as calculus without limits, it became popular after research publications of Albert Einstein in 1905. Later on, in 1910, F. Jackson thorough study of the subject developed many areas of q-calculus. Many researchers believe that quantum calculus is a bridge between mathematics and physics. It has lot of applications in field of number theory, combinatorics, cryptography, hypergeometric functions, mechanics, theory of quantum, theory of relativity etc. [12, 13]. In 1966, Al-Salam [14] introduced a q-analogue of the q-fractional integrals and q-Riemann-Liouville fractional. Thenceforth, the work provided foundation to latest area of research and increased the research in field of quantum fractional analysis. In particular, in 2013, Tariboon and Ntouyas introduced the left quantum difference operator and left quantum integral in [15]. In 2020, Bermudo et al. introduced the notion of right quantum derivative and right quantum integral in [3].

The post quantum calculus is the generalized version of quantum calculus. Quantum and post quantum integral inequalities have been explored by many researchers for different types of convexities. For example, in [16–22], the authors proved Hermite-Hadamard integral inequalities and their left-right estimates for integrals. In [23], the generalized version of q-integral inequalities was presented by Noor et al. In [24] Nwaeze et al. proved certain partametrized quantum integral inequalities for generalized quasi convex functions. Khan et al.proved Hermite-Hadamard inequality using the green function in [25]. For convex and co-ordinated convex functions, Budak et al. [26], Ali et al. [27, 28] and Vivas-Cortez et al. [29] developed new quantum Simpson's and Newton's type inequalities. For quantum Ostrowski's type inequalities for convex and co-ordinated convex functions, please refer to [30–32].

Motivated by ongoing research, we have developed some new variants of Midpoint and Trapezoid type inequalities for (α, m)-convex functions by utilizing $(p, q)^{\rho}$ integral of post quantum calculus . The obtained post quantum inequalities can be turned into quantum Midpoint and trapezoid type inequalities for convex functions [33]. Moreover, these quantum inequalities can be further reduced to classical Midpoint type [34] and the classical Trapezoid type inequalities for convex functions [35] without proving each of them separately.

The structure of this paper is as follows: Section 2 provides a brief overview of the fundamentals of (p, q)-calculus as well as related results of the field. Section 3 focuses on two major identities that are crucial in establishing the main results of the paper. The Midpoint and Trapezoid type integral inequalities for (p, q)-differentiable functions via (p, q)-integrals are presented in section 4 and section 5. The applications of main results to special means are discussed in section 6. Section 7 concludes the article with some feasible research suggestions for the future.

2. Preliminaries and definitions of (p, q)-calculus

In this section, we first present the definitions and some properties of quantum integrals. We also mention some well known inequalities for quantum integrals. Throughout this paper, we assumed $0 < q < 1$ and $0 < p \leq 1$ be the parameters.

The (p, q)-number or (p, q)-analogue of $n \in \mathbb{N}$ is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

The classical (p,q)- integral defined in [36] from 0 to ρ as follows:

$$\int_0^{\rho} F(\varkappa)d_{p,q}\varkappa = (p-q)\rho\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F(\rho\frac{q^n}{p^{n+1}})$$

provided the sum converges absolutely.

Definition 2.1. *[36] Let $F: I \to \mathbb{R}$ be a continuous function and let $\varkappa \in I$. Then the $(p,q)_\sigma$ derivative on I of F at $\varkappa$ is defined as*

$$_{\sigma}D_{p,q}F(\varkappa) = \frac{F(p\varkappa+(1-p)\sigma)-F(q\varkappa+(1-q)\sigma)}{(p-q)(\varkappa-\sigma)} \quad \varkappa \neq \sigma, \tag{1}$$

$$_{\sigma}D_{p,q}F(\sigma) = lim_{\varkappa\to\sigma}\ _{\sigma}D_{p,q}F(\varkappa).$$

If $\sigma = 0$ in (1), then we get classical (p,q)-derivative of $F(\varkappa)$ at $\varkappa \in I$, given by

$$_{0}D_{p,q}F(\varkappa) = D_{p,q}F(\varkappa) = \frac{F(p\varkappa)-F(q\varkappa)}{(p-q)\varkappa}.$$

Definition 2.2. *[36] Let $F: I \to \mathbb{R}$ be a continuous function. Then the $(p,q)_\sigma$-integral on I is defined as*

$$\int_{\sigma}^{\varkappa} F(\xi)_{\sigma}d_{p,q}\xi = (p-q)(\varkappa-\sigma)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F(\frac{q^n}{p^{n+1}}\varkappa+(1-\frac{q^n}{p^{n+1}})\sigma) \tag{2}$$

for $\varkappa \in I$. If $\sigma = 0$ in (2), then

$$\int_{0}^{\varkappa} F(\xi)_{0}d_{p,q}\xi = \int_{0}^{\varkappa} F(\xi)d_{p,q}\xi,$$

where $\int_0^{\varkappa} F(\xi)d_{p,q}\xi$ is familiar classical (p,q)-definite integral on $[0,\varkappa]$ defined by the expression

$$\int_{0}^{\varkappa} F(\xi)_{0}d_{p,q}\xi = \int_{0}^{\varkappa} F(\xi)d_{p,q}\xi = (p-q)\varkappa\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F(\frac{q^n}{p^{n+1}}\varkappa).$$

Moreover, if $c \in (\sigma,\varkappa)$, then the (p,q)-integral on I is defined as

$$\int_{c}^{\varkappa} F(\xi)_{\sigma}d_{p,q}\xi = \int_{\sigma}^{\varkappa} F(\xi)_{\sigma}d_{p,q}\xi - \int_{\sigma}^{c} F(\xi)_{\sigma}d_{p,q}\xi.$$

Theorem 2.3. *[36]If $F: [\sigma,\rho] \to \mathbb{R}$ is a continuous function and $z \in [\sigma,\rho]$, then the following identities hold:*

(i) $_{\sigma}D_{p,q}\int_{\sigma}^{z} F(\varkappa)_{\sigma}d_{p,q}\varkappa = F(z)$

(ii) $\int_{\sigma}^{z}\ _{\sigma}D_{p,q}F(\varkappa)_{\sigma}d_{p,q}\varkappa = F(z)$

(iii) $\int_{c}^{z}\ _{\sigma}D_{p,q}F(\varkappa)_{\sigma}d_{p,q}\varkappa = F(z)-F(c)$ *for* $c \in (\sigma,z)$

Definition 2.4. *[37] Let $F: I \to \mathbb{R}$ be a continuous function and let $\varkappa \in I$. Then the $(p,q)^\rho$ derivative on I of F at $\varkappa$ is defined as*

$$^{\rho}D_{p,q}F(\varkappa) = \frac{F(p\varkappa+(1-p)\rho)-F(q\varkappa+(1-q)\rho)}{(p-q)(\varkappa-\rho)} \quad \varkappa \neq \rho,$$

$$^{\rho}D_{p,q}F(\rho) = lim_{\varkappa\to\rho}\ ^{\rho}D_{p,q}F(\varkappa).$$

Definition 2.5. *[37] Let $F : I \rightarrow \mathbb{R}$ be a continuous function. Then the $(p,q)^{\rho}$-integral on I is defined as*

$$\int_{\varkappa}^{\rho} F(\xi)^{\rho} d_{p,q}\xi = (p-q)(\rho-\varkappa)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F(\frac{q^n}{p^{n+1}}\varkappa+(1-\frac{q^n}{p^{n+1}})\rho) \tag{3}$$

for $\varkappa \in I$. If $\rho = 1$ in (3), then

$$\int_{\varkappa}^{1} F(\xi)^{1} d_{p,q}\xi = \int_{\varkappa}^{1} F(\xi) d_{p,q}\xi,$$

where $\int_0^{\varkappa} F(\xi)d_{p,q}\xi$ is familiar (p,q)-definite integral on $[0,\varkappa]$ defined by the expression

$$\int_0^{\varkappa} F(\xi)_0 d_{p,q}\xi = \int_0^{\varkappa} F(\xi) d_{p,q}\xi = (p-q)\varkappa\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}F(\frac{q^n}{p^{n+1}}\varkappa).$$

Moreover, if $c \in (\sigma,\varkappa)$, then the (p,q)-integral on I is defined as

$$\int_c^{\varkappa} F(\xi)_{\sigma} d_{p,q}\xi = \int_{\sigma}^{\varkappa} F(\xi)_{\sigma} d_{p,q}\xi - \int_{\sigma}^{c} F(\xi)_{\sigma} d_{p,q}\xi.$$

In [38], M. Kunt et al. proved the corresponding Hermite-Hadamard inequalities for convex functions by using $(p,q)_{\sigma}$-integrals, which is given by

Theorem 2.6. *[38] If $F : [\sigma,\rho] \rightarrow \mathbb{R}$ be a convex differentiable function on $[\sigma,\rho]$ and $0 < q < p \leq 1$. Then, Hermite-Hadamard inequalities is given by*

$$F\left(\frac{q\sigma+p\rho}{[2]_{p,q}}\right) \leq \frac{1}{p(\rho-\sigma)}\int_{\sigma}^{p\rho+(1-p)\sigma} F(\xi) \ {}_{\sigma}d_q\xi \ \leq \frac{qF(\sigma)+pF(\rho)}{[2]_{p,q}}. \tag{4}$$

M.A. Ali et al. proved the corresponding Hermite-Hadamard inequalities for convex functions by using $(p,q)^{\rho}$- integrals, as follows:

Theorem 2.7. *[39] If $F : [\sigma,\rho] \rightarrow \mathbb{R}$ be a convex differentiable function on $[\sigma,\rho]$ and $0 < q < p \leq 1$. Then, $(p,q)^{\rho}$-Hermite-Hadamard inequalities*

$$F\left(\frac{p\sigma+q\rho}{[2]_{p,q}}\right) \leq \frac{1}{p(\rho-\sigma)}\int_{\sigma p+(1-p)\rho}^{\rho} F(\xi) \ {}^{\rho}d_{p,q}\xi \ \leq \frac{pF(\sigma)+qF(\rho)}{[2]_{p,q}}. \tag{5}$$

Now, we present a new lemma for post quantum calculus which is significant in proving the upcoming lemmas.

Lemma 2.8. *For continuous functions $F, g : [\sigma,\rho] \rightarrow \mathbb{R}$, the following equality true:*

$$\int_0^c g(\xi)^{\rho} D_{p,q}F(\xi\sigma+(1-\xi)\rho)d_{p,q}\xi = -\left.\frac{g\left(\frac{\xi}{p}\right)F(\xi\sigma+(1-\xi)\rho)}{\rho-\sigma}\right|_0^c + \frac{1}{p(\rho-\sigma)}\int_0^c D_{p,q}g\left(\frac{\xi}{p}\right)F(q\xi\sigma+(1-q\xi)\rho)\,d_{p,q}\xi$$

Proof. The lemma can be demonstrated using simple calculations, hence it is omitted. □

Remark 2.9. *If we take $p = 1$ in Lemma 2.8, we get relevant result in quantum calculus [40, Lemma 1].*

3. Crucial Identities

In this section, we prove two major post quantum integral identities utilizing the integration by parts method for post quantum integrals, which are helpful to obtain our main outcomes.

Lemma 3.1. *For $F : I \subset \mathbb{R} \to \mathbb{R}$ be a (p,q) differentiable function on I, and $\sigma, \rho \in I$ with $\sigma < \rho$. If ${}^{m\rho}D_{p,q}F$ is continious and integrable on I, then one has the identity for $m \in [0,1]$:*

$$\begin{aligned} & q(m\rho-\sigma)\left[\int_0^{\frac{p}{[2]_{p,q}}} \xi \ {}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi + \int_{\frac{p}{[2]_{p,q}}}^1 (\xi-\frac{1}{q}) \ {}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi\right] \\ = \ & \frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma+mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\ {}^{m\rho}d_{p,q}\varkappa. \end{aligned} \tag{6}$$

Proof. From fundamental properties of post quantum integrals, we have

$$\begin{aligned} & \left[\int_0^{\frac{p}{[2]_{p,q}}} \xi^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi + \int_{\frac{p}{[2]_{p,q}}}^1 (\xi-\frac{1}{q}) \ {}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi\right] \\ = \ & \left[\int_0^{\frac{p}{[2]_{p,q}}} \xi \ {}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi + \int_0^1 (\xi-\frac{1}{q})^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi \right. \\ & \left. -\int_0^{\frac{p}{[2]_{p,q}}} (\xi-\frac{1}{q}) \ {}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi\right] \\ = \ & I_1 + I_2 - I_3. \end{aligned}$$

Using the Lemma 2.8, we have

$$\begin{aligned} I_1 &= \int_0^{\frac{p}{[2]_{p,q}}} \xi \ {}^{\rho m}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi \\ &= -\frac{\xi}{p}\frac{F(\xi\sigma+m(1-\xi)\rho}{m\rho-\sigma}\Big|_0^{\frac{p}{[2]_{p,q}}} + \frac{1}{p^2(m\rho-\sigma)}\int_0^{\frac{p}{[2]_{p,q}}} F(\xi q\sigma+m(1-\xi q)\rho)d_{p,q}\xi \\ &= -\frac{1}{[2]_{p,q}(m\rho-\sigma)}F\left(\frac{p\sigma+mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho-\sigma)}\int_0^{\frac{p}{[2]_{p,q}}} F(\xi q\sigma+m(1-\xi q)\rho)d_{p,q}\xi. \end{aligned} \tag{7}$$

Similarly, we have

$$\begin{aligned} I_2 &= \int_0^1 (\xi-\frac{1}{q}) \ {}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi \\ &= \frac{p-q}{pq(m\rho-\sigma)}F(\sigma) - \frac{1}{q(m\rho-\sigma)}F(m\rho) + \frac{1}{p^2(m\rho-\sigma)}\int_0^1 F(\xi q\sigma+m(1-\xi q)\rho)d_{p,q}\xi \\ &= \frac{p-q}{pq(m\rho-\sigma)}F(\sigma) - \frac{1}{q(m\rho-\sigma)}F(m\rho) + \frac{1}{p^2q(m\rho-\sigma)^2}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\ {}^{m\rho}d_{p,q}\varkappa - \frac{p-q}{p^2q(m\rho-\sigma)}F(\sigma) \end{aligned} \tag{8}$$

and

$$\begin{aligned} I_3 &= \int_0^{\frac{p}{[2]_{p,q}}} (\xi-\frac{1}{q})^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi \qquad (9) \\ &= \frac{p}{q\,[2]_{p,q}(m\rho-\sigma)}F(\frac{p\sigma+mq\rho}{[2]_{p,q}}) - \frac{1}{q(m\rho-\sigma)}F(m\rho) + \frac{1}{p^2(m\rho-\sigma)}\int_0^{\frac{p}{[2]_{p,q}}} F(\xi q\sigma+m(1-\xi q)\rho)d_{p,q}\xi. \end{aligned}$$

Thus from (7), (8) and (9), we have

$$I_1 + I_2 - I_3 = \frac{(p-q)(p-1)}{p^2q(m\rho-\sigma)}F(\sigma) - \frac{1}{q(m\rho-\sigma)}F(\frac{p\sigma+mq\rho}{[2]_{p,q}}) + \frac{1}{p^2q(m\rho-\sigma)^2}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa \tag{10}$$

and we obtained required equality (6) by multiplying $q(m\rho-\sigma)$ on both sides of (10). Thus, the proof is accomplished. □

Remark 3.2. *In Lemma 3.1,*

(i) If we set $m = 1$, then we obtain new variant of the identity.

(ii) If we set $p = 1$ and $\alpha = m = 1$, then identity reduces to quantum calculus [33, Lemma 2].

(iii) If we set $p = 1$, $\alpha = m = 1$ and take $q \to 1^-$, then we obtain [34, Lemma 2.1].

Lemma 3.3. *Assume that $F : I \subset \mathbb{R} \to \mathbb{R}$ be a (p,q) differentiable function on I, and $\sigma, \rho \in I$ with $\sigma < \rho$. If ${}^{m\rho}D_{p,q}F$ is continious and integrable on I, then one has the identity for $m \in [0,1]$*

$$\begin{aligned}
&\frac{(p^2+pq^2-q^2)F(\sigma)+p^2qF(m\rho)}{[2]_{p,q}} - \frac{1}{(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)^{m\rho}d_{p,q}\varkappa \\
= \;& \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\int_0^1 (1-[2]_{p,q}\,\xi)\,{}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi.
\end{aligned} \tag{11}$$

Proof. From fundamental properties of post quantum integral, we have

$$\begin{aligned}
&\int_0^1 (1-[2]_{p,q}\,\xi)\,{}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi \\
= \;& -\left.\frac{(1-[2]_{p,q}\frac{\xi}{p})F(\xi\sigma+m(1-\xi)\rho)}{m\rho-\sigma}\right|_0^1 - \frac{[2]_{p,q}}{p^2(m\rho-\sigma)}\int_0^1 F(q\xi\sigma+m(1-q\xi)\rho)d_{p,q}\xi \\
= \;& \frac{qF(\sigma)+pF(m\rho)}{p(m\rho-\sigma)} - \frac{[2]_{p,q}}{p^2(m\rho-\sigma)}\int_0^1 F(q\xi\sigma+m(1-q\xi)\rho)d_{p,q}\xi \\
= \;& \frac{qF(\sigma)+pF(m\rho)}{p(m\rho-\sigma)} - \frac{[2]_{p,q}\,(p-q)}{p^2q(m\rho-\sigma)}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}F\left(\frac{q^{n+1}}{p^{n+1}}\sigma+m(1-\frac{q^{n+1}}{p^{n+1}})\rho\right) \\
= \;& \frac{qF(\sigma)+pF(m\rho)}{p(m\rho-\sigma)} - \frac{[2]_{p,q}\,(p-q)}{p^2q(m\rho-\sigma)}\left(\sum_{k=0}^{\infty}\frac{q^{k}}{p^{k}}F\left(\frac{q^{k}}{p^{k}}\sigma+m(1-\frac{q^{k}}{p^{k}})\rho\right)-F(\sigma)\right) \\
= \;& \frac{qF(\sigma)+pF(m\rho)}{p(m\rho-\sigma)} - \frac{[2]_{p,q}}{p^2q(m\rho-\sigma)^2}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)^{m\rho}d_{p,q}\varkappa + \frac{[2]_{p,q}\,(p-q)}{p^2q(m\rho-\sigma)}F(\sigma) \\
= \;& \frac{(p^2+pq^2-q^2)F(\sigma)+p^2qF(m\rho)}{p^2q(m\rho-\sigma)} - \frac{[2]_{p,q}}{p^2q(m\rho-\sigma)^2}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)^{m\rho}d_{p,q}\varkappa.
\end{aligned} \tag{12}$$

and we obtain the required equality (11) by multiplying $\frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}$ on both sides of (12). □

Thus, the proof is accomplished.

Remark 3.4. *In Lemma 3.3,*

(i) If we set $m = 1$, then we obtain new variant of the identity.

(ii) If we set $p = 1$ and $\alpha = m = 1$, then identity reduces to quantum calculus [33, Lemma 1].

(iii) If we set $p = 1$, $\alpha = m = 1$ and take limit as $q \to 1^-$, then we find [35, Lemma 2.1].

4. Midpoint Type inequalities for (α, m)-convex functions

In this section, we will derive Midpoint type inequalities for differentiable (α, m)-convex functions.

Theorem 4.1. *Under the assumption of Lemma 3.1, if $|^{m\rho}D_{p,q}F|$ is (α, m)-convex function over $[\sigma, \rho]$, then we find the following midpoint type inequality:*

$$\begin{aligned}
&\left|\frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma + mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho - \sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq\ & q(m\rho - \sigma)[(\Psi_1(p,q) + \Psi_3(p,q))\ |^{m\rho}D_{p,q}F(\sigma)| + m\ (\Psi_2(p,q) + \Psi_4(p,q))\ |^{m\rho}D_{p,q}F(\rho)|], \qquad (13)
\end{aligned}$$

where

$$\begin{aligned}
\Psi_1(p,q) &= \int_0^{\frac{p}{[2]_{p,q}}} \xi^{\alpha+1} d_{p,q}\xi = \frac{p^{\alpha+2}}{[2]_{p,q}^{\alpha+2}\,[\alpha+2]_{p,q}} \\
\Psi_2(p,q) &= \int_0^{\frac{p}{[2]_{p,q}}} \xi(1-\xi^\alpha)\ d_{p,q}\xi = \frac{p^2}{[2]_{p,q}^3} - \frac{p^{\alpha+2}}{[2]_{p,q}^{\alpha+2}\,[\alpha+2]_{p,q}} \\
\Psi_3(p,q) &= \int_{\frac{p}{[2]_{p,q}}}^1 \xi^\alpha\left(\frac{1}{q} - \xi\right)\ d_{p,q}\xi = \frac{1}{q\,[\alpha+1]_{p,q}} - \frac{1}{[\alpha+2]_{p,q}} \\
&- \frac{p^{\alpha+1}}{q\,[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}} + \frac{p^{\alpha+2}}{[2]_{p,q}^{\alpha+2}\,[\alpha+2]_{p,q}} \\
\Psi_4(p,q) &= \int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{1}{q} - \xi\right)(1-\xi^\alpha)\ d_{p,q}\xi = \frac{p^2}{[2]_{p,q}^3} + \frac{1}{[\alpha+2]_{p,q}} - \frac{p^{\alpha+2}}{[2]_{p,q}^{\alpha+2}\,[\alpha+2]_{p,q}} \\
&+ \frac{p^{\alpha+1}}{q[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}} - \frac{1}{q[\alpha+1]_{p,q}}.
\end{aligned}$$

Proof. *By taking modulus in (6), and using (α, m)-convexity of $|^{m\rho}D_{p,q}F|$, we have*

$$\begin{aligned}
&\left|\frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma + mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho - \sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq\ & q(m\rho - \sigma)\left[\int_0^{\frac{p}{[2]_{p,q}}} \xi\ |^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)|\ d_{p,q}\xi + \int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{1}{q} - \xi\right)\ |^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)|\ d_{p,q}\xi\right] \\
\leq\ & q(m\rho - \sigma)\left[\int_0^{\frac{p}{[2]_{p,q}}} \xi^{\alpha+1}\ |^{m\rho}D_{p,q}F(\sigma)|\ d_{p,q}\xi + \int_0^{\frac{p}{[2]_{p,q}}} m(\xi - \xi^{\alpha+1})\ |^{m\rho}D_{p,q}F(\rho)|\ d_{p,q}\xi\right. \\
&\left. + \int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{\xi^\alpha}{q} - \xi^{\alpha+1}\right)\ |^{m\rho}D_{p,q}F(\sigma)|\ d_{p,q}\xi + \int_{\frac{p}{[2]_{p,q}}}^1 m\left(\frac{1}{q} - \xi\right)(1-\xi^\alpha)\ |^{m\rho}D_{p,q}F(\rho)|\ d_{p,q}\xi\right] \\
=\ & q(m\rho - \sigma)\left[\Psi_1(p,q)\ |^{m\rho}D_{p,q}F(\sigma)| + m\ \Psi_2(p,q)\ |^{m\rho}D_{p,q}F(\rho)| + \Psi_3(p,q)\ |^{m\rho}D_{p,q}F(\sigma)| + m\ \Psi_4(p,q)\ |^{m\rho}D_{p,q}F(\rho)|\right] \\
=\ & q(m\rho - \sigma)[(\Psi_1(p,q) + \Psi_3(p,q))\ |^{m\rho}D_{p,q}F(\sigma)| + m\ (\Psi_2(p,q) + \Psi_4(p,q))\ |^{m\rho}D_{p,q}F(\rho)|].
\end{aligned}$$

Thus, the proof is accomplished. □

Remark 4.2. *In Theorem 4.1, we have*

(i) If we set $m = \alpha = 1$, then we get new result for convex function in (p,q)-calculus.

(ii) If we set $\alpha = m = 1$ and $p = 1$, then we get [33, Theorem 1].

(iii) If we set $\alpha = m = 1$, $p = 1$ and take limit as $q \to 1^-$, then we find [34, Theorem 2.2].

Theorem 4.3. *Under the assumption of Lemma 3.1, If $|{}^{m\rho}D_{p,q}F(\varkappa)|^r$, $r \geqslant 1$ is (α, m)-convex function over $[\sigma, \rho]$, then we find the following midpoint type inequality:*

$$\begin{aligned}
&\left|\frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma + mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq\ & q(m\rho-\sigma)\left(\frac{p^2}{[2]_{p,q}^3}\right)^{1-\frac{1}{r}} \Big[\left(\Psi_1(q)\ |{}^{m\rho}D_qF(\sigma)|^r + m\ \Psi_2(q)\ |{}^{m\rho}D_qF(\rho)|^r\right)^{\frac{1}{r}} \\
&+\left(\Psi_3(q)\ |{}^{m\rho}D_qF(\sigma)|^r + m\ \Psi_4(q)\ |{}^{m\rho}D_qF(\rho)|^r\right)^{\frac{1}{r}}\Big].
\end{aligned} \tag{14}$$

Proof. *By taking modulus in (6), and using power mean inequality, we have*

$$\begin{aligned}
&\left|\frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma + mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq\ & q(m\rho-\sigma)\left[\int_0^{\frac{p}{[2]_{p,q}}} |\xi\ {}^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)|\ d_{p,q}\xi + \int_{\frac{p}{[2]_{p,q}}}^1 \left|\left(\xi - \frac{1}{q}\right)\ {}^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)\right|\ d_{p,q}\xi\right] \\
\leq\ & q(m\rho-\sigma)\left[\left(\int_0^{\frac{p}{[2]_{p,q}}} \xi d_{p,q}\xi\right)^{1-\frac{1}{r}} \left(\int_0^{\frac{p}{[2]_{p,q}}} \xi\ |{}^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)|^r\ d_{p,q}\xi\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{1}{q} - \xi\right)\ d_{p,q}\xi\right)^{1-\frac{1}{r}} \left(\int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{1}{q} - \xi\right)\ \left|{}^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)\right|^r\ d_{p,q}\xi\right)^{\frac{1}{r}}\right].
\end{aligned}$$

By applying (α, m)-convexity of $|{}^{m\rho}D_{p,q}F(\varkappa)|^r$, we have

$$\begin{aligned}
&\left|\frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma + mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq\ & q(m\rho-\sigma)\left(\frac{p^2}{[2]_{p,q}^3}\right)^{1-\frac{1}{r}} \\
&\times\left[\left(\int_0^{\frac{p}{[2]_{p,q}}} \xi^{\alpha+1}\ |{}^{m\rho}D_{p,q}F(\sigma)|^r\ d_{p,q}\xi + \int_0^{\frac{p}{[2]_{p,q}}} m\ (\xi - \xi^{\alpha+1})|{}^{m\rho}D_{p,q}F(\rho)|^r\ d_{p,q}\xi\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{\xi^\alpha}{q} - \xi^{\alpha+1}\right)\ |{}^{m\rho}D_{p,q}F(\sigma)|^r\ d_{p,q}\xi + \int_{\frac{p}{[2]_{p,q}}}^1 m\left(\frac{1}{q} - \xi\right)(1 - \xi^\alpha)\ \left|{}^{m\rho}D_{p,q}F(\sigma)\right|^r\ d_{p,q}\xi\right)^{\frac{1}{r}}\right] \\
=\ & q(m\rho-\sigma)\left(\frac{p^2}{[2]_{p,q}^3}\right)^{1-\frac{1}{r}} \Big[\left(\Psi_1(p,q)\ |{}^{m\rho}D_{p,q}F(\sigma)|^r + m\ \Psi_2(q)\ |{}^{m\rho}D_{p,q}F(\sigma)|^r\right)^{\frac{1}{r}} \\
&+\left(\Psi_3(p,q)\ |{}^{m\rho}D_{p,q}F(\rho)|^r + m\ \Psi_4(p,q)\ |{}^{m\rho}D_{p,q}F(\rho)|^r\right)^{\frac{1}{r}}\Big].
\end{aligned}$$

Thus, the proof is accomplished. □

Remark 4.4. *In Theorem 4.3,*

(i) If we set $p = 1$ and $\alpha = m = 1$, then we obtain [33, Theorem 2].

(ii) If we set $\alpha = m = 1$, $p = 1$ and take limit as $q \to 1^-$, then we find [11, Corollary 17].

Theorem 4.5. *Under the assumption of Lemma 3.1, if $r > 1$ is a real number, if $|^{m\rho}D_{p,q}F(\varkappa)|^r$ is (α, m)-convex function over $[\sigma, \rho]$, then we find the following midpoint type inequality, where $r^{-1} + s^{-1} = 1$.*

$$\begin{aligned}
&\left|\frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma + mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho - \sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq\ & q(m\rho - \sigma) \\
&\times\left[\left(\frac{1}{[2]_{p,q}^{s+1}\,[s+1]_{p,q}}\right)^{\frac{1}{s}}\left(\Phi_1(p,q)|^{m\rho}D_{p,q}F(\sigma)|^r + mC_1(p,q)|^{m\rho}D_{p,q}F(\rho)|^r\right)^{\frac{1}{r}}\right. \\
&\left.+(\eta(p,q))^{\frac{1}{s}}\left(\Phi_2(p,q)|^{m\rho}D_{p,q}F(\sigma)|^r + mC_2(p,q)|^{m\rho}D_{p,q}F(\rho)|^r\right)^{\frac{1}{r}}\right], \qquad (15)
\end{aligned}$$

where

$$\begin{aligned}
\Phi_1(p,q) &= \int_0^{\frac{p}{[2]_{p,q}}} \xi^\alpha d_{p,q}\xi = \frac{p^{\alpha+1}}{[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}} \\
\Phi_2(p,q) &= \int_{\frac{p}{[2]_{p,q}}}^1 \xi^\alpha d_{p,q}\xi = \frac{1}{[\alpha+1]_{p,q}} - \frac{p^{\alpha+1}}{[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}} \\
C_1(p,q) &= \int_0^{\frac{p}{[2]_{p,q}}} (1-\xi^\alpha)\, d_{p,q}\xi = \frac{p}{[2]_{p,q}} - \frac{p^{\alpha+1}}{[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}} \\
C_2(p,q) &= \int_{\frac{p}{[2]_{p,q}}}^1 (1-\xi^\alpha)\, d_{p,q}\xi = \frac{q}{[2]_{p,q}} - \frac{1}{[\alpha+1]_{p,q}} + \frac{p^{\alpha+1}}{[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}} \\
\eta(p,q) &= \int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{1}{q} - \xi\right)^s d_{p,q}\xi.
\end{aligned}$$

Proof. *Taking absolute value of (6) and using the Hölder's inequality, we have*

$$\begin{aligned}
&\left|\frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma + mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho - \sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq\ & q(m\rho - \sigma)\left[\int_0^{\frac{p}{[2]_{p,q}}} |\xi\ {}^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)|\ d_{p,q}\xi\right. \\
&\left.+\int_{\frac{p}{[2]_{p,q}}}^1 \left|\left(\xi - \frac{1}{q}\right)\ {}^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)\right|\ d_{p,q}\xi\right] \\
\leq\ & q(m\rho - \sigma)\left[\left(\int_0^{\frac{p}{[2]_{p,q}}} \xi^s d_{p,q}\xi\right)^{\frac{1}{s}}\left(\int_0^{\frac{p}{[2]_{p,q}}} |^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)|^r\ d_{p,q}\xi\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{1}{q} - \xi\right)^s d_{p,q}\xi\right)^{\frac{1}{s}}\left(\int_{\frac{p}{[2]_{p,q}}}^1 \left|^{m\rho}D_{p,q}F(\xi\sigma + m(1-\xi)\rho)\right|^r\ d_{p,q}\xi\right)^{\frac{1}{r}}\right].
\end{aligned}$$

By applying (α, m)- convexity of $|^{m\rho}D_{p,q}F(\varkappa)|^r$, we have

$$\begin{aligned}
&\left|\frac{(p-q)(p-1)}{p^2}F(\sigma) - F\left(\frac{p\sigma + mq\rho}{[2]_{p,q}}\right) + \frac{1}{p^2(m\rho - \sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq\ & q(m\rho - \sigma)
\end{aligned}$$

$$\times\left[\left(\int_0^{\frac{p}{[2]_{p,q}}} \xi^s d_{p,q}\xi\right)^{\frac{1}{s}} \left(\int_0^{\frac{p}{[2]_{p,q}}} \xi^\alpha \ |^{m\rho}D_{p,q}F(\sigma)|^r \ d_{p,q}\xi + \int_0^{\frac{p}{[2]_{p,q}}} m \ (1-\xi^\alpha)|^{m\rho}D_{p,q}F(\rho)|^r \ d_{p,q}\xi\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(\int_{\frac{p}{[2]_{p,q}}}^1 \left(\frac{1}{q}-\xi\right)^s \ d_{p,q}\xi\right)^{\frac{1}{s}} \left(\int_{\frac{p}{[2]_{p,q}}}^1 \xi^\alpha \ |^{m\rho}D_{p,q}F(\sigma)|^r \ d_{p,q}\xi + \int_{\frac{p}{[2]_{p,q}}}^1 m \ (1-\xi^\alpha)|^{m\rho}D_{p,q}F(\rho)|^r \ d_{p,q}\xi\right)^{\frac{1}{r}}\right]$$

$$= q(m\rho-\sigma)$$

$$\times\left[\left(\frac{1}{[2]_{p,q}^{s+1}\,[s+1]_{p,q}}\right)^{\frac{1}{s}} \left(\Phi_1(p,q)|^{m\rho}D_{p,q}F(\sigma)|^r + mC_1(p,q)|^{m\rho}D_{p,q}F(\rho)|^r\right)^{\frac{1}{r}}\right.$$

$$\left.+(\eta(p,q))^{\frac{1}{s}}\left(\Phi_2(p,q)|^{m\rho}D_{p,q}F(\sigma)|^r + mC_2(q)|^{m\rho}D_{p,q}F(\sigma)|^r\right)^{\frac{1}{r}}\right]. \tag{16}$$

Thus, the proof is accomplished. □

Remark 4.6. *In Theorem 4.5, we have*

(i) If we set $\alpha = m = 1$, then we obtain new result for convex function in (p,q)-calculus.

(ii) If we set $p = 1$ and $\alpha = m = 1$, then we find quantum mid point type inequality for convex function.

(iii) If we set $p = 1$, $\alpha = m = 1$ and take the limit as $q \to 1^-$, then we find [34, Theorem 2.3].

5. Trapezoid type inequalities for (α, m)-convex functions

In this section, we will derive Trapezoid type inequalities for differentiable (α, m)-convex functions.

Theorem 5.1. *Under the assumption of Lemma 3.3, if $|^{m\rho}D_{p,q}F|$ is (α, m)-convex function over $[\sigma, \rho]$, then we have the following trapezoid type inequality:*

$$\left|\frac{(p^2+pq^2-q^2)F(\sigma)+p^2qF(m\rho)}{[2]_{p,q}} - \frac{1}{(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho} F(\varkappa)^{m\rho}d_{p,q}\varkappa\right|$$

$$\leq \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\left[\left|^{m\rho}D_{p,q}F(\sigma)\right|(K_1(p,q)-K_2(p,q)) + m\left|^{m\rho}D_{p,q}F(\rho)\right|(L_1(p,q)-L_2(p,q))\right], \tag{17}$$

where

$$K_1(p,q) = \int_0^{\frac{1}{[2]_{p,q}}} (\xi^\alpha - [2]_{p,q}\,\xi^{\alpha+1})d_{p,q}\xi = \frac{q^{\alpha+1}}{[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}\,[\alpha+2]_{p,q}}$$

$$K_2(p,q) = \int_{\frac{1}{[2]_{p,q}}}^1 (\xi^\alpha - [2]_{p,q}\,\xi^{\alpha+1})d_{p,q}\xi = \frac{1}{[\alpha+1]_{p,q}} - \frac{[2]_{p,q}}{[\alpha+2]_{p,q}} - \frac{q^{\alpha+1}}{[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}\,[\alpha+2]_{p,q}}$$

$$L_1(p,q) = \int_0^{\frac{1}{[2]_{p,q}}} (1-[2]_{p,q}\,\xi)(1-\xi^\alpha)d_{p,q}\xi = \frac{q}{[2]_{p,q}^2} - \frac{q^{\alpha+1}}{[2]_{p,q}^{\alpha+1}\,[\alpha+1]_{p,q}\,[\alpha+2]_{p,q}}$$

$$L_2(p,q) = \int_{\frac{1}{[2]_{p,q}}}^1 (1-[2]_{p,q}\,\xi)(1-\xi^\alpha)d_{p,q}\xi$$

$$= \frac{q[\alpha]_{p,q}}{[\alpha+1]_{p,q}\,[\alpha+2]_{p,q}} - \frac{q}{[2]_{p,q}^2} + \frac{1}{[\alpha+1]_{p,q}\,[2]_{p,q}^{\alpha+1}} - \frac{1}{[\alpha+2]_{p,q}\,[2]_{p,q}^{\alpha+1}}.$$

Proof. By taking modulus in (11), and using (α, m)-convexity of $|{}^{m\rho}D_{p,q}F|$, we have

$$\begin{aligned}
&\left|\frac{(p^2+pq^2-q^2)F(\sigma)+p^2qF(m\rho)}{[2]_{p,q}}-\frac{1}{(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho}F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
= & \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\left|\int_0^1(1-[2]_{p,q}\,\xi)\,{}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi\right| \\
\leq & \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\int_0^1\left|(1-[2]_{p,q}\,\xi)\right|\left|{}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)\right|d_{p,q}\xi \\
\leq & \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\left(\int_0^1\left|(1-[2]_{p,q}\,\xi)\right|\xi^{\alpha}\left|{}^{m\rho}D_{p,q}F(\sigma)\right|d_{p,q}\xi+\int_0^1\left|(1-[2]_{p,q}\,\xi)\right|m(1-\xi^{\alpha})\left|{}^{m\rho}D_{p,q}F(\rho)\right|d_{p,q}\xi\right) \\
= & \frac{p^2q(m\rho-\sigma)}{p+q}\left[\left|{}^{m\rho}D_{p,q}F(\sigma)\right|(K_1(p,q)-K_2(p,q))\ +m\left|{}^{m\rho}D_{p,q}F(\rho)\right|(L_1(p,q)-L_2(p,q))\right].
\end{aligned}$$

Thus, the proof is accomplished. □

Remark 5.2. *In Theorem 5.1, we have*

(i) If we set $\alpha = m = 1$, then a new result for convex function is obtained in (p, q)-calculus.

(ii) If we set $p = 1$ and $\alpha = m = 1$, then we get [33, Theorem 1].

(iii) If we set $p = 1$, $\alpha = m = 1$ and take the limit as $q \to 1^-$, then we find [35, Theorem 2.2].

Theorem 5.3. *Under the assumption of Lemma 3.3, if $|{}^{m\rho}D_{p,q}F(\varkappa)|^r$, $r \geqslant 1$ is (α, m)-convex function over $[\sigma, \rho]$, then we have the following trapezoid type inequality:*

$$\begin{aligned}
&\left|\frac{(p^2+pq^2-q^2)F(\sigma)+p^2qF(m\rho)}{[2]_{p,q}}-\frac{1}{(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho}F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq & \frac{q(m\rho-\sigma)}{[2]_{p,q}}\left(\frac{2q}{[2]_{p,q}^2}\right)^{1-\frac{1}{r}}\left[\left|{}^{m\rho}D_{p,q}F(\sigma)\right|^r(K_1(p,q)-K_2(p,q))+m\left|{}^{m\rho}D_{p,q}F(\rho)\right|^r(L_1(p,q)-L_2(p,q))\right]^{\frac{1}{r}}, \quad (18)
\end{aligned}$$

Proof. By taking modulus in (11) and using power mean inequality, we have

$$\begin{aligned}
&\left|\frac{(p^2+pq^2-q^2)F(\sigma)+p^2qF(m\rho)}{[2]_{p,q}}-\frac{1}{(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho}F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
= & \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\left|\int_0^1(1-[2]_{p,q}\,\xi)\,{}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)d_{p,q}\xi\right| \\
\leq & \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\left(\int_0^1\left|(1-[2]_{p,q}\,\xi)\right|d_{p,q}\xi\right)^{1-\frac{1}{r}}\left(\int_0^1\left|(1-[2]_{p,q}\,\xi)\right|\left|{}^{m\rho}D_{p,q}F(\xi\sigma+m(1-\xi)\rho)\right|^r d_{p,q}\xi\right)^{\frac{1}{r}}.
\end{aligned}$$

By applying (α, m)-convexity of $|{}^{m\rho}D_{p,q}F(\varkappa)|^r$, we have

$$\begin{aligned}
&\left|\frac{(p^2+pq^2-q^2)F(\sigma)+p^2qF(m\rho)}{[2]_{p,q}}-\frac{1}{(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho}F(\varkappa)\,{}^{m\rho}d_{p,q}\varkappa\right| \\
\leq & \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\left(\int_0^1\left|(1-[2]_{p,q}\,\xi)\right|d_{p,q}\xi\right)^{1-\frac{1}{r}} \\
&\times\left(\int_0^1\left|(1-[2]_{p,q}\,\xi)\right|\xi^{\alpha}\left|{}^{m\rho}D_{p,q}F(\sigma)\right|^r d_{p,q}\xi+\int_0^1\left|(1-[2]_{p,q}\,\xi)\right|m(1-\xi^{\alpha})\left|{}^{m\rho}D_{p,q}F(\rho)\right|^r d_{p,q}\xi\right)^{\frac{1}{r}}
\end{aligned}$$

$$= \frac{p^2q(m\rho-\sigma)}{[2]_{p,q}}\left(\frac{2q}{[2]_{p,q}^2}\right)^{1-\frac{1}{r}}$$
$$\times\left[\left|{}^{m\rho}D_{p,q}F(\sigma)\right|^r(K_1(p,q)-K_2(p,q))+m\left|{}^{m\rho}D_{p,q}F(\rho)\right|^r(L_1(p,q)-L_2(p,q))\right]^{\frac{1}{r}}.$$

Thus, the proof is accomplished. □

Remark 5.4. *In Theorem 5.3, we have*

(i) If we set $\alpha = m = 1$, then a new result for convex function is obtained in (p, q)-calculus.

(ii) If we set $p = 1$ and $\alpha = m = 1$, then we get [33, Theorem 2].

(iii) If we set $p = 1$, $\alpha = m = 1$ and take the limit as $q \to 1^-$, then we find [41, Theorem 1].

6. Application to special means

For any positive number $\sigma, \rho \in \mathbb{R}$, we consider the following means:

(i) The Arithmetic mean

$$A(\sigma,\rho) = \frac{\sigma+\rho}{2}$$

(ii) The Harmonic mean

$$H(\sigma,\rho) = \frac{2\sigma\rho}{\sigma+\rho}$$

(iii) The Geometric mean

$$G(\sigma,\rho) = \sqrt{\sigma\rho}\ .$$

Proposition 6.1. *Let $\sigma, \rho \in \mathbb{R}$, $\sigma < \rho$, $\alpha \in [0,1]$, $m \in [0,1]$ and $0 < q < p \leq 1$. Then we find*

$$\left|\frac{2}{G^2(\sigma p.p)}[A(p^2,q)-A(pq,p)]-\frac{A(p,q)}{A(p\sigma,mq\rho)}+\frac{1}{G^2(p,p)}\Upsilon_1\right| \leq \frac{q(m\rho-\sigma)}{p-q} \tag{19}$$
$$\times\left[\begin{array}{c}\left|\frac{2}{(\sigma-m\rho)H(q\sigma+(1-q)m\rho,m\rho(p-1)-p\sigma)}\right|(\Psi_1(p,q)+\Psi_3(p,q)) \\ +m\left|\frac{2}{(\rho-m\rho)H(q\rho+(1-q)m\rho,m\rho(p-1)-p\rho)}\right|(\Psi_2(p,q)+\Psi_4(p,q))\end{array}\right]$$

where

$$\Upsilon_1 = \frac{1}{(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho}\frac{1}{\varkappa}\ {}^{m\rho}d_{p,q}\varkappa = (p-q)\sum_{n=0}^{\infty}\frac{\frac{q^n}{p^n}}{\frac{q^n}{p^n}\sigma+m(1-\frac{q^n}{p^n})\rho}. \tag{20}$$

Proof. The inequality (13) for function $F(\varkappa) = \frac{1}{\varkappa}$ leads to required result. □

Proposition 6.2. *If we take $\sigma = 1, \rho = 2, q = 0.7, m = 0.9, p = 0.8$ and $\alpha = 0$ in (19), we get*

$$\left|\frac{2}{G^2(\sigma p.p)}[A(p^2,q)-A(pq,p)]-\frac{A(p,q)}{A(\sigma,m(p+q-1)\rho)}+\frac{1}{G^2(p,p)}\Upsilon_1\right| \approx 0.1815789774$$

$$\frac{q(m\rho-\sigma)}{p-q}\left[\begin{array}{c}\left|\frac{2}{(\sigma-m\rho)H(q\sigma+(1-q)m\rho,m\rho(p-1)-p\sigma)}\right|(\Psi_1(p,q)+\Psi_3(p,q))\\+m\left|\frac{2}{(\rho-m\rho)H(q\rho+(1-q)m\rho,m\rho(p-1)-p\rho)}\right|(\Psi_2(p,q)+\Psi_4(p,q))\end{array}\right]=1.226707537.$$

Hence,
$0.1815789774 \le 1.226707537.$

Proposition 6.3. *Let $\sigma, \rho \in \mathbb{R}$, $\sigma < \rho$, $\alpha \in [0,1]$, $m \in [0,1]$ and $0 < q < p \le 1$. Then we find*

$$\left|\frac{A(G^2(p^2,m\rho)+G^2(pq,qm\rho),G^2(p^2,\sigma q)-G^2(q,qm\rho))}{G^2(m\rho,\sigma)A(p,q)}-\Upsilon_1\right| \le -\frac{G^2(q,p^2)(m\rho-\sigma)}{2(p-q)A(p,q)} \tag{21}$$
$$\times\left[\begin{array}{c}\left|\frac{2}{(\sigma-m\rho)H(q\sigma+(1-q)m\rho,m\rho(p-1)-p\sigma)}\right|(K_1(p,q)-K_2(p,q))\\+m\left|\frac{2}{(\rho-m\rho)H(q\rho+(1-q)m\rho,m\rho(p-1)-p\rho)}\right|(L_1(p,q)-L_2(p,q))\end{array}\right]$$

where

$$\Upsilon_1=\frac{1}{(m\rho-\sigma)}\int_{p\sigma+m(1-p)\rho}^{m\rho}\frac{1}{\varkappa}\ {}^{m\rho}d_{p,q}\varkappa=(p-q)\sum_{n=0}^{\infty}\frac{\frac{q^n}{p^n}}{\frac{q^n}{p^n}\sigma+m(1-\frac{q^n}{p^n})\rho}. \tag{22}$$

Proof. The inequality (17) for function $F(\varkappa)=\frac{1}{\varkappa}$ leads to required result. □

Proposition 6.4. *If we take $\sigma = 1, \rho = 2, q = 0.9, m = 0.8, p = 0.95$ and $\alpha = 0.1$ in (21), we get*

$$\left|\frac{A(G^2(p^2,m\rho)+G^2(pq,qm\rho),G^2(p^2,\sigma q)-G^2(q,qm\rho))}{G^2(m\rho,\sigma)A(p,q)}-\Upsilon_1\right|\approx 0.0094180663,$$

$$-\frac{G^2(q,p^2)(m\rho-\sigma)}{2(p-q)\Psi(p,q)}\left[\begin{array}{c}\left|\frac{2}{(\sigma-m\rho)H(q\sigma+(1-q)m\rho,m\rho(p-1)-p\sigma)}\right|(K_1(p,q)-K_2(p,q))\\+m\left|\frac{2}{(\rho-m\rho)H(q\rho+(1-q)m\rho,m\rho(p-1)-p\rho)}\right|(L_1(p,q)-L_2(p,q))\end{array}\right]=2.425109248.$$

Hence,
$0.0094180663 \le 2.425109248.$

7. Conclusion

In this paper, we have presented two novel post quantum identities for midpoint and trapezoid type inequalities. Utilizing these identities, we have developed some midpoint and trapezoid type integral inequalities for (α, m) convexity. The major motivation for this research was to propose some novel right post quantum Midpoint and Trapezoid type inequalities for (α, m) differentiable convex functions. The determined results of reseach have been reduced to the outcomes of previously published articles, ensuring the validity of the results. In future, similar inequalities can be developed for co-ordinated (α, m) convex functions or any other generalized convexity.

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