Filomat 37:14 (2023), 4481–4491
https://doi.org/10.2298/FIL2314481A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a Riesz basis of diagonally generalized subordinate operator matrices and application to a Gribov operator matrix in Bargmann space

Boulbeba Abdelmoumen[a,*], Alaeddine Damergi[a], Yousra Krichene[a]

[a]*Département de Mathématiques, Faculté des Sciences de Sfax, B. P. 1171, 3000, Sfax, Tunisie*

Abstract. In this paper, we study the change of spectrum and the existence of Riesz bases of specific classes of $n \times n$ unbounded operator matrices, called: diagonally and off-diagonally generalized subordinate block operator matrices. An application to a $n \times n$ Gribov operator matrix acting on a sum of Bargmann spaces, illustrates the abstract results. As example, we consider a particular Gribov operator matrix by taking special values of the real parameters of Pomeron.

1. Introduction

Block operator matrices are matrices with entries that are linear operators between Banach or Hilbert spaces. They play a crucial role in several active fields of research: in control and systems theory, in differential equations, in evolution problems and in mathematical physics with its various applications, such as hydrodynamics, magnetohydrodynamics, elasticity theory, optimal control, mechanics and quantum mechanics. The study of spectral proprieties of this type of operators is then of great importance, but also of complexity, and the situation becomes more complicated for $n \geq 3$. In effect, most of the works that have been carried out on this subject have been done in the particular case $n = 2$, and we can quote for instance [1, 3, 4, 8, 9, 17, 18, 20, 21, 28]. Among the works that have dealt with the general case $n \geq 2$, we can cite [23] and [29]. In this latter, Tretter et al. introduced the block numerical range of bounded $n \times n$ operator matrices and proved the spectral inclusion property. In [23], the results obtained in [28] and [29] was generalized to unbounded *diagonally* (respectively *off-diagonally*) *dominant $n \times n$ operator matrices*, which are matrices such that the off-diagonal entries of each column are relatively bounded to the diagonal entry of the same column (respectively the diagonal entry of each column is relatively bounded to each off-diagonal entry of the same column).

In this paper, we introduce specific matrices of the last classes called *diagonally* and *off-diagonally generalized subordinate matrices*, for which we will characterize the location of the spectrum and study the existence of Riesz bases of generalized eigenvectors.

The results of this paper, are of importance for application to a $n \times n$ Gribov operator matrix, which is an operator matrix whose entries are Gribov operators.

2020 *Mathematics Subject Classification*. 47A55; 47B06; 30H20.
Keywords. Perturbation theory; Riesz basis; Bargmann space; Gribov operator.
Received: 21 July 2022; Accepted: 03 January 2023
Communicated by Dragan S. Djordjević
* Corresponding author: Boulbeba Abdelmoumen
Email addresses: Boulbeba.Abdelmoumen@ipeis.rnu.tn (Boulbeba Abdelmoumen), damergialaeddine1981@gmail.com (Alaeddine Damergi), yousraakrichene@gmail.com (Yousra Krichene)

Notice that the Gribov operators form a family of non-self-adjoint operators that govern the Reggeon field theory. This theory was introduced by Gribov [11] in 1967, in order to study strong interactions, i.e., the interaction between protons and neutrons among other less stable particles. These operators was originally constructed as polynomials in the standard creation and annihilation operators. A representation space of this family is the Bargmann space of analytic functions on $\mathbb{C}^d$. In zero transverse dimension ($d = 1$), the Bargmann space is the following:

$$\mathcal{B} = \left\{\varphi : \mathbb{C} \longrightarrow \mathbb{C} \text{ entire such that } \int_{\mathbb{C}} e^{-|z|^2}|\varphi(z)|^2 dxdy < \infty \text{ and } \varphi(0) = 0\right\}.$$

It forms with the inner product

$$\langle \varphi, \psi \rangle \longrightarrow \frac{1}{\pi}\int_{\mathbb{C}} e^{-|z|^2}\varphi(z)\overline{\psi}(z)dxdy,$$

a Hilbert space. An orthonormal basis for $\mathcal{B}$, is $\{z \mapsto e_k(z) = \dfrac{z^k}{\sqrt{k!}}\}_{k\in\mathbb{N}}$.

In this representation, the standard annihilation operator A and the standard creation operator A^*, are defined by:

$$\begin{cases} A : \mathcal{D}(A) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow A\varphi = \dfrac{d\varphi}{dz} \\ \mathcal{D}(A) = \{\varphi \in \mathcal{B} \text{ such that } A\varphi \in \mathcal{B}\} \end{cases}$$

and

$$\begin{cases} A^* : \mathcal{D}(A^*) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow A^*\varphi : z \mapsto z\varphi(z) \\ \mathcal{D}(A^*) = \{\varphi \in \mathcal{B} \text{ such that } A^*\varphi \in \mathcal{B}\}. \end{cases}$$

The following operators play a crucial role in Reggeon field theory.

$$\begin{cases} G : \mathcal{D}(G) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow G\varphi = A^{*3}A^3\varphi \\ \mathcal{D}(G) = \{\varphi \in \mathcal{B} \text{ such that } G\varphi \in \mathcal{B}\}, \end{cases}$$

$$\begin{cases} S : \mathcal{D}(S) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow S\varphi = A^{*2}A^2\varphi \\ \mathcal{D}(S) = \{\varphi \in \mathcal{B} \text{ such that } S\varphi \in \mathcal{B}\}, \end{cases}$$

$$\begin{cases} H_0 : \mathcal{D}(H_0) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow H_0\varphi = A^*A\varphi \\ \mathcal{D}(H_0) = \{\varphi \in \mathcal{B} \text{ such that } H_0\varphi \in \mathcal{B}\} \end{cases}$$

and

$$\begin{cases} H_1 : \mathcal{D}(H_1) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow H_1\varphi = A^*(A^* + A)A\varphi \\ \mathcal{D}(H_1) = \{\varphi \in \mathcal{B} \text{ such that } H_1\varphi \in \mathcal{B}\}. \end{cases}$$

En effect, a third degree representative of this theory, is the following Gribov operator:

$$H_{\lambda'',\lambda',\mu,\lambda} \quad = \quad \lambda'' G + \lambda' S + \mu H_0 + i\lambda H_1, \tag{1}$$

where $\mathbf{i}^2 = -1$, the real parameters λ'', λ' and λ are respectively the magic, the four and the triple coupling of Pomeron and the real μ is the Pomeron intercept.

The paper is organized as follows: In Section 2, we study the location of spectrum and the existence of Riesz bases of diagonally and off-diagonally generalized subordinate $n \times n$ operator matrices. In section 3, the main results are illustrated by an application to a problem of a $n \times n$ Gribov operator matrix acting on a sum of n identical Bargmann spaces.

2. Main results

Let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be Hilbert spaces. In the Hilbert space $\mathcal{H} := \bigoplus_{j=1}^{n} \mathcal{H}_j$, we consider a $n \times n$ operator matrix $\mathcal{M} = (A_{ij})_{1 \le i,j \le n}$, where the entries A_{ij} are unbounded linear operators from $\mathcal{D}(A_{ij}) \subset \mathcal{H}_j$ to $\mathcal{H}_i$. Obviously, $\mathcal{M}$ has as natural domain

$$\mathcal{D}(\mathcal{M}) := \bigoplus_{j=1}^{n} \bigcap_{i=1}^{n} \mathcal{D}(A_{ij})$$

and if the entries are densely defined, then $\mathcal{M}$ is too.

For the sequel, we adopt the following decomposition:

$$\mathcal{M} = \mathfrak{D} + \mathcal{R}$$

with

$$\mathfrak{D} = diag(A_{11}, ..., A_{nn}) \text{ and } \mathcal{R} = ((1 - \delta_{ij})A_{ij})_{1 \le i,j \le n},$$

where δ_{ij} denote the Kronecker delta of (i, j). For the sake of simplicity, we denote by Ω_n, the following set:

$$\Omega_n := \{(i, j) \in \{1, \dots, n\}^2 \text{ such that } i \neq j\}.$$

The new concepts of diagonally and off-diagonally generalized subordinate operator matrices are given in the following definition.

Definition 2.1. *The operator matrix $\mathcal{M}$ is called*

1. *diagonally generalized subordinate, if for all $(i, j) \in \Omega_n$,*
   $$A_{ij} \text{ is generalized subordinate to } A_{jj},$$
   i.e, $\mathcal{D}(A_{jj}) \subset \mathcal{D}(A_{ij})$ and there exist $N^{(i,j)} \in \mathbb{N}^*$, $\{p_k^{(i,j)}\}_{k=1}^{N^{(i,j)}} \subset [0, 1]$ and $\{b_k^{(i,j)}\}_{k=1}^{N^{(i,j)}} \subset \mathbb{R}_+$, such that
   $$\|A_{ij}U_j\| \le \sum_{k=1}^{N^{(i,j)}} b_k^{(i,j)} \|A_{jj}U_j\|^{p_k^{(i,j)}} \|U_j\|^{1-p_k^{(i,j)}}, \ \forall \ U_j \in \mathcal{D}(A_{jj}).$$
2. *off-diagonally generalized subordinate, if for all $(i, j) \in \Omega_n$,*
   $$A_{jj} \text{ is generalized subordinate to } A_{ij},$$
   i.e, $\mathcal{D}(A_{ij}) \subset \mathcal{D}(A_{jj})$ and there exist $N'^{(i,j)} \in \mathbb{N}^*$, $\{p_k'^{(i,j)}\}_{k=1}^{N'^{(i,j)}} \subset [0, 1]$ and $\{b_k'^{(i,j)}\}_{k=1}^{N'^{(i,j)}} \subset \mathbb{R}_+$, such that
   $$\|A_{jj}U_j\| \le \sum_{k=1}^{N'^{(i,j)}} b_k'^{(i,j)} \|A_{ij}U_j\|^{p_k'^{(i,j)}} \|U_j\|^{1-p_k'^{(i,j)}}, \forall \ U_j \in \mathcal{D}(A_{ij}).$$

Remark 2.2. *Note that the concept of* generalized subordinate perturbations *was introduced in [2] as a natural generalization of the notion of p-*subordinate perturbations *studied by Krein (see [15]), Markus (see[19]) and Wyss (see [30] and [31]).*

Now, we will state a crucial result that we will use in the rest of this paper.

Lemma 2.3. *The following hold:*

(i) *Suppose that $\mathcal{M}$ is diagonally generalized subordinate, then $\mathcal{R}$ is generalized subordinate to $\mathfrak{D}$, i.e, $\mathcal{D}(\mathfrak{D}) \subset \mathcal{D}(\mathcal{R})$ and $\exists N \in \mathbb{N}^*$, $\{p_k\}_{k=1}^N \subset [0,1]$ and $\{b_k\}_{k=1}^N \subset \mathbb{R}_+$, such that*

$$\|\mathcal{R}U\| \leq \sum_{k=1}^{N} b_k \|\mathfrak{D}U\|^{p_k} \|U\|^{1-p_k}, \ \forall U \in \mathcal{D}(\mathfrak{D}).$$

(ii) *Suppose that $\mathcal{M}$ is off-diagonally generalized subordinate, then $\mathfrak{D}$ is generalized subordinate to $\mathcal{R}$, i.e, $\mathcal{D}(\mathcal{R}) \subset \mathcal{D}(\mathfrak{D})$ and $\exists N' \in \mathbb{N}^*$, $\{p_k'\}_{k=1}^{N'} \subset [0,1]$ and $\{b_k'\}_{k=1}^{N'} \subset \mathbb{R}_+$, such that*

$$\|\mathfrak{D}U\| \leq \sum_{k=1}^{N'} b_k' \|\mathcal{R}U\|^{p_k'} \|U\|^{1-p_k'}, \ \forall U \in \mathcal{D}(\mathcal{R}).$$

Proof. We prove **(i)**, the proof of **(ii)** is analogue. Let $U = (U_1, ..., U_n) \in \bigoplus_{j=1}^{n} \mathcal{D}(A_{jj})$,

$$\begin{aligned}
\|\mathcal{R}U\| &= \sum_{i=1}^{n} \|\sum_{\substack{j=1\\ j\neq i}}^{n} A_{ij} U_j\| \\
&\leq \sum_{(i,j)\in\Omega_n} \|A_{ij} U_j\| \\
&\leq \sum_{(i,j)\in\Omega_n} \sum_{k=1}^{N^{(i,j)}} b_k^{(i,j)} \|A_{jj} U_j\|^{p_k^{(i,j)}} \|U_j\|^{1-p_k^{(i,j)}} \\
&\leq \sum_{(i,j)\in\Omega_n} \sum_{k=1}^{N^{(i,j)}} b_k^{(i,j)} \|\mathfrak{D}U\|^{p_k^{(i,j)}} \|U\|^{1-p_k^{(i,j)}}.
\end{aligned}$$

It suffices to take

$$\begin{cases}
N = \displaystyle\sum_{(i,j)\in\Omega_n} N^{(i,j)}, \\
\{b_k\}_{k=1}^N = \displaystyle\bigcup_{(i,j)\in\Omega_n} \{b_k^{(i,j)}\}_{k=1}^{N^{(i,j)}}, \\
\{p_k\}_{k=1}^N = \displaystyle\bigcup_{(i,j)\in\Omega_n} \{p_k^{(i,j)}\}_{k=1}^{N^{(i,j)}}.
\end{cases} \tag{2}$$

This ends the proof. □

The following theorem provides a detailed description of the changed spectrum of diagonally generalized subordinate $n \times n$ block operator matrices. For off-diagonally generalized subordinate $n \times n$ block operator matrices, the result is similar.

Theorem 2.4. *Suppose that the operator matrix $\mathcal{M}$ is diagonally generalized subordinate, with $\{p_k\}_{1\leq k\leq N} \subset [0,1[$. Suppose moreover, that for all $j = 1,\ldots,n$, A_{jj} is normal with compact resolvent and its spectrum lies on finitely many rays:*

$$\sigma(A_{jj}) \subset \bigcup_{l=1}^{m_j} e^{i\theta_{jl}}\mathbb{R}_{\geq 0};\ 0 \leq \theta_{jl} \leq 2\pi,\ l = 1,\cdots,m_j.$$

Then $\mathcal{M}$ is with compact resolvent and there exists $m \in \mathbb{N}^*$, such that for every $\alpha > \sum_{k=1}^{N} b_k$, there exists $r_0 > 0$ satisfying

$$\sigma(\mathcal{M}) \subset B_{r_0}(0) \cup \bigcup_{l=1}^{m}\{e^{\boldsymbol{i}\theta_l}(x+\boldsymbol{i}y);\ x \geq 0 \text{ and } |y| \leq \max_{1\leq k\leq N} \alpha x^{p_k}\}.$$

Proof. The entries $(A_{jj})_{1\leq j\leq n}$ are normals with compact resolvents. So, $\mathfrak{D}$ is too and

$$\sigma(\mathfrak{D}) = \bigcup_{s=1}^{n}\sigma(A_{ss}) \subset \bigcup_{s=1}^{n}\bigcup_{l=1}^{m_s} e^{\mathbf{i}\theta_{sl}}\mathbb{R}_{\geq 0}.$$

Apply Lemma 2.3 and [2, Theorem 2.1] to the decomposition $\mathcal{M} = \mathfrak{D} + \mathcal{R}$, we obtain the required result. □

Since for non-normal operators there is no analogue of the spectral theorem, the existence of Riesz bases is of great importance. Recently, some progress has been made for the existence of the Riesz bases of eigenvectors of 2×2 block operator matrices, (see [6, 13, 16, 30, 31]) and 3×3 block operator matrices (see [2]). For $n > 3$, up until now, there have been no results on the existence of Riesz bases for block operator matrices. The following theorem provides different necessary conditions in terms of the spectrum in order to prove the existence of Riesz bases with parentheses for diagonally generalized subordinate $n\times n$ block operator matrices.

Theorem 2.5. *Under the hypotheses of Theorem 2.4, suppose moreover that*

$$\forall\, s \in \{1,\ldots,n\},\ \liminf_{r\to+\infty} \frac{N(r,A_{ss})}{r^{1-p}} < +\infty,\ \textit{with } p = \max_{1\leq k\leq N}\{p_k\},$$

where $N(r,A_{ss})$ denotes the sum of multiplicities of all eigenvalues of A_{ss} contained in $B_r(0)$, the open disk with radius r and center 0 in $\mathbb{C}$. Then the operator matrix $\mathcal{M}$ admits a Riesz basis with parentheses.

Proof. We have $N(r,\mathfrak{D}) = \sum_{s=1}^{n} N(r,A_{ss})$. Hence, the result follows from Lemma 2.3 and [2, Theorem 2.2]. □

3. Gribov Operator Matrix in Bargmann space

The Gribov operator matrices are matrices whose entries are Gribov operators between subspaces of the Bargmann space $\mathcal{B}$. To illustrate the applicability of abstract results described above, we give an application to a representative of the $n\times n$ Gribov operator matrices in zero transverse dimension. To define this representative, we must start by giving some fundamental results and introducing some basic notations.

We first begin by recalling basic properties of the operators G, S and H_0, that follow from [12, Lemma 3, p 112] and [12, Proposition 4, p 112].

Proposition 3.1. **(i)** *The operators G, S and H_0 are self-adjoint and with compact resolvents.*

(ii) $\{e_k := \frac{z^k}{\sqrt{k!}}\}_{k\geq 1}$ *is a system of eigenvectors for* H_0 *associated with the eigenvalues* $\{n\}_{n\geq 1}$ *and its spectral decomposition is the following:*

$$H_0\varphi = \sum_{k=1}^{\infty} k\langle\varphi, e_k\rangle e_k, \ \forall\ \varphi \in \mathcal{D}(H_0).$$

(iii) $\{e_k := \frac{z^k}{\sqrt{k!}}\}_{k\geq 1}$ *is a system of eigenvectors for* S *associated with the eigenvalues* $\{k(k-1)\}_{k\geq 1}$ *and its spectral decomposition is the following:*

$$S\varphi = \sum_{k=2}^{\infty} k(k-1)\langle\varphi, e_k\rangle e_k, \ \forall\ \varphi \in \mathcal{D}(S).$$

(iv) $\{e_k := \frac{z^k}{\sqrt{k!}}\}_{k\geq 2}$ *is a system of eigenvectors for* G *associated with the eigenvalues*

$\{k(k-1)(k-2)\}_{k\geq 2}$ *and its spectral decomposition is the following:*

$$G\varphi = \sum_{k=2}^{\infty}(k-2)(k-1)k\langle\varphi, e_k\rangle e_k, \ \forall\ \varphi \in \mathcal{D}(G).$$

Now, for $\lambda'' \in \mathbb{R}^*$, $(\lambda', \lambda, \mu) \in \mathbb{R}^3$ and $\beta > 0$, we introduce the following operators:

$$\begin{cases} G_{\lambda''} : \mathcal{D}(G_{\lambda''}) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow G_{\lambda''}\varphi = \lambda'' G\varphi \\ \mathcal{D}(G_{\lambda''}) = \mathcal{D}(G), \end{cases}$$

$$\begin{cases} H_0^{\beta} : \mathcal{D}(H_0^{\beta}) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow H_0^{\beta}\varphi := \sum_{k=1}^{\infty} k^{\beta}\langle\varphi, e_k\rangle e_k \\ \mathcal{D}(H_0^{\beta}) = \left\{\varphi \in \mathcal{B} \text{ such that } \sum_{k=1}^{\infty} k^{2\beta}|\langle\varphi, e_k\rangle|^2 < \infty\right\}, \end{cases}$$

and

$$\begin{cases} H_{\lambda',\lambda,\mu}^{\beta} : \mathcal{D}(H_{\lambda',\lambda,\mu}^{\beta}) \subset \mathcal{B} \longrightarrow \mathcal{B} \\ \qquad\qquad \varphi \longrightarrow H_{\lambda',\lambda,\mu}^{\beta}\varphi = [\lambda' S + i\lambda H_1 + \mu H_0^{\beta}]\varphi \\ \mathcal{D}(H_{\lambda',\lambda,\mu}^{\beta}) = \left\{\varphi \in \mathcal{B} \text{ such that } H_{\lambda',\lambda,\mu}^{\beta}\varphi \in \mathcal{B}\right\}. \end{cases}$$

The following lemma provides several useful inequalities for the operators we defined previously.

Lemma 3.2. *Put* $c_1 = 5$, $c_2 = \sqrt{1 + 2^6}$ *and* $c_3 = 1 + 2\sqrt{2}$. *The following inequalities hold trues:*

(i) $||H_0^3\varphi|| \leq c_1||G\varphi|| + c_2||\varphi||$, $\forall\varphi \in \mathcal{D}(G)$.

(ii) $||H_0^{\beta}\varphi|| \leq ||H_0^3\varphi||^{\frac{\beta}{3}}||\varphi||^{1-\frac{\beta}{3}} \leq c_1^{\frac{\beta}{3}}||G\varphi||^{\frac{\beta}{3}}||\varphi||^{1-\frac{\beta}{3}} + c_2^{\frac{\beta}{3}}||\varphi||$, $\forall\beta \in]0, 3[$ *and* $\forall\varphi \in \mathcal{D}(H_0^{\beta})$.

(iii) $\|S\varphi\| \le \|H_0^3\varphi\|^{\frac{2}{3}}\|\varphi\|^{\frac{1}{3}} \le c_1^{\frac{2}{3}}\|G\varphi\|^{\frac{2}{3}}\|\varphi\|^{\frac{1}{3}} + c_2^{\frac{2}{3}}\|\varphi\|, \forall\varphi \in \mathcal{D}(S).$

(iv) $\|H_1\varphi\| \le c_3\|H_0^{\frac{3}{2}}\varphi\| \le c_3\|H_0^3\varphi\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}} \le c_3c_1^{\frac{1}{2}}\|G\varphi\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}} + c_3c_2^{\frac{1}{2}}\|\varphi\|, \forall\varphi \in \mathcal{D}(H_1).$

Proof. Firstly, for all $(a,b) \in \mathbb{R}_+^2\backslash\{(0,0)\}$, we have the following inequalities:

$$(a+b)^\alpha \le a^\alpha + b^\alpha, \qquad \text{if } 0 \le \alpha \le 1, \tag{3}$$

$$a^\alpha + b^\alpha \le (a+b)^\alpha, \qquad \text{if } \alpha \ge 1. \tag{4}$$

(i) Let $\varphi \in \mathcal{D}(G)$. By using the spectral decomposition of H_0^3 and G and the fact that $k^2 \le 5(k-2)(k-1)$ for all $k \ge 3$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} k^6 \mid \langle\varphi, e_k\rangle \mid^2 &\le \mid \langle\varphi, e_1\rangle \mid^2 + 2^6 \mid \langle\varphi, e_2\rangle \mid^2 + \sum_{k=3}^{\infty}[5(k-2)(k-1)k]^2 \mid \langle\varphi, e_k\rangle \mid^2 \\ &\le (1+2^6)\|\varphi\|^2 + 25\|G\varphi\|^2.\end{aligned}$$

Apply (3), the inequality follows.

(ii) Let $\varphi \in \mathcal{D}(H_0^\beta)$. The first inequality follows from the application of the Hôlder inequality:

$$\sum_{k=1}^{\infty} k^{2\beta}|\langle\varphi, e_k\rangle|^2 \le \Big(\sum_{k=1}^{\infty} k^6|\langle\varphi, e_k\rangle|^2\Big)^{\frac{\beta}{3}}\Big(\sum_{k=1}^{\infty} k^{2\beta}|\langle\varphi, e_k\rangle|^2\Big)^{1-\frac{\beta}{3}}.$$

For the second inequality, it suffices to apply **(i)** and inequality (3).

Concerning the first two inequalities of **(iii)** and **(iv)**, one can see [6, Lemme 4.1]. The other two inequalities, result from **(ii)**, (3) and (4). □

The next proposition follows from Lemma 3.2.

Proposition 3.3. *Let $(\lambda'', (\lambda', \lambda, \mu), \beta) \in \mathbb{R}^* \times \mathbb{R}^3 \times]0, 3[$ and put $a = \frac{|\mu|c_1^{\frac{\beta}{3}}}{|\lambda''|^{\frac{\beta}{3}}}, b = \frac{|\lambda|c_3c_1^{\frac{1}{2}}}{|\lambda''|^{\frac{1}{2}}}$ and $c = \frac{|\lambda'|c_1^{\frac{2}{3}}}{|\lambda''|^{\frac{2}{3}}}$. Then $H^\beta_{\lambda',\mu,\lambda}$ is $\{\frac{\beta}{3}, \frac{1}{2}, \frac{2}{3}\}$-generalized subordinate to $G_{\lambda''}$ with bound $\{a, b, c\}$, i.e;*

$$\|H^\beta_{\lambda',\lambda,\mu}\varphi\| \le a\|G_{\lambda''}\varphi\|^{\frac{\beta}{3}}\|\varphi\|^{1-\frac{\beta}{3}} + b\|G_{\lambda''}\varphi\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}} + c\|G_{\lambda''}\varphi\|^{\frac{2}{3}}\|\varphi\|^{\frac{1}{3}},\ \forall\,\varphi \in \mathcal{D}(G_{\lambda''}).$$

Proof.

$$\begin{aligned}\|H^\beta_{\lambda',\lambda,\mu}\varphi\| &\le \mid\mu\mid \|H_0^\beta\varphi\| + \mid\lambda\mid \|H_1\varphi\| + \mid\lambda'\mid \|S\varphi\| \\ &\le \frac{\mid\mu\mid c_1^{\frac{\beta}{3}}}{\mid\lambda''\mid^{\frac{\beta}{3}}}\|G_{\lambda''}\varphi\|^{\frac{\beta}{3}}\|\varphi\|^{1-\frac{\beta}{3}} + \frac{\mid\lambda\mid c_3c_1^{\frac{1}{2}}}{\mid\lambda''\mid^{\frac{1}{2}}}\|G_{\lambda''}\varphi\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}} + \frac{\mid\lambda'\mid c_1^{\frac{2}{3}}}{\mid\lambda''\mid^{\frac{2}{3}}}\|G_{\lambda''}\varphi\|^{\frac{2}{3}}\|\varphi\|^{\frac{1}{3}} \\ &\quad + c_2^{\frac{1}{3}} \mid\mu\mid \|\varphi\| + c_3c_2^{\frac{1}{2}} \mid\lambda\mid \|\varphi\| + \mid\lambda'\mid c_2^{\frac{2}{3}}\|\varphi\|.\end{aligned}$$

□

Now, for $(\lambda''_j)_{1\le j\le n} \subset \mathbb{R}^*$ and $\big((\lambda'_{ij}, \lambda_{ij}, \mu_{ij}), \beta_{ij})\big)_{(i,j)\in\Omega_n} \subset \mathbb{R}^3\times]0, 3[$, we consider the $n \times n$ Gribov operator matrix $\mathcal{M}^\beta_{\lambda'',\lambda',\lambda,\mu}$ defined by:

$$\mathcal{M}^\beta_{\lambda'',\lambda',\lambda,\mu} := \mathfrak{D}_{\lambda''} + \mathcal{R}^\beta_{\lambda',\lambda,\mu}, \tag{5}$$

where

$$\mathfrak{D}_{\lambda''} = Diag(G_{\lambda''_1}, \ldots, G_{\lambda''_n}) \text{ and } R^{\beta}_{\lambda',\lambda,\mu} = \Big((1-\delta_{ij})H^{\beta_{ij}}_{\lambda'_{ij},\lambda_{ij},\mu_{ij}}\Big)_{1\le i,j\le n}.$$

Remark 3.4. *The Gribov operator matrix $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ can be written as follows:*

$$\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu} := \mathfrak{D}_{\lambda''} + \mathfrak{S}_{\lambda'} + \mathfrak{H}_{\lambda} + \mathfrak{H}^{\beta}_{\mu}, \tag{6}$$

where

$$\mathfrak{S}_{\lambda'} = \mathcal{R}^{\beta}_{\lambda',0,0},\ \mathfrak{H}_{\lambda} = \mathcal{R}^{\beta}_{0,\lambda,0} \text{ and } \mathfrak{H}^{\beta}_{\mu} = \mathcal{R}^{\beta}_{0,0,\mu}.$$

According to this decomposition, $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ can be considered as a representative for the $n\times n$ Gribov operator matrices.

For the sequel, we put

$$b_1^{(i,j)} = \frac{|\,\mu_{ij}\,|\,c_1^{\frac{\beta_{ij}}{3}}}{|\,\lambda''_j\,|^{\frac{\beta_{ij}}{3}}},\quad b_2^{(i,j)} = \frac{|\,\lambda_{ij}\,|\,c_3c_1^{\frac{1}{2}}}{|\,\lambda''_j\,|^{\frac{1}{2}}} \quad\text{and}\quad b_3^{(i,j)} = \frac{|\,\lambda'_{ij}\,|\,c_1^{\frac{2}{3}}}{|\,\lambda''_j\,|^{\frac{2}{3}}},\ \text{for all } (i,j)\in\Omega_n.$$

Now, we state a straightforward, but useful result.

Proposition 3.5. *The Gribov operator matrix $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ is diagonally generalized subordinate.*

Proof. According to Proposition 3.3, $H^{\beta_{ij}}_{\lambda'_{ij},\lambda_{ij},\mu_{ij}}$ is $\{\frac{\beta_{ij}}{3}, \frac{1}{2}, \frac{2}{3}\}$-generalized subordinate to $G_{\lambda''_j}$ for all $(i,j)\in\Omega_n$. $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ is then diagonally generalized subordinate. □

As a consequence, we obtain the following result.

Proposition 3.6. *$\mathcal{R}^{\beta}_{\lambda',\lambda,\mu}$ is $\{\frac{\beta_{ij}}{3}\}_{(i,j)\in\Omega_n} \cup \{\frac{1}{2}, \frac{2}{3}\}$-generalized subordinate to $\mathfrak{D}_{\lambda''}$ with bound*

$$\{b_1^{(i,j)}\}_{(i,j)\in\Omega_n} \bigcup \{\sum_{(i,j)\in\Omega_n} b_2^{(i,j)}, \sum_{(i,j)\in\Omega_n} b_3^{(i,j)}\}.$$

Proof. Follows immediately from Lemma 2.3, Proposition 3.5 and the decomposition (5). □

Remark 3.7. *Note that 0 is an eigenvalue of the operator matrix $\mathfrak{D}_{\lambda''}$ and that $\mathcal{R}^{\beta}_{\lambda',\lambda,\mu}$ is generalized subordinate but not p-subordinate to $\mathfrak{D}_{\lambda''}$. Therefore, the results of [19] and [31] are not applicable to the Gribov operator matrix $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$.*

Below, we provide sufficient conditions for the closedness of the Gribov operator matrix $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$.

Proposition 3.8. *Suppose that $\sum_{(i,j)\in\Omega_n} \Big(\frac{1}{3}b_1^{(i,j)}\beta_{ij} + \frac{1}{2}b_2^{(i,j)} + \frac{2}{3}b_3^{(i,j)}\Big) < 1$. Then the Gribov operator matrix $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ is closed.*

Proof. $\mathfrak{D}_{\lambda''}$ is closed and $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ is diagonally generalized subordinate. So, it suffices to apply [14, IV Theorem 1.1]. □

The self-adjointness or non-self-adjointness of $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ depends on the non-symmetrical operators $H_{\lambda_{ij}} := \lambda_{ij}H_1$. This has of course an effect on the location of the spectrum as well as on the existence of bases.

Theorem 3.9. *The Gribov operator matrix $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ has compact resolvent. Moreover, we have:*

1. *If for all $(i,j) \in \Omega_n$, $\lambda_{ij} = 0$ and $\sum_{(i,j)\in\Omega_n}\left(\frac{1}{3}b_1^{(i,j)}\beta_{ij} + \frac{2}{3}b_3^{(i,j)}\right) < 1$, then $\mathcal{M}^{\beta}_{\lambda'',\lambda',0,\mu}$ is self adjoint and hence $\sigma(\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}) \subset \mathbb{R}$.*
2. *If not for all $(i,j) \in \Omega_n$, $\lambda_{ij} = 0$, then for every $\alpha > \sum_{(i,j)\in\Omega_n}\left(b_1^{(i,j)} + b_2^{(i,j)} + b_3^{(i,j)}\right)$, there exists $r_0 > 0$ satisfying*

$$\sigma(\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}) \subset B_{r_0}(0) \cup \{(x+\boldsymbol{i}y);\ x \geq 0 \text{ and } |y| \leq \alpha \max_{(i,j)\in\Omega_n}\{x^{\frac{2}{3}}, x^{\beta_{ij}}\}\}.$$

Proof. Firstly, since $\mathfrak{D}_{\lambda''}$ has compact resolvent, then $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ has so. For the first case, $\mathfrak{D}_{\lambda''}$ is self-adjoint and $\mathcal{R}^{\beta}_{\lambda'',\lambda',0,\mu}$ is symmetric. Then the result follows from Kato-Rellich Theorem [25]. For the second case, $\mathfrak{D}_{\lambda''}$ is self-adjoint with compact resolvent and $\mathcal{R}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ is non-symmetrical. So, we should apply Proposition 3.6 and Theorem 2.4. □

Theorem 3.10. *We have the two following assertions:*

1. *If for all $(i,j) \in \Omega_n$, $\lambda_{ij} = 0$ and $\sum_{(i,j)\in\Omega_n}\left(\frac{1}{3}b_1^{(i,j)}\beta_{ij} + \frac{2}{3}b_3^{(i,j)}\right) < 1$, then $\mathcal{M}^{\beta}_{\lambda'',\lambda',0,\mu}$ admits an orthonormal basis of eigenvectors.*
2. *If not for all $(i,j) \in \Omega_n$, $\lambda_{ij} = 0$, then $\mathcal{M}^{\beta}_{\lambda'',\lambda',\lambda,\mu}$ admits a Riesz basis with parentheses.*

Proof. For the first case, since $\mathcal{M}^{\beta}_{\lambda'',\lambda',0,\mu}$ is self-adjoint with compact resolvent, then we apply the spectral theorem. For the second case, let $1 \leq j \leq n$ and $\xi_k := \lambda''_j(k-2)(k-1)k$ the eigenvalue number k of $G_{\lambda''_j}$. Put $r_k = \frac{\xi_k+\xi_{k+1}}{2}$. We have

$$\frac{N(r_k, G_{\lambda''_j})}{r_k^{1-\frac{2}{3}}} = \frac{2^{\frac{1}{3}}}{{\lambda''_j}^{\frac{1}{3}}}\frac{k}{(k(k-1)(2k-1))^{1-\frac{2}{3}}} \underset{r_k\longrightarrow\infty}{\sim} \frac{1}{{\lambda''_j}^{\frac{1}{3}}} < +\infty.$$

Hence, To finish, it suffices to apply Proposition 3.6 and Theorem 2.5. □

Remark 3.11. *By virtue of Theorems 3.9 and 3.10, the operator matrices $\mathfrak{D}_{\lambda''}$, $\mathfrak{S}_{\lambda'}$, $\mathfrak{H}_{\lambda}$ and $\mathfrak{H}^{\beta}_{\mu}$, have the same spectral properties of their corresponding entries.*

We close this paper by a simple example of Gribov operator matrix by taking particular values of the real parameters of Pomeron.

Example 3.12. *We consider the Gribov matrix:*

$$\mathcal{M}^{3p}_{\lambda'',\lambda',0,\mu} = \lambda'' Diag(G,\dots,G) + \left((1-\delta_{ij})p_{ij}H_0^{3p_{ij}}\right)_{1\leq i,j\leq n},$$

where $\{p_{ij}\}_{(i,j)\in\Omega_n}$ is a double indexed and decreased sequence defined by:

$$\begin{cases} p_{12} = p_{21} = \frac{1}{3}, \\ p_{ij} = \frac{1}{a^{i+j}}, \text{ for } (i,j) \in \Omega_n \setminus \{(1,2),(2,1)\} \ (a \geq \frac{7}{5}). \end{cases}$$

Proposition 3.13. *Suppose that* $\gamma := \frac{c_1}{|\lambda''|} < 1$, *then* $\sigma\Big(\mathcal{M}^{3p}_{\lambda'',\lambda',0,\mu}\Big) \subset \mathbb{R}$ *and* $\mathcal{M}^{3p}_{\lambda'',\lambda',0,\mu}$ *admits an orthonormal basis of eigenvectors.*

Proof. According to Theorems 3.9 and 3.10, it suffices to prove that $\displaystyle\sum_{(i,j)\in\Omega_n} \gamma^{p_{ij}} p_{ij}^2 < 1$.

We have

$$\sum_{(i,j)\in\Omega_n} \gamma^{p_{ij}} p_{ij}^2 \le 2 \sum_{1\le i<j\le n} p_{ij}^2 = 2p_{12}^2 + 2 \sum_{2\le i<j\le n} p_{ij}^2 = \frac{2}{9} + 2 \sum_{2\le i<j\le n} p_{ij}^2.$$

Put $S := \displaystyle\sum_{2\le i<j\le n} p_{ij}^2$. Then $S = \displaystyle\sum_{i=2}^{n-1} \frac{S_i}{a^{2i}}$, where $S_i = \displaystyle\sum_{j=i+1}^{n} \frac{1}{a^{2j}}$. Thus,

$$S_i = \frac{1}{a^{2(i+1)}} \frac{1-(\frac{1}{a^2})^{n-i}}{1-\frac{1}{a^2}} < \frac{1}{a^{2i}(a^2-1)}.$$

Hence,

$$S < \frac{1}{a^2-1} \sum_{i=2}^{n-1} \frac{1}{a^{4i}} < \frac{1}{a^4(a^4-1)(a^2-1)}.$$

Since $a \ge \frac{7}{5}$, then $S < \frac{7}{18}$ and therefore, $\displaystyle\sum_{(i,j)\in\Omega_n} \gamma^{p_{ij}} p_{ij}^2 < 1$. □

References

[1] B. Abdelmoumen, *Essential Spectra of Some Matrix Operators by Means of Measures of weak noncompactness*. Oper. Matr, **8**, 205-216, (2014). doi:10.7153/oam-08-11.
[2] B. Abdelmoumen and A. Lefi, On an unconditional basis with parenthese for generalized subordinate perturbations and application to Gribov operators in Bargmann space. Ind. Math., **28**, 1002-1018, (2017).
[3] B. Abdelmoumen, S. Chakroun and M. Mnif, Perturbation results in the Fredholm theory and M-essential spectra of some matrix operators, Filomat **35**, 2141-2149, (2021).
[4] F. V. Atkinson, H. Langer, R. Mennicken and A. Shkalikov, The essential spectrum of some matrix operators, Math. Nachr., **167**, 5-20, (1994).
[5] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*. Commun. Pure Appl. Math., **14**, 187-214, (1961).
[6] S. Charfi, A. Damergi and A. Jeribi, *On a riesz basis of finite-dimensionanl invariant subspaces and application to Gribov operators in Bargmann space*. Linear and Multilinear Algebra, **61**, 1577-1591, (2013).
[7] N. Dunford and J.T. Schwartz, *Linear operators III*. Wiley-Interscience, New-York. (1971).
[8] K-J. Engel, *Matrix representation of linear operators on product spaces*. Rend. Circ. Mat. Palermo Series II, Suppl **56**, 219-224, (1998).
[9] K-J. Engel, *Spectral theory and generator property for one-sided coupled operators matrices*. Semigroup Forum., **58**, 267-295, (1999).
[10] I. C. Gohberg and M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators in Hilbert space.* Amer. Math. Soc., Providence, (1969).
[11] V. N. Gribov, *A reggon diagram technique.* J. E. T. P. Sov. Phys. **26**, 414-423, (1968).
[12] A. Intissar, *Analyse fonctionnelle et théorie spectrale pour les opérateurs compacts non-autoadjoints*. Editions CEPADUES, (1997).
[13] B. Jacob, C. Trunk, M. Winklmeier, *Analyticity and Riesz basis property of semigroups associated to damped vibrations*. J. Evol. Equ., **8**, 263-281, (2008).
[14] T. Kato, *Perturbation Theory for Linear Operators*. Classics Math., Springer, Berlin, 1995, reprint of the 1980 edition.
[15] S.G. Krein, *Linear differential equations in Banach space*, Amer. Math. Soc., Providence, (1971).
[16] C. R. Kuiper, H. J. Zwart, *Connection between the algebraic Riccati equation and the Hamiltonian for Riesz- spectral systems*. J. Math. Systems Estim. Control., **6**, 1-48, (1996).
[17] H. Langer, A.S. Markus, V.I. Matsaev and C. Tretter, *A new concept for block operator matrices: The quadratic numerical range.* Linear. Alg. Appl., **330**, 89-112, (2001).
[18] H. Langer and C. Tretter, *Spectral decomposition of some nonselfadjoint block operator matrix*. J. Oper. Th., **39**, 339-359, (1998).
[19] A. S. Markus, *Introduction to the spectral theory of polynomial operators pencils*. Trans. Math. Monogr., **71**, Amer. Math. Soc., Providence, RI, (1988).
[20] R. Nagel, *Towards a "matrix theory" for unbounded operator matrices*. Math. Z., **201**, 57-68. (1989).
[21] R. Nagel, *The spectrum of unbounded operator matrices with no-diagonal domain*. J. Funct. Anal., **89**, 291-302, (1990).
[22] B. Sz. Nagy, *Perturbations des transformations linéaires fermées.* Acta Sci. Math. Szeged, **14**, 125-137, (1951).

[23] T. H. Rasulov and C. Tretter *Spectral inclusion for unbounded diagonally dominant $n \times n$ operator matrices.* Rocky Mt. J. Math. **48**, 279-324, (2018).
[24] M. Reed and B. Simon, *Methods of modeern mathematical,* vol IV: Analysis of operators, Academic Press, New York, (1978).
[25] F. Rellich, *Storungstheorie der Spektral zerlegung,* I. Mitt. Math. Ann., **113**, 600-619, (1937).
[26] I. Singer, *Bases in Banach Spaces II,* Springer, Berlin, (1981).
[27] C. Tretter, *Spectral Theory of block operator matrices and applications.* Imperial College Press, London, (2008).
[28] C. Tretter, *Spectral inclusion for unbounded block operator matrices*. J. Funct. Anal., **256**, 3806-3829, (2009).
[29] C. Tretter and M. Wagenhofer, *The block numerical range of an $n \times n$ block operator matrix.* SIAM J. Matrix Anal. Appl. **24**, 1003-1017, (2003).
[30] C. Wyss, *Perturbation theory for Hamiltonian operator matrices and Riccati equations.* PhD thesis. University of Bern, (2008).
[31] C. Wyss, *Riesz bases for p-subordinate perturbations of normal operator.* J. Funct. Anal. **258**, 208-240, (2010).