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Complete convergence for weighted sums of widely negative dependent random variables under the sub-linear expectations

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Abstract. In the paper, the complete convergence for weighted sums of arrays of rowwise widely negative dependent random variables under the sub-linear expectations is established. The main result partially extend some known results both in the classical probability space and sub-linear expectation space. In addition, it weaken their conditions.

1. Introduction

The classical complete convergence, which is one of the fundamental limit theorems in probability theory, plays a major role in the development of probability theory and its applications. The initial concept of complete convergence of sequence of random variables was introduced by Hsu and Robbins [7]. After that, a great deal of domestic and foreign scholars had shown in-depth and extended researches on it, such as Wang and Hu [18], Bai et al. [3] and Hu et al. [8]. It is well known that the analysis of weighted sums was also important in the statistic, such as jackknife estimate, nonparametric regression function estimate and so on. Many authors studied the complete convergence for the weighted sums of random variables, we can refer to An and Yuan [2], Ahmed et al. [1] and Liang [10].

However, in practice, many problems such as uncertainties in statistics, measures of risks, superhedging in finance and non-linear stochastic calculus can not be resolved by the classical probability theories. Motivated by this phenomena, the general sub-linear expectations and related non-additive probabilities generated by them were considered. The general framework of the sub-linear expectation was introduced by Peng [14–17] in a general function space by relaxing the linear property of the classical expectation to the sub-additivity and positive homogeneity (cf. Definition 2.1 below). Under Peng's framework, many limit theorems have been established recently, for example, Liu and zhang [12] studied the law of the iterated logarithm for linear processes generated by a sequence of stationary independent random variables, Li [9] obtained the central limit theorem for m-dependent sequences, Guo and Li [6] showed the laws of large numbers for pseduo-independent random variables, Zhang [19] revealed the Donsker's invariance principle

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and Chung's law of the iterated logarithm, Chen and Xiong [5] acquired the large deviation principle for diffusion processes, Zhong and Wu [24] gained the complete convergence and complete moment convergence for extended negatively dependent random variables, Zhang [20] received the self-normalized moderate deviation and laws of the iterated logarithm under G-expectation and so on. Besides these, many important and fundamental inequalities serve as tools in the probability theory such as Rosenthal type inequality, Fuk and Nagave type inequalities, Kolmogorov type exponential inequalities, L$\acute{e}$vy maximal inequality in Zhang [22] and Zhang [23] and so on. In this paper, we study the complete convergence for weighted sums of widely negative dependent random variables under association assumptions by combining the properties of sublinear expectations and using local Lipschitz function. We can extend some known complete convergence conclusions in the traditional probability space to the case of sub-linear expectation space is of great significance in the theory and application.

This paper proceeds as follows. In Section 2, we recall some basic notions and definitions under sub-linear expectations which will be used in this article. In Section 3, we state the main results of this article. In Section 4, we give some useful lemmas and the proof of the complete convergence which is the main result in this paper. Throughout this paper, the symbol C denotes a positive constant which may take different values whenever it appears in different expressions and the symbol c_n' denotes the derivative with respect to n .Let $I(A)$ denote the indicator function of the event A and $\log x = \max \log\{x, e\}$. Denote $x \vee y = \max(x, y)$ for $x, y \in \mathbb{R}$.

2. Basic Settings

We use the framework and notations of Peng [17]. Let $(\Omega, \mathcal{F})$ be a given measurable space. Let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$, such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz continuous functions φ satisfying

$$|\varphi(x) - \varphi(y)| \le c(1 + |x|^m + |y|^m)|x - y|, \quad \text{for all} \quad x, y \in \mathbb{R}^n,$$

for some $c > 0$ and $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ contains all I_A where $A \in \mathcal{F}$. We also denote $\varphi \in C_{b,Lip}(\mathbb{R}^n)$ as the linear space of bounded Lipschitz continuous functions φ satisfying

$$|\varphi(x) - \varphi(y)| \le c|x - y|,$$

for all $x, y \in \mathbb{R}^n$ and some $c > 0$.

Definition 2.1. A function $\hat{\mathbb{E}} : \mathcal{H} \to [-\infty, \infty]$ is said to be a sub-linear expection if it satisfies for any $X, Y \in \mathcal{H}$,

(a) Monotonicity: $X \ge Y$ implies $\hat{\mathbb{E}}[X] \ge \hat{\mathbb{E}}[Y]$;

(b) Constant preserving: $\hat{\mathbb{E}}[c] = c, \ \forall c \in \mathbb{R}$;

(c) Sub-additivity: $\hat{\mathbb{E}}[X + Y] \le \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;

(d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \ \forall \lambda \ge 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space. Give a sub-linear expectation $\hat{\mathbb{E}}$, let us denote conjugate expectation $\hat{\mathcal{E}}$ of $\hat{\mathbb{E}}$ by

$$\hat{\mathcal{E}}[X] := -\hat{\mathbb{E}}[-X], \ \forall X \in \mathcal{H}.$$

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$.

Definition 2.2. ([22]) A function $\mathbb{V} : \mathcal{G} \to [0, 1]$ is called a capacity if

(i) $\mathbb{V}(\emptyset) = 0, \mathbb{V}(\Omega) = 1$;

(ii) $\mathbb{V}(A) \leq \mathbb{V}(B), \ \forall A \subset B, \ A, B \in \mathcal{G}$.

It is called to be sub-additive if $\mathbb{V}(A \bigcup B) \leq \mathbb{V}(A) + \mathbb{V}(B)$ for all $A, B \in \mathcal{G}$ with $A \bigcup B \in \mathcal{G}$. A pair $(\mathbb{V}, \mathcal{V})$ of capacity were generated by a pair $(\hat{\mathbb{E}}, \hat{\mathcal{E}})$ of expectation denoted by:

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \ \ \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \text{for all} \quad A \in \mathcal{F},$$

where A^c is the complement set of A. Then

$$\begin{aligned} \mathbb{V}(A) := \hat{\mathbb{E}}[I_A], \ \ \mathcal{V}(A) := \hat{\mathcal{E}}[I_A], \quad \text{if} \quad I_A \in \mathcal{H}. \\ \hat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}[g], \ \ \hat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \hat{\mathcal{E}}[g], \quad \text{if} \quad f \leq I_A \leq g, \ \ f, \ g \in \mathcal{H}. \end{aligned} \tag{2.1}$$

This implies $\mathbb{V}$ is sub-additive from $\mathbb{V}(A \cup B) := \hat{\mathbb{E}}[I_{A\cup B}] \leq \hat{\mathbb{E}}[I_A + I_B] \leq \hat{\mathbb{E}}[I_A] + \hat{\mathbb{E}}[I_B] = \mathbb{V}(A) + \mathbb{V}(B)$ and Markov's inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \hat{\mathbb{E}}[|X|^p]/x^p, \quad \text{for all} \quad x > 0, \ p > 0$$

from $I(|X| \geq x) \leq |X|^p/x^p \in \mathcal{H}$. By Lemma 2.2 in Lin and Feng [11], we have Jensen's inequality: Let $f(x)$ be a convex function on $\mathbb{R}$. Suppose that $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}[f(X)]$ exist, then $f(\hat{\mathbb{E}}[X]) \leq \hat{\mathbb{E}}[f(X)]$.

In addition, a pair $(C_{\mathbb{V}}, C_{\mathcal{V}})$ of the Choquet integrals/expectations denoted by

$$C_V[X] = \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 [V(X \geq t) - 1]dt$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

If $\lim\limits_{c\to\infty} \hat{\mathbb{E}}[(|X| - c)^+] = 0$ or $\hat{\mathbb{E}}$ is countably sub-additive, then $\hat{\mathbb{E}}(|X|) \leq C_{\mathbb{V}}(|X|)$(cf. Zhang [22, Lemma 4.5 (iii)]).

Definition 2.3. ([21])

(I) A sub-linear expectation $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ is called to be countably sub-linear if it satisfies

  (i) Countably sub-additivity: $\hat{\mathbb{E}}[X] \leq \sum_{n=1}^\infty \hat{\mathbb{E}}[X_n]$, whenever $X \leq \sum_{n=1}^\infty X_n, X, X_n \in \mathcal{H}$ and $X \geq 0, X_n \geq 0, n = 1, 2, \ldots$.
  It is called continuous if it satisfies:

  (ii) Continuity from below: $\hat{\mathbb{E}}[X_n] \uparrow \hat{\mathbb{E}}[X]$ if $0 \leq X_n \uparrow X$, where $X_n, X \in \mathcal{H}$.

  (iii) Continuity from above: $\hat{\mathbb{E}}[X_n] \downarrow \hat{\mathbb{E}}[X]$ if $0 \leq X_n \downarrow X$, where $X_n, X \in \mathcal{H}$.

(II) A function $\mathbb{V} : \mathcal{F} \to [0, 1]$ is called to be countably sub-additive if

$$\mathbb{V}\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mathbb{V}(A_n), \quad \text{for all} \quad A_n \in \mathcal{F}.$$

(III) A capacity $\mathbb{V} : \mathcal{F} \to [0, 1]$ is called a continuous capacity if it satisfies

  (i) Continuity from below: $\mathbb{V}(A_n) \uparrow \mathbb{V}(A)$ if $A_n \uparrow A$, where $A_n, A \in \mathcal{F}$.

  (ii) Continuity from above: $\mathbb{V}(A_n) \downarrow \mathbb{V}(A)$ if $A_n \downarrow A$, where $A_n, A \in \mathcal{F}$.

Because $\mathbb{V}$ may be not countably sub-additive in general, we define an outer capacity $\mathbb{V}^*$ by

$$\mathbb{V}^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mathbb{V}(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n\right\}, \ \ \mathcal{V}^*(A) = 1 - \mathbb{V}^*(A^c), \quad A \in \mathcal{F}.$$

Then it can be shown that $\mathbb{V}^*(A)$ is a countably sub-additive capacity with $\mathbb{V}^*(A) \le \mathbb{V}(A)$ and if $\mathbb{V}$ is countably sub-additive, then $\mathbb{V}^* \equiv \mathbb{V}$(cf. Zhang [21]).

Definition 2.4. ([13]) Let $X_1, X_2, \ldots, X_{n+1}$ be real measurable random variables of $(\Omega, \mathcal{F})$. X_{n+1} is called widely negative dependent of $(X_1, X_2, \ldots, X_n)$ under $\hat{\mathbb{E}}$, if for every nonnegative measurable function $\varphi_i(\cdot)$ with the same monotonicity on $\mathbb{R}$ and $\hat{\mathbb{E}}[\varphi_i(X_i)] < \infty, i = 1, \ldots, n+1$, there exists a positive finite real number $g(n+1)$ such that

$$\hat{\mathbb{E}}\left[\prod_{i=1}^{n+1} \varphi_i(X_i)\right] \le g(n+1)\hat{\mathbb{E}}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right]\hat{\mathbb{E}}\left[\varphi_{n+1}(X_{n+1})\right].$$

$\{X_i\}_{i=1}^{\infty}$ is said to be a sequence of widely negative dependent random variables, if for any $n \ge 1, X_{n+1}$ is widely negative dependent of $(X_1, X_2, \ldots, X_n)$.

$\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ is said to be an array of rowwise widely negative dependent random variables, if for any $n \ge 1$, $\{X_{nk}, 1 \le k \le k_n\}$ is a sequence of widely negative dependent random variables.

Remark 2.1. *For a sequence of widely negative dependent random variables $\{X_i\}_{i=1}^{\infty}$, we have*

$$\hat{\mathbb{E}}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right] \le \tilde{g}(n) \prod_{i=1}^{n} \hat{\mathbb{E}}\left[\varphi_i(X_i)\right], \ \ \textit{where} \ \ \tilde{g}(n) := \prod_{i=1}^{n} g(i),$$

for any $n \ge 1$ and every nonnegative measurable function $\varphi_i(\cdot)$ with the same monotonicity on $\mathbb{R}$ and $\hat{\mathbb{E}}\varphi_i(X_i) < \infty$, $i = 1, \ldots, n$. Without loss of generality, we will assume that $g(n) \ge 1$ for any $n \ge 1$ in the sequel. If $\{X_i\}_{i=1}^{\infty}$ is a sequence of extend negatively dependent random variables, then $\tilde{g}(n) = K$, where $K \ge 1$ is a dominating constant by Definition 2.1 in [24], if $\{X_i\}_{i=1}^{\infty}$ is a sequence of negatively dependent random variables, then $\tilde{g}(n) = 1$ by Definition 2.3 in [22].

3. Preliminaries

Before proving the main results, we state the following several useful lemmas.

Lemma 3.1. *([11, Lemma 2.6]) Suppose that $\{X_i\}_{i=1}^{\infty}$ is a sequence of widely negative dependent random variables under $\hat{\mathbb{E}}$, and $\{\psi_i(x)\}_{i=1}^{\infty}$ is a sequence of measurable function with the same monotonicity. Then $\{\psi_i(X_i)\}_{i=1}^{\infty}$ is also a sequence of widely negative dependent random variables.*

Lemma 3.2. *Let $\{X_n, n \ge 1\}$ be an sequence of widely negative dependent random variables such that $\hat{\mathbb{E}}[X_n] \le 0$ and $\hat{\mathbb{E}}[X_n^2] < \infty$. Then for all x, y and $r > 0$,*

$$\mathbb{V}\left(\sum_{k=1}^{n} X_k \ge x\right) \le \sum_{k=1}^{n} \mathbb{V}\left(|X_k| \ge \frac{2x}{r}\right) + C\tilde{g}(n)x^{-r}B_n^{r/2}, \tag{3.1}$$

where $B_n = \sum_{k=1}^{n} \hat{\mathbb{E}}[X_k^2]$.

Proof. Let $Y_k = X_k \wedge y$, $T_n = \sum_{k=1}^{n} Y_k$. Then $X_k - Y_k = (X_k - y)^+ \ge 0$ and $\hat{\mathbb{E}}[Y_k] \le \hat{\mathbb{E}}[X_k] \le 0$. Note that $\varphi(x) := e^{t(x \vee y)}$ is a bounded non-decreasing function and belongs to $C_{l,Lip}(\mathbb{R})$ since $0 \le \varphi'(x) \le te^{ty}$ if $t \ge 0$. Then by the definition of widely negative dependent and Markov inequality, we have for any $t > 0$,

$$\mathbb{V}(S_n \ge x) \le \mathbb{V}(T_n \ne S_n) + \mathbb{V}(T_n \ge x)$$

$$\leq \sum_{k=1}^{n} \mathbb{V}(|X_k| > y) + e^{-tx}\hat{\mathbb{E}}[e^{tT_n}]$$
$$\leq \sum_{k=1}^{n} \mathbb{V}(|X_k| > y) + \tilde{g}(n)e^{-tx}\prod_{k=1}^{n}\hat{\mathbb{E}}[e^{tY_k}]. \tag{3.2}$$

Note that the function $h(x) = (e^{tx} - 1 - tx)/x^p$ is increasing for all $x > 0$, with $t > 0$, $0 < p \leq 2$. Therefore

$$\begin{aligned}\hat{\mathbb{E}}[e^{tY_k}] &\leq t\hat{\mathbb{E}}[Y_k] + 1 + \frac{e^{ty} - ty - 1}{y^2}\hat{\mathbb{E}}[|Y_k|^2]\\ &\leq 1 + \frac{e^{ty} - ty - 1}{y^2}\hat{\mathbb{E}}[|X_k|^2]\\ &\leq \exp\left\{\frac{e^{ty} - ty - 1}{y^2}\hat{\mathbb{E}}[|X_k|^2]\right\}.\end{aligned} \tag{3.3}$$

Choosing $t = \frac{1}{y}\ln(1 + \frac{xy}{B_n})$ yields

$$\begin{aligned}&\mathbb{V}\left(\sum_{k=1}^{n} X_k \geq x\right)\\ &\leq \sum_{k=1}^{n} \mathbb{V}(|X_k| \geq y) + \tilde{g}(n)\exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy}{B_n}\right)\right\}.\end{aligned}$$

Let $y = 2x/r$, we get

$$\mathbb{V}\left(\sum_{k=1}^{n} X_k \geq x\right) \leq \sum_{k=1}^{n} \mathbb{V}\left(|X_k| \geq \frac{2x}{r}\right) + C\tilde{g}(n)x^{-r}B_n^{r/2}. \tag{3.4}$$

So the proof of Lemma 3.2 is completed. □

Lemma 3.3. (*[21, Lemma 3.6]*)

(i) If $\hat{\mathbb{E}}$ is continuous from below, then it is countably sub-additive. Similarly, if $\mathbb{V}$ is continuous from below, then it is countably sub-additive.

(ii) Set $\mathbb{H} = \{A : I_A \in \mathcal{H}\}$, then $\mathbb{V}$ is a countably sub-additive capacity in $\mathbb{H}$ if $\hat{\mathbb{E}}$ is countably sub-additive in $\mathcal{H}$, and $(\mathbb{V}, \mathcal{V})$ is a pair of continuous capacities in $\mathbb{H}$ if $\hat{\mathbb{E}}$ is continuous in $\mathcal{H}$.

4. Main results

4.1. A general theorem

The following Theorem 4.1 is our main result, which can induce some known results.

Theorem 4.1. *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise widely negative dependent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, there exists a random variable $X \in \mathcal{H}$ and a constant C satisfying*

$$\frac{1}{n}\sum_{i=1}^{n} \hat{\mathbb{E}}[f(X_{ni})] \leq C\hat{\mathbb{E}}[f(X)], \tag{4.1}$$

for all $1 \le i \le n$, $n \ge 1$ *and for any* $0 < f \in C_{l,Lip}(\mathbb{R})$. *Suppose that for* $r, p > 0$, $\hat{\mathbb{E}}[|X|^p] \vee C_{\mathbb{V}}[|X|^p] < \infty$ *and for* $p \ge 1$, $\hat{\mathbb{E}}[X_{ni}] = \hat{\mathbb{E}}[-X_{ni}] = 0$. *Assume that* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *is an array of real numbers,* $\{b_n, n \ge 1\}$ *and* $\{c_n, n \ge 1\}$ *be two sequences of positive constants, the following conditions hold:*

$$c_n^{-p} n \max_{1 \le i \le n} |a_{ni}| = o(1),$$

$$b_n n = O\left(c_n' c_n^{p-1}\right),$$

$$b_n n \max_{1 \le i \le n} |a_{ni}|^r = O\left(c_n' c_n^{p-1}\right),$$

$$\sum_{j=k}^{\infty} \theta^j \left(b_n c_n^{-r} n \max_{1 \le i \le n} |a_{ni}|^r\right)\Big|_{n=\theta^j} = O\left((c_{\theta^k})' (c_{\theta^k})^{p-r-1}\right) \quad \text{for any} \quad \theta > 1,$$

$$\sum_{n=1}^{\infty} b_n c_n^{-r} \tilde{g}(n) n^{\frac{r}{2}} \max_{1 \le i \le n} |a_{ni}|^r < \infty \quad \text{for} \quad p \ge 2$$

and

$$\sum_{n=1}^{\infty} b_n c_n^{-\frac{rp}{2}} \tilde{g}(n) n^{\frac{r}{2}} \max_{1 \le i \le n} |a_{ni}|^r < \infty \quad \text{for} \quad 0 \le p < 2.$$

Then for any $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} b_n \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon c_n\right) < \infty \tag{4.2}$$

and

$$\sum_{n=1}^{\infty} b_n \mathbb{V}\left(-\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon c_n\right) < \infty. \tag{4.3}$$

Remark 4.1. *Theorem 4.1 can be applied many random sequences in the sub-linear expectation space, such as extended negatively dependent sequence, negatively dependent sequence, independent and identically distributed sequence, widely acceptable sequence, strictly stationary independent sequence, extended independence and so on.*

Corollary 4.1. *Let* $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of rowwise widely negative dependent random variables in* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. *Suppose that* $\hat{\mathbb{E}}$ *is continuous from below, for* $r, p > 0$, $C_{\mathbb{V}}[|X|^p] < \infty$ *and for* $p \ge 1$, $\hat{\mathbb{E}}[X_{ni}] = \hat{\mathbb{E}}[-X_{ni}] = 0$, *dominated condition (4.1) and the six conditions in Theorem 4.1 remain unchanged. Then for any* $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} b_n \mathbb{V}^*\left(\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon c_n\right) < \infty \tag{4.4}$$

and

$$\sum_{n=1}^{\infty} b_n \mathbb{V}^*\left(-\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon c_n\right) < \infty. \tag{4.5}$$

The following is the sufficient part of the Hsu-Robbins-Erd*ös* strong law generalized to the weighted sums in the sub-linear expectation space.

Corollary 4.2. *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise independent and dominated condition (4.1) random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_{ni}] = \hat{\mathbb{E}}[-X_{ni}] = 0$ and $\hat{\mathbb{E}}[|X|^2] \vee C_{\mathbb{V}}[|X|^2] < \infty$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max\limits_{1 \le i \le n} |a_{ni}|^r \le C$ for some $r > 2$. Then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni} X_{ni}\right| > \varepsilon n\right) < \infty. \tag{4.6}$$

Remark 4.2. *The sufficient part of Hsu-Robbins-Erdös strong law in the classical probability space is that: Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d random variables with zero mean and $\mathbb{E}X_1^2 < \infty$, then*

$$\sum_{i=1}^{n} \mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| > \varepsilon n\right) < \infty. \tag{4.7}$$

It is obvious that the dominated condition (4.1) is weaker than identically distributed and we extend the conclusion to weighted sums under the sub-linear expectation, so it is a useful supplement for (4.7).

The next one is Baum-Katz type complete convergence, we expand it from the classical probability space to sub-linear expectation space and optimize the conclusions by Theorem 4.1.

Corollary 4.3. *Let $\alpha > 1/2$, $\alpha p > 1$ and $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise widely negative dependent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_{ni}] = \hat{\mathbb{E}}[-X_{ni}] = 0$ when $p \ge 1$ and dominated condition (4.1). Assume that $\tilde{g}(n)$ is regularly varying function with index αt for some $t > 0$ and for some $r > \max\{(\alpha p + \alpha t - 1)/(\alpha - \frac{1}{2}), 2(\alpha p + \alpha t - 1)/(\alpha p - 1)\}$, $\hat{\mathbb{E}}[|X|^p] \vee C_{\mathbb{V}}[|X|^p] < \infty$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max\limits_{1 \le i \le n} |a_{ni}| = O(\log^{-1} n)$. Then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon n^{\alpha}\right) < \infty. \tag{4.8}$$

Remark 4.3. *Let $\alpha > 1/2$, $\alpha p > 1$ and $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise widely negative dependent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Assume that $\tilde{g}(n)$ is regularly varying function with index αt for some $t > 0$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\sum_{i=1}^{n} |a_{ni}|^r = O(n)$, where $r > \max\{p, (\alpha p + \alpha t - 1)/(\alpha - \frac{1}{2}), 2(\alpha p + \alpha t - 1)/(\alpha p - 1)\}$ and $\hat{\mathbb{E}}[|X|^p] \vee C_{\mathbb{V}}[|X|^p] < \infty$. Moreover, we assume that $\hat{\mathbb{E}}[X_{ni}] = \hat{\mathbb{E}}[-X_{ni}] = 0$ and there exist a random variable $X \in \mathcal{H}$ and a constant C, satisfying*

$$\hat{\mathbb{E}}[f(X_{ni})] \le C\hat{\mathbb{E}}[f(X)], \tag{4.9}$$

for all $1 \le i \le n$, $n \ge 1$ and for any $0 < f \in C_{l,Lip}(\mathbb{R})$. Then for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon n^{\alpha}\right) < \infty. \tag{4.10}$$

Lu and Meng [13] proved the above conclusion (4.10). By comparing with the result in Lu and Meng [13], it is clear that if X dominates the sequence $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ in the condition (4.9), then it also dominates the sequence in the condition (4.1), so we weaken the dominated condition for X.

Corollary 4.4. *Let $1 < \alpha < 2$, $\alpha < p$ and $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise negative dependent random variables in$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_{ni}] = \hat{\mathbb{E}}[-X_{ni}] = 0$ and the dominated condition (4.1). Assume that $\hat{\mathbb{E}}[|X|^p] \vee C_{\mathbb{V}}[|X|^p] < \infty$ and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying*

$$\max_{1 \le i \le n} |a_{ni}|^r = O\left(n^{p/\alpha - 1} (\log n)^{1-r}\right),$$

where $r > \max\{2, p, 2(p - \alpha)/(2 - \alpha)\}$. Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{p}}\right) < \infty. \tag{4.11}$$

Remark 4.4. *Let $1 < \alpha < 2$, $\alpha < p$, $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed negatively orthant dependent random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$. Then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\left(\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{p}} \right) < \infty. \tag{4.12}$$

Chen and Sung [4] have shown the above complete conclusion in the classical probability space, it is obvious that if $\tilde{g}(n) = 1$ and $\mathcal{P} = \{P\}$, Corollary 4.4 is an interesting supplement for Theorem 1.1 in Chen and Sung [4] in the sub-linear expectation space because the two $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ controls don't contain each other and the condition (4.1) is weaker than identically distributed.

5. proofs of main results

Proof. **[Proof of Theorem 4.1]** We only need to prove (4.2), because of considering $\{-X_{ni}, 1 \leq i \leq n, n \geq 1\}$ instead of $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ in (4.2), we get (4.3). For fixed $n \geq 1$, denote for $1 \leq i \leq n$ that

$$Y_{ni} = X_{ni} I(|X_{ni}| \leq c_n) + c_n I(X_{ni} > c_n) - c_n I(X_{ni} < -c_n).$$

Note that for all $\varepsilon > 0$

$$\left\{ \sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon c_n \right\} \subset \bigcup_{i=1}^{n} \{|X_{ni}| > c_n\} \cup \left\{ \sum_{i=1}^{n} a_{ni} Y_{ni} > \varepsilon c_n \right\},$$

which yields

$$\begin{aligned}
&\sum_{n=1}^{\infty} b_n \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon c_n \right) \\
&\leq \sum_{n=1}^{\infty} b_n \sum_{i=1}^{n} \mathbb{V}(|X_{ni}| > c_n) + \sum_{n=1}^{\infty} b_n \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} Y_{ni} > \varepsilon c_n \right) \\
&= \sum_{n=1}^{\infty} b_n \sum_{i=1}^{n} \mathbb{V}(|X_{ni}| > c_n) \\
&\quad + \sum_{n=1}^{\infty} b_n \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]) > \varepsilon c_n - \sum_{i=1}^{n} a_{ni} \hat{\mathbb{E}}[Y_{ni}] \right) \\
&:= I + J.
\end{aligned}$$

We note that (4.1) does not imply $\sum_{i=1}^{n} \mathbb{V}(f(X_{ni}) \in A) \leq Cn\mathbb{V}(f(X) \in A)$. Hence, to deal with $\mathbb{V}(f(X_{ni}) \in A)$, we need to convert $\mathbb{V}$ to $\hat{\mathbb{E}}$ via (2.1). It is easy to see that in the classical probability space, $\mathbb{E}I(|X| \leq a) = \mathbb{P}(|X| \leq a)$, however, it is false since indicator function $I(|X| \leq a) \notin C_{l,Lip}(\mathbb{R})$ but $\hat{\mathbb{E}}$ is defined through functions in $C_{l,Lip}$. We need to modify the indicator function by functions in $C_{l,Lip}$. To this end, we define the function $g(x) \in C_{l,Lip}$ as follows.

For $0 < \mu < 1$, let $g(x) \in C_{l,Lip}(\mathbb{R})$, $0 \leq g(x) \leq 1$ for all x, $g(x) = 1$ if $|x| \leq \mu$, $g(x) = 0$ if $|x| > 1$ and $g(x)$ is not non-increasing function when $x > 0$. Then

$$I(|x| \leq \mu) \leq g(x) \leq I(|x| \leq 1), I(|x| > 1) \leq 1 - g(x) \leq I(|x| > \mu). \tag{5.1}$$

For any $m > 0$, by C_r-inequality, (5.1) and the definition of Y_{ni}, we have

$$\begin{aligned}
|Y_{ni}|^m &\leq C|X_{ni}|^m I(|X_{ni}| \leq c_n) + Cc_n^m I(|X_{ni}| > c_n) \\
&\leq C|X_{ni}|^m g\left(\frac{\mu X_{ni}}{c_n}\right) + Cc_n^m \left(1 - g\left(\frac{X_{ni}}{c_n}\right)\right),
\end{aligned}$$

whence

$$\frac{1}{n}\sum_{i=1}^{n}\hat{\mathbb{E}}[|Y_{ni}|^m] \le C\hat{\mathbb{E}}\Big[|X|^m g\Big(\frac{\mu X}{c_n}\Big)\Big] + Cc_n^m\hat{\mathbb{E}}\Big[\Big(1-g\Big(\frac{X}{c_n}\Big)\Big]$$

$$\le C\hat{\mathbb{E}}\Big[|X|^m g\Big(\frac{\mu X}{c_n}\Big)\Big] + Cc_n^m\mathbb{V}(|X| > \mu c_n) \tag{5.2}$$

from (4.1) and (5.1). Let $g_j(x) \in C_{l.Lip}(\mathbb{R})$, $j \ge 1$ such that $0 \le g_j(x) \le 1$ for all x and $g_j\left(\frac{x}{b_j}\right) = 1$, if $b_{j-1} < |x| \le b_j$, $g_j\left(\frac{x}{b_j}\right) = 0$, if $|x| \le \mu b_{j-1}$ or $|x| > (1+\mu)b_j$. Here we let $b_0 = 1$. Then

$$g_j\left(\frac{X}{b_j}\right) \le I(\mu b_{j-1} < |X| \le (1+\mu)b_j), \quad |X|^m g\Big(\frac{X}{b_k}\Big) \le \sum_{j=1}^{k}|X|^m g_j\Big(\frac{X}{b_j}\Big). \tag{5.3}$$

Firstly, we will show that

$$c_n^{-1}\left|\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}[Y_{ni}]\right| \to 0 \text{ as } n \to \infty. \tag{5.4}$$

For the case of $0 < p < 1$, by Markov inequality and (5.2) with $m = 1$, thus

$$\begin{aligned}
&c_n^{-1}\left|\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}[Y_{ni}]\right| \\
&\le c_n^{-1}\max_{1\le i\le n}|a_{ni}|\sum_{i=1}^{n}\hat{\mathbb{E}}[|Y_{ni}|] \\
&\le Cc_n^{-1}n\max_{1\le i\le n}|a_{ni}|\hat{\mathbb{E}}\left[|X|g\left(\frac{\mu X}{c_n}\right)\right] + Cc_n^{-1}n\max_{1\le i\le n}|a_{ni}|c_n\mathbb{V}(|X| > \mu c_n) \\
&\le Cc_n^{-p}n\max_{1\le i\le n}|a_{ni}|\hat{\mathbb{E}}[|X|^p] \to 0.
\end{aligned}$$

For the case of $p \ge 1$, by $\hat{\mathbb{E}}[X_{ni}] = 0$, (5.1) and condition (4.1), thus

$$\begin{aligned}
c_n^{-1}\left|\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}[Y_{ni}]\right| &\le c_n^{-1}\max_{1\le i\le n}|a_{ni}|\sum_{i=1}^{n}\left|\hat{\mathbb{E}}[X_{ni}] - \hat{\mathbb{E}}[Y_{ni}]\right| \\
&\le c_n^{-1}\max_{1\le i\le n}|a_{ni}|\sum_{i=1}^{n}\hat{\mathbb{E}}[|X_{ni} - Y_{ni}|] \\
&\le c_n^{-1}\max_{1\le i\le n}|a_{ni}|\sum_{i=1}^{n}\hat{\mathbb{E}}[|X_{ni}|I(|X_{ni}| > c_n)] \\
&\le c_n^{-1}\max_{1\le i\le n}|a_{ni}|\sum_{i=1}^{n}\hat{\mathbb{E}}\left[|X_{ni}|\left(1-g\left(\frac{X_{ni}}{c_n}\right)\right)\right] \\
&\le Cc_n^{-1}n\max_{1\le i\le n}|a_{ni}|\hat{\mathbb{E}}\left[|X|\left(1-g\left(\frac{X}{c_n}\right)\right)\right] \\
&\le Cc_n^{-p}n\max_{1\le i\le n}|a_{ni}|\hat{\mathbb{E}}[|X|^p] \to 0.
\end{aligned}$$

So, when $p > 0$, for any $\varepsilon > 0$ and for all n large enough, we have

$$c_n^{-1}\left|\sum_{i=1}^{n}a_{ni}\hat{\mathbb{E}}[Y_{ni}]\right| \le \frac{\varepsilon}{2}. \tag{5.5}$$

For the term I, we get

$$\begin{aligned}
I &\leq \sum_{n=1}^{\infty} b_n \sum_{i=1}^{n} \hat{\mathbb{E}}\left[\left(1-g\left(\frac{X_{ni}}{c_n}\right)\right)\right] \\
&\leq C \sum_{n=1}^{\infty} b_n n \hat{\mathbb{E}}\left[\left(1-g\left(\frac{X}{c_n}\right)\right)\right] \\
&\leq C \sum_{n=1}^{\infty} b_n n \mathbb{V}(|X| > \mu c_n) \\
&\leq C \sum_{n=1}^{\infty} c_n' c_n^{p-1} \mathbb{V}(|X| > \mu c_n) \\
&\leq C C_{\mathbb{V}}[|X|^p] < \infty.
\end{aligned}$$

By Lemma 3.1, $\{a_{ni}(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]), 1 \leq i \leq n, n \geq 1\}$ is also an array of widely negative dependent random variables by the fact that $f_c(x) = xI(|x| \leq c) + cI(x > c) - cI(x < -c) \in C_{l,Lip}(\mathbb{R})$ for any $c > 0$ and $f_c(x)$ being non-decreasing. Then by Lemma 3.2, we have

$$\begin{aligned}
J &\leq \sum_{n=1}^{\infty} b_n \mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\right) > \frac{\varepsilon}{2} c_n\right) \\
&\leq \sum_{n=1}^{\infty} b_n \sum_{i=1}^{n} \mathbb{V}\left(\left|a_{ni}\left(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\right)\right| > \frac{\varepsilon c_n}{r}\right) \\
&\quad + C \sum_{n=1}^{\infty} b_n \tilde{g}(n) c_n^{-r} \left(\sum_{i=1}^{n} \hat{\mathbb{E}}\left[\left(a_{ni}\left(Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\right)\right)^2\right]\right)^{\frac{r}{2}} \\
&:= J_1 + J_2.
\end{aligned}$$

By Markov inequality, C_r-inequality and (5.2) with $m = r$, we get

$$\begin{aligned}
J_1 &\leq C \sum_{n=1}^{\infty} b_n c_n^{-r} \max_{1\leq i\leq n} |a_{ni}|^r \sum_{i=1}^{n} \hat{\mathbb{E}}\left[\left|Y_{ni} - \hat{\mathbb{E}}[Y_{ni}]\right|^r\right] \\
&\leq C \sum_{n=1}^{\infty} b_n c_n^{-r} \max_{1\leq i\leq n} |a_{ni}|^r \sum_{i=1}^{n} \hat{\mathbb{E}}\left[|Y_{ni}|^r\right] \\
&\leq C \sum_{n=1}^{\infty} b_n c_n^{-r} n \max_{1\leq i\leq n} |a_{ni}|^r \hat{\mathbb{E}}\left[|X|^r g\left(\frac{\mu X}{c_n}\right)\right] + C \sum_{n=1}^{\infty} b_n n \max_{1\leq i\leq n} |a_{ni}|^r \mathbb{V}(|X| > \mu c_n) \\
&:= J_{11} + J_{12}.
\end{aligned}$$

By the proof of term I, it is easy to get that $J_{12} < \infty$. As for term J_{11}, via (5.3),

$$\begin{aligned}
J_{11} &\leq \sum_{j=1}^{\infty} \sum_{n=\theta^{j-1}}^{\theta^j} b_n c_n^{-r} n \max_{1\leq i\leq n} |a_{ni}|^r \hat{\mathbb{E}}\left[|X|^r g\left(\frac{X}{c_n}\right)\right] \\
&\leq C \sum_{j=1}^{\infty} \theta^j \left(b_n c_n^{-r} n \max_{1\leq i\leq n} |a_{ni}|^r\right)\Big|_{n=\theta^j} \hat{\mathbb{E}}\left[|X|^r g\left(\frac{X}{c_{\theta^j}}\right)\right] \\
&\leq C \sum_{j=1}^{\infty} \theta^j \left(b_n c_n^{-r} n \max_{1\leq i\leq n} |a_{ni}|^r\right)\Big|_{n=\theta^j} \sum_{k=1}^{j} \hat{\mathbb{E}}\left[|X|^r g_k\left(\frac{X}{c_{\theta^k}}\right)\right]
\end{aligned}$$

$$\leq C\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\theta^j\left(b_nc_n^{-r}n\max_{1\leq i\leq n}|a_{ni}|^r\right)\Big|_{n=\theta^j}(c_{\theta^k})^r\mathbb{V}(|X|>\mu c_{\theta^{k-1}})$$
$$\leq C\sum_{k=1}^{\infty}(c_{\theta^k})'(c_{\theta^k})^{p-1}\mathbb{V}\left(|X|>\mu c_{\theta^{k-1}}\right)$$
$$\leq CC_{\mathbb{V}}[|X|^p]<\infty.$$

Lastly, we consider J_2, for the case of $p\geq 2$, by condition (4.1)

$$J_2\leq C\sum_{n=1}^{\infty}b_nc_n^{-r}\tilde{g}(n)\max_{1\leq i\leq n}|a_{ni}|^r\left(\sum_{i=1}^{n}\hat{\mathbb{E}}\left[Y_{ni}^2\right]\right)^{\frac{r}{2}}$$
$$\leq C\sum_{n=1}^{\infty}b_nc_n^{-r}n^{\frac{r}{2}}\tilde{g}(n)\max_{1\leq i\leq n}|a_{ni}|^r\left(\hat{\mathbb{E}}[|X|^2]\right)^{\frac{r}{2}}$$
$$\leq C\sum_{n=1}^{\infty}b_nc_n^{-r}\tilde{g}(n)n^{\frac{r}{2}}\max_{1\leq i\leq n}|a_{ni}|^r<\infty.$$

for the case of $0<p<2$, by Markov inequality and (5.2) with $m=2$, we have

$$J_2\leq C\sum_{n=1}^{\infty}b_nc_n^{-r}n^{\frac{r}{2}}\tilde{g}(n)\max_{1\leq i\leq n}|a_{ni}|^r\left(c_n^{2-p}\hat{\mathbb{E}}[|X|^p]\right)^{\frac{r}{2}}$$
$$\leq C\sum_{n=1}^{\infty}b_nc_n^{-\frac{pr}{2}}n^{\frac{r}{2}}\tilde{g}(n)\max_{1\leq i\leq n}|a_{ni}|^r<\infty.$$

The proof of Theorem (4.1) is completed. □

Proof. **[Proof of Corollary 4.1]** In view of Definition 2.1, we know that $\hat{\mathbb{E}}$ have the sub-additivity, by (*i*) in Lemma 3.3 and $\hat{\mathbb{E}}$ is continuous from below, $\hat{\mathbb{E}}$ is countably sub-additive, therefore $\hat{\mathbb{E}}[|X|^p]\leq C_{\mathbb{V}}[|X|^p]$. By (*ii*) in Lemma 3.3, we get $\mathbb{V}$ is countably sub-additive, employ the definition of $\mathbb{V}^*$, we get $\mathbb{V}\equiv\mathbb{V}^*$. So we need only to prove

$$\sum_{n=1}^{\infty}b_n\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_{ni}>\varepsilon c_n\right)<\infty.$$

By Theorem 4.1, the proof of Corollary 4.1 is completed. □

Proof. **[Proof of Corollary 4.2]** It is easy for us to confirm the conditions in Theorem 4.1 with $\tilde{g}(n)=1$, $b_n=1$, $c_n=n$, so the proof of Corollary 4.2 is obmited. □

Proof. **[Proof of Corollary 4.3]** We only need to check the conditions in Theorem 4.1. Let $b_n=n^{\alpha p-2}$, $c_n=n^{\alpha}$, it is easy to find that

$$c_n^{-p}n\max_{1\leq i\leq n}|a_{ni}|\leq Cn^{-\alpha p+1}(\log n)^{-1}=o(1),$$
$$b_nn=n^{\alpha p-1}=O\left(c_n'c_n^{p-1}\right),$$
$$b_nn\max_{1\leq i\leq n}|a_{ni}|^r\leq Cn^{\alpha p-1}=O\left(c_n'c_n^{p-1}\right),$$
$$\sum_{j=k}^{\infty}\theta^j\left(b_nc_n^{-r}n\max_{1\leq i\leq n}|a_{ni}|^r\right)\Big|_{n=\theta^j}\leq C\sum_{j=k}^{\infty}\theta^{j(\alpha p-\alpha r)}j^{-r}\leq C\theta^{\alpha k(p-r)}k^{-r+1}$$
$$\leq\theta^{\alpha k(p-r)}=O\left((c_{\theta^k})'(c_{\theta^k})^{p-r-1}\right),$$

$$\sum_{n=1}^{\infty} b_n c_n^{-r} \tilde{g}(n) n^{\frac{r}{2}} \max_{1\le i\le n} |a_{ni}|^r \le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha r-2+\frac{r}{2}} (\log n)^{-r} \tilde{g}(n), \text{ for } p \ge 2,$$

since $r > (\alpha p+\alpha t-1)/(\alpha-\frac{1}{2})$, then the function $x^{\alpha p-\alpha r-2+\frac{r}{2}}(\log x)^{-r}\tilde{g}(x)$ corresponding to $n^{\alpha p-\alpha r-2+\frac{r}{2}}(\log n)^{-r}\tilde{g}(n)$ is regularly varying at infinity with index $\alpha p-\alpha r-2+\frac{r}{2}+\alpha t < -1$. Since the index is less than -1, so we can get

$$\sum_{n=1}^{\infty} b_n c_n^{-r} \tilde{g}(n) n^{\frac{r}{2}} \max_{1\le i\le n} |a_{ni}|^r < \infty, \text{ for } p \ge 2,$$

$$\sum_{n=1}^{\infty} b_n c_n^{-\frac{rp}{2}} \tilde{g}(n) n^{\frac{r}{2}} \max_{1\le i\le n} |a_{ni}|^r \le C n^{\alpha p-\frac{\alpha pr}{2}-2+\frac{r}{2}} (\log n)^{-r} \tilde{g}(n), \text{ for } 0 \le p < 2,$$

since $r > 2(\alpha p+\alpha t-1)/(\alpha p-1)$, then the function $x^{\alpha p-\frac{\alpha pr}{2}-2+\frac{r}{2}}(\log x)^{-r}\tilde{g}(x)$ corresponding to $n^{\alpha p-\frac{\alpha pr}{2}-2+\frac{r}{2}}(\log n)^{-r}\tilde{g}(n)$ is regularly varying at infinity with index $\alpha p-\frac{\alpha pr}{2}-2+\frac{r}{2}+\alpha t < -1$. Since the index is less than -1, so we can get

$$\sum_{n=1}^{\infty} b_n c_n^{-\frac{rp}{2}} \tilde{g}(n) n^{\frac{r}{2}} \max_{1\le i\le n} |a_{ni}|^r < \infty, \text{for } 0 \le p < 2.$$

Therefore, based on the above tests, the proof of Corollary 4.3 is completed. □

Proof. [**Proof of Corollary 4.4**] As the similar proof as Corollary 4.3, we shall check the conditions in Theorem 4.1, we know that for negative dependent sequence, $\tilde{g}(n) = 1$. Let $b_n = n^{-1}$, $c_n = n^{1/\alpha}(\log n)^{1/p}$, then it is easy to check that

$$\begin{aligned} c_n^{-p} n \max_{1\le i\le n} |a_{ni}| &\le C\left(n^{-\frac{p}{\alpha}}(\log n)^{-1} n n^{\left(\frac{p}{\alpha}-1\right)\frac{1}{r}} (\log n)^{\frac{1-r}{r}}\right) \\ &\le C\left(n^{-\left(\frac{p}{\alpha}-1\right)\left(1-\frac{1}{r}\right)}(\log n)^{\frac{1}{r}-2}\right) = o(1), \end{aligned}$$

$$b_n n = 1 \le C n^{\frac{p}{\alpha}-1}\alpha \log n = O\left(c_n' c_n^{p-1}\right),$$

$$b_n n \max_{1\le i\le n} |a_{ni}|^r \le C n^{\frac{p}{\alpha}-1}(\log n)^{1-r} \le C n^{\frac{p}{\alpha}-1}(\log n) = O\left(c_n' c_n^{p-1}\right),$$

$$\begin{aligned} \sum_{j=k}^{\infty} \theta^j \left(b_n c_n^{-r} n \max_{1\le i\le n} |a_{ni}|^r\right)\Big|_{n=\theta^j} &\le C \sum_{j=k}^{\infty} \theta^{\frac{j}{\alpha}(p-r)} j^{1-r-\frac{r}{p}} \le C\theta^{\frac{k}{\alpha}(p-r)} k^{2-r-\frac{r}{p}} \\ &\le C\theta^{\frac{k}{\alpha}(p-r)} k^{-\frac{r}{p}}(k+1) = O\left((c_{\theta^k})'(c_{\theta^k})^{p-r-1}\right), \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} b_n c_n^{-r} \tilde{g}(n) n^{\frac{r}{2}} \max_{1\le i\le n} |a_{ni}|^r &\le C \sum_{n=1}^{\infty} n^{-1} n^{-\frac{r}{\alpha}} (\log n)^{-\frac{r}{p}} n^{\frac{r}{2}} n^{\frac{p}{\alpha}-1} (\log n)^{1-r} \\ &\le C \sum_{n=1}^{\infty} n^{-\frac{r}{\alpha}+\frac{r}{2}+\frac{p}{\alpha}-2} (\log n)^{1-r-\frac{r}{p}} < \infty \text{ for } p \ge 2 \end{aligned}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} b_n c_n^{-\frac{rp}{2}} \tilde{g}(n) n^{\frac{r}{2}} \max_{1\le i\le n} |a_{ni}|^r &\le C \sum_{n=1}^{\infty} n^{-1} n^{-\frac{rp}{2\alpha}} (\log n)^{-\frac{r}{2}} n^{\frac{r}{2}} n^{\frac{p}{\alpha}-1} (\log n)^{1-r} \\ &\le C \sum_{n=1}^{\infty} n^{-\frac{rp}{2\alpha}+\frac{r}{2}+\frac{p}{\alpha}-2} (\log n)^{1-\frac{3r}{2}} < \infty \text{ for } 0 \le p < 2. \end{aligned}$$

Hence, based on the above inequalities, the desired results can be obtained.

□

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