Filomat 37:13 (2023), 4443–4451
https://doi.org/10.2298/FIL2313443K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cumulative residual Fisher information of a general k-th order autoregressive process

Omid Kharazmi[a,*], Narayanaswamy Balakrishnan[b]

[a]*Department of Statistics, Faculty of Mathematical Sciences, Vali-e-Asr University of Rafsanjan, Iran*
[b]*Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada*

Abstract. In this work, we consider cumulative residual Fisher information and Bayes-cumulative residual Fisher information for a general k-th order autoregressive and their corresponding stationary equilibrium distributions and develop some associated results. We establish several interesting connections between these measures and some known informational measures such as chi-square divergence, Gini's mean difference, cumulative residual Kullback-Leibler, Jeffreys and Jensen-cumulative residual entropy divergences.

1. Introduction

Several criteria have been discussed in the literature for measuring the uncertainty of a probabilistic model. Moreover, various divergences measures have also been developed in the literature for measuring similarity (closeness) between two probability distributions. Shannon entropy, Fisher information and Gini's mean difference are three most important information measures that have been used extensively in many different fields. For more details, one may see Shannon (1948), Fisher (1929) and Gini (1912). Recently, these information measures have been generalized based on Jensen inequality, which have come to be known as Jensen-Shannon, Jensen-Fisher and Jensen-Gini information measures, respectively. For pertinent details, one may refer to Lin (1991), Sánchez-Moreno et al. (2012) and Mehrali et al. (2018). Let X be an absolutely continuous random variable with cumulative distribution function (CDF) F_θ and density function f_θ, where $\theta \in \Theta \subseteq R$. Then, the Fisher information of a random variable X, about parameter θ, is defined as

$$I(\theta) \quad = \quad E\left(\frac{\partial \log f_\theta(X)}{\partial \theta}\right)^2 = \int_X \left(\frac{\partial \log f_\theta(x)}{\partial \theta}\right)^2 f_\theta(x)dx, \tag{1}$$

where "log" corresponds to the natural logarithm.

Recently, Kharazmi and Blakrishnan (2021) defined the cumulative residual Fisher (CRF) information, by replacing the probability density function (PDF) in (1) by the survival function, as

2020 *Mathematics Subject Classification*. Primary 94A15; Secondary 62B10.
Keywords. Autoregressive AR(k) process, Cumulative residual Fisher information, Stationary equilibrium distribution, cumulative residual Kullback-Leibler divergence, Bayescumulative Fisher information.
Received: 07 March 2022; Revised: 10 April 2022; Accepted: 13 April 2022
Communicated by Aleksandar Nastić
* Correspondng author: Omid Kharazmi
Email addresses: omid.kharazmi@vru.ac.ir (Omid Kharazmi), bala@mcmaster.ca (Narayanaswamy Balakrishnan)

$$\mathcal{CI}(\theta) = \int \left(\frac{\partial \log \bar{F}_\theta(x)}{\partial \theta} \right)^2 \bar{F}_\theta(x) dx, \tag{2}$$

where $\bar{F}_\theta(x) = 1 - F(x)$ is the survival function of X. It can be seen readily that this information measure gives decreasing weights for larger values of X due to the term $\bar{F}_\theta$ in the integrand, which suggests that this information measure will naturally be robust to the presence of outliers. In order to simplify the notation, we have suppressed $\mathcal{X}$ for the integration with respect to x in (2), as well as in the rest of this paper, unless a distinction becomes necessary.

Recently, many authors have developed various autoregressive models with minification process. Autoregressive models have been the backbone of time series analysis for time-dependent data sets. Specifically, these have been developed for modeling the current value of the series, X_t, as a function of past values $X_{t-1}, ..., X_{t-m}$, where m is the time lag of dependence. A generalized k-th order autoregressive structure, with mixing parameter vector $\boldsymbol{p} = (p_0, ..., p_{k-1})$, is given by

$$X_n = \begin{cases} \epsilon_n, & w.p. \ \frac{p_0}{k} \\ min(X_{n-1}, \epsilon_n), & w.p. \ \frac{p_1}{k} \\ min(X_{n-2}, \epsilon_n), & w.p. \ \frac{p_2}{k} \\ \quad . & \quad . \\ \quad . & \quad . \\ \quad . & \quad . \\ min(X_{n-k}, \epsilon_n), & w.p. \ \frac{p_k}{k}, \end{cases} \tag{3}$$

where $0 \le p_i \le 1$, $i = 0, 1, ..., k$, $\sum_{i=1}^{k-1} p_i \le 1$ and $\{\epsilon_n\}$ is an innovation process of independent and identically distributed random variables chosen so chosen that $\{X_n\}$ is a stationary Markov process with a specified marginal distribution function $F_X(x)$. The survival function of the autoregressive model X_n in (3) is then

$$\bar{F}_{X_n}(x) = \frac{p_0}{k} \bar{F}_{\epsilon_n}(x) + \frac{1}{k} \sum_{j=1}^{k-1} p_j \bar{F}_{X_{n-j}}(x) \bar{F}_{\epsilon_n}(x) + \Big(1 - \frac{\sum_{j=0}^{k-1} p_j}{k}\Big) \bar{F}_{X_{n-k}}(x) \bar{F}_{\epsilon_n}(x). \tag{4}$$

The stationary equilibrium distribution of the autoregressive model X_n in (3) is known to be a Marshall-Olkin distribution with survival function as

$$\bar{F}_X(x) = \frac{p_0 \bar{F}_{\epsilon_n}(x)}{1 - (1 - p_0) \bar{F}_{\epsilon_n}(x)}; \tag{5}$$

see Marshall and Olkin (1997) for more details on the model in (5). Here, our main interest here is to discuss cumulative residual Fisher information and Bayes-cumulative residual Fisher information measures for the survival functions in (4) and (5). The purpose of this work is two-fold. In the first part, we study the cumulative residual Fisher information of a general k-th order autoregressive process and its corresponding stationary equilibrium distribution (Marshall-Olkin family). In the second part, the Bayes-cumulative residual Fisher information of these models is studied under different prior distributions for the parameter $\boldsymbol{p}$. The results display several interesting connections between cumulative residual Fisher information and Bayes-cumulative residual Fisher information of mixing parameter vector and some known informational divergences.

The rest of this paper is organized as follows. In Section 2, we briefly describe some key information and entropy measures. Next, in Section 3, we consider an autoregressive AR(k) process and its corresponding stationary equilibrium distribution and establish some results for the cumulative measure of mixing parameter vector. We show specifically that the cumulative residual Fisher information of the mixing parameter vector of AR(k) is connected to chi-square divergence. In addition, an analogous result is established for the CRF information measure of the Marshall-Olkin distribution. Next, in Section 4, we discuss the Bayes-CRF information for the mixing parameter vector of the survival function of AR(k) under some prior distributions for the mixing parameter. We show that this measure is connected to Jensen-cumulative residual Shannon entropy, cumulative residual Kullback-Leibler and Jeffreys divergences. We also show that the Bayes-CRF information Marshall-Oklin distribution under beta prior distribution is connected to Gini's mean difference. Finally, in Section 5, we state some concluding remarks.

2. Preliminaries

We briefly introduce some informational measures that will be used in the sequel. Let X and Y be two continuous random variables with survival functions $\bar{F}$ and $\bar{G}$, respectively. Then, the cumulative residual Kullback-Leibler (CRKL) divergence between X and Y (or $\bar{F}$ and $\bar{G}$) is defined as

$$CRKL(X\|Y) = CRKL(\bar{F}, \bar{G}) = \int \bar{F}(x) \log \frac{\bar{F}(x)}{\bar{G}(x)} dx - \Big(E(X) - E(Y)\Big).$$

The CRKL discrimination between Y and X can be defined in an analogous manner. For more details, one may refer to Park et al. (2012).

Another important diversity measure between two survival functions $\bar{F}$ and $\bar{G}$ is the chi-square divergence, defined as

$$\chi^2(\bar{F}, \bar{G}) = \int \frac{(\bar{F}(x) - \bar{G}(x))^2}{\bar{G}(x)} dx.$$

In a similar manner, we can define $\chi^2(\bar{G}, \bar{F})$. For more details, see Kharazmi and Balakrishnan (2021).

The Fisher information of a random variable X, or its PDF $f(x;\theta)$, about the parameter θ is defined as

$$\mathcal{I}(\theta) = \int \left[\frac{\partial \log f(x;\theta)}{\partial \theta}\right]^2 f(x;\theta) dx.$$

In Bayesian statistics, one assumes that the parameter θ is endowed with a prior $\pi(\theta)$. Then, the expected cumulative residual Fisher information,

$$\tilde{C\mathcal{I}}(\theta) = E_\pi[C\mathcal{I}(\Theta)] = \int C\mathcal{I}(\theta)\pi(\theta) d\theta,$$

is called the *Bayes-cumulative residual Fisher information*. For more details, see Kharazmi and Balakrishnan (2021) and the references therein.

Rao et al. (2004) defined the cumulative residual entropy (CRE) as

$$\xi(X) = \xi(\bar{F}) = -\int \bar{F}(x) \log \bar{F}(x) dx, \tag{6}$$

where $\bar{F}(x) = 1 - F(x)$ is the survival function of X. Subsequently, Asadi and Zohrevand (2007) provided an interesting representation for (6), based on the mean residual lifetime function, as

$$\xi(X) = E[m_F(X)], \tag{7}$$

where $m_F(t)$ is the mean residual lifetime of X defined by $m_F(t) = \frac{\int_t^\infty \bar{F}(u) du}{\bar{F}(t)}$. Recently, Kharazmi and Balakrishan (2021) introduced an extension of (6), called Jensen-cumulative residual entropy. Let $X_1, .., X_n$ be variables with survival functions $\bar{F}_1, ..., \bar{F}_n$, respectively, and $\alpha_1, ..., \alpha_n$ be non-negative real numbers such that $\sum_{i=1}^n \alpha_i = 1$. Then, the Jensen-cumulative residual entropy (JCRE) information measure is defined as

$$\begin{aligned}\mathcal{JCE}(\bar{F}_1, ..., \bar{F}_n, \boldsymbol{\alpha}) &= \xi\Big(\sum_{i=1}^n \alpha_i \bar{F}_i\Big) - \sum_{i=1}^n \alpha_i \xi(\bar{F}_i) \\ &= \sum_{i=1}^n \alpha_i CRKL(\bar{F}_i, \bar{F}_T),\end{aligned}$$

where $\bar{F}_T = \sum_{i=1}^n \alpha_i \bar{F}_i$ is the weighted survival function.

3. CRF information measure for AR(k) process and its stationary equilibrium distribution

In this section, we first study the cumulative residual Fisher information for parameter $\boldsymbol{p} = (p_0, ..., p_{k-1})$ of the AR(k) process in (3). We then examine this type of information measure for Marshall-Olkin distribution as stationary equilibrium distribution of the AR(k) process. The following theorem gives a representation for the CRF information measure in (2), denoted by $\mathcal{CI}(p_i), i = 0, ..., k-1$, for the k-component finite mixture survival function in (4).

3.1. *CRF information measure for AR(k) process*

Theorem 3.1. *The CRF information measure of the SF in (4) about parameter $p_i, i = 0, ..., k-1$, is given by*

$$\mathcal{CI}(p_i) = \frac{1}{\big(p_i - (k-1)\big)^2}\begin{cases} \chi^2(\bar{F}_{X_n}^{-i}, \bar{F}_{X_n}), & i = 1, ..., k-1, \\ \chi^2(\bar{F}_{X_n}^{-0}, \bar{F}_{X_n}), & i = 0, \end{cases}$$

where

$$\begin{aligned}\bar{F}_{X_n}^{-i}(x) &= \frac{p_0}{k}\bar{F}_{\epsilon_n}(x) + \frac{k-1}{k}\bar{F}_{X_{n-i}}(x)\bar{F}_{\epsilon_n}(x) + \frac{1}{k}\sum_{j=1, j\neq i}^{k-1} p_j \bar{F}_{X_{n-j}}(x)\bar{F}_{\epsilon_n}(x) \\ &+ \frac{1}{k}\Big(1 - \sum_{j=0}^{k-1} p_j\Big)\bar{F}_{X_{n-k}}(x)\bar{F}_{\epsilon_n}(x)\end{aligned}$$

and

$$\bar{F}_{X_n}^{-0}(x) = \frac{k-1}{k}\bar{F}_{\epsilon_n}(x) + \frac{1}{k}\sum_{j=1}^{k-1} p_j \bar{F}_{X_{n-j}}(x)\bar{F}_{\epsilon_n}(x) + \frac{1}{k}\Big(1 - \sum_{j=1}^{k-1} p_j\Big)\bar{F}_{X_{n-k}}(x)\bar{F}_{\epsilon_n}(x).$$

Proof: From the definition of the CRF information measure in (2), for $i = 1, ..., k-1$, we have

$$\begin{aligned}\mathcal{CI}(p_i) &= E\Big[\frac{\partial \log \bar{F}_{X_n}(X)}{\partial p_i}\Big]^2 \\ &= \frac{1}{k^2}\int \bar{F}_{\epsilon_n}^2 \frac{\big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\big)^2}{\bar{F}_{X_n}(x)} dx \\ &= \frac{1}{\big(p_i - (k-1)\big)^2}\int \frac{\big(\bar{F}_{X_n}^{-i}(x) - \bar{F}_{X_n}(x)\big)^2}{\bar{F}_{X_n}(x)} dx \\ &= \frac{1}{\big(p_i - (k-1)\big)^2}\chi^2(\bar{F}_{X_n}^{-i}(x), \bar{F}_{X_n}(x)), \end{aligned} \tag{8}$$

where the third equality follows from the fact that, for $i = 1, ..., k-1$,

$$\bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big) = \frac{k}{p_i - (k-1)}\Big(\bar{F}_{X_n}(x) - \bar{F}^{-i}_{X_n}(x)\Big).$$

The proof for $i = 0$ is quite similar. □

3.2. CRF information measure for Marshall-Olkin family

In this section, we obtain the CRF information measure for Marshall-Olkin family as the stationary equilibrium distribution of AR(k) process. Let $S(x)$ be the survival function of a given distribution. Then, the Marshall-Olkin distribution, obtained by introducing a new parameter p, has its survival function as

$$S_p(x) = \frac{pS(x)}{1-(1-p)S(x)},\ 0 < p < 1. \tag{9}$$

Clearly, when $p = 1$, we get the original survival function $S(x)$.

Theorem 3.2. *The cumulative residual Fisher information for the SF in (9), about parameter p, is given by*

$$\mathcal{CI}(p) = \frac{1}{p^2}\chi^2(S_p^2, S_p), \tag{10}$$

where S_p^2 is the corresponding proportional hazards survival function of order 2 of S_p.

Proof: From the defintion of $\mathcal{CI}(p)$ in (2), we have

$$\begin{aligned}
\mathcal{CI}(p) &= \int_0^\infty \left\{\frac{\partial \log S_p(x)}{\partial p}\right\}^2 S_p(x)dx \\
&= \int_0^\infty \left\{\frac{\partial \log \frac{pS(x)}{1-(1-p)S(x)}}{\partial p}\right\}^2 S_p(x)dx \\
&= \int_0^\infty \left\{\frac{S(x)-S^2(x)}{\Big(1-(1-p)S(x)\Big)^2}\right\}^2 S_p^{-1}(x)dx \\
&= \frac{1}{p^2}\int_0^\infty \frac{\Big(S_p(x)-S_p^2(x)\Big)^2}{S_p(x)}dx \\
&= \frac{1}{p^2}\chi^2(S_p^2, S_p),
\end{aligned}$$

as required. □

4. Bayes-CRF information for AR(k) process and Marshall-Olkin distribution

In this section, we first study the Bayes-CRF information for the mixing parameter vector $\boldsymbol{p}$ of the mixture SF in (4) under some prior distributions for the mixing parameter vector and then examine the Bayes-CRF information of Marshall-Olkin distribution under beta prior distribution. We now introduce two notations that will be used in the sequel. For the parameter vector $\boldsymbol{p} = (p_0, ..., p_{k-1})$, we define $(0_i, \boldsymbol{p}) = (p_0, ..., p_{i-1}, 0, p_{i+1}, ..., p_{k-1})$ and $(1_i, \boldsymbol{p}) = (p_0, ..., p_{i-1}, 1, p_{i+1}, ..., p_{k-1})$.

Theorem 4.1. *The Bayes-cumulative residual Fisher information for the mixture SF in (4), for parameter p_i, $i = 0, ..., k-1$, under the uniform prior on* $[0, 1]$, *is given by*

$$\tilde{C\mathcal{I}}(p_i) = \begin{cases} J\Big(\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}, \bar{F}^{(1_i,\boldsymbol{p})}_{X_n}\Big), & i = 1, ..., k-1, \\ \\ J\Big(\bar{F}^{(0,\boldsymbol{p})}_{X_n}, \bar{F}^{(1,\boldsymbol{p})}_{X_n}\Big), & i = 0, \end{cases}$$

where

$$\begin{aligned} \bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x) &= \frac{p_0}{k}\bar{F}_{\epsilon_n}(x) + \frac{1}{k}\bar{F}_{X_{n-i}}(x)\bar{F}_{\epsilon_n}(x) + \frac{1}{k}\sum_{j=1, j\neq i}^{k-1} p_j \bar{F}_{X_{n-j}}(x)\bar{F}_{\epsilon_n}(x) \\ &\quad + \Big(1 - \frac{1 + \sum_{j=0, j\neq i}^{k-1} p_j}{k}\Big)\bar{F}_{X_{n-k}}(x)\bar{F}_{\epsilon_n}(x), \end{aligned} \tag{11}$$

$$\begin{aligned} \bar{F}^{(0_i,\boldsymbol{p})}_{X_n}(x) &= \frac{p_0}{k}\bar{F}_{\epsilon_n}(x) + \frac{1}{k}\sum_{j=1, j\neq i}^{k-1} p_j \bar{F}_{X_{n-j}}(x)\bar{F}_{\epsilon_n}(x) \\ &+ \Big(1 - \frac{\sum_{j=0, j\neq i}^{k-1} p_j}{k}\Big)\bar{F}_{X_{n-k}}(x)\bar{F}_{\epsilon_n}(x), \end{aligned} \tag{12}$$

$$\bar{F}^{(1,\boldsymbol{p})}_{X_n}(x) = \frac{1}{k}\bar{F}_{\epsilon_n}(x) + \frac{1}{k}\sum_{j=1}^{k-1} p_j \bar{F}_{X_{n-j}}(x)\bar{F}_{\epsilon_n}(x) + \Big(1 - \frac{1 + \sum_{j=1}^{k-1} p_j}{k}\Big)\bar{F}_{X_{n-k}}(x)\bar{F}_{\epsilon_n}(x),$$

$$\bar{F}^{(0,\boldsymbol{p})}_{X_n}(x) = \frac{1}{k}\sum_{j=1}^{k-1} p_j \bar{F}_{X_{n-j}}(x)\bar{F}_{\epsilon_n}(x) + \Big(1 - \frac{\sum_{j=1}^{k-1} p_j}{k}\Big)\bar{F}_{X_{n-k}}(x)\bar{F}_{\epsilon_n}(x),$$

with J denoting Jeffreys' divergence.

Proof: By definition and from (2), for $i = 1, ..., k-1$, we have

$$\begin{aligned} \tilde{C\mathcal{I}}(p_i) &= E[C\mathcal{I}(P_i)] \\ &= \frac{1}{k^2}\int_0^1 \Big\{\int \bar{F}^2_{\epsilon_n}\frac{\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)^2}{\bar{F}_{X_n}(x)}dx\Big\}dp_i \\ &= \frac{1}{k}\int \bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)\Big\{\int_0^1 \frac{1}{k}\frac{\bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)}{\bar{F}_{X_n}(x)}dp_i\Big\}dx \\ &= \frac{1}{k}\int \bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)\Big\{\log\Big(\bar{F}_{X_n}(x)\Big)\Big|_0^1\Big\}dx \\ &= \frac{1}{k}\int \bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)\log\Big\{\frac{\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x)}{\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}(x)}\Big\}dx. \end{aligned} \tag{13}$$

On the other hand, we have

$$\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x) - \bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x) = \frac{1}{k}\bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big). \tag{14}$$

Hence, by substituting (14) into (13), we obtain

$$\begin{aligned}
\tilde{C\mathcal{I}}(p_i) &= \frac{1}{k}\int \bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x)-\bar{F}_{X_{n-k}}(x)\Big)\log\Big\{\frac{f_{(1_i,\boldsymbol{\theta})}(x)}{f_{(0_i,\boldsymbol{\theta})}(x)}\Big\}dx \\
&= \int \Big(\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x)-\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x)\Big)\log\Big\{\frac{f_{(1_i,\boldsymbol{\theta})}(x)}{f_{(0_i,\boldsymbol{\theta})}(x)}\Big\}dx \\
&= CRKL\Big(\bar{F}^{(1_i,\boldsymbol{p})}_{X_n},\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}\Big)+CRKL\Big(\bar{F}^{(0_i,\boldsymbol{p})}_{X_n},\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}\Big) \\
&= J\Big(\bar{F}^{(0_i,\boldsymbol{p})}_{X_n},\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}\Big),
\end{aligned}$$

as required. The proof for $i = 0$ is similar. □

Let us now consider the following general triangular prior for the parameter $p_i, i = 1, ..., k-1$:

$$\pi_\alpha(p_i) = \begin{cases} \frac{2p_i}{\alpha}, & 0 < p_i \le \alpha, \\ \frac{2(1-p_i)}{1-\alpha}, & \alpha \le p_i < 1, \end{cases} \tag{15}$$

for some $\alpha \in (0, 1)$.

Theorem 4.2. *The Bayes-cumulative residual Fisher information for parameter $p_i, i = 1, ..., k-1$, with the general triangular prior with density $\pi_\alpha(p_i)$ in (15), we have*

$$\begin{aligned}
\tilde{C\mathcal{I}}(p_i) &= \frac{2}{\alpha(1-\alpha)}\left[\alpha CRKL(\bar{F}^{(1_i,\boldsymbol{p})}_{X_n},\bar{F}^{\alpha}_{X_n})+(1-\alpha)CRKL(\bar{F}^{(0_i,\boldsymbol{p})}_{X_n},\bar{F}^{\alpha}_{X_n})\right] \\
&= \frac{2}{\alpha(1-\alpha)}\Big[JCRE\Big(\bar{F}^{(0_i,\boldsymbol{p})}_{X_n},\bar{F}^{(1_i,\boldsymbol{p})}_{X_n};\alpha\Big)+\alpha E(X_1)+(1-\alpha)E(X_0)-E(X_\alpha)\Big],
\end{aligned}$$

where the variables X_0 and X_1 have the survival functions $\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}$ and $\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}$, given in (12) and (11), respectively and $\bar{F}^{\alpha}_{X_n}(x)$ is the mixture survival function of the variable X_α, given by

$$\begin{aligned}
\bar{F}^{\alpha}_{X_n}(x) &= \alpha\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x)+(1-\alpha)\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}(x) \\
&= \frac{p_0}{k}\bar{F}_{\epsilon_n}(x)+\frac{\alpha}{k}\bar{F}_{X_{n-i}}(x)\bar{F}_{\epsilon_n}(x)+\frac{1}{k}\sum_{j=1,j\neq i}^{k-1}p_j\bar{F}_{X_{n-j}}(x)\bar{F}_{\epsilon_n}(x) \\
&+ \Big[1-\frac{1}{k}\Big(\alpha+\sum_{j=0,j\neq i}^{k-1}p_j\Big)\Big]\bar{F}_{X_{n-k}}(x)\bar{F}_{\epsilon_n}(x).
\end{aligned}$$

Proof: For $i = 1, ..., k-1$, we have

$$
\begin{aligned}
\tilde{C\mathcal{I}}(p_i) &= E\left[C\mathcal{I}(P_i)\right] \\
&= \int_0^{\alpha} C\mathcal{I}(p_i)\pi_\alpha dp_i + \int_\alpha^1 C\mathcal{I}(p_i)\pi_\alpha dp_i \\
&= \frac{2}{k\alpha}\int \bar{F}^2_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)\left[\int_0^\alpha \frac{p_i}{k}\frac{\left(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\right)}{\bar{F}_{X_n}(x)}dp_i\right]dx \\
&\quad + \frac{2}{k(1-\alpha)}\int \bar{F}^2_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)\left[\int_\alpha^1 \frac{1-p_i}{k}\frac{\left(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\right)}{\bar{F}_{X_n}(x)}dp_i\right]dx \\
&= \frac{2}{k\alpha}\int_{-\infty}^{\infty} \bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)\left[\int_0^\alpha \Big(1 - \frac{\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}(x)}{\bar{F}_{X_n}(x)}\Big)dp_i\right]dx \\
&\quad - \frac{2}{k(1-\alpha)}\int \bar{F}_{\epsilon_n}(x)\Big(\bar{F}_{X_{n-i}}(x) - \bar{F}_{X_{n-k}}(x)\Big)\left[\int_0^\alpha \Big(1 - \frac{\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x)}{\bar{F}_{X_n}(x)}\Big)dp_i\right]dx \\
&= -\frac{2}{\alpha}\int \bar{F}^{(0_i,\boldsymbol{p})}_{X_n}(x)\log\left\{\frac{\bar{F}^{\alpha}_{X_n}(x)}{\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}(x)}\right\}dx + \frac{2}{1-\alpha}\int \bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x)\log\left\{\frac{\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x)}{\bar{F}^{\alpha}_{X_n}(x)}\right\}dx \\
&= \frac{2}{\alpha(1-\alpha)}\left[\alpha CRKL(\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}, \bar{F}^{\alpha}_{X_n}) + (1-\alpha)CRKL(\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}, \bar{F}^{\alpha}_{X_n})\right] \\
&= \frac{2}{\alpha(1-\alpha)}\Big[JCRE\Big(\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}, \bar{F}^{(1_i,\boldsymbol{p})}_{X_n}(x);\alpha\Big) + \alpha E(X_1) + (1-\alpha)E(X_0) - E(X_\alpha)\Big],
\end{aligned}
$$

as required. □

From (15), it is evident that by setting $\alpha = 1$ and $\alpha = 0$, we obtain Beta(2,1) and Beta(1,2) prior distributions, respectively. The following corollary gives limiting cases of Theorem 4.2 with respect to α.

Corollary 4.3. *From Theorem 4.2, for p_i, $i = 1, ..., k-1$, we have*

(a) $\lim_{\alpha\to 1} \tilde{C\mathcal{I}}(p_i) = 2CRKL(\bar{F}^{(0_i,\boldsymbol{p})}_{X_n}, \bar{F}^{(1_i,\boldsymbol{p})}_{X_n})$;

(b) $\lim_{\alpha\to 0} \tilde{C\mathcal{I}}(p_i) = 2CRKL(\bar{F}^{(1_i,\boldsymbol{p})}_{X_n}, \bar{F}^{(0_i,\boldsymbol{p})}_{X_n})$.

Next, we show that the Bayes-CRF information measure of the Marshall-Olkin distribution is connected to Gini's mean difference.

Theorem 4.4. *Let X have its SF as in (9). Then, the Bayes risk of $C\mathcal{I}(p)$ in (3.2), under Beta$(3,1)$ prior distribution for the parameter p, is given by*

$$E\Big[C\mathcal{I}(P)\Big] = \frac{3}{4}GMD(S),$$

where GMD is Gini's mean difference associated with X defined by

$$GMD(S) = 2\int S(x)(1 - S(x))dx.$$

Proof: From (9) and (10), we have

$$
\begin{aligned}
E\Big[\mathcal{CI}(P)\Big] &= \int_0^1 \mathcal{CI}(p)\, 3p^2 dp \\
&= 3\int_0^1 \int \frac{1}{p^2} \frac{S(x)(1-S(x))^2}{\Big\{1-(1-p^{-1})(1-S(x))\Big\}^3} dx dp \\
&= 3\int S(x)\Big(1-S(x)\Big)\Big\{\int_0^1 \frac{1}{p^2} \frac{1-S(x)}{\Big\{1-(1-p^{-1})(1-S(x))\Big\}^3} dp\Big\} dx \\
&= 3\int S(x)\Big(1-S(x)\Big)\Big\{\int_1^\infty \frac{1-S(x)}{\Big\{1-(1-u)(1-S(x))\Big\}^3} du\Big\} dx \\
&= \frac{3}{2}\int S(x)\Big(1-S(x)\Big) dx \\
&= \frac{3}{4}\, GMD(S),
\end{aligned}
$$

as required. □

5. Concluding remarks

In this paper, we have considered a general autoregressive model of order k and Marshall-Olkin family as its corresponding stationary equilibrium distribution and have derived their cumulative residual Fisher information and Bayes-cumulative residual Fisher information. We have also examined the Fisher information measure for the parameter of the Marshall-Olkin family as stationary equilibrium distribution of the considered autoregressive model. We have shown that the cumulative residual Fisher information of both these models is connected to the chi-square divergence. We have also studied the Bayes-cumulative residual Fisher information for the mixing parameter of the autoregressive model and the Marshall-Olkin distribution. These results provide connections between these two information measures and some known informational measures such as chi-square divergence, Gini's mean difference, cumulative residual Kullback-Leibler, Jeffreys and Jensen-cumulative residual entropy divergence measures.

References

[1] M. Asadi, and Y. Zohrevand, *On the dynamic cumulative residual entropy*, J Stat Plan Inference. **137** (2007), 1931-1941.
[2] R. A. Fisher, *Tests of significance in harmonic analysis*, Proc R Soc Lond A Math Phys Sci, **125** (1929), 54-59.
[3] C. Gini, *Variabilità e mutabilità*. Reprinted in Memorie di metodologica statistica (Ed. Pizetti E, Salvemini, T). Rome: Libreria Eredi Virgilio Veschi, 1912.
[4] O. Kharazmi, N. Balakrishnan, *Cumulative residual and relative cumulative residual Fisher information and their properties*, IEEE Trans Inf Theory, **67**(2021), 6306-6312.
[5] J. Lin, *Divergence measures based on the Shannon entropy*, IEEE Trans Inf Theory, **37** (1991), 145-151.
[6] A. W. Marshall, I. Olkin, *A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families*. Biometrika, **84** (1997), 641-652.
[7] Y. Mehrali, M. Asadi, & Kharazmi, *A Jensen-Gini measure of divergence with application in parameter estimation*, Metron, **76** (2018), 115-131.
[8] S. Park, M. Rao, D. W. Shin, *On cumulative residual Kullback-Leibler information*, Stat Probab Lett, **82** (2012), 2025-2032.
[9] M. Rao, Y. Chen, B. C. Vemuri, F. Wang, *Cumulative residual entropy: a new measure of information*, IEEE Trans Inf Theory, **50** (2004), 1220-1228.
[10] P. Sánchez-Moreno, A. Zarzo, J.S. Dehesa, *Jensen divergence based on Fisher's information*, J Phys A Math Theor, **45** (2012), 125305.
[11] C. E. Shannon, *A mathematical theory of communication*, Bell Syst. tech. j, **27** (1948), 379-423.