Filomat 37:13 (2023), 4425–4441
https://doi.org/10.2298/FIL2313425I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Power inequalities for log-convex functions with applications

**Mohamed Amine Ighachane**[a], **Mohamed Akkouchi**[b], **Mohammad Sababheh**[c]

[a]*Sciences and Technologies Team (ESTE), Higher School of Education and Training of El Jadida, Chouaib Doukkali University, El Jadida, Morocco*
[b]*Department of Mathematics, Faculty of Sciences-Semlalia, University Cadi Ayyad, Av. Prince My. Abdellah, BP: 2390, Marrakesh (40.000-Marrakech), Morocco (Maroc)*
[c]*Department of Basic Sciences, Princess Sumaya University for Technology, Amman 11941, Jordan*

**Abstract.** In this paper, we present new inequalities for log-convex functions, with some applications to operator means. The significance of the obtained results is two folded; the results themselves and the way they extend many known results in the literature.

## 1. Introduction and Preliminaries

Let $B(\mathcal{H})$ denote the $C^*-$algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is called positive, denoted by $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. The set of all positive operators is denoted by $B(\mathcal{H})^+$. The set of all invertible operators in $B(\mathcal{H})^+$ is denoted by $B(\mathcal{H})^{++}$. When $\mathcal{H}$ is finite dimensional, we identify $B(\mathcal{H})$ with the algebra $\mathbf{M}_n$ of all $n \times n$ complex matrices.

A binary operation $\sigma : (A, B) \in B(\mathcal{H})^{++} \times B(\mathcal{H})^{++} \mapsto A\sigma B \in B(\mathcal{H})^{++}$ is called a connection if the following conditions hold:

1. Monotonicity: $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$, for all $A, B, C, D \in B(\mathcal{H})^{++}$.
2. Upper continuity: $A_k \downarrow A$ and $B_k \downarrow B$ imply $A_k\sigma B_k \downarrow A\sigma B$ in the strong operator topology, for all $A_k, B_k, A, B \in B(\mathcal{H})^{++}$.
3. Transformer inequality: $T^*(A\sigma B)T \leq (T^*AT)\sigma(T^*BT)$ for every invertible operator $T \in B(\mathcal{H})$.

An operator mean is a normalized connection in the sense that

(4) Normalization: $I_{\mathcal{H}}\sigma I_{\mathcal{H}} = I_{\mathcal{H}}$, where $\mathcal{I}_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$.

Among the most important operator means there are the arithmetic, geometric, harmonic and power means defined respectively for $A, B \in B(\mathcal{H})^{++}$ and $\mu \in [0, 1]$, as follows:
$A\nabla_\mu B := (1-\mu)A + \mu B$, $A\sharp_\mu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\mu A^{1/2}$, $A!_\mu B := ((1-\mu)A^{-1} + \mu B^{-1})^{-1}$ and

$$A\sharp_{p,\mu} B := A^{1/2}\left((1-\mu)I + \mu(A^{-1/2}BA^{-1/2})^p\right)^{\frac{1}{p}} A^{1/2};\ p \in \mathbb{R}\backslash\{0\},$$

2020 *Mathematics Subject Classification*. Primary 15A39, 15B48; Secondary 26A51, 47A30, 47A63.
*Keywords*. Log-convex functions, operator means, matrix inequalities.
Received: 03 July 2022; Revised: 09 September 2022; Accepted: 11 September 2022
Communicated by Fuad Kittaneh
*Email addresses:* mohamedamineighachane@gmail.com (Mohamed Amine Ighachane), akkm555@yahoo.fr ( Mohamed Akkouchi ), sababheh@psut.edu.jo; sababheh@yahoo.com ( Mohammad Sababheh )

and

$$A\sharp_{0,\mu}B := \lim_{p\longrightarrow 0} A\sharp_{p,\mu}B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\mu}A^{1/2}.$$

Thus, the value $p \to 0$ gives the geometric mean, while the values $p = 1, -1$ give the arithmetic and harmonic means, respectively.

For $a, b > 0$, $\mu \in [0, 1]$ and $p \in \mathbb{R}$ the power maen is defined by

$$a\sharp_{p,\mu}b := (\mu a^p + (1-\mu)b^p))^{\frac{1}{p}} \text{ if } p \neq 0,$$

and

$$a\sharp_{\mu}b := a\sharp_{0,\mu}b := \lim_{p\longrightarrow 0} a\sharp_{p,\mu}b = a^{\mu}b^{1-\mu}.$$

Further, the values $p = 1, -1$ give the arithmetic and harmonic means.
The harmonic-geometric-arithmetic mean inequalities state

$$a!_{\mu}b \leq a\sharp_{\mu}b \leq a\nabla_{\mu}b, \tag{1}$$

for $\mu \in [0, 1]$, with equality if and only if $a = b$, where $a!_{\mu}b = (\mu a^{-1} + (1-\mu)b^{-1})^{-1}$ and $a\nabla_{\mu}b = \mu a + (1-\mu)b$. Here the second inequality in (1) is the classical Young's inequality.
Though simple, (1) has numerous applications in mathematical and operator inequalities. In particular, a considerable amount of research has been done trying to refine (1). In particular F. Kittaneh and Y. Manasrah [13] obtained the following interesting refinement of Young's inequality

$$a^{\mu}b^{1-\mu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \mu a + (1-\mu)b, \tag{2}$$

where $r_0 = \min\{\mu, 1-\mu\}$. We refer the reader to [13] to see how this inequality was used to obtain some operator inequalities that are refinements of some known inequalities back then.

In [4], H. Alzer et al. proved an important refinement of Young's inequality that happened to be better than (2):

**Theorem 1.1 (Alzer-Fonseca-Kovačec).** *Let $a, b > 0$ and let $\lambda, \mu, \nu$ be real numbers with $\lambda \geq 1$ and $0 \leq \mu < \nu \leq 1$. Then*

$$\left(\frac{\mu}{\nu}\right)^{\lambda} \leq \frac{(a\nabla_{\mu}b)^{\lambda} - (a\sharp_{\mu}b)^{\lambda}}{(a\nabla_{\nu}b)^{\lambda} - (a\sharp_{\nu}b)^{\lambda}} \leq \left(\frac{1-\mu}{1-\nu}\right)^{\lambda}.$$

The significance of Theorem 1.1 is: when $\lambda = 1$ and $\mu = \frac{1}{2}$ or $\nu = \frac{1}{2}$, then Theorem 1.1 retrieves (2). In fact, Theorem 1.1 implies better estimates than the main result in [3], as one can see in [17, Subsection 2.4].

J. Liao and J. Wu [21] replicated Theorem 1.1 as follows.

**Theorem 1.2 ([21]).** *Let $a, b > 0$ and let $\lambda, \mu, \nu$ be real numbers with $\lambda \geq 1$ and $0 \leq \mu < \nu \leq 1$. Then*

$$\left(\frac{\mu}{\nu}\right)^{\lambda} \leq \frac{(a\nabla_{\mu}b)^{\lambda} - (a!_{\mu}b)^{\lambda}}{(a\nabla_{\nu}b)^{\lambda} - (a!_{\nu}b)^{\lambda}} \leq \left(\frac{1-\mu}{1-\nu}\right)^{\lambda}.$$

M. Khosravi [12] presented the inequalities between the arithmetic and power means as follows.

**Theorem 1.3 ([12]).** *Let $a, b > 0$, $p \in (-1, 1)$ and let $\mu, \nu$ be real numbers with $0 \leq \mu < \nu \leq 1$. Then*

$$\frac{\mu}{\nu} \leq \frac{(a\nabla_{\mu}b) - (a\sharp_{\mu,p}b)}{(a\nabla_{\nu}b) - (a\sharp_{\nu,p}b)} \leq \frac{1-\mu}{1-\nu}.$$

Interestingly, Theorems 1.1, 1.2 and 1.3 happened to be special cases of a more general result obtained by M. Sababheh via convexity:

**Theorem 1.4 ([15]).** *Let $f : [0,1] \longrightarrow [0,+\infty)$ be convex and let $\lambda, \mu, \nu$ be real numbers with $\lambda \geq 1$ and $0 \leq \mu < \nu \leq 1$. Then*

$$\Big(\frac{\mu}{\nu}\Big)^{\lambda} \leq \frac{(\mu f(1)+(1-\mu)f(0))^{\lambda}-f^{\lambda}(\mu)}{(\nu f(1)+(1-\nu)f(0))^{\lambda}-f^{\lambda}(\nu)} \leq \Big(\frac{1-\mu}{1-\nu}\Big)^{\lambda}.$$

The interested reader is referred to [1, 2, 5, 7–10, 13, 17, 18, 18–20] as a sample of recent progress related to the above discussion.

The above inequalities have been used to obtain new bounds for operator means, trace, determinant, singular values and norm inequalities of matrices. This summarizes part of their significance.

Our main goal in this paper is to find a new extended inequality for log-convex functions, that refine the aforementioned results with applications to operator means and matrix inequalities.

## 2. Log-convex functions

In this section, we present our main new result for log-convex functions. First, we recall the arithmetic-geometric mean inequality.

**Lemma 2.1.** *Let $n$ be a positive integer. For $k = 1, 2, \ldots, n$, let $x_k > 0$ and let $\mu_k \geq 0$ satisfy $\sum_{k=1}^{n} \mu_k = 1$. Then*

$$\prod_{k=1}^{n} x_k^{\mu_k} \leq \sum_{k=1}^{n} \mu_k x_k. \tag{3}$$

We need also the following two lemmas.

**Lemma 2.2 ([9]).** *Let $m$ be a positive integer and let $\mu$ be a positive number, such that $0 \leq \mu \leq 1$. Then*

$$\sum_{k=1}^{m} \binom{m}{k} k \mu^{k}(1-\mu)^{m-k} = m\mu, \tag{4}$$

*and*

$$\sum_{k=0}^{m-1} \binom{m}{k} (m-k) \mu^{k}(1-\mu)^{m-k} = m(1-\mu), \tag{5}$$

*where $\binom{m}{k}$ is the binomial coefficient.*

**Lemma 2.3 ([11]).** *Let $\mu$ and $\nu$ be two positive numbers such that $0 \leq \mu \leq \nu \leq 1$ and $m$ be a positive integer.*

1. *For $0 \leq k \leq m$, we have*
$$\mu^{k}(1-\mu)^{m-k} - \Big(\frac{\mu}{\nu}\Big)^{m} \nu^{k}(1-\nu)^{m-k} \geq 0.$$
2. *If $0 \leq \mu \leq \frac{\nu}{2}$, then*
$$(1-\mu)^{m} - (1-\nu)^{m}\Big(\frac{\mu}{\nu}\Big)^{m} - \Big(\frac{\mu}{\nu}\Big)^{m} \geq 0.$$
3. *If $\frac{\nu}{2} \leq \mu \leq \nu$, then*
$$(1-\mu)^{m} - (1-\nu)^{m}\Big(\frac{\mu}{\nu}\Big)^{m} - \Big(1-\frac{\mu}{\nu}\Big)^{m} \geq 0.$$

Now we are ready to prove our main result about log-convex functions. Due to the delicate calculations, we will present the results in multiple theorems. Also, we will present the significance of these results in Remark 2.7 below.

**Theorem 2.4.** *Let $f:[0,1]\longrightarrow[0,+\infty)$ be log-convex, $0\le\mu\le\nu\le1$ and $m$ be a positive integer. If $0\le\mu\le\frac{\nu}{2}$, then*

$$\begin{aligned}&\Big(\frac{\mu}{\nu}\Big)^m\Big((f(1)\nabla_\nu f(0))^m-f^m(\nu)\Big)+\Big(\frac{\mu}{\nu}\Big)^m\Big(f^{\frac{m}{2}}(0)-f^{\frac{m}{2}}(\nu)\Big)^2\\&+r_m\Big(f^{\frac{m}{2}}(0)-f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)^2\le\Big(f(1)\nabla_\mu f(0)\Big)^m-f^m(\mu),\end{aligned}\tag{6}$$

*where* $r_m=\min\Big\{2\Big(\frac{\mu}{\nu}\Big)^m,(1-\mu)^m-\Big(\frac{\mu}{\nu}\Big)^m\Big((1-\nu)^m+1\Big)\Big\}\ge0.$

*Proof.* Suppose that $0\le\mu\le\frac{\nu}{2}$. We show that

$$\begin{aligned}&\Big(f(1)\nabla_\mu f(0)\Big)^m-\Big(\frac{\mu}{\nu}\Big)^m\Big((f(1)\nabla_\nu f(0))^m-f^m(\nu)\Big)-\Big(\frac{\mu}{\nu}\Big)^m\Big(f^{\frac{m}{2}}(0)-f^{\frac{m}{2}}(\nu)\Big)^2\\&-r_m\Big(f^{\frac{m}{2}}(0)-f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)^2\ge f^m(\mu).\end{aligned}$$

We have the following identities

$$\begin{aligned}&\Big(f(1)\nabla_\mu f(0)\Big)^m-\Big(\frac{\mu}{\nu}\Big)^m\Big((f(1)\nabla_\nu f(0))^m-f^m(\nu)\Big)\\&-\Big(\frac{\mu}{\nu}\Big)^m\Big(f^{\frac{m}{2}}(0)-f^{\frac{m}{2}}(\nu)\Big)^2-r_m\Big(f^{\frac{m}{2}}(0)-f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)^2\\=&\sum_{k=0}^m\binom{m}{k}\mu^k(1-\mu)^{m-k}f^k(1)f^{m-k}(0)\\&-\Big(\frac{\mu}{\nu}\Big)^m\Big(\sum_{k=0}^m\binom{m}{k}\nu^k(1-\nu)^{m-k}f^k(1)f^{m-k}(0)-f^m(\nu)\Big)\\&-\Big(\frac{\mu}{\nu}\Big)^m\Big(f^m(0)+f^m(\nu)-2f^{\frac{m}{2}}(0)f^{\frac{m}{2}}(\nu)\Big)\\&-r_m\Big(f^m(0)+f^{\frac{m}{2}}(0)f^{\frac{m}{2}}(\nu)-2f^{\frac{m}{2}+\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)\\=&\sum_{k=1}^m\binom{m}{k}\Big(\mu^k(1-\mu)^{m-k}-\Big(\frac{\mu}{\nu}\Big)^m\nu^k(1-\nu)^{m-k}\Big)f^k(1)f^{m-k}(0)\\&+\Big((1-\mu)^m-(1-\nu)^m\Big(\frac{\mu}{\nu}\Big)^m-\Big(\frac{\mu}{\nu}\Big)^m-r_m\Big)f^m(0)\\&+\Big(2\Big(\frac{\mu}{\nu}\Big)^m-r_m\Big)f^{\frac{m}{2}}(0)f^{\frac{m}{2}}(\nu)+2r_mf^{\frac{m}{2}+\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\\=&\sum_{k=0}^{m+2}\mu_kx_k,\end{aligned}$$

where

$$x_k=\begin{cases}f^m(0), & k=0\\ f^k(1)f^{m-k}(0), & 1\le k\le m\\ f^{\frac{m}{2}}(0)f^{\frac{m}{2}}(\nu), & k=m+1\\ f^{\frac{m}{2}+\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu), & k=m+2\end{cases}$$

and

$$\mu_k=\begin{cases}(1-\mu)^m-(1-\nu)^m\Big(\frac{\mu}{\nu}\Big)^m-\Big(\frac{\mu}{\nu}\Big)^m-r_m, & k=0\\ \binom{m}{k}\Big(\mu^k(1-\mu)^{m-k}-\Big(\frac{\mu}{\nu}\Big)^m\nu^k(1-\nu)^{m-k}\Big), & 1\le k\le m\\ 2\Big(\frac{\mu}{\nu}\Big)^m-r_m, & k=m+1\\ 2r_m, & k=m+2\end{cases}.$$

Lemma 2.3 implies that $\mu_k \geq 0$ for all $k \in \{0, 1, \ldots, m+1, m+2\}$. Further, direct calculations show that $\sum_{k=0}^{m+2} \mu_k = 1$.

Hence by Lemma 2.1, we get

$$\begin{aligned}
&\Big(f(1)\nabla_\mu f(0)\Big)^m - \Big(\frac{\mu}{\nu}\Big)^m\Big((f(1)\nabla_\nu f(0))^m - f^m(\nu)\Big) - \Big(\frac{\mu}{\nu}\Big)^m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\nu)\Big)^2 \\
&- r_m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)^2 \\
&= \sum_{k=0}^{m+2} \mu_k x_k \geq \prod_{k=0}^{m+2} x_k^{\mu_k} = f(1)^{\alpha(m)} f(0)^{\beta(m)} f^{\gamma(m)}(\nu),
\end{aligned}$$

where

$$\begin{aligned}
\alpha(m) &= \sum_{k=1}^{m} \binom{m}{k} k\Big(\mu^k(1-\mu)^{m-k} - \Big(\frac{\mu}{\nu}\Big)^m \nu^k(1-\nu)^{m-k}\Big) \\
&= m\mu - m\nu\Big(\frac{\mu}{\nu}\Big)^m \quad \text{(by Lemma 2.2)},
\end{aligned}$$

$$\begin{aligned}
\beta(m) &= \sum_{k=1}^{m-1} \binom{m}{k}(m-k)\Big(\mu^k(1-\mu)^{m-k} - \Big(\frac{\mu}{\nu}\Big)^m \nu^k(1-\nu)^{m-k}\Big) \\
&\quad + m\Big((1-\mu)^m - (1-\nu)^m\Big(\frac{\mu}{\nu}\Big)^m - \Big(\frac{\mu}{\nu}\Big)^m - r_m\Big) \\
&\quad + \frac{m}{2}\Big(2\Big(\frac{\mu}{\nu}\Big)^m - r_m\Big) + \Big(\frac{m}{2} + \frac{m}{4}\Big)2r_m \\
&= \sum_{k=0}^{m-1} \binom{m}{k}(m-k)\Big(\mu^k(1-\mu)^{m-k} - \Big(\frac{\mu}{\nu}\Big)^m \nu^k(1-\nu)^{m-k}\Big) \\
&= m(1-\mu) - m(1-\nu)\Big(\frac{\mu}{\nu}\Big)^m \quad \text{(by Lemma 2.2)},
\end{aligned}$$

and

$$\gamma(m) = \frac{m}{2}\Big(2\Big(\frac{\mu}{\nu}\Big)^m - r_m\Big) + \frac{m}{2} r_m = m\Big(\frac{\mu}{\nu}\Big)^m.$$

Consequently,

$$\begin{aligned}
&\Big(f(1)\nabla_\mu f(0)\Big)^m - \Big(\frac{\mu}{\nu}\Big)^m\Big((f(1)\nabla_\nu f(0))^m - f^m(\nu)\Big) - \Big(\frac{\mu}{\nu}\Big)^m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\nu)\Big)^2 \\
&- r_m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)^2 \geq \Big(f(1)^{\alpha_1(m)} f(0)^{\beta_1(m)} f^{\gamma_1(m)}(\nu)\Big)^m,
\end{aligned}$$

where $\alpha_1(m) = \mu - \nu\Big(\frac{\mu}{\nu}\Big)^m$, $\beta_1(m) = (1-\mu) - (1-\nu)\Big(\frac{\mu}{\nu}\Big)^m$ and $\gamma_1(m) = \Big(\frac{\mu}{\nu}\Big)^m$. Under the condition $0 \leq \mu \leq \frac{\nu}{2}$, we have $\alpha_1(m), \beta_1(m), \gamma_1(m) \geq 0$ and $\alpha_1(m) + \beta_1(m) + \gamma_1(m) = 1$. Applying the log-convexity of $f$, we get

$$\Big(f(1)^{\alpha_1(m)} f(0)^{\beta_1(m)} f^{\gamma_1(m)}(\nu)\Big)^m \geq f^m\Big(\alpha_1(m) + \nu\gamma_1(m)\Big) = f^m(\mu).$$

This completes the proof of the theorem. □

Now we present the other version of Theorem 2.4 for the values $\frac{\nu}{2} \leq \mu \leq \nu$.

**Theorem 2.5.** *Let $f : [0,1] \longrightarrow [0,+\infty)$ be log-convex, $0 \le \mu \le \nu \le 1$ and $m$ be a positive number. If $\frac{\nu}{2} \le \mu \le \nu$, then*

$$\begin{aligned}
&\Big(\frac{\mu}{\nu}\Big)^m\Big((f(1)\nabla_\nu f(0))^m - f^m(\nu)\Big) + \Big(1-\frac{\mu}{\nu}\Big)^m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\nu)\Big)^2\\
&+ r_m\Big(f^{\frac{m}{2}}(\nu) - f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)^2\\
\le&\Big(f(1)\nabla_\mu f(0)\Big)^m - f^m(\mu),
\end{aligned} \tag{7}$$

*where $r_m = \min\Big\{2\Big(1-\frac{\mu}{\nu}\Big)^m, \Big(\frac{\mu}{\nu}\Big)^m - \Big(1-\frac{\mu}{\nu}\Big)^m\Big\} \ge 0$.*

*Proof.* Suppose that $\frac{\nu}{2} \le \mu \le \nu$. We show that

$$\begin{aligned}
&\Big(f(1)\nabla_\mu f(0)\Big)^m - \Big(\frac{\mu}{\nu}\Big)^m\Big((f(1)\nabla_\nu f(0))^m - f^m(\nu)\Big) - \Big(1-\frac{\mu}{\nu}\Big)^m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\nu)\Big)^2\\
&- r_m\Big(f^{\frac{m}{2}}(\nu) - f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)^2 \ge f^m(\mu).
\end{aligned}$$

We have the following identities

$$\begin{aligned}
&\Big(f(1)\nabla_\mu f(0)\Big)^m - \Big(\frac{\mu}{\nu}\Big)^m\Big((f(1)\nabla_\nu f(0))^m - f^m(\nu)\Big) - \Big(1-\frac{\mu}{\nu}\Big)^m\Big(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\nu)\Big)^2\\
&- r_m\Big(f^{\frac{m}{2}}(\nu) - f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\Big)^2\\
=&\sum_{k=0}^{m}\binom{m}{k}\mu^k(1-\mu)^{m-k}f^k(1)f^{m-k}(0)\\
&-\Big(\frac{\mu}{\nu}\Big)^m\Big(\sum_{k=0}^{m}\binom{m}{k}\nu^k(1-\nu)^{m-k}f^k(1)f^{m-k}(0) - f^m(\nu)\Big)\\
&-\Big(1-\frac{\mu}{\nu}\Big)^m\Big(f^m(0) + f^m(\nu) - 2f^{\frac{m}{2}}(0)f^{\frac{m}{2}}(\nu)\Big)\\
&- r_m\Big(f^m(\nu) + f^{\frac{m}{2}}(0)f^{\frac{m}{2}}(\nu) - 2f^{\frac{m}{2}+\frac{m}{4}}(\nu)f^{\frac{m}{4}}(0)\Big)\\
=&\sum_{k=1}^{m}\binom{m}{k}\Big(\mu^k(1-\mu)^{m-k} - \Big(\frac{\mu}{\nu}\Big)^m\nu^k(1-\nu)^{m-k}\Big)f^k(1)f^{m-k}(0)\\
&+\Big((1-\mu)^m - (1-\nu)^m\Big(\frac{\mu}{\nu}\Big)^m - \Big(1-\frac{\mu}{\nu}\Big)^m\Big)f^m(0)\\
&+\Big(\Big(\frac{\mu}{\nu}\Big)^m - \Big(1-\frac{\mu}{\nu}\Big)^m - r_m\Big)f^m(\nu)\\
&+\Big(2\Big(1-\frac{\mu}{\nu}\Big)^m - r_m\Big)f^{\frac{m}{2}}(\nu)f^{\frac{m}{2}}(0) + 2r_m f^{\frac{m}{2}+\frac{m}{4}}(\nu)f^{\frac{m}{4}}(0)\\
=&\sum_{k=0}^{m+3}\mu_k x_k,
\end{aligned}$$

where

$$x_k = \begin{cases} f^m(0), & k=0\\ f^k(1)f^{m-k}(0), & 1\le k\le m\\ f^m(\nu), & k=m+1\\ f^{\frac{m}{2}}(0)f^{\frac{m}{2}}(\nu), & k=m+2\\ f^{\frac{m}{2}+\frac{m}{4}}(\nu)f^{\frac{m}{4}}(0), & k=m+3 \end{cases}$$

and

$$\mu_k = \begin{cases} (1-\mu)^m - (1-\nu)^m\left(\frac{\mu}{\nu}\right)^m - \left(1-\frac{\mu}{\nu}\right)^m, & k=0 \\ \binom{m}{k}\left(\mu^k(1-\mu)^{m-k} - \left(\frac{\mu}{\nu}\right)^m \nu^k(1-\nu)^{m-k}\right), & 1 \le k \le m \\ \left(\frac{\mu}{\nu}\right)^m - \left(1-\frac{\mu}{\nu}\right)^m - r_m, & k=m+1 \\ 2\left(1-\frac{\mu}{\nu}\right)^m - r_m, & k=m+2 \\ 2r_m, & k=m+3 \end{cases}.$$

Lemma 2.3 implies that
$\mu_k \geq 0$ for all $k \in \{0,1,\dots,m+2,m+3\}$. Further $\sum_{k=0}^{m+3} \mu_k = 1$.
Hence by Lemma 2.1, we get

$$\begin{aligned} &\left(f(1)\nabla_\mu f(0)\right)^m - \left(\frac{\mu}{\nu}\right)^m\left((f(1)\nabla_\nu f(0))^m - f^m(\nu)\right) - \left(1-\frac{\mu}{\nu}\right)^m\left(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\nu)\right)^2 \\ &- r_m\left(f^{\frac{m}{2}}(\nu) - f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\right)^2 \\ &= \sum_{k=0}^{m+3} \mu_k x_k \geq \prod_{k=0}^{m+3} x_k^{\mu_k} = f(1)^{\alpha(m)} f(0)^{\beta(m)} f^{\gamma(m)}(\nu), \end{aligned}$$

where

$$\begin{aligned} \alpha(m) &= \sum_{k=1}^{m}\binom{m}{k}k\left(\mu^k(1-\mu)^{m-k} - \left(\frac{\mu}{\nu}\right)^m \nu^k(1-\nu)^{m-k}\right) \\ &= m\mu - m\nu\left(\frac{\mu}{\nu}\right)^m \quad \text{(by Lemma 2.2)}, \end{aligned}$$

$$\begin{aligned} \beta(m) &= \sum_{k=0}^{m-1}\binom{m}{k}(m-k)\left(\mu^k(1-\mu)^{m-k} - \left(\frac{\mu}{\nu}\right)^m \nu^k(1-\nu)^{m-k}\right) \\ &\quad -m\left(1-\frac{\mu}{\nu}\right)^m + \frac{m}{2}\left(2\left(1-\frac{\mu}{\nu}\right)^m - r_m\right) + \frac{m}{2}r_m \\ &= \sum_{k=0}^{m-1}\binom{m}{k}(m-k)\left(\mu^k(1-\mu)^{m-k} - \left(\frac{\mu}{\nu}\right)^m \nu^k(1-\nu)^{m-k}\right) \\ &= m(1-\mu) - m(1-\nu)\left(\frac{\mu}{\nu}\right)^m \quad \text{(by Lemma 2.2)}, \end{aligned}$$

and

$$\gamma(m) = m\left(\left(\frac{\mu}{\nu}\right)^m - \left(1-\frac{\mu}{\nu}\right)^m - r_m\right) + \frac{m}{2}\left(2\left(1-\frac{\mu}{\nu}\right)^m - r_m\right) + \frac{3m}{2}r_m = m\left(\frac{\mu}{\nu}\right)^m.$$

So,

$$\begin{aligned} &\left(f(1)\nabla_\mu f(0)\right)^m - \left(\frac{\mu}{\nu}\right)^m\left((f(1)\nabla_\nu f(0))^m - f^m(\nu)\right) - \left(1-\frac{\mu}{\nu}\right)^m\left(f^{\frac{m}{2}}(0) - f^{\frac{m}{2}}(\nu)\right)^2 \\ &- r_m\left(f^{\frac{m}{2}}(\nu) - f^{\frac{m}{4}}(0)f^{\frac{m}{4}}(\nu)\right)^2 \geq \prod_{k=0}^{m+3} x_k^{\mu_k} = \left(f(1)^{\alpha_1(m)} f(0)^{\beta_1(m)} f^{\gamma_1(m)}(\nu)\right)^m, \end{aligned}$$

where $\alpha_1(m) = \mu - \nu\left(\frac{\mu}{\nu}\right)^m$, $\beta_1(m) = (1-\mu) - (1-\nu)\left(\frac{\mu}{\nu}\right)^m$ and $\gamma_1(m) = \left(\frac{\mu}{\nu}\right)^m$. Under the condition $\frac{\nu}{2} \leq \mu \leq \nu$, we have $\alpha_1(m), \beta_1(m), \gamma_1(m) \geq 0$ and $\alpha_1(m) + \beta_1(m) + \gamma_1(m) = 1$. Applying the log-convexity of the function $f$, we get

$$\left(f(1)^{\alpha_1(m)} f(0)^{\beta_1(m)} f^{\gamma_1(m)}(\nu)\right)^m \geq f^m\left(\alpha_1(m) + \nu\gamma_1(m)\right) = f^m(\mu).$$

This completes the proof of the theorem. □

As a consequence of Theorems 2.4 and 2.5 we present the following result.

**Theorem 2.6.** *Let $f:[0,1]\longrightarrow[0,+\infty)$ be log-convex, $0\le\mu\le\nu\le 1$ and $m$ be a positive number.*

1. *If $\mu\le\nu\le\frac{1+\mu}{2}$, then*

$$\begin{aligned}&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big((f(1)\nabla_\mu f(0))^m-f^m(\mu)\Big)+\Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big(f^{\frac{m}{2}}(1)-f^{\frac{m}{2}}(\mu)\Big)^2\\&+r_m\Big(f^{\frac{m}{2}}(\mu)-f^{\frac{m}{4}}(1)f^{\frac{m}{4}}(\mu)\Big)^2\\\le&\Big(f(1)\nabla_\nu f(0)\Big)^m-f^m(\nu),\end{aligned}\tag{8}$$

*where* $r_m=\min\Big\{2\Big(1-\frac{1-\nu}{1-\mu}\Big)^m,\Big(\frac{1-\nu}{1-\mu}\Big)^m-\Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big\}\ge 0.$

2. *If $\frac{1+\mu}{2}\le\nu\le 1$, then*

$$\begin{aligned}&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big((f(1)\nabla_\mu f(0))^m-f^m(\mu)\Big)+\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(f^{\frac{m}{2}}(1)-f^{\frac{m}{2}}(\mu)\Big)^2\\&+r_m\Big(f^{\frac{m}{2}}(1)-f^{\frac{m}{4}}(1)f^{\frac{m}{4}}(\mu)\Big)^2\\\le&\Big(f(1)\nabla_\nu f(0)\Big)^m-f^m(\nu),\end{aligned}\tag{9}$$

*where* $r_m=\min\Big\{2\Big(\frac{1-\nu}{1-\mu}\Big)^m,\nu^m-\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(\mu^m+1\Big)\Big\}\ge 0.$

*Proof.*
1. If $0\le\mu\le\nu\le 1$, then we have $0\le 1-\nu\le 1-\mu\le 1$. Suppose that $\mu\le\nu\le\frac{1+\mu}{2}$, then $\frac{1-\mu}{2}\le 1-\nu\le 1-\mu$. So by changing $f(x)$, $\mu$ and $\nu$ by $f(1-x)$, $1-\nu$ and $1-\mu$, respectively in the inequality (7), the desired inequality (8) is obtained.
2. Suppose that $\frac{1+\mu}{2}\le\nu\le 1$, then $0\le 1-\nu\le\frac{1-\mu}{2}$. So by changing $f(x)$, $\mu$ and $\nu$ by $f(1-x)$, $1-\nu$ and $1-\mu$, respectively in the inequality (6), the desired inequality (9) is obtained.
□

**Remark 2.7.** *Before proceeding to further results, we explain a little about the relation between Theorems 2.4, 2.5, 2.6 and Theorem 1.4.*

*Notice that the first inequality in Theorem 1.4 can be written as*

$$\Big(\frac{\mu}{\nu}\Big)^m\Big[(f(1)\nabla_\nu f(0))^m-f^m(\nu)\Big]\le\Big(f(1)\nabla_\mu f(0)\Big)^m-f^m(\mu);0\le\mu<\nu\le 1;m=1,2,\cdots,\tag{10}$$

*while the second inequality in the same theorem can be stated as*

$$\Big(f(1)\nabla_\mu f(0)\Big)^m-f^m(\mu)\le\Big(\frac{1-\mu}{1-\nu}\Big)^m\Big[(f(1)\nabla_\nu f(0))^m-f^m(\nu)\Big];0\le\mu<\nu\le 1;m=1,2,\cdots.\tag{11}$$

*Consequently, Theorems 2.4 and 2.5 present two refining terms of* (10)*, while Theorem 2.6 presents two refining terms of* (11)*.*
*Consequently the three Theorems 2.4, 2.5 and 2.6 give a considerable refinement of Theorem 1.4.*

*However, it should be noted that these refinements have been shown for integer powers $m$, and for log-convex functions.*
*We also notice that the assumption $f$ being log-convex was essential in the proof.*

*Since Theorem 1.4 was a generalization of the results in [3, 4, 12, 21], it follows that our results in this section provide better new estimates than the results in these references. This is the main significance of our results. In the next sections, we present explicit examples of refined inequalities for both scalars and operators.*

## 3. Scalar inequalities

When $a, b > 0$ and $p \in (-\infty, 0)$ the function $f(x) = a\sharp_{p,x}b := (xa^p + (1-x)b^p)^{\frac{1}{p}}$ is log-convex. Applying Theorems 2.4, 2.5 and 2.6, we obtain the following new bounds for the difference between the arithmetic and power means.

**Corollary 3.1.** *Let $a, b > 0$, $0 \le \mu \le \nu \le 1$, $m$ be a positive integer and $p \in (-\infty, 0)$.*

1. *If $0 \le \mu \le \frac{\nu}{2}$, then*

$$\begin{aligned}&\Big(\frac{\mu}{\nu}\Big)^m\Big((a\nabla_\nu b)^m - (a\sharp_{p,\nu}b)^m\Big) + \Big(\frac{\mu}{\nu}\Big)^m\Big(b^{\frac{m}{2}} - (a\sharp_{p,\nu}b)^{\frac{m}{2}}\Big)^2 + r_m\Big(b^{\frac{m}{2}} - b^{\frac{m}{4}}(a\sharp_{p,\nu}b)^{\frac{m}{4}}\Big)^2\\&\le\Big(a\nabla_\mu b\Big)^m - \Big(a\sharp_{p,\mu}b\Big)^m,\end{aligned}\tag{12}$$

*where* $r_m = \min\Big\{2\Big(\frac{\mu}{\nu}\Big)^m, (1-\mu)^m - \Big(\frac{\mu}{\nu}\Big)^m\Big((1-\nu)^m + 1\Big)\Big\}$.

2. *If $\frac{\nu}{2} \le \mu \le \nu$, then*

$$\begin{aligned}&\Big(\frac{\mu}{\nu}\Big)^m\Big((a\nabla_\nu b)^m - (a\sharp_{p,\nu}b)^m\Big) + \Big(1 - \frac{\mu}{\nu}\Big)^m\Big(b^{\frac{m}{2}} - (a\sharp_{p,\nu}b)^{\frac{m}{2}}\Big)^2\\&+ r_m\Big((a\sharp_{p,\nu}b)^{\frac{m}{2}} - b^{\frac{m}{4}}(a\sharp_{p,\nu}b)^{\frac{m}{4}}\Big)^2\\&\le\Big(a\nabla_\mu b\Big)^m - \Big(a\sharp_{p,\mu}b\Big)^m,\end{aligned}\tag{13}$$

*where* $r_m = \min\Big\{2\Big(1 - \frac{\mu}{\nu}\Big)^m, \Big(\frac{\mu}{\nu}\Big)^m - \Big(1 - \frac{\mu}{\nu}\Big)^m\Big\}$.

3. *If $\mu \le \nu \le \frac{1+\mu}{2}$, then*

$$\begin{aligned}&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big((a\nabla_\mu b)^m - (a\sharp_{p,\mu}b)^m\Big) + \Big(1 - \frac{1-\nu}{1-\mu}\Big)^m\Big(a^{\frac{m}{2}} - (a\sharp_{p,\mu}b)^{\frac{m}{2}}\Big)^2\\&+ r_m\Big((a\sharp_{p,\mu}b)^{\frac{m}{2}} - a^{\frac{m}{4}}(a\sharp_{p,\mu}b)^{\frac{m}{4}}\Big)^2\\&\le\Big(a\nabla_\nu b\Big)^m - \Big(a\sharp_{p,\nu}b\Big)^m,\end{aligned}\tag{14}$$

*where* $r_m = \min\Big\{2\Big(1 - \frac{1-\nu}{1-\mu}\Big)^m, \Big(\frac{1-\nu}{1-\mu}\Big)^m - \Big(1 - \frac{1-\nu}{1-\mu}\Big)^m\Big\}$.

4. *If $\frac{1+\mu}{2} \le \nu \le 1$, then*

$$\begin{aligned}&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big((a\nabla_\mu b)^m - (a\sharp_{p,\mu}b)^m\Big) + \Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(a^{\frac{m}{2}} - (a\sharp_{p,\mu}b)^{\frac{m}{2}}\Big)^2\\&+ r_m\Big(a^{\frac{m}{2}} - a^{\frac{m}{4}}(a\sharp_{p,\mu}b)^{\frac{m}{4}}\Big)^2\\&\le\Big(a\nabla_\nu b\Big)^m - \Big(a\sharp_{p,\nu}b\Big)^m,\end{aligned}\tag{15}$$

*where* $r_m = \min\Big\{2\Big(\frac{1-\nu}{1-\mu}\Big)^m, \nu^m - \Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(\mu^m + 1\Big)\Big\}$.

If we let $p \longrightarrow 0$ in Corollary 3.1, we get the following bounds for the difference between the arithmetic and geometric means obtained in [11].

**Corollary 3.2.** *Let $a, b > 0$, $0 \le \mu \le \nu \le 1$ and $m$ be a positive integer.*

1. *If* $0 \le \mu \le \frac{\nu}{2}$, *then*

$$\left(\frac{\mu}{\nu}\right)^m\left((a\nabla_\nu b)^m - (a\sharp_\nu b)^m\right) + \left(\frac{\mu}{\nu}\right)^m\left(b^{\frac{m}{2}} - (a\sharp_\nu b)^{\frac{m}{2}}\right)^2 + r_m\left(b^{\frac{m}{2}} - b^{\frac{m}{4}}(a\sharp_\nu b)^{\frac{m}{4}}\right)^2$$
$$\le\left(a\nabla_\mu b\right)^m - \left(a\sharp_\mu b\right)^m, \tag{16}$$

*where* $r_m = \min\left\{2\left(\frac{\mu}{\nu}\right)^m, (1-\mu)^m - \left(\frac{\mu}{\nu}\right)^m\left((1-\nu)^m + 1\right)\right\}$.

2. *If* $\frac{\nu}{2} \le \mu \le \nu$, *then*

$$\left(\frac{\mu}{\nu}\right)^m\left((a\nabla_\nu b)^m - (a\sharp_\nu b)^m\right) + \left(1 - \frac{\mu}{\nu}\right)^m\left(b^{\frac{m}{2}} - (a\sharp_\nu b)^{\frac{m}{2}}\right)^2$$
$$+ r_m\left((a\sharp_\nu b)^{\frac{m}{2}} - b^{\frac{m}{4}}(a\sharp_\nu b)^{\frac{m}{4}}\right)^2$$
$$\le\left(a\nabla_\mu b\right)^m - \left(a\sharp_\mu b\right)^m, \tag{17}$$

*where* $r_m = \min\left\{2\left(1 - \frac{\mu}{\nu}\right)^m, \left(\frac{\mu}{\nu}\right)^m - \left(1 - \frac{\mu}{\nu}\right)^m\right\}$.

3. *If* $\mu \le \nu \le \frac{1+\mu}{2}$, *then*

$$\left(\frac{1-\nu}{1-\mu}\right)^m\left((a\nabla_\mu b)^m - (a\sharp_\mu b)^m\right) + \left(1 - \frac{1-\nu}{1-\mu}\right)^m\left(a^{\frac{m}{2}} - (a\sharp_\mu b)^{\frac{m}{2}}\right)^2$$
$$+ r_m\left((a\sharp_\mu b)^{\frac{m}{2}} - a^{\frac{m}{4}}(a\sharp_\mu b)^{\frac{m}{4}}\right)^2$$
$$\le\left(a\nabla_\nu b\right)^m - \left(a\sharp_\nu b\right)^m, \tag{18}$$

*where* $r_m = \min\left\{2\left(1 - \frac{1-\nu}{1-\mu}\right)^m, \left(\frac{1-\nu}{1-\mu}\right)^m - \left(1 - \frac{1-\nu}{1-\mu}\right)^m\right\}$.

4. *If* $\frac{1+\mu}{2} \le \nu \le 1$, *then*

$$\left(\frac{1-\nu}{1-\mu}\right)^m\left((a\nabla_\mu b)^m - (a\sharp_\mu b)^m\right) + \left(\frac{1-\nu}{1-\mu}\right)^m\left(a^{\frac{m}{2}} - (a\sharp_\mu b)^{\frac{m}{2}}\right)^2$$
$$+ r_m\left(a^{\frac{m}{2}} - a^{\frac{m}{4}}(a\sharp_\mu b)^{\frac{m}{4}}\right)^2$$
$$\le\left(a\nabla_\nu b\right)^m - \left(a\sharp_\nu b\right)^m, \tag{19}$$

*where* $r_m = \min\left\{2\left(\frac{1-\nu}{1-\mu}\right)^m, \nu^m - \left(\frac{1-\nu}{1-\mu}\right)^m\left(\mu^m + 1\right)\right\}$.

On the other hand, letting $p = -1$ in Corollary 3.1, we have the following bounds for the difference between the arithmetic and harmonic means. This provides new refinements and reverses for this difference.

**Corollary 3.3.** *Let* $a, b > 0$, $0 \le \mu \le \nu \le 1$ *and* $m$ *be a positive integer.*

1. *If* $0 \le \mu \le \frac{\nu}{2}$, *then*

$$\left(\frac{\mu}{\nu}\right)^m\left((a\nabla_\nu b)^m - (a!_\nu b)^m\right) + \left(\frac{\mu}{\nu}\right)^m\left(b^{\frac{m}{2}} - (a!_\nu b)^{\frac{m}{2}}\right)^2 + r_m\left(b^{\frac{m}{2}} - b^{\frac{m}{4}}(a!_\nu b)^{\frac{m}{4}}\right)^2$$
$$\le\left(a\nabla_\mu b\right)^m - \left(a!_\mu b\right)^m, \tag{20}$$

*where* $r_m = \min\left\{2\left(\frac{\mu}{\nu}\right)^m, (1-\mu)^m - \left(\frac{\mu}{\nu}\right)^m\left((1-\nu)^m + 1\right)\right\}$.

2. *If* $\frac{\nu}{2} \le \mu \le \nu$, *then*

$$\begin{aligned}&\Big(\frac{\mu}{\nu}\Big)^m\Big((a\nabla_\nu b)^m - (a!_\nu b)^m\Big) + \Big(1-\frac{\mu}{\nu}\Big)^m\Big(b^{\frac{m}{2}} - (a!_\nu b)^{\frac{m}{2}}\Big)^2\\ &+ r_m\Big((a!_\nu b)^{\frac{m}{2}} - b^{\frac{m}{4}}(a!_\nu b)^{\frac{m}{4}}\Big)^2\\ \le&\Big(a\nabla_\mu b\Big)^m - \Big(a!_\mu b\Big)^m,\end{aligned} \tag{21}$$

*where* $r_m = \min\Big\{2\Big(1-\frac{\mu}{\nu}\Big)^m, \Big(\frac{\mu}{\nu}\Big)^m - \Big(1-\frac{\mu}{\nu}\Big)^m\Big\}$.

3. *If* $\mu \le \nu \le \frac{1+\mu}{2}$, *then*

$$\begin{aligned}&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big((a\nabla_\mu b)^m - (a!_\mu b)^m\Big) + \Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big(a^{\frac{m}{2}} - (a!_\mu b)^{\frac{m}{2}}\Big)^2\\ &+ r_m\Big((a!_\mu b)^{\frac{m}{2}} - a^{\frac{m}{4}}(a!_\mu b)^{\frac{m}{4}}\Big)^2\\ \le&\Big(a\nabla_\nu b\Big)^m - \Big(a!_\nu b\Big)^m,\end{aligned} \tag{22}$$

*where* $r_m = \min\Big\{2\Big(1-\frac{1-\nu}{1-\mu}\Big)^m, \Big(\frac{1-\nu}{1-\mu}\Big)^m - \Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big\}$.

4. *If* $\frac{1+\mu}{2} \le \nu \le 1$, *then*

$$\begin{aligned}&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big((a\nabla_\mu b)^m - (a!_\mu b)^m\Big) + \Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(a^{\frac{m}{2}} - (a!_\mu b)^{\frac{m}{2}}\Big)^2\\ &+ r_m\Big(a^{\frac{m}{2}} - a^{\frac{m}{4}}(a!_\mu b)^{\frac{m}{4}}\Big)^2\\ \le&\Big(a\nabla_\nu b\Big)^m - \Big(a!_\nu b\Big)^m,\end{aligned} \tag{23}$$

*where* $r_m = \min\Big\{2\Big(\frac{1-\nu}{1-\mu}\Big)^m, \nu^m - \Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(\mu^m + 1\Big)\Big\}$.

## 4. Inequalities for operators

In this section, we present operator versions of the above scalar inequalities. The following lemma is essential in this regard.

**Lemma 4.1 ([14, p. 3]).** *Let* $T \in B(\mathcal{H})$ *be self-adjoint. If* $f$ *and* $g$ *are both continuous real valued functions with* $f(t) \ge g(t)$ *for* $t \in Sp(T)$ (*where the sign* $Sp(T)$ *denotes the spectrum of* $T$), *then* $f(T) \ge g(T)$.

The following theorem presents the operator version of Theorems 2.4, 2.5 and 2.6, providing new refinements and reverses for the difference between operator arithmetic and power means.

**Theorem 4.2.** *Let* $A, B \in B(\mathcal{H})^{++}$, $0 \le \mu \le \nu \le 1$ *and let* $m$ *be a positive integer.*

1. *If* $0 \le \mu \le \frac{\nu}{2}$, *then*

$$\begin{aligned}&\Big(\frac{\mu}{\nu}\Big)^m\Big(A\sharp_m(A\nabla_\nu B) - A\sharp_m(A\sharp_{p,\nu}B)\Big)\\ &+\Big(\frac{\mu}{\nu}\Big)^m\Big(A + A\sharp_m(A\sharp_{p,\nu}B) - 2A\sharp_{\frac{m}{2}}(A\sharp_{p,\nu}B)\Big)\\ &+ r_m\Big(A + A\sharp_{\frac{m}{2}}(A\sharp_{p,\nu}B) - 2A\sharp_{\frac{m}{4}}(A\sharp_{p,\nu}B)\Big)\\ \le&\ A\sharp_m(A\nabla_\mu B) - A\sharp_m(A\sharp_{p,\mu}B),\end{aligned} \tag{24}$$

*where* $r_m = \min\Big\{2\Big(\frac{\mu}{\nu}\Big)^m, (1-\mu)^m - \Big(\frac{\mu}{\nu}\Big)^m\Big((1-\nu)^m + 1\Big)\Big\}$.

2. *If* $\frac{\nu}{2} \le \mu \le \nu$*, then*

$$\begin{aligned}&\Big(\frac{\mu}{\nu}\Big)^m\Big(A\sharp_m(A\nabla_\nu B) - A\sharp_m(A\sharp_{p,\nu}B)\Big)\\&+\Big(1-\frac{\mu}{\nu}\Big)^m\Big(A + A\sharp_m(A\sharp_{p,\nu}B) - 2A\sharp_{\frac{m}{2}}(A\sharp_{p,\nu}B)\Big)\\&+r_m\Big(A\sharp_m(A\sharp_{p,\nu}B) + A\sharp_{\frac{m}{2}}(A\sharp_{p,\nu}B) - 2A\sharp_{\frac{3m}{4}}(A\sharp_{p,\nu}B)\Big)\\&\le A\sharp_m(A\nabla_\mu B) - A\sharp_m(A\sharp_{p,\mu}B),\end{aligned}\tag{25}$$

*where* $r_m = \min\Big\{2\Big(1-\frac{\mu}{\nu}\Big)^m, \Big(\frac{\mu}{\nu}\Big)^m - \Big(1-\frac{\mu}{\nu}\Big)^m\Big\}$.

3. *If* $\mu \le \nu \le \frac{1+\mu}{2}$*, then*

$$\begin{aligned}&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(A\sharp_m(A\nabla_\mu B) - A\sharp_m(A\sharp_{p,\mu}B)\Big)\\&+\Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big(A\sharp_m B + A\sharp_m(A\sharp_{p,\mu}B) - 2(A\sharp_{\frac{m}{2}}B)A^{-1}A\sharp_{\frac{m}{2}}(A\sharp_{p,\mu}B)\Big)\\&+r_m\Big(A\sharp_m(A\sharp_{p,\mu}B) + (A\sharp_{\frac{m}{2}}B)A^{-1}(A\sharp_{\frac{m}{2}}(A\sharp_{p,\mu}B)) - 2(A\sharp_{\frac{m}{4}}B)A^{-1}(A\sharp_{\frac{3m}{4}}(A\sharp_{p,\mu}B))\Big)\\&\le A\sharp_m(A\nabla_\nu B) - A\sharp_m(A\sharp_{p,\nu}B),\end{aligned}\tag{26}$$

*where* $r_m = \min\Big\{2\Big(1-\frac{1-\nu}{1-\mu}\Big)^m, \Big(\frac{1-\nu}{1-\mu}\Big)^m - \Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big\}$.

4. *If* $\frac{1+\mu}{2} \le \nu \le 1$*, then*

$$\begin{aligned}&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(A\sharp_m(A\nabla_\mu B) - A\sharp_m(A\sharp_{p,\mu}B)\Big)\\&+\Big(A\sharp_m B + A\sharp_m(A\sharp_{p,\mu}B) - 2(A\sharp_{\frac{m}{2}}B)A^{-1}A\sharp_{\frac{m}{2}}(A\sharp_{p,\mu}B)\Big)\\&+r_m\Big(A\sharp_m B + (A\sharp_{\frac{m}{2}}B)A^{-1}(A\sharp_{\frac{m}{2}}(A\sharp_{p,\mu}B)) - 2(A\sharp_{\frac{3m}{4}}B)A^{-1}(A\sharp_{\frac{m}{4}}(A\sharp_{p,\mu}B))\Big)\\&\le A\sharp_m(A\nabla_\nu B) - A\sharp_m(A\sharp_{p,\nu}B),\end{aligned}\tag{27}$$

*where* $r_m = \min\Big\{2\Big(\frac{1-\nu}{1-\mu}\Big)^m, \nu^m - \Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(\mu^m+1\Big)\Big\}$.

*Proof.* We prove the first inequality. The other inequalities can be shown similarly. Let $b = 1$ in inequality (12). Then

$$\begin{aligned}\Big(\frac{\mu}{\nu}\Big)^m\Big((\nu a + (1-\nu))^m \;&-\; (\nu a^p + (1-\nu))^{\frac{m}{p}}\Big)\\&+\;\Big(\frac{\mu}{\nu}\Big)^m\Big(1 + (\nu a^p + (1-\nu))^{\frac{m}{p}} - 2(\nu a^p + (1-\nu))^{\frac{m}{2p}}\Big)\\&+\;r_m\Big(1 + (\nu a^p + (1-\nu))^{\frac{m}{2p}} - 2(\nu a^p + (1-\nu))^{\frac{m}{4p}}\Big)\\&\le\;(\mu a + (1-\mu))^m - (\mu a^p + (1-\mu))^{\frac{m}{p}}\end{aligned}\tag{28}$$

Since the operator $C = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ has a positive spectrum, Lemma 4.1 and inequality (28) imply

$$\begin{aligned}\Big(\frac{\mu}{\nu}\Big)^m\Big((\nu C + (1-\nu)I)^m \;&-\; (\nu C^p + (1-\nu)I)^{\frac{m}{p}}\Big)\\&+\;\Big(\frac{\mu}{\nu}\Big)^m\Big(I + (\nu C^p + (1-\nu)I)^{\frac{m}{p}} - 2(\nu C^p + (1-\nu)I)^{\frac{m}{2p}}\Big)\\&+\;r_m\Big(I + (\nu C^p + (1-\nu)I)^{\frac{m}{2p}} - 2(\nu C^p + (1-\nu)I)^{\frac{m}{4p}}\Big)\\&\le\;(\mu C + (1-\mu)I)^m - (\mu C^p + (1-\mu)I)^{\frac{m}{p}}.\end{aligned}\tag{29}$$

Finally, multiplying inequality (29) by $A^{\frac{1}{2}}$ on the left and right sides, we get

$$\begin{aligned}
&\Big(\frac{\mu}{\nu}\Big)^m\Big(A\sharp_m(A\nabla_\nu B)-A\sharp_m(A\sharp_{p,\nu}B)\Big)\\
&+\Big(\frac{\mu}{\nu}\Big)^m\Big(A+A\sharp_m(A\sharp_{p,\nu}B)-2A\sharp_{\frac{m}{2}}(A\sharp_{p,\nu}B)\Big)\\
&+r_m\Big(A+A\sharp_{\frac{m}{2}}(A\sharp_{p,\nu}B)-2A\sharp_{\frac{m}{4}}(A\sharp_{p,\nu}B)\Big)\\
\leq\ &A\sharp_m(A\nabla_\mu B)-A\sharp_m(A\sharp_{p,\mu}B),
\end{aligned}$$

□

If we let $p\longrightarrow 0$ in Theorem 4.2, we get the following result obtained in [11].

**Theorem 4.3.** *Let $A,B\in B(\mathcal{H})^{++}, 0\leq\mu\leq\nu\leq 1$ and let $m$ be a positive integer.*

1. *If $0\leq\mu\leq\frac{\nu}{2}$, then*

$$\begin{aligned}
&\Big(\frac{\mu}{\nu}\Big)^m\Big(A\sharp_m(A\nabla_\nu B)-A\sharp_{m\nu}B\Big)+\Big(\frac{\mu}{\nu}\Big)^m\Big(A+A\sharp_{m\nu}B-2A\sharp_{\frac{m\nu}{2}}B\Big)\\
&+r_m\Big(A+A\sharp_{\frac{m\nu}{2}}B-2A\sharp_{\frac{m\nu}{4}}B\Big)\\
\leq\ &A\sharp_m(A\nabla_\mu B)-A\sharp_{m\mu}B,
\end{aligned}\tag{30}$$

*where* $r_m=\min\Big\{2\big(\frac{\mu}{\nu}\big)^m,(1-\mu)^m-\big(\frac{\mu}{\nu}\big)^m\big((1-\nu)^m+1\big)\Big\}$.

2. *If $\frac{\nu}{2}\leq\mu\leq\nu$, then*

$$\begin{aligned}
&\Big(\frac{\mu}{\nu}\Big)^m\Big(A\sharp_m(A\nabla_\nu B)-A\sharp_{m\nu}B\Big)+\Big(1-\frac{\mu}{\nu}\Big)^m\Big(A+A\sharp_{m\nu}B-2A\sharp_{\frac{m\nu}{2}}B\Big)\\
&+r_m\Big(A\sharp_{m\nu}B+A\sharp_{\frac{m\nu}{2}}B-2A\sharp_{\frac{m\nu}{2}+\frac{m\nu}{4}}B\Big)\\
\leq\ &A\sharp_m(A\nabla_\mu B)-A\sharp_{m\mu}B,
\end{aligned}\tag{31}$$

*where* $r_m=\min\Big\{2\big(1-\frac{\mu}{\nu}\big)^m,\big(\frac{\mu}{\nu}\big)^m-\big(1-\frac{\mu}{\nu}\big)^m\Big\}$.

3. *If $\mu\leq\nu\leq\frac{1+\mu}{2}$, then*

$$\begin{aligned}
&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(A\sharp_m(A\nabla_\mu B)-A\sharp_{m\mu}B\Big)+\Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big(A\sharp_m B+A\sharp_{m\mu}B-2A\sharp_{\frac{m}{2}+\frac{m\mu}{2}}B\Big)\\
&+r_m\Big(A\sharp_m B+A\sharp_{\frac{m}{2}+\frac{m\nu}{2}}B-2A\sharp_{\frac{3m}{4}+\frac{m\nu}{4}}B\Big)\\
\leq\ &A\sharp_m(A\nabla_\nu B)-A\sharp_{m\nu}B,
\end{aligned}\tag{32}$$

*where* $r_m=\min\Big\{2\big(1-\frac{1-\nu}{1-\mu}\big)^m,\big(\frac{1-\nu}{1-\mu}\big)^m-\big(1-\frac{1-\nu}{1-\mu}\big)^m\Big\}$.

4. *If $\frac{1+\mu}{2}\leq\nu\leq 1$, then*

$$\begin{aligned}
&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(A\sharp_m(A\nabla_\mu B)-A\sharp_{m\mu}B\Big)+\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(A\sharp_m B+A\sharp_{m\mu}B-2A\sharp_{\frac{m}{2}+\frac{m\mu}{2}}B\Big)\\
&+r_m\Big(A+A\sharp_{\frac{m\nu}{2}}B-2A\sharp_{\frac{m\nu}{4}}B\Big)\\
\leq\ &A\sharp_m(A\nabla_\nu B)-A\sharp_{m\nu}B,
\end{aligned}\tag{33}$$

*where* $r_m=\min\Big\{2\big(\frac{1-\nu}{1-\mu}\big)^m,\nu^m-\big(\frac{1-\nu}{1-\mu}\big)^m\big(\mu^m+1\big)\Big\}$.

If we take $p = -1$ in Theorem 4.2, we get the following theorem

**Theorem 4.4.** *Let $A, B \in B(\mathcal{H})^{++}, 0 \le \mu \le \nu \le 1$ and let $m$ be a positive integer.*

1. *If $0 \le \mu \le \frac{\nu}{2}$, then*

$$\begin{aligned}&\left(\frac{\mu}{\nu}\right)^m\Big(A\sharp_m(A\nabla_\nu B) - A\sharp_m(A!_\nu B)\Big)\\&\quad+\left(\frac{\mu}{\nu}\right)^m\Big(A + A\sharp_m(A!_\nu B) - 2A\sharp_{\frac{m}{2}}(A!_\nu B)\Big)\\&\quad+ r_m\Big(A + A\sharp_{\frac{m}{2}}(A!_\nu B) - 2A\sharp_{\frac{m}{4}}(A!_\nu B)\Big)\\&\le A\sharp_m(A\nabla_\mu B) - A\sharp_m(A!_\mu B),\end{aligned} \tag{34}$$

*where* $r_m = \min\left\{2\left(\frac{\mu}{\nu}\right)^m, (1-\mu)^m - \left(\frac{\mu}{\nu}\right)^m\Big((1-\nu)^m + 1\Big)\right\}$.

2. *If $\frac{\nu}{2} \le \mu \le \nu$, then*

$$\begin{aligned}&\left(\frac{\mu}{\nu}\right)^m\Big(A\sharp_m(A\nabla_\nu B) - A\sharp_m(A!_\nu B)\Big)\\&\quad+\left(1-\frac{\mu}{\nu}\right)^m\Big(A + A\sharp_m(A!_\nu B) - 2A\sharp_{\frac{m}{2}}(A!_\nu B)\Big)\\&\quad+ r_m\Big(A\sharp_m(A!_\nu B) + A\sharp_{\frac{m}{2}}(A!_\nu B) - 2A\sharp_{\frac{3m}{4}}(A!_\nu B)\Big)\\&\le A\sharp_m(A\nabla_\mu B) - A\sharp_m(A!_\mu B),\end{aligned} \tag{35}$$

*where* $r_m = \min\left\{2\left(1-\frac{\mu}{\nu}\right)^m, \left(\frac{\mu}{\nu}\right)^m - \left(1-\frac{\mu}{\nu}\right)^m\right\}$.

3. *If $\mu \le \nu \le \frac{1+\mu}{2}$, then*

$$\begin{aligned}&\left(\frac{1-\nu}{1-\mu}\right)^m\Big(A\sharp_m(A\nabla_\mu B) - A\sharp_m(A!_\mu B)\Big)\\&\quad+\left(1-\frac{1-\nu}{1-\mu}\right)^m\Big(A\sharp_m B + A\sharp_m(A!_\mu B) - 2(A\sharp_{\frac{m}{2}}B)A^{-1}A\sharp_{\frac{m}{2}}(A!_\mu B)\Big)\\&\quad+ r_m\Big(A\sharp_m(A!_\mu B) + (A\sharp_{\frac{m}{2}}B)A^{-1}(A\sharp_{\frac{m}{2}}(A!_\mu B)) - 2(A\sharp_{\frac{m}{4}}B)A^{-1}(A\sharp_{\frac{3m}{4}}(A!_\mu B))\Big)\\&\le A\sharp_m(A\nabla_\nu B) - A\sharp_m(A!_\nu B),\end{aligned} \tag{36}$$

*where* $r_m = \min\left\{2\left(1-\frac{1-\nu}{1-\mu}\right)^m, \left(\frac{1-\nu}{1-\mu}\right)^m - \left(1-\frac{1-\nu}{1-\mu}\right)^m\right\}$.

4. *If $\frac{1+\mu}{2} \le \nu \le 1$, then*

$$\begin{aligned}&\left(\frac{1-\nu}{1-\mu}\right)^m\Big(A\sharp_m(A\nabla_\mu B) - A\sharp_m(A!_\mu B)\Big)\\&\quad+\Big(A\sharp_m B + A\sharp_m(A!_\mu B) - 2(A\sharp_{\frac{m}{2}}B)A^{-1}A\sharp_{\frac{m}{2}}(A!_\mu B)\Big)\\&\quad+ r_m\Big(A\sharp_m B + (A\sharp_{\frac{m}{2}}B)A^{-1}(A\sharp_{\frac{m}{2}}(A!_\mu B)) - 2(A\sharp_{\frac{3m}{4}}B)A^{-1}(A\sharp_{\frac{m}{4}}(A!_\mu B))\Big)\\&\le A\sharp_m(A\nabla_\nu B) - A\sharp_m(A!_\nu B),\end{aligned} \tag{37}$$

*where* $r_m = \min\left\{2\left(\frac{1-\nu}{1-\mu}\right)^m, \nu^m - \left(\frac{1-\nu}{1-\mu}\right)^m\Big(\mu^m + 1\Big)\right\}$.

## 5. Some matrix inequalities

In this part of the paper, by selecting some appropriate log-convex functions, we obtain multiple term refinements of some results in the literature, related to matrices.

Let $\mathbf{M}_n$ be the algebra of all complex matrices of order $n \times n$ and let $\mathbf{M}_n^{++}$ be the class of positive definite matrices in $\mathbf{M}_n$. On $\mathbf{M}_n$, a unitarily invariant norm $|||.|||$ is a matrix norm that satisfies the invariance property $|||UAV||| = |||A|||$ for every $A \in \mathbf{M}_n$ and for all unitary matrices $U, V \in \mathbf{M}_n$.

The classical Young inequality $a\sharp_\mu b \le a\nabla_\mu b$ has been extended to matrices as follows

$$|||A^\mu X B^{1-\mu}||| \le \mu|||AX||| + (1-\mu)|||XB|||, 0 \le \mu \le 1, \tag{38}$$

for $A, B \in \mathbf{M}_n^+$ and $X \in \mathbf{M}_n$, where $|||.|||$ is a unitarily invariant norm.

It is known that when $A, B \in \mathbf{M}_n^{++}$ and $X \in \mathbf{M}_n$, the function $f(\nu) = |||A^\nu X B^{1-\nu}|||$ is log-convex on $[0, 1]$, (see [16]) for any unitarily invariant norm $|||.|||$ on $\mathbf{M}_n$. Then by using Theorems 2.4, 2.5 and 2.6, we obtain the following new result refining and reversing (38). This extends the corresponding results in [15].

**Theorem 5.1.** *Let $A, B \in \mathbf{M}_n^{++}$, $X \in \mathbf{M}_n$ and $0 \le \mu \le \nu \le 1$. Then for all positive integer $m$,*

1. *If $0 \le \mu \le \frac{\nu}{2}$, then*

$$\begin{aligned}
&\left(\frac{\mu}{\nu}\right)^m\Big((|||AX|||\nabla_\nu|||XB|||)^m - (|||A^\nu XB^{1-\nu}|||)^m\Big)\\
&+\left(\frac{\mu}{\nu}\right)^m\Big(|||XB|||^{\frac{m}{2}} - |||A^\nu XB^{1-\nu}|||^{\frac{m}{2}}\Big)^2\\
&+r_m\Big(|||XB|||^{\frac{m}{2}} - |||XB|||^{\frac{m}{4}}|||A^\nu XB^{1-\nu}|||^{\frac{m}{4}}\Big)^2\\
&\le\Big(\mu|||AX||| + (1-\mu)|||XB|||\Big)^m - \Big(|||A^\mu XB^{1-\mu}|||\Big)^m,
\end{aligned} \tag{39}$$

*where $r_m = \min\left\{2\left(\frac{\mu}{\nu}\right)^m, (1-\mu)^m - \left(\frac{\mu}{\nu}\right)^m\Big((1-\nu)^m + 1\Big)\right\}$.*

2. *If $\frac{\nu}{2} \le \mu \le \nu$, then*

$$\begin{aligned}
&\left(\frac{\mu}{\nu}\right)^m\Big((|||AX|||\nabla_\nu|||XB|||)^m - (|||A^\nu XB^{1-\nu}|||)^m\Big)\\
&+\left(1-\frac{\mu}{\nu}\right)^m\Big(|||XB|||^{\frac{m}{2}} - |||A^\nu XB^{1-\nu}|||^{\frac{m}{2}}\Big)^2\\
&+r_m\Big(|||A^\nu XB^{1-\nu}|||^{\frac{m}{2}} - |||XB|||^{\frac{m}{4}}|||A^\nu XB^{1-\nu}|||^{\frac{m}{4}}\Big)^2\\
&\le\Big(\mu|||AX||| + (1-\mu)|||XB|||\Big)^m - \Big(|||A^\mu XB^{1-\mu}|||\Big)^m,
\end{aligned} \tag{40}$$

*where $r_m = \min\left\{2\left(1-\frac{\mu}{\nu}\right)^m, \left(\frac{\mu}{\nu}\right)^m - \left(1-\frac{\mu}{\nu}\right)^m\right\}$.*

3. *If $\mu \le \nu \le \frac{1+\mu}{2}$, then*

$$\begin{aligned}
&\left(\frac{1-\nu}{1-\mu}\right)^m\Big((|||AX|||\nabla_\mu|||XB|||)^m - (|||A^\mu XB^{1-\mu}|||)^m\Big)\\
&+\left(1-\frac{1-\nu}{1-\mu}\right)^m\Big(|||AX|||^{\frac{m}{2}} - |||A^\mu XB^{1-\mu}|||^{\frac{m}{2}}\Big)^2\\
&+r_m\Big(|||A^\mu XB^{1-\mu}|||^{\frac{m}{2}} - |||AX|||^{\frac{m}{4}}|||A^\mu XB^{1-\mu}|||^{\frac{m}{4}}\Big)^2\\
&\le\Big(\nu|||AX||| + (1-\nu)|||XB|||\Big)^m - \Big(|||A^\nu XB^{1-\nu}|||\Big)^m,
\end{aligned} \tag{41}$$

*where $r_m = \min\left\{2\left(1-\frac{1-\nu}{1-\mu}\right)^m, \left(\frac{1-\nu}{1-\mu}\right)^m - \left(1-\frac{1-\nu}{1-\mu}\right)^m\right\}$.*

4. *If* $\frac{1+\mu}{2} \leq \nu \leq 1$, *then*

$$\begin{aligned}
&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big((|||AX|||\nabla_\mu|||XB|||)^m - (|||A^\mu XB^{1-\mu}|||)^m\Big)\\
+&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(|||AX|||^{\frac{m}{2}} - |||A^\mu XB^{1-\mu}|||^{\frac{m}{2}}\Big)^2\\
+&r_m\Big(|||AX|||^{\frac{m}{2}} - |||AX|||^{\frac{m}{4}}|||A^\mu XB^{1-\mu}|||^{\frac{m}{4}}\Big)^2\\
\leq&\Big(\nu|||AX||| + (1-\nu)|||XB|||\Big)^m - \Big(|||A^\nu XB^{1-\nu}|||\Big)^m,
\end{aligned} \tag{42}$$

*where* $r_m = \min\Big\{2\Big(\frac{1-\nu}{1-\mu}\Big)^m, \nu^m - \Big(\frac{1-\nu}{1-\mu}\Big)^m\Big(\mu^m + 1\Big)\Big\}$.

It is known that when $A, B \in \mathbf{M}_n^{++}$ the function $f(\nu) = tr(A^\nu B^{1-\nu})$ is log-convex on $[0, 1]$, (see [16]). Then by using Theorems 2.4, 2.5 and 2.6, we obtain the following new trace inequalities, refining and reversing the corresponding results in the literature.

**Theorem 5.2.** *Let* $A, B \in \mathbf{M}_n^{++}$ *and* $0 \leq \mu \leq \nu \leq 1$. *Then for all positive integer* $m$,

1. *If* $0 \leq \mu \leq \frac{\nu}{2}$, *then*

$$\begin{aligned}
&\Big(\frac{\mu}{\nu}\Big)^m\Big((tr(A)\nabla_\nu tr(B))^m - (tr(A^\nu B^{1-\nu}))^m\Big)\\
+&\Big(\frac{\mu}{\nu}\Big)^m\Big(tr(B)^{\frac{m}{2}} - tr(A^\nu B^{1-\nu})^{\frac{m}{2}}\Big)^2\\
+&r_m\Big(tr(B)^{\frac{m}{2}} - tr(B)^{\frac{m}{4}}tr(A^\nu B^{1-\nu})^{\frac{m}{4}}\Big)^2\\
\leq&\Big(tr(\mu A + (1-\mu)B)\Big)^m - \Big(tr(A^\mu B^{1-\mu})\Big)^m,
\end{aligned} \tag{43}$$

*where* $r_m = \min\Big\{2\Big(\frac{\mu}{\nu}\Big)^m, (1-\mu)^m - \Big(\frac{\mu}{\nu}\Big)^m\Big((1-\nu)^m + 1\Big)\Big\}$.

2. *If* $\frac{\nu}{2} \leq \mu \leq \nu$, *then*

$$\begin{aligned}
&\Big(\frac{\mu}{\nu}\Big)^m\Big((tr(A)\nabla_\nu tr(B))^m - (tr(A^\nu B^{1-\nu}))^m\Big)\\
+&\Big(1-\frac{\mu}{\nu}\Big)^m\Big(tr(B)^{\frac{m}{2}} - tr(A^\nu B^{1-\nu})^{\frac{m}{2}}\Big)^2\\
+&r_m\Big(tr(A^\nu B^{1-\nu})^{\frac{m}{2}} - tr(B)^{\frac{m}{4}}tr(A^\nu B^{1-\nu})^{\frac{m}{4}}\Big)^2\\
\leq&\Big(tr(\mu A + (1-\mu)B)\Big)^m - \Big(tr(A^\mu B^{1-\mu})\Big)^m,
\end{aligned} \tag{44}$$

*where* $r_m = \min\Big\{2\Big(1-\frac{\mu}{\nu}\Big)^m, \Big(\frac{\mu}{\nu}\Big)^m - \Big(1-\frac{\mu}{\nu}\Big)^m\Big\}$.

3. *If* $\mu \leq \nu \leq \frac{1+\mu}{2}$, *then*

$$\begin{aligned}
&\Big(\frac{1-\nu}{1-\mu}\Big)^m\Big((tr(A)\nabla_\mu tr(B))^m - (tr(A^\mu B^{1-\mu}))^m\Big)\\
+&\Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big(tr(A)^{\frac{m}{2}} - tr(A^\mu B^{1-\mu})^{\frac{m}{2}}\Big)^2\\
+&r_m\Big(tr(A^\mu B^{1-\mu})^{\frac{m}{2}} - tr(A)^{\frac{m}{4}}tr(A^\mu B^{1-\mu})^{\frac{m}{4}}\Big)^2\\
\leq&\Big(tr(\nu A + (1-\nu)B)\Big)^m - \Big(tr(A^\nu B^{1-\nu})\Big)^m,
\end{aligned} \tag{45}$$

*where* $r_m = \min\Big\{2\Big(1-\frac{1-\nu}{1-\mu}\Big)^m, \Big(\frac{1-\nu}{1-\mu}\Big)^m - \Big(1-\frac{1-\nu}{1-\mu}\Big)^m\Big\}$.

4. *If* $\frac{1+\mu}{2} \leq \nu \leq 1$, *then*

$$\begin{aligned}
&\Big(\frac{1-\nu}{1-\mu}\Big)^{m}\Big((tr(A)\nabla_{\mu}tr(B))^{m} - (tr(A^{\mu}B^{1-\mu}))^{m}\Big)\\
&+\Big(\frac{1-\nu}{1-\mu}\Big)^{m}\Big(tr(A)^{\frac{m}{2}} - tr(A^{\mu}B^{1-\mu})^{\frac{m}{2}}\Big)^{2}\\
&+r_{m}\Big(tr(A)^{\frac{m}{2}} - tr(A)^{\frac{m}{4}}tr(A^{\mu}B^{1-\mu})^{\frac{m}{4}}\Big)^{2}\\
&\leq\Big(tr(\nu A + (1-\nu)B)\Big)^{m} - \Big(tr(A^{\nu}B^{1-\nu})\Big)^{m}, \qquad (46)
\end{aligned}$$

*where* $r_m = \min\Big\{2\Big(\frac{1-\nu}{1-\mu}\Big)^{m}, \nu^{m} - \Big(\frac{1-\nu}{1-\mu}\Big)^{m}\Big(\mu^{m}+1\Big)\Big\}$.

## Acknowledgment

The authors would like to express there sincere thanks to the anonymous reviewer, whose comments have significantly improved the paper.

## References

[1] Y. Al-Manasrah and F. Kittaneh, *A generalization of two refined Young inequalities,* Positivity **19** (2015), 757–768.
[2] Y. Al-Manasrah and F. Kittaneh, *Further generalization refinements and reverses of the Young and Heinz inequalities,* Results Math. **71** (2017), 1063–1072.
[3] Y. Manasrah and F. Kittaneh, *A generalization of two refined Young inequalities*, Positivity **19** (2015), 757-768.
[4] H. Alzer, C. M. Fonseca and A. Kovačes, *Young-type inequalities and their matrix analogues*, Linear Multilinear Algebra **63**(3) (2015), 622–635.
[5] D. Choi, *A generalization of Young-type inequalities,* Math. Inequal. Appl. **21** (2018), 99–106.
[6] O. Hirzallah and F. Kittaneh, *Matrix Young inequalities for the Hilbert-Schmidt norm,* Linear Algebra Appl. **308** (2000), 77–84.
[7] M. A. Ighachane and M. Akkouchi, *A new generalization of two refined Young inequalities and applications,* Moroccan J. Pure Appl. Anal. **6**(2) (2020), 155–167.
[8] M. A. Ighachane, M. Akkouchi, *Further generalized refinement of Young's inequalities for ν-measurable operators,* Moroccan J. Pure Appl. Anal. **7**(2) (2021), 214–226.
[9] M. A. Ighachane, M. Akkouchi and E. H. Benabdi, *A new generalized refinement of the weighted arithmetic-geometric mean inequality,* Math. Ineq. Appl. **23**(3) (2020), 1079–1085.
[10] M. A. Ighachane and M. Akkouchi, *Further refinement of Young's type inequality for positive linear maps,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** Article number: 94(2021).
[11] M. A. Ighachane, M. Akkouchi and E. H. Benabdi, *Further refinement of Alzer-Fonseca-Kovačec's inequalities and applications,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. **115** Article number: 152(2021).
[12] M. Khosravi, *Some matrix inequalities for weighted power mean,* Ann. Func. Anal. **7** (2016), 348–357.
[13] F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrices,* J. Math. Anal. Appl. **361**(1) (2010), 292–269.
[14] J. Pečarić, T. Furuta, T. Mićić and T. Seo, *Mond-Pečarić method in operator inequalities,* Element, Zagreb (2005).
[15] M. Sababheh, *Convexity and matrix means,* Linear Algebra Appl. **506** (2016), 588–602.
[16] M. Sababheh, *Log and harmonically log-convex functions related to matrix norms,* Oper. Matrices **10**(2) (2016), 453–465.
[17] M. Sababheh, *Graph indices via the AM-GM inequality*, Discrete Appl. Math. **230**(4) (2017), 100–111.
[18] M. Sababheh, *Means refinements via convexity*, Mediterr. J. Math. **14**, 125 (2017).
[19] M. Sababheh *Convex functions and means of matrices*, Math. Inequal. Appl. **20**(1) (2017), 29–47.
[20] M. Sababheh, *Interpolated inequalities for unitarily invariant norms*, Linear Algebra Appl. **475** (2015), 240–250.
[21] J. Liao and J. Wu, *Matrix inequalities for the difference between arithmetic mean and harmonic mean*, Ann. Func. Anal. **6** (2015), 191–202.