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An algorithmic approach for a class of set-valued variational inclusion problems

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Abstract. The main goal of this paper is twofold. Our first objective is to prove the Lipschitz continuity of the proximal-point mapping associated with an H-accretive operator and to compute an estimate of its Lipschitz constant under some new appropriate conditions imposed on the parameter and mappings involved in it. Using the notion of proximal-point mapping, a new iterative algorithm is constructed for solving a new class of set-valued variational inclusion problems in the setting of q-uniformly smooth Banach spaces. As an application, the strong convergence of the sequences generated by our proposed iterative algorithm to a solution of our considered problem is proved. The second objective of this paper is to investigate and analyze the notion of $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping introduced and studied in [S. Gupta, S. Husain, V.N. Mishra, Variational inclusion governed by $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping, Filomat 31(20)(2017) 6529–6542]. Some comments concerning $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping and related conclusions appeared in the above-mentioned paper are also pointed out.

1. Introduction

In the past five decades, the theory of variational inequalities has grown tremendously and the mathematical literature dedicated to this field is extensive. This is mainly because a large number of problems in different branches of science lead to mathematical models expressed in terms of variational inequalities. This fact has motivated many authors to make an effort to generalize and extend variational inequalities in many different directions using novel and innovative techniques; see, for example, [2, 4, 5, 7, 11, 25] and the references therein. Variational inclusions, as a useful and important generalization of variational inequalities, have been widely studied in recent decades. It is worthwhile to stress that variational inclusions include variational inequalities, quasi-variational and variational-like inequalities as special cases. The development of an efficient and implementable iterative algorithm is one of the most interesting and important problems in the theory of variational inequalities/inclusions. The study of various kinds of iterative methods for the approximation of solutions of different classes of variational inclusion problems has been flourishing areas of research for many researchers. Among these methods, the method based on the proximal-point mappings (resolvent operators) has received a great extent of attention. For more details

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and further information, we refer the reader to [1, 3, 6, 12–15, 18, 21–23, 26, 27, 29–32, 34, 37, 38] and the references contained therein.

It should be pointed out that the notions of monotonicity and accretivity constitute a valuable tool in studying important operators, such as the gradient or subdifferential of a convex function, which appears in various types of problems in equilibrium, optimization, variational inequality problems and differential equations, see, for example, [8, 36] and the references therein. It is worth noting that in the setting of Hilbert spaces, the two classes of monotone and accretive operators coincide. In recent years, considerable efforts have been made and several extensions and generalizations of maximal monotone operators and m-accretive mappings have been introduced in the setting of different spaces. For instance, the notion of generalized m-accretive mapping was initially introduced by Huang and Fang [19] and a definition of the proximal-point mapping associated with it was given in the framework of Banach spaces. In the last eighteen years, the attempts have been continued and another interesting classes of generalized maximal monotone operators and generalized m-accretive mappings such as maximal η-monotone operators [18], η-subdifferential operators [23], H-monotone operators [13], generalized H-monotone operators [34], H-accretive (to avoid confusion, throughout the paper we call it $\widehat{H}$-accretive) mappings [12], (H,η)-monotone operators [14], P-η-accretive (also referred to as (H,η)-accretive) mappings [29], A-monotone operators [31], (A,η)-monotone operators [32] and (A,η)-accretive (also referred to as A-maximal m-relaxed η-accretive) mappings [22] have been appeared in the literature. By the same taken, in 2008, Sun et al. [30] succeeded to introduce another class of generalized maximal monotone operators in the setting of Hilbert spaces the so-called M-monotone operators. They defined the resolvent operator associated with an M-monotone operator and constructed with the aid of it a proximal-point algorithm for solving a class of variational inequality problems. In the same year, Zou and Huang [38] introduced the class of $H(.,.)$-accretive operators in the framework of Banach spaces as a generalization of the notions of H-monotone, $\widehat{H}$-accretive and M-monotone operators. By defining the resolvent operator associated with an $H(.,.)$-accretive operator, they derived some properties relating to it. Afterwards, Lou and Huang [26] introduced the concept of B-monotone operator and proved the Lipschitz continuity of the proximal mapping associated with it. Using the proximal-point mapping, they proposed an iterative algorithm for solving a class of variational inclusion problems in Banach spaces. With the purpose of presenting a generalization of the notions of M-monotone operator [30] and $H(.,.)$-accretive mapping [38], Kazmi et al. [21] introduced an extension of m-accretive mapping called generalized $H(.,.)$-accretive mapping and defined the proximal-point mapping associated with it. They considered a system of generalized variational inclusions involving generalized $H(.,.)$-accretive mappings and proposed an iterative algorithm for approximating its solution. Besides, they studied the convergence analysis of the sequence generated by their iterative algorithm under some appropriate conditions.

Recently, Gupta et al. [15] introduced and studied another class of accretive mappings called $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings as a generalization of the generalized accretive mappings appeared in [12, 13, 20, 21, 26, 30, 38]. They defined the proximal-point mapping associated with an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping and verified its Lipschitz continuity under some suitable conditions. They considered a class of generalized set-valued variational inclusion problems involving $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings and constructed an iterative algorithm for finding its approximate solution. At the same time, they proved the strong convergence of the sequence generated by their proposed iterative algorithm to the solution of their considered problem.

The paper is organized in the following manner. Section 2 provides the basic definitions and properties concerning $\widehat{H}$-accretive mapping and its associated proximal-point mapping in a q-uniformly smooth Banach space setting. This section is ended with a new conclusion, in which the Lipschitz continuity of the proximal-point mapping associated with a $\widehat{H}$-accretive mapping is proved and a new estimate of its Lipschitz constant is computed. In Sect. 3, a new class of set-valued variational inclusion problems (for short, SVIP) is considered and its equivalence with a fixed point problem is demonstrated. By using the obtained equivalence, an iterative algorithm for finding an approximate solution of the SVIP is constructed. As an application of the defined algorithm, at the end of Sect. 3, under some suitable assumptions imposed on the parameters, the strong convergence of the sequences generated by our proposed iterative algorithm to the solution of the SVIP is proved. Section 4 is devoted to the investigation and analysis of the notion of $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping introduced and studied in [15]. We point out that under the conditions imposed on $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping in [15], every $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping is actually $\widehat{H}$-accretive and is not a new one. Moreover, we review and investigate the results appeared in [15] and by pointing out some comments regarding them, we show that one can deduce all conclusions existing in [15] with the aid of the results given in the previous sections.

2. Notation, basic definitions and fundamental properties

In addition to standard conventions, the following notation will be used throughout the paper. We assume that E is a real Banach space with a norm $\|.\|$, that E^* is the dual of E containing all bounded linear functionals on E, and that E and E^* are paired by $\langle .,. \rangle$. As usual, x^* will stand for the weak star topology in E^*, and the value of a functional $x^* \in E^*$ at $x \in E$ is denoted by either $\langle x, x^* \rangle$ or $x^*(x)$, as is convenient. For sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$. The notation $CB(E)$ is used for the family of all the closed and bounded subsets of E. We further use the symbols S_E and B_E to represent the unit sphere and the unit ball in E, respectively.

For a given set-valued mapping $\widehat{M} : E \multimap E$, its effective domain, graph and range are the sets

$$\mathrm{Dom}(\widehat{M}) := \{x \in E : \exists y \in E : y \in \widehat{M}(x)\} = \{x \in E : \widehat{M}(x) \neq \emptyset\},$$

$$\mathrm{Graph}(\widehat{M}) := \{(x, y) \in E \times E : y \in \widehat{M}(x)\},$$

and

$$\mathrm{Range}(\widehat{M}) := \{y \in E : \exists x \in E : (x, y) \in \mathrm{Graph}(\widehat{M})\},$$

respectively. $\mathrm{Dom}(\widehat{M}) = E$, shall denote the full domain of $\widehat{M}$ and the inverse $\widehat{M}^{-1}$ of $\widehat{M}$ is $\{(y, x) : (x, y) \in \mathrm{Graph}(\widehat{M})\}$. For an arbitrary real constant ρ and set-valued mappings $\widehat{M}, \widehat{N} : E \multimap E$, we define $\rho\widehat{M}$ and $\widehat{M} + \widehat{N}$ by

$$\rho\widehat{M} = \{(x, \rho y) : (x, y) \in \mathrm{Graph}(\widehat{M})\}$$

and

$$\widehat{M} + \widehat{N} = \{(x, y + z) : (x, y) \in \mathrm{Graph}(\widehat{M}), (x, z) \in \mathrm{Graph}(\widehat{N})\},$$

respectively. Let us recall that (the norm of) a Banach space E is said to be strictly convex if $\|x + y\| < 2$ when x and y are different points of S_E, and that (the norm of) E is smooth if for every $x \in S_E$ there is exactly one $x^* \in S_{E^*}$ such that $x^*(x) = 1$. It is well known truth that E is smooth if E^* is strictly convex, and that E is strictly convex if E^* is smooth.

Definition 2.1. *A normed space E is said to be uniformly convex if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in B_E$ with $\|x - y\| \geq 2\varepsilon$, then the average $(x + y)/2$ has norm at most $1 - \delta$.*

The function $\delta_E : [0, 2] \to [0, 1]$ defined by the formulation

$$\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in B_E : \|x - y\| \geq 2\varepsilon\}$$

is called the modulus of convexity of E.

It should be remarked that in the definition of $\delta_E(\varepsilon)$ we can as well take the infimum over all vectors $x, y \in S_E$ with $\|x - y\| = 2\varepsilon$.

The function δ_E is continuous and increasing on the interval $[0, 2]$ and $\delta_E(0) = 0$. In the light of the definition of the function δ_E, a normed space E is uniformly convex if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$.

Definition 2.2. *A normed space E is said to be uniformly smooth if, for all $\varepsilon > 0$ there is a $\tau > 0$ such that for all $x, y \in B_E$ with $\|x - y\| \leq 2\tau$, then the average $(x + y)/2$ has norm at least $1 - \varepsilon\tau$.*

The function $\rho_E : [0, +\infty) \to [0, +\infty)$ defined by the formula

$$\rho_E(\tau) = \sup\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| \leq \tau\}$$

is called the modulus of smoothness of the space E. Similarly, here it is to be noted that in the definition of $\rho_E(\tau)$ we may as well take the supremum over all vectors $x, y \in E$ with $\|x\| = 1$ and $\|y\| = \tau$. It is also remarkable that the function ρ_E is convex, continuous and increasing on the interval $[0, +\infty) = \mathbb{R}^+$ and $\rho_E(0) = 0$. In addition, $\rho_E(\tau) \leq \tau$ for all $\tau \in \mathbb{R}^+$. Thanks to the definition of the function ρ_E, it is significant to mention that a normed space E is uniformly smooth if $\lim\limits_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$.

It should be pointed out that a Banach space E is uniformly convex (resp., uniformly smooth) if and only if E^* is uniformly smooth (resp., uniformly convex). The spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, are uniformly convex as well as uniformly smooth, see, for example, [10, 16, 24]. At the same time, the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W^p_m, $1 < p < \infty$, $m \in \mathbb{N}$, can be found in [10, 16, 24].

For an arbitrary but fixed real number $q > 1$, the set-valued mapping $J_q : E \multimap E^*$ given by

$$J_q(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

is called the *generalized duality mapping* of E. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$, for all $x \neq 0$. Let us emphasize that J_q is single-valued if E is uniformly smooth or equivalently E^* is strictly convex. Note, in particular, that if E is a Hilbert space, then J_2 becomes the identity mapping on E.

For a real constant $q > 1$, a Banach space E is called q-uniformly smooth if there exists a constant $C > 0$ such that $\rho_E(\tau) \leq Ct^q$ for all $\tau \in \mathbb{R}^+$.

It is well known that (see e.g. [35]) L_q (or l_q) is q-uniformly smooth for $1 \leq q \leq 2$ and is 2-uniformly smooth if $q > 2$.

Concerned with the characterization inequalities in q-uniformly smooth Banach spaces, Xu [35] proved the following result.

Lemma 2.3. *Let E be a real uniformly smooth Banach space. For a real constant $q > 1$, E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Throughout the rest of the paper, unless otherwise specified, we assume that E is a real q-uniformly smooth Banach space for a real constant $q > 1$.

Let us also recall some required definitions, concepts and well known results which shall be used in the sequel.

Definition 2.4. *A single-valued mapping $\widehat{H} : E \to E$ is said to be*

(i) *accretive if*

$$\langle \widehat{H}(x) - \widehat{H}(y), J_q(x - y)\rangle \geq 0, \quad \forall x, y \in E;$$

(ii) *strictly accretive if $\widehat{H}$ is accretive and equality holds if and only if $x = y$;*

(iii) *k-strongly accretive if there exists a constant $k > 0$ such that*

$$\langle \widehat{H}(x) - \widehat{H}(y), J_q(x - y)\rangle \geq k\|x - y\|^q, \quad \forall x, y \in E;$$

(iv) *γ-Lipschitz continuous if there exists a constant $\gamma > 0$ such that*

$$\|\widehat{H}(x) - \widehat{H}(y)\| \leq \gamma\|x - y\|, \quad \forall x, y \in E;$$

(v) *ς-expansive if there exists a constant $\varsigma > 0$ such that*

$$\|\widehat{H}(x) - \widehat{H}(y)\| \geq \varsigma\|x - y\|, \quad \forall x, y \in E.$$

Clearly, it is expansive if and only if $\varsigma = 1$.

Definition 2.5. *Let $\widehat{H} : E \to E$ be a single-valued mapping and $\widehat{M} : E \multimap E$ be a set-valued mapping. $\widehat{M}$ is said to be*

(i) *accretive if*

$$\langle u - v, J_q(x - y)\rangle \geq 0, \quad \forall (x, u), (y, v) \in \mathrm{Graph}(M);$$

(ii) *r-strongly accretive if there exists a constant $r > 0$ such that*

$$\langle u - v, J_q(x - y)\rangle \geq r\|x - y\|^q, \quad \forall (x, u), (y, v) \in \mathrm{Graph}(M);$$

(iii) *m-accretive if M is accretive and* $\mathrm{Range}(I + \rho\widehat{M}) = E$ *holds for every real constant $\rho > 0$, where I is the identity mapping on E.*

We note that $\widehat{M}$ is an m-accretive mapping if and only if $\widehat{M}$ is accretive and there is no other accretive mapping whose graph contains strictly $\mathrm{Graph}(\widehat{M})$. The m-accretivity is to be understood in terms of inclusion of graphs. If $\widehat{M} : E \multimap E$ is an m-accretive mapping, then adding anything to its graph so as to obtain the graph of a new set-valued mapping, destroys the accretivity. In fact, the extended mapping is no longer accretive. In other words, for every pair $(x, u) \in E \times E\backslash\,\mathrm{Graph}(\widehat{M})$ there exits $(y, v) \in \mathrm{Graph}(\widehat{M})$ such that $\langle u - v, J_q(x - y)\rangle < 0$. Taking into account the arguments mentioned above, a necessary and sufficient condition for set-valued mapping $\widehat{M} : E \multimap E$ to be m-accretive is that the property

$$\langle u - v, J_q(x - y)\rangle \geq 0, \quad \forall (y, v) \in \mathrm{Graph}(\widehat{M})$$

is equivalent to $u \in \widehat{M}(x)$. The above characterization of generalized m-accretive mappings provides us a useful and manageable way for recognizing that an element u belongs to $\widehat{M}(x)$.

The introduction of the notion $\widehat{H}$-accretive mapping was first made by Fang and Huang [12] which can be viewed as a unifying framework for maximal monotone operators and m-accretive operators.

Definition 2.6. [12] *Let $\widehat{H} : E \to E$ be a single-valued mapping and $\widehat{M} : E \multimap E$ be a set-valued mapping. $\widehat{M}$ is said to be $\widehat{H}$-accretive if $\widehat{M}$ is accretive and* $\mathrm{Range}(\widehat{H} + \rho\widehat{M}) = E$ *holds for every $\rho > 0$.*

The following example illustrates that for given single-valued mapping $\widehat{H} : E \to E^*$, an m-accretive mapping need not be $\widehat{H}$-accretive.

Example 2.7. Let p and n be two arbitrary but fixed natural numbers such that n is even and let $M_{p\times n}(\mathbb{R})$ be the vector space of all $p \times n$ matrices with real entries over $\mathbb{R}$. Then

$$M_{p\times n}(\mathbb{R}) = \{A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, p; j = 1, 2, \ldots, n\}$$

is a Hilbert space with the inner product $\langle A, B\rangle := tr(AB^*)$, for all $A, B \in M_{p\times n}(\mathbb{R})$, where tr denotes the trace, that is, the sum of diagonal entries, and B^* denotes the transpose of the matrix B. The inner product defined above induces a norm on $M_{p\times n}(\mathbb{R})$ as follows:

$$\|A\| = \Big(\sum_{i=1}^{p}\sum_{j=1}^{n} a_{ij}^2\Big)^{\frac{1}{2}}, \quad \forall A = \begin{pmatrix} a_{ij} \end{pmatrix} \in M_{p\times n}(\mathbb{R}).$$

Taking into account that every finite-dimensional normed space is a Banach space, it follows that the Hilbert space $(M_{p\times n}(\mathbb{R}), \|.\|)$ is a 2-uniformly smooth Banach space.

For any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in M_{p\times n}(\mathbb{R})$, we have $A = \sum_{i=1}^{p}\sum_{j\in\Gamma}(A_{i(2j-1)(2j+1)} + A_{i(2j)(2j+2)})$, where $\Gamma = \{1, 3, \dots, \frac{n-2}{2}\}$, that is, every $p \times n$ matrix $A \in M_{p\times n}(\mathbb{R})$ can be written as a linear combination of $\frac{pn}{2}$ matrices $A_{i(2j-1)(2j+1)}$ and $A_{i(2j)(2j+2)}$, where for each $i \in \{1, 2, \dots, p\}$ and $j \in \Gamma$, $A_{i(2j-1)(2j+1)}$ is a $p\times n$ matrix with the $(i, 2j-1)$-entry $a_{i(2j-1)}$, the $(i, 2j+1)$-entry $a_{i(2j+1)}$, and all other entries equal to zero, and $A_{i(2j)(2j+2)}$ is a $p\times n$ matrix such that for each $i \in \{1, 2, \dots, p\}$ and $j \in \Gamma$, the $(i, 2j)$-entry equals to $a_{i(2j)}$, $(i, 2j+2)$-entry equals to $a_{i(2j+2)}$, and all other entries equal to zero. For each $i \in \{1, 2, \dots, p\}$ and $j \in \Gamma$, there are four real numbers $b_{i(2j-1)}, b_{i(2j)}, b_{i(2j+1)}$ and $b_{i(2j+2)}$ such that $b_{i(2j-1)} + b_{i(2j+1)} = a_{i(2j-1)}$, $b_{i(2j-1)} - b_{i(2j+1)} = a_{i(2j+1)}$, $b_{i(2j)} + b_{i(2j+2)} = a_{i(2j)}$ and $b_{i(2j)} - b_{i(2j+2)} = a_{i(2j+2)}$. Then, for each $i \in \{1, 2, \dots, p\}$ and $j \in \Gamma$, we have

$$
\begin{aligned}
A_{i(2j-1)(2j+1)} + A_{i(2j)(2j+2)} &= \begin{pmatrix}
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & a_{i(2j-1)} & 0 & a_{i(2j+1)}\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0
\end{pmatrix}\\
&+ \begin{pmatrix}
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & a_{i(2j)} & 0 & a_{i(2j+2)}\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0
\end{pmatrix}\\
&= \begin{pmatrix}
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & b_{i(2j-1)} + b_{i(2j-1)} & 0 & b_{i(2j-1)} - b_{i(2j+1)}\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0
\end{pmatrix}\\
&+ \begin{pmatrix}
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & b_{i(2j)} + b_{i(2j+2)} & 0 & b_{l(2j)} - b_{i(2j+2)}\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0
\end{pmatrix}\\
&= b_{i(2j-1)} \begin{pmatrix}
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & 1_{i(2j-1)} & 0 & 1_{i(2j+1)}\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0
\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+ b_{i(2j+1)}\begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 1_{i(2j-1)} & 0 & -1_{i(2j+1)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \\
&+ b_{i(2j)}\begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 1_{i(2j)} & 0 & 1_{i(2j+2)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \\
&+ b_{i(2j+2)}\begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 1_{i(2j)} & 0 & -1_{i(2j+2)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \\
&= b_{i(2j-1)}Q_{i(2j-1)(2j+1)} + b_{i(2j+1)}Q'_{i(2j-1)(2j+1)} + b_{i(2j)}Q_{i(2j)(2j+2)} + b_{i(2j+2)}Q'_{i(2j)(2j+2)},
\end{aligned}
$$

where for each $i \in \{1, 2, \dots, p\}$ and $j \in \Gamma$, $Q_{i(2j-1)(2j+1)}$ is a $p \times n$ matrix with the $(i, 2j-1)$ and $(i, 2j+1)$-entries equal to 1 and all other entries equal to zero, $Q'_{i(2j-1)(2j+1)}$ is a $p \times n$ matrix with the $(i, 2j-1)$-entry 1, the $(i, 2j+1)$-entry 1, and all other entries equal to zero, $Q_{i(2j)(2j+2)}$ is a $p \times n$ matrix with the $(i, 2j)$ and $(i, 2j+2)$-entries 1, and all other entries equal to zero, and $Q'_{i(2j)(2j+2)}$ is a $p \times n$ matrix in which the $(i, 2j)$ and $(i, 2j+2)$-entries equal to 1 and -1, respectively, and all other entries equal to zero. Accordingly, for any $A \in M_{p\times n}(\mathbb{R})$, we have

$$
\begin{aligned}
A = \sum_{i=1}^{p}\sum_{j\in\Gamma}(A_{i(2j-1)(2j+1)} + A_{i(2j)(2j+2)}) = \sum_{i=1}^{p}\sum_{j\in\Gamma}\Big(&b_{i(2j-1)}Q_{i(2j-1)(2j+1)} + b_{i(2j+1)}Q'_{i(2j-1)(2j+1)} \\
&+ b_{i(2j)}Q_{i(2j)(2j+2)} + b_{i(2j+2)}Q'_{i(2j)(2j+2)}\Big).
\end{aligned}
$$

Thus, the set

$$
\Big\{Q_{i(2j-1)(2j+1)}, Q'_{i(2j-1)(2j+1)}, Q_{i(2j)(2j+2)}, Q'_{i(2j)(2j+2)} : i = 1, 2, \dots, p; j = 1, 3, \dots, \frac{n-2}{2}\Big\}
$$

spans the Hilbert space $M_{p\times n}(\mathbb{R})$. Taking $E_{i(2j-1)(2j+1)} := \frac{1}{\sqrt{2}}Q_{i(2j-1)(2j+1)}$, $E'_{i(2j-1)(2j+1)} := \frac{1}{\sqrt{2}}Q'_{i(2j-1)(2j+1)}$, $E_{i(2j)(2j+2)} := \frac{1}{\sqrt{2}}Q_{i(2j)(2j+2)}$ and $E'_{i(2j)(2j+2)} := \frac{1}{\sqrt{2}}Q'_{i(2j)(2j+2)}$, for each $i \in \{1, 2, \dots, p\}$ and $j \in \Gamma$, it follows that the set

$$
\mathfrak{B} = \Big\{E_{l(2j-1)(2j+1)}, E'_{i(2j-1)(2j+1)}, E_{i(2j)(2j+2)}, E'_{i(2j)(2j+2)} : i = 1, 2, \dots, p; j = 1, 3, \dots, \frac{n-2}{2}\Big\}
$$

also spans the Hilbert space $M_{p\times n}(\mathbb{R})$. It can be easily proved that the set $\mathfrak{B}$ is linearly independent and orthonormal and so $\mathfrak{B}$ is an orthonormal basis for the Hilbert space $M_{p\times n}(\mathbb{R})$.

Let the mappings $\widehat{H}, \widehat{M} : M_{p\times n}(\mathbb{R}) \to M_{p\times n}(\mathbb{R})$ be defined, respectively, by $\widehat{H}(A) = -\gamma A + E_{s(2l)(2l+2)} + E'_{s(2l)(2l+2)}$ and $\widehat{M}(A) = \gamma A + E_{s(2l-1)(2l+1)} + E'_{s(2l-1)(2l+1)}$, for all $A \in M_{p\times n}(\mathbb{R})$, where $\gamma > 0$ is an arbitrary real constant, and $s \in \{1, 2, \ldots, p\}$ and $l \in \Gamma$ are arbitrary but fixed natural numbers.

In virtue of the fact that every finite-dimensional normed space is a Banach space, it follows that $(M_{p\times n}(\mathbb{R}), \|.\|)$ is a 2-uniformly smooth Banach space. Then, for all $A, B \in M_{p\times n}(\mathbb{R})$, we have

$$\begin{aligned}
&\langle \widehat{M}(A) - \widehat{M}(B), J_2(A - B)\rangle = \langle \widehat{M}(A) - \widehat{M}(B), A - B\rangle \\
&= \langle \gamma A + E_{s(2l-1)(2l+1)} + E'_{s(2l-1)(2l+1)} - \gamma B - E_{s(2l-1)(2l+1)} - E'_{s(2l-1)(2l+1)}, A - B\rangle \\
&= \gamma\langle A - B, A - B\rangle = \gamma\|A - B\|^2 = \gamma\sqrt{\sum_{i=1}^{p}\sum_{j=1}^{n}(a_{ij} - b_{ij})^2} \geq 0,
\end{aligned}$$

which ensures that $\widehat{M}$ is an accretive mapping.

Since for any $A \in M_{p\times n}(\mathbb{R})$,

$$\|(\widehat{H} + \widehat{M})(A)\| = \|E_{s(2l-1)(2l+1)} + E'_{s(2l-1)(2l+1)} + E_{s(2l)(2l+2)} + E'_{s(2l)(2l+2)}\| = 2 > 0,$$

it follows that $\mathbf{0} \notin (\widehat{H} + \widehat{M})(M_{p\times n}(\mathbb{R}))$, i.e., $\widehat{H} + \widehat{M}$ is not surjective and so the mapping $\widehat{M}$ is not $\widehat{H}$-accretive. Let $\rho > 0$ be an arbitrary real constant. In the light of the fact that for any $A \in M_{p\times n}(\mathbb{R})$,

$$\begin{aligned}
(I + \rho\widehat{M})(A) &= A + \gamma\rho A + \rho E_{s(2l-1)(2l+1)} + \rho E'_{s(2l-1)(2l+1)} \\
&= (1 + \gamma\rho)A + \rho E_{s(2l-1)(2l+1)} + \rho E'_{s(2l-1)(2l+1)},
\end{aligned}$$

where I is the identity mapping on $M_{p\times n}(\mathbb{R})$, we conclude that $(I + \rho\widehat{M})(M_{p\times n}(\mathbb{R})) = M_{p\times n}(\mathbb{R})$, that is, the mapping $I + \rho\widehat{M}$ is surjective. Taking into account the arbitrariness in the choice of $\rho > 0$, it follows that $\widehat{M}$ is an m-accretive mapping.

Example 2.8. Let $H_2(\mathbb{C})$ be the set of all Hermitian matrices with complex entries. We recall that a square matrix A is said to be Hermitian (or self-adjoint) if it is equal to its own Hermitian conjugate, i.e., $A^* = \overline{A^t} = A$. In view of the definition of a Hermitian 2×2 matrix, the condition $A^* = A$ implies that the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is Hermitian iff $a, d \in \mathbb{R}$ and $b = \bar{c}$. Therefore,

$$H_2(\mathbb{C}) = \Big\{\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} | x, y, z, w \in \mathbb{R}\Big\}.$$

Then, $H_2(\mathbb{C})$ is a subspace of $M_2(\mathbb{C})$, the space of all 2×2 matrices with complex entries, with respect to the operations of addition and scalar multiplication defined on $M_2(\mathbb{C})$, when $M_2(\mathbb{C})$ is considered as a real vector space. In other words, $H_2(\mathbb{C})$ together with the mentioned operations is a vector space over $\mathbb{R}$. By introducing the scalar product on $H_2(\mathbb{C})$ as $\langle A, B\rangle := \frac{1}{2}tr(AB)$, for all $A, B \in H_2(\mathbb{C})$, it is easy to check that $\langle ., .\rangle$ is an inner product, that is, $(H_2(\mathbb{C}), \langle ., .\rangle)$ is an inner product space. The inner product defined above induces a norm on $H_2(\mathbb{C})$ as follows:

$$\begin{aligned}
\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\frac{1}{2}tr(AA)} &= \Big\{\frac{1}{2}tr\Big(\begin{pmatrix} x^2 + y^2 + z^2 & (z + w)(x - iy) \\ (z + w)(x + iy) & x^2 + y^2 + w^2 \end{pmatrix}\Big)\Big\}^{\frac{1}{2}} \\
&= \sqrt{x^2 + y^2 + \frac{1}{2}(z^2 + w^2)}, \quad \forall A \in H_2(\mathbb{C}).
\end{aligned}$$

Since $(H_2(\mathbb{C}), \|.\|)$ is a finite-dimensional normed space, we infer that it is a 2-uniformly smooth Banach space. Define now the mappings $\widehat{H}_1, \widehat{H}_2, \widehat{M} : H_2(\mathbb{C}) \to H_2(\mathbb{C})$, respectively, as follows:

$$\widehat{H}_1(A) = \widehat{H}_1\Big(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}\Big) = \begin{pmatrix} 2^{-z} - \alpha z^m + \beta & x^{2l} - iy^{2l} \\ x^{2l} + iy^{2l} & (\frac{1}{2})^{|w|-1} - \gamma w^n \end{pmatrix},$$

$$\widehat{H}_2(A) = \widehat{H}_2\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\Big) = \begin{pmatrix} \theta z^q & x-iy \\ x+iy & \varrho w^k \end{pmatrix}$$

and

$$\widehat{M}(A) = \widehat{M}\Big(\begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix}\Big) = \begin{pmatrix} \alpha z^m & x^l-iy^l \\ x^l+iy^l & \gamma w^n \end{pmatrix},$$

for all $A = \begin{pmatrix} z & x-iy \\ x+iy & w \end{pmatrix} \in H_2(\mathbb{C})$, where β, θ and ϱ are arbitrary real constants, α and γ are arbitrary positive real constants, m, n and l are three arbitrary but fixed odd natural numbers, and q and k are two arbitrary but fixed even natural numbers such that $m > q$ and $n > k$. Then, for any $A = \begin{pmatrix} z_1 & x_1-iy_1 \\ x_1+iy_1 & w_1 \end{pmatrix}, B = \begin{pmatrix} z_2 & x_2-iy_2 \\ x_2+iy_2 & w_2 \end{pmatrix} \in H_2(\mathbb{C})$, it yields

$$\begin{aligned}
&\langle \widehat{M}(A) - \widehat{M}(B), J_2(A-B)\rangle = \langle \widehat{M}(A) - \widehat{M}(B), A-B\rangle \\
&= \Big\langle \begin{pmatrix} \alpha(z_1^m - z_2^m) & x_1^l - x_2^l - i(y_1^l - y_2^l) \\ x_1^l - x_2^l + i(y_1^l - y_2^l) & \gamma(w_1^n - w_2^n) \end{pmatrix}, \\
&\quad \begin{pmatrix} z_1 - z_2 & x_1 - x_2 - i(y_1 - y_2) \\ x_1 - x_2 + i(y_1 - y_2) & w_1 - w_2 \end{pmatrix}\Big\rangle \\
&= \frac{1}{2}tr\Big(\begin{pmatrix} \Psi_{11}(x_1,x_2,y_1,y_1,z_1,z_2) & \Psi_{12}(x_1,x_2,y_1,y_1,z_1,z_2) \\ \Psi_{21}(x_1,x_2,y_1,y_1,z_1,z_2) & \Psi_{22}(x_1,x_2,y_1,y_1,z_1,z_2) \end{pmatrix}\Big) \\
&= \frac{\alpha}{2}(z_1 - z_2)(z_1^m - z_2^m) + \frac{\gamma}{2}(w_1 - w_2)(w_1^n - w_2^n) \\
&\quad + (x_1 - x_2)(x_1^l - x_2^l) + (y_1 - y_2)(y_1^l - y_2^l) \\
&= \frac{\alpha}{2}(z_1 - z_2)^2 \sum_{i=1}^{m} z_1^{m-i} z_2^{i-1} + \frac{\gamma}{2}(w_1 - w_2)^2 \sum_{i'=1}^{n} w_1^{n-i'} w_2^{i'-1} \\
&\quad + (x_1 - x_2)^2 \sum_{i''=1}^{l} x_1^{l-i''} x_2^{i''-1} + (y_1 - y_2)^2 \sum_{i'''=1}^{l} y_1^{l-i'''} y_2^{i'''-1},
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{11}(x_1,x_2,y_1,y_1,z_1,z_2) &= \alpha(z_1 - z_2)(z_1^m - z_2^m) + (x_1 - x_2)(x_1^l - x_2^l) + (y_1 - y_2)(y_1^l - y_2^l) \\
&\quad + i(x_1^l - x_2^l)(y_1 - y_2) - i(x_1 - x_2)(y_1^l - y_2^l), \\
\Psi_{12}(x_1,x_2,y_1,y_1,z_1,z_2) &= \alpha(z_1^m - z_2^m)(x_1 - x_2 - i(y_1 - y_2)) + (x_1^l - x_2^l - i(y_1^l - y_2^l))(w_1 - w_2), \\
\Psi_{21}(x_1,x_2,y_1,y_1,z_1,z_2) &= (x_1^l - x_2^l + i(y_1^l - y_2^l))(z_1 - z_2) + \gamma(w_1^n - w_2^n)(x_1 - x_2 + i(y_1 - y_2)), \\
\Psi_{22}(x_1,x_2,y_1,y_1,z_1,z_2) &= (x_1 - x_2)(x_1^l - x_2^l) + (y_1 - y_2)(y_1^l - y_2^l) - i(x_1^l - x_2^l)(y_1 - y_2) \\
&\quad + i(x_1 - x_2)(y_1^l - y_2^l) + \gamma(w_1 - w_2)(w_1^n - w_2^n).
\end{aligned}$$

In virtue of the fact that m, n, l are natural numbers, it is easy to see that $\sum\limits_{i=1}^{m} z_1^{m-i} z_2^{i-1} \geq 0$, $\sum\limits_{i'=1}^{n} w_1^{n-i'} w_2^{i'-1} \geq 0$, $\sum\limits_{i''=1}^{l} x_1^{l-i''} x_2^{i''-1} \geq 0$ and $\sum\limits_{i'''=1}^{l} y_1^{l-i'''} y_2^{i'''-1} \geq 0$. Since $\alpha, \gamma > 0$, the preceding relation implies that

$$\langle \widehat{M}(A) - \widehat{M}(B), J_2(A-B)\rangle = \langle \widehat{M}(A) - \widehat{M}(B), A-B\rangle \geq 0,$$

which means that $\widehat{M}$ is an accretive mapping.

Let us now define the functions $f, g, h : \mathbb{R} \to \mathbb{R}$, respectively, as follows:

$$f(s) := 2^{-s} + \beta,\ g(s) := (\frac{1}{2})^{|s|-1} \text{ and } h(s) := s^{2l} + s^l, \quad \forall s \in \mathbb{R}.$$

Then, for any $A = \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \in H_2(\mathbb{C})$, we obtain

$$\begin{aligned}(\widehat{H}_1 + \widehat{M})(A) &= (\widehat{H}_1 + \widehat{M})\Big(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}\Big) \\ &= \begin{pmatrix} 2^{-z} + \beta & x^{2l} + x^l - i(y^{2l} + y^l) \\ x^{2l} + x^l + i(y^{2l} + y^l) & (\frac{1}{2})^{|w|-1} \end{pmatrix} \\ &= \begin{pmatrix} f(z) & h(x) - ih(y) \\ h(x) + ih(y) & g(w) \end{pmatrix}.\end{aligned}$$

It can be easily seen that $f(s) > \beta$ and $0 < g(s) \leq 1$, for all $s \in \mathbb{R}$. At the same time, for all $s \in \mathbb{R}$, we have

$$h(s) = s^{2l} + s^l = (s^l + \frac{1}{2})^2 - \frac{1}{4} \geq -\frac{1}{4}.$$

Taking into account that $f(\mathbb{R}) = (\beta, +\infty)$, $g(\mathbb{R}) = (0, 2]$ and $h(\mathbb{R}) = [-\frac{1}{4}, +\infty)$, it follows that $(\widehat{H}_1 + \widehat{M})(H_2(\mathbb{C})) \neq H_2(\mathbb{C})$. Hence, $\widehat{H}_1 + \widehat{M}$ is not surjective and so $\widehat{M}$ is not $\widehat{H}_1$-accretive.

Now, let $\rho > 0$ be an arbitrary real constant and let the functions $\widetilde{f}, \widetilde{g}, \widetilde{h} : \mathbb{R} \to \mathbb{R}$ be defined, respectively, by

$$\widetilde{f}(x) := \theta x^q + \rho\alpha x^m,\ \widetilde{g}(x) := \varrho x^k + \rho\gamma x^n \text{ and } \widetilde{h}(x) := x + \rho x^l, \quad \forall x \in \mathbb{R}.$$

Then, for any $A = \begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix} \in H_2(\mathbb{C})$, we get

$$\begin{aligned}(\widehat{H}_2 + \rho\widehat{M})(A) &= (\widehat{H}_2 + \rho\widehat{M})\Big(\begin{pmatrix} z & x - iy \\ x + iy & w \end{pmatrix}\Big) \\ &= \begin{pmatrix} \theta z^q + \rho\alpha z^m & x + \rho x^l - i(y + \rho y^l) \\ x + \rho x^l + i(y + \rho y^l) & \varrho w^k + \rho\gamma w^n \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{f}(z) & \widetilde{h}(x) - i\widetilde{h}(y) \\ \widetilde{h}(x) + i\widetilde{h}(y) & \widetilde{g}(w) \end{pmatrix}.\end{aligned}$$

Relying on the fact that q and k are even natural numbers and m, n, l are odd natural numbers such that $m > q$ and $n > k$, it is easy to see that $\widetilde{f}(\mathbb{R}) = \widetilde{g}(\mathbb{R}) = \widetilde{h}(\mathbb{R}) = \mathbb{R}$. This fact implies that $(\widehat{H}_2 + \rho\widehat{M})(H_2(\mathbb{C})) = H_2(\mathbb{C})$, that is, $\widehat{H}_2 + \rho\widehat{M}$ is surjective. Since $\rho > 0$ was arbitrary, we conclude that $\widehat{M}$ is a $\widehat{H}_2$-accretive mapping.

Remark 2.9. If $\widehat{H} = I$, the identity mapping on E, then the definition of $\widehat{H}$-accretive mappings is that of m-accretive mappings. In fact, the class of $\widehat{H}$-accretive mappings has close relation with that of m-accretive mappings. This fact is illustrated by the following assertion.

Lemma 2.10. [12, Theorem 2.1] *Let $\widehat{H} : E \to E$ be a strictly accretive operator, $\widehat{M} : E \multimap E$ be a $\widehat{H}$-accretive operator, and $x, u \in E$ be given points. If $\langle u - v, J_q(x - y)\rangle \geq 0$ holds, for all $(y, v) \in \mathrm{Graph}(\widehat{M})$, then $u \in \widehat{M}(x)$.*

Theorem 2.11. *Let $\widehat{H} : E \to E$ be a strictly accretive mapping and $\widehat{M} : E \multimap E$ be an accretive mapping. Then the mapping $(\widehat{H} + \rho\widehat{M})^{-1} : \mathrm{Range}(\widehat{H} + \rho\widehat{M}) \to E$ is single-valued for every constant $\rho > 0$.*

Proof. Choose positive real constant ρ and point $u \in \mathrm{Range}(\widehat{H} + \rho\widehat{M})$ arbitrarily but fixed. Then for any $x, y \in (\widehat{H} + \rho\widehat{M})^{-1}(u)$, we have

$$u = (\widehat{H} + \rho\widehat{M})(x) = (\widehat{H} + \rho\widehat{M})(y), \tag{1}$$

from which we deduce that

$$\rho^{-1}(u - \widehat{H}(x)) \in \widehat{M}(x) \text{ and } \rho^{-1}(u - \widehat{H}(x)) \in \widehat{M}(y). \tag{2}$$

Since $\widehat{M}$ is accretive, we infer that

$$0 \leq \langle \rho^{-1}(u - \widehat{H}(x)) - \rho^{-1}(u - \widehat{H}(y)), J_q(x - y)\rangle = -\rho^{-1}\langle \widehat{H}(x) - \widehat{H}(y), J_q(x - y)\rangle.$$

Making use of the last inequality and the strict accretiveness of $\widehat{H}$ it follows that $x = y$. This fact ensures that the mapping $(\widehat{H} + \rho\widehat{M})^{-1}$ is single-valued. The proof is finished. □

Theorem 2.12. *Suppose that $\widehat{H} : E \to E$ is an accretive mapping and $\widehat{M} : E \multimap E$ is a k-strongly accretive mapping. Then the mapping $(\widehat{H} + \rho\widehat{M})^{-1} : \mathrm{Range}(\widehat{H} + \rho\widehat{M}) \to E$ is single-valued for every constant $\rho > 0$.*

Proof. Let constat $\rho > 0$ be chosen arbitrarily but fixed. For any given $u \in \mathrm{Range}(\widehat{H} + \rho\widehat{M})$, letting $x, y \in (\widehat{H} + \rho\widehat{M})^{-1}(u)$, we have (1) which implies (2). Taking into account that $\widehat{H}$ is accretive and $\widehat{M}$ is k-strongly accretive, it follows that

$$\rho k\|x - y\|^q \leq \rho\langle \rho^{-1}(u - \widehat{H}(x)) - \rho^{-1}(u - \widehat{H}(y)), J_q(x - y)\rangle + \langle \widehat{H}(x) - \widehat{H}(y), J_q(x - y)\rangle = 0.$$

In virtue of the preceding inequality and the fact that $\rho, k > 0$, we conclude that $x = y$, which guarantees that the mapping $\widehat{H} + \rho\widehat{M}$ from $\mathrm{Range}(\widehat{H} + \rho\widehat{M})$ into E is single-valued. This completes the proof. □

It should be pointed out that in the rest of the paper, we say that $\widehat{M}$ is a $\widehat{H}$-strongly accretive mapping, means that $\widehat{M}$ is a k-strongly accretive mapping and $(\widehat{H} + \rho\widehat{M})(E) = E$, for every real constant $\rho > 0$.

As immediate corollaries of the last results, we obtain the following assertions, respectively.

Corollary 2.13. [12, Theorem 2.2] *Let $\widehat{H} : E \to E$ be a strictly accretive operator, and $\widehat{M} : E \multimap E$ be a $\widehat{H}$-accretive operator. Then, the operator $(\widehat{H} + \rho\widehat{M})^{-1} : E \to E$ is single-valued, where $\rho > 0$ is a real constant.*

Corollary 2.14. *Let $\widehat{H} : E \to E$ be an accretive mapping and $\widehat{M} : E \multimap E$ be a $\widehat{H}$-k-strongly accretive mapping. Then, the mapping $(\widehat{H} + \rho\widehat{M})^{-1} : E \to E$ is single-valued for every constant $\rho > 0$.*

Based on Corollary 2.13, the proximal-point mapping (or resolvent operator) $R^{\widehat{H}}_{\rho,\widehat{M}}$ associated with $\widehat{H}, \widehat{M}$ and $\rho > 0$ is defined in [12] as follows.

Definition 2.15. [12, Definition 2.4] *Let $\widehat{H} : E \to E$ be a strictly accretive operator, $\widehat{M} : E \multimap E$ be a $\widehat{H}$-accretive operator, and $\rho > 0$ be an arbitrary real constant. The proximal-point mapping (resolvent operator) $R^{\widehat{H}}_{\rho,\widehat{M}} : E \to E$ associated with $\widehat{H}, \widehat{M}$ and $\rho > 0$ is defined by*

$$R^{\widehat{H}}_{\rho,\widehat{M}}(u) = (\widehat{H} + \rho\widehat{M})^{-1}(u), \quad \forall u \in E.$$

We now ready to present the main result of this section in which the conditions that guarantee the Lipschitz continuity of the proximal-point mapping $R^{\widehat{H}}_{\rho,\widehat{M}}$ are stated and its Lipschitz constant is also calculated.

Theorem 2.16. *Let $\widehat{H}: E \to E$ be a l-strongly accretive mapping and let $\widehat{M}: E \multimap E$ be a $\widehat{H}$-k-strongly accretive mapping. Then, the proximal-point mapping $R^{\widehat{H}}_{\rho,\widehat{M}}: E \to E$ is $\frac{1}{l+\rho k}$-Lipschitz continuous, i.e.,*

$$\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\| \leq \frac{1}{l+\rho k}\|u-v\|, \quad \forall u,v \in E.$$

Proof. Since $\widehat{M}$ is a $\widehat{H}$-accretive mapping, for any given $u,v \in E$ with $\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\| \neq 0$, we have

$$R^{\widehat{H}}_{\rho,\widehat{M}}(u) = (\widehat{H} + \rho\widehat{M})^{-1}(u) \text{ and } R^{\widehat{H}}_{\rho,\widehat{M}}(v) = (\widehat{H} + \rho\widehat{M})^{-1}(v),$$

and so

$$\rho^{-1}(u - \widehat{H}(R^{\widehat{H}}_{\rho,\widehat{M}}(u))) \in \widehat{M}(R^{\widehat{H}}_{\rho,\widehat{M}}(u)) \text{ and } \rho^{-1}(v - \widehat{H}(R^{\widehat{H}}_{\rho,\widehat{M}}(v))) \in \widehat{M}(R^{\widehat{H}}_{\rho,\widehat{M}}(v)).$$

From k-strongly accretiveness of $\widehat{M}$ it follows that

$$\begin{aligned}&\rho^{-1}\langle u - \widehat{H}(R^{\widehat{H}}_{\rho,\widehat{M}}(u)) - (v - \widehat{H}(R^{\widehat{H}}_{\rho,\widehat{M}}(v))), J_q(R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v))\rangle \\ &\geq k\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\|^q,\end{aligned}$$

from which yields

$$\begin{aligned}&\langle u - v, J_q(R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v))\rangle \geq \rho k\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\|^q \\ &+ \langle \widehat{H}(R^{\widehat{H}}_{\rho,\widehat{M}}(u)) - \widehat{H}(R^{\widehat{H}}_{\rho,\widehat{M}}(v)), J_q(R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v))\rangle.\end{aligned} \tag{3}$$

Making use of (3) and taking into account that the mapping $\widehat{H}$ is l-strongly accretive, we derive that

$$\begin{aligned}\|u-v\|\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\|^{q-1} &= \|u-v\|\|J_q(R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v))\| \\ &\geq \rho k\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\|^q \\ &\quad + \langle \widehat{H}(R^{\widehat{H}}_{\rho,\widehat{M}}(u)) - \widehat{H}(R^{\widehat{H}}_{\rho,\widehat{M}}(v)), J_q(R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v))\rangle \\ &\geq \rho k\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\|^q + l\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\|^q \\ &= (l+\rho k)\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\|^q.\end{aligned}$$

In view of the fact that $\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\| \neq 0$, it follows that

$$\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\| \leq \frac{1}{l+\rho k}\|u-v\|.$$

This completes the proof. □

As a direct consequence of the above theorem, we obtain the following conclusion for the special case when $E = \mathcal{H}$ is a real Hilbert space. Let us, before proceeding to it, recall the following assertion.

Lemma 2.17. [37, Theorem 2.1] *Let $\widehat{H}: \mathcal{H} \to \mathcal{H}$ be a strongly monotone, continuous and single-valued operator. Then a set-valued operator $\widehat{M}: \mathcal{H} \multimap \mathcal{H}$ is $\widehat{H}$-monotone if and only if $\widehat{M}$ is maximal monotone.*

Corollary 2.18. [37, Theorem 2.2] *Let $\widehat{H} : \mathcal{H} \to \mathcal{H}$ be a continuous and strongly monotone operator with constant γ and let $\widehat{M} : \mathcal{H} \multimap \mathcal{H}$ be maximal strongly monotone with constant η. Then, the resolvent operator $R^{\widehat{H}}_{\rho,\widehat{M}} : \mathcal{H} \to \mathcal{H}$ is $\frac{1}{\gamma+\rho\eta}$-Lipschitz continuous, i.e.,*

$$\|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\| \leq \frac{1}{\gamma + \rho\eta}\|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

Proof. Taking into account that $\widehat{H} : \mathcal{H} \to \mathcal{H}$ is strongly monotone and continuous, Lemma 2.17 implies that $\widehat{M}$ is $\widehat{H}$-monotone and so the desired result follows from Theorem 2.16 immediately. □

3. Formulation of the problem, iterative algorithms and convergence result

Let $G, \widehat{H} : E \to E$, $F : E \times E \to E$, $S, T : E \multimap CB(E)$ and $\widehat{M} : E \multimap E$ be the mappings such that $\widehat{M}$ is $\widehat{H}$-accretive. For given $\lambda \in E$, we consider the problem of finding $u \in E$, $v \in S(u)$ and $w \in T(u)$ such that

$$\lambda \in G(u) + F(v, w) + \widehat{M}(u), \tag{4}$$

which is called a set-valued variational inclusion problem (SVIP) with $\widehat{H}$-accretive mappings in real q-uniformly smooth Banach spaces.

If $G \equiv 0$, then the SVIP (4) reduces to the set-valued variational inclusion problem of finding $u \in E$, $v \in S(u)$ and $w \in T(u)$ such that

$$\lambda \in F(v, w) + \widehat{M}(u). \tag{5}$$

For the case when $F \equiv 0$ and $\lambda = 0$, the SVIP (4) collapses to the variational inclusion problem of finding $u \in E$ such that

$$0 \in G(u) + \widehat{M}(u),$$

which was considered and studied by Fang and Huang [12].

It is worthwhile to stress that for appropriate and suitable choices of the mappings $G, \widehat{H}, F, S, T, \widehat{M}$, element $\lambda \in E$ and the space E, the SVIP (4) reduces to various classes of variational inclusions and variational inequalities, see, for example, [13, 17, 22, 33, 38] and the references therein.

The next assertion provides us a characterization of a solution of the SVIP (4) and plays a prominent role in proving of our main results in the sequel.

Lemma 3.1. *Let $F, G, S, T, \widehat{M}, \lambda$ be the same as in the* SVIP (4) *and let $\widehat{H} : E \to E$ be a strictly accretive mapping. Then $(u, v, w) \in E \times S(u) \times T(u)$ is a solution of the* SVIP (4) *if and only if (u, v, w) satisfies the relation*

$$u = R^{\widehat{H}}_{\rho,\widehat{M}}[\widehat{H}(u) - \rho G(u) - \rho F(v, w) + \rho\lambda], \tag{6}$$

where $\rho > 0$ is an arbitrary constant and $R^{\widehat{H}}_{\rho,\widehat{M}} = (\widehat{H} + \rho\widehat{M})^{-1}$.

Proof. Making use of Definition 2.15, for given constant $\rho > 0$,

$$\begin{aligned}
\lambda \in G(u) + F(v, w) + \widehat{M}(u) &\Leftrightarrow (\widehat{H}(u) - \rho G(u) - \rho F(v, w) + \rho\lambda) \in \widehat{H}(u) + \rho\widehat{M}(u) \\
&\Leftrightarrow (\widehat{H}(u) - \rho G(u) - \rho F(v, w) + \rho\lambda) \in (\widehat{H} + \rho\widehat{M})(u) \\
&\Leftrightarrow u = (\widehat{H} + \rho\widehat{M})^{-1}(\widehat{H}(u) - \rho G(u) - \rho F(v, w) + \rho\lambda) \\
&\Leftrightarrow u = R^{\widehat{H}}_{\rho,\widehat{M}}[\widehat{H}(u) - \rho G(u) - \rho F(v, w) + \rho\lambda].
\end{aligned}$$

□

As a direct consequence of Lemma 3.1, we obtain the following assertion which gives a characterization of a solution of problem (5).

Lemma 3.2. *Assume that $F, S, T, \widehat{M}, \widehat{H}, \lambda$ are the same as in Lemma 3.1. Then $(u, v, w) \in E \times S(u) \times T(u)$ is a solution to problem* (5) *if and only if (u, v, w) satisfies the relation*

$$u = R^{\widehat{H}}_{\rho,\widehat{M}}[\widehat{H}(u) - \rho F(v, w) + \rho\lambda],$$

where $\rho > 0$ is a constant and $R^{\widehat{H}}_{\rho,\widehat{M}} = (\widehat{H} + \rho\widehat{M})^{-1}$.

Remark 3.3. *Equality* (6) *can be written as follows:*

$$\begin{cases} u = R^{\widehat{H}}_{\rho,\widehat{M}}(z), \\ z = \widehat{H}(u) - \rho G(u) - \rho F(v, w) + \rho\lambda. \end{cases} \tag{7}$$

The fixed point formulation (7) and Nadler's theorem [28] enable us to construct an iterative algorithm for finding an approximate solution of the SVIP (4) as follows.

Algorithm 3.4. *Suppose that $F, G, S, T, \widehat{H}, \widehat{M}, \lambda$ are the same as in the* SVIP (4) *such that $\widehat{H}$ is a strictly accretive mapping and $\widehat{M}$ is a $\widehat{H}$-accretive mapping. For any given $z_0 \in E$, we can compute the sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ in E by the iterative schemes*

$$\begin{cases} u_n = R^{\widehat{H}}_{\rho,\widehat{M}}(z_n), \\ v_n \in S(u_n) : \|v_n - v_{n+1}\| \le (1 + \frac{1}{n+1})D(S(u_n), S(u_{n+1})), \\ w_n \in T(u_n) : \|w_n - w_{n+1}\| \le (1 + \frac{1}{n+1})D(T(u_n), T(u_{n+1})), \\ z_{n+1} = \widehat{H}(u_n) - \rho G(u_n) - \rho F(v_n, w_n) + \rho\lambda + e_n, \end{cases} \tag{8}$$

where $n = 0, 1, 2, \ldots$; $\rho > 0$ is a constant; $D(.,.)$ is the Huasdorff metric on $CB(E)$ and $\{e_n\}_{n=0}^{\infty} \subset E$ is an error to take into account a possible inexact computation of the proximal-point mapping point satisfying the following conditions:

$$\sum_{j=1}^{\infty} \|e_j - e_{j-1}\|\varpi^{j-1} < \infty, \ \forall \varpi \in (0, 1); \lim_{n\to\infty} e_n = 0. \tag{9}$$

If $G \equiv 0$, then Algorithm 3.4 reduces to the following iterative algorithm.

Algorithm 3.5. *Assume that $F, G, S, T, \widehat{H}, \widehat{M}, \lambda$ are the same as in Algorithm 3.4. For any given $z_0 \in E$, define the sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ in E by the iterative schemes*

$$\begin{cases} u_n = R^{\widehat{H}}_{\rho,\widehat{M}}(z_n), \\ v_n \in S(u_n) : \|v_n - v_{n+1}\| \le (1 + \frac{1}{n+1})D(S(u_n), S(u_{n+1})), \\ w_n \in T(u_n) : \|w_n - w_{n+1}\| \le (1 + \frac{1}{n+1})D(T(u_n), T(u_{n+1})), \\ z_{n+1} = \widehat{H}(u_n) - \rho F(v_n, w_n) + \rho\lambda + e_n, \end{cases}$$

where $n = 0, 1, 2, \ldots$; $\rho > 0$ is a constant; $D(.,.)$ and $\{e_n\}_{n=0}^{\infty} \subset E$ are the same as in Algorithm 3.4.

Before to deal with the convergence analysis of our proposed iterative algorithms, we need to recall the following concepts.

Definition 3.6. [15, Definition 2.4] *Let $\widehat{H} : E \to E$ and $F : E \times E \to E$ be two single-valued mappings and $S : E \multimap E$ be a set-valued mapping. Then F is said to be*

(i) *r-strongly accretive with respect to S in the first argument if there exists a constant $r > 0$ such that*

$$\langle F(w_1,.) - F(w_2,.), J_q(u-v)\rangle \geq r\|u-v\|^q, \quad \forall u,v \in E, w_1 \in S(u), w_2 \in S(v);$$

(ii) *k-strongly accretive with respect to S in the second argument if there exists a constant $k > 0$ such that*

$$\langle F(.,w_1) - F(.,w_2), J_q(u-v)\rangle \geq k\|u-v\|^q, \quad \forall u,v \in E, w_1 \in S(u), w_2 \in S(v);$$

(iii) *γ-strongly accretive with respect to S and $\widehat{H}$ in the first argument if there exits a constant $\gamma > 0$ such that*

$$\langle F(w_1,.) - F(w_2,.), J_q(\widehat{H}(u) - \widehat{H}(v))\rangle \geq \gamma\|\widehat{H}(u) - \widehat{H}(v)\|^q, \quad \forall u,v \in E, w_1 \in S(u), w_2 \in S(v);$$

(iv) *ς-strongly accretive with respect to S and $\widehat{H}$ in the second argument if there exits a constant $\varsigma > 0$ such that*

$$\langle F(.,w_1) - F(.,w_2), J_q(\widehat{H}(u) - \widehat{H}(v))\rangle \geq \varsigma\|\widehat{H}(u) - \widehat{H}(v)\|^q, \quad \forall u,v \in E, w_1 \in S(u), w_2 \in S(v);$$

(v) *ζ-Lipschitz continuous in the first argument if there exists a constant $\zeta > 0$ such that*

$$\|F(u,w) - F(v,w)\| \leq \zeta\|u-v\|, \quad \forall u,v,w \in E;$$

(vi) *ϱ-Lipschitz continuous in the second argument if there exists a constant $\varrho > 0$ such that*

$$\|F(w,u) - F(w,v)\| \leq \varrho\|u-v\|, \quad \forall u,v,w \in E.$$

It is significant to emphasize that for a given single-valued mapping $\widehat{H} : E \to E$ and a set-valued mapping $S : E \multimap E$, a strong accretive mapping with respect to S and $\widehat{H}$ in the first (resp., second) argument need not be strong accretive with respect to S in the first (resp., second) argument. This fact is illustrated in the following example.

Example 3.7. Consider $E = \mathbb{R}$ with the Euclidean norm $\|.\| = |.|$ and let the set-valued mapping $S : E \multimap E$ be defined by

$$S(x) = \begin{cases} k - x, & x < l, \\ [-k-l, k-l], & x = l, \\ -k - x, & x > l, \end{cases}$$

where $k > l > 0$ are arbitrary but fixed real numbers. Moreover, define the mappings $F : E \times E \to E$ and $\widehat{H} : E \to E$, respectively, by $F(x,y) = \frac{x+y}{2k}$ and $\widehat{H}(x) = \gamma x$ for all $x, y \in E$, where $\gamma < 0$ is an arbitrary real constant. Since E is a Hilbert space, it follows that E is a 2-uniformly smooth Banach space. Then, for all $u, v \in E$, $w_1 \in S(u)$, $w_2 \in S(v)$ and $t \in E$, we obtain

$$\begin{aligned} \langle F(w_1,t) - F(w_2,t), J_2(\widehat{H}(u) - \widehat{H}(v))\rangle &= \langle F(w_1,t) - F(w_2,t), \widehat{H}(u) - \widehat{H}(v)\rangle \\ &= (\frac{w_1 + t}{2k} - \frac{w_2 + t}{2k})(\gamma u - \gamma v) \\ &= -\gamma(\frac{w_1 - w_2}{2k})(v - u). \end{aligned}$$

If $u, v > l$, then $S(u) = -k - u$ and $S(v) = -k - v$. Then taking $w_1 = -k - u$ and $w_2 = -k - v$, it yields

$$\begin{aligned} -\gamma(\frac{w_1 - w_2}{2k})(v - u) &= -\gamma(\frac{-k - u + k + v}{2k})(v - u) \\ &= -\frac{\gamma(v-u)^2}{2k} \\ &\geq \frac{1}{2k}|\widehat{H}(u) - \widehat{H}(v)|^2. \end{aligned}$$

If $u > l$ and $v < l$, then $S(u) = -k - u$ and $S(v) = k - v$. Then picking $w_1 = -k - u$ and $w_2 = k - v$ and in virtue of the fact that $v < u$, we get

$$\begin{aligned}
-\gamma(\frac{w_1 - w_2}{2k})(v-u) &= -\gamma(\frac{-k-u-k+v}{2k})(v-u)\\
&= -\gamma(\frac{v-u}{2k} - 1)(v-u)\\
&= -\frac{\gamma(v-u)^2}{2k} + \gamma(v-u)\\
&> -\frac{\gamma(v-u)^2}{2k}\\
&= \frac{1}{2k}|\widehat{H}(u) - \widehat{H}(v)|^2.
\end{aligned}$$

For the case when $u, v < l$, we have $S(u) = k - u$ and $S(v) = k - v$. Then, setting $w_1 = k - u$ and $w_2 = k - v$, it follows that

$$\begin{aligned}
-\gamma(\frac{w_1 - w_2}{2k})(v-u) &= -\gamma(\frac{k-u-k+v}{2k})(v-u)\\
&= -\frac{\gamma(v-u)^2}{2k}\\
&\geq \frac{1}{2k}|\widehat{H}(u) - \widehat{H}(v)|^2.
\end{aligned}$$

If $u > l$ and $v = l$, then $S(u) = -k - u$ and $S(v) = [-k - l, k - l]$. Taking into account that $-\frac{w_2}{2k} \in [\frac{l-k}{2k}, \frac{k+l}{2k}]$ for all $w_2 \in [-k-l, k-l]$, picking $w_1 = -k-u$ and thanks to the fact that $u > l$, it follows that for all $w_2 \in [-k-l, k-l]$,

$$\begin{aligned}
-\gamma(\frac{w_1 - w_2}{2k})(v-u) &= -\gamma(\frac{-k-u-w_2}{2k})(l-u)\\
&\geq -\gamma(\frac{-k-u+k+l}{2k})(l-u)\\
&= -\frac{\gamma(l-u)^2}{2k}\\
&= \frac{1}{2k}|\widehat{H}(u) - \widehat{H}(v)|^2.
\end{aligned}$$

In the case where $u < l$ and $v = l$, we have $S(u) = k - u$ and $S(v) = [-k - l, k - l]$. Since $u < l$, taking $w_1 = k - u$, for all $w_2 \in [-k - l, k - l]$, we infer that

$$\begin{aligned}
-\gamma(\frac{w_1 - w_2}{2k})(v-u) &= -\gamma(\frac{k-u-w_2}{2k})(l-u)\\
&\geq -\gamma(\frac{k-u+l-k}{2k})(l-u)\\
&= -\frac{\gamma(l-u)^2}{2k}\\
&= \frac{1}{2k}|\widehat{H}(u) - \widehat{H}(v)|^2.
\end{aligned}$$

The above-mentioned discussions ensure that for all $u, v \in E$, $w_1 \in S(u)$, $w_2 \in S(v)$ and $t \in E$,

$$\langle F(w_1, t) - F(w_2, t), J_2(\widehat{H}(u) - \widehat{H}(v))\rangle \geq \frac{1}{2k}|\widehat{H}(u) - \widehat{H}(v)|^2,$$

i.e., F is a $\frac{1}{2k}$-strongly accretive mapping with respect to S and $\widehat{H}$ in the first argument. In view of the fact that $F(x, y) = F(y, x)$ for all $x, y \in E$, following the same arguments, we can show that F is $\frac{1}{2k}$-strongly accretive with respect to S and $\widehat{H}$ in the second argument. However, F is not $\frac{1}{2k}$-strongly accretive with respect to S in the first (second) argument. In fact, if $v < l < u$, then we have $S(u) = -k - u$ and $S(v) = k - v$. Then, taking $w_1 = -k - u$ and $w_2 = k - v$, from the fact that $u > v$, it follows that for all $t \in E$,

$$\begin{aligned}
\langle F(w_1,t) - F(w_2,t), J_2(u-v)\rangle &= \langle F(w_1,t) - F(w_2,t), u-v\rangle \\
&= \frac{w_1 - w_2}{2k}(u-v) \\
&= (\frac{-k-u-k+v}{2k})(u-v) \\
&= (\frac{u-v}{2k} - 1)(u-v) \\
&< 0 < \frac{1}{2k}|u-v|^2,
\end{aligned}$$

which implies that F is not $\frac{1}{2k}$-strongly accretive with respect to S in the first argument. Relying on the fact that $F(x,y) = F(y,x)$ for all $x, y \in E$, in a similar fashion to the preceding analysis, one can show that F is not $\frac{1}{2k}$-strongly accretive with respect to S in the second argument. In the light of the arguments mentioned above, we observe that in Definition 3.6, the concepts given in parts (iii) and (iv) are generalizations of the notions presented in parts (i) and (ii), respectively.

Definition 3.8. *A set-valued mapping $S : E \multimap CB(E)$ is said to be σ-D-Lipschitz continuous* (*or D-Lipschitz continuous with constant σ*) *if there exists a constant $\sigma > 0$ such that*

$$D(S(u), S(v)) \leq \sigma\|u - v\|, \quad \forall u, v \in E.$$

Theorem 3.9. *Let $E, F, G, \widehat{H}, S, T, \widehat{M}$ and λ be the same as in the* SVIP (4) *such that $\widehat{H}$ is a l-strongly accretive, τ-Lipschitz continuous and ς-expansive mapping, and $\widehat{M}$ is a $\widehat{H}$-k-strongly accretive mapping. Suppose that the mapping F is ϵ_1, ϵ_2-Lipschitz continuous in the first and second arguments, respectively, σ-strongly accretive with respect to S and $\widehat{H}$ in the first argument, and δ-strongly accretive with respect to T and $\widehat{H}$ in the second argument. Let the mapping G be α-Lipschitz continuous, and the mappings S and T be l_1, l_2-D-Lipschitz continuous, respectively. Assume further that there exists a constant $\rho > 0$ such that*

$$\sqrt[q]{\tau^q - q\rho(\sigma+\delta)\varsigma^q + c_q\rho^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q} < l + (k-\alpha)\rho, \tag{10}$$

where c_q is a constant guaranteed by Lemma 2.3, and for the case when q is an even natural number, in addition to (10), *the constant ρ satisfies the following condition:*

$$q\rho(\sigma+\delta)\varsigma^q < \tau^q + c_q\rho^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q. \tag{11}$$

Then, the iterative sequences $\{u_n\}_{n=0}^\infty, \{v_n\}_{n=0}^\infty$ and $\{w_n\}_{n=0}^\infty$ generated by Algorithm 3.4 converge strongly to u, v and w, respectively, and (u, v, w) is a solution of the SVIP (4).

Proof. Using (8), Theorem 2.16 and the assumptions, we conclude that

$$\|u_{n+1} - u_n\| = \|R^{\widehat{H}}_{\rho,\widehat{M}}(z_{n+1}) - R^{\widehat{H}}_{\rho,\widehat{M}}(z_n)\| \leq \frac{1}{l+\rho k}\|z_{n+1} - z_n\|. \tag{12}$$

Making use of (8), we get

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|\widehat{H}(u_n) - \rho G(u_n) - \rho F(v_n, w_n) + \rho\lambda + e_n - (\widehat{H}(u_{n-1}) \\
&\quad - \rho G(u_{n-1}) - \rho F(v_{n-1}, w_{n-1}) + \rho\lambda + e_{n-1})\| \\
&\leq \|\widehat{H}(u_n) - \widehat{H}(u_{n-1}) - \rho(F(v_n, w_n) - F(v_{n-1}, w_{n-1}))\| \\
&\quad + \rho\|G(u_n) - G(u_{n-1})\| + \|e_n - e_{n-1}\|.
\end{aligned} \tag{13}$$

Using Lemma 2.3, it yields

$$\begin{aligned}\|\widehat{H}(u_n) - \widehat{H}(u_{n-1}) - \rho(F(v_n, w_n) - F(v_{n-1}, w_{n-1}))\|^q &\leq \|\widehat{H}(u_n) - \widehat{H}(u_{n-1})\|^q \\ &- q\rho\langle F(v_n, w_n) - F(v_{n-1}, w_{n-1}), J_q(\widehat{H}(u_n) - \widehat{H}(u_{n-1}))\rangle \\ &+ c_q\rho^q\|F(v_n, w_n) - F(v_{n-1}, w_{n-1})\|^q.\end{aligned} \tag{14}$$

Since $\widehat{H}$ is a τ-Lipschitz continuous mapping, it follows that

$$\|\widehat{H}(u_n) - \widehat{H}(u_{n-1})\| \leq \tau\|u_n - u_{n-1}\|. \tag{15}$$

Taking into account that the mapping F is ϵ_1, ϵ_2-Lipschitz continuous in the first and second arguments, respectively, and the mappings S and T are l_1, l_2-D-Lipschitz continuous, respectively, by (8), we obtain

$$\begin{aligned}\|F(v_n, w_n) - F(v_{n-1}, w_{n-1})\| &\leq \|F(v_n, w_n) - F(v_{n-1}, w_n)\| \\ &\quad + \|F(v_{n-1}, w_n) - F(v_{n-1}, w_{n-1})\| \\ &\leq \epsilon_1\|w_n - w_{n-1}\| + \epsilon_2\|v_n - v_{n-1}\| \\ &\leq \epsilon_1(1 + \frac{1}{n})D(S(u_n), S(u_{n-1})) \\ &\quad + \epsilon_2(1 + \frac{1}{n})D(T(u_n), T(u_{n-1})) \\ &\leq (\epsilon_1 l_1 + \epsilon_2 l_2)(1 + \frac{1}{n})\|u_n - u_{n-1}\|.\end{aligned} \tag{16}$$

In virtue of the facts that F is σ-strongly accretive with respect to S and $\widehat{H}$ in the first argument, and δ-strongly accretive with respect to T and $\widehat{H}$ in the second argument, and the mapping $\widehat{H}$ is ς-expansive, it yields

$$\begin{aligned}&\langle F(v_n, w_n) - F(v_{n-1}, w_{n-1}), J_q(\widehat{H}(u_n) - \widehat{H}(u_{n-1}))\rangle \\ &= \langle F(v_n, w_n) - F(v_{n-1}, w_n), J_q(\widehat{H}(u_n) - \widehat{H}(u_{n-1}))\rangle \\ &\quad + F(v_{n-1}, w_n) - F(v_{n-1}, w_{n-1}), J_q(\widehat{H}(u_n) - \widehat{H}(u_{n-1}))\rangle \\ &\geq (\sigma + \delta)\|\widehat{H}(u_n) - \widehat{H}(u_{n-1})\|^q \\ &\geq (\sigma + \delta)\varsigma^q\|u_n - u_{n-1}\|^q.\end{aligned} \tag{17}$$

Employing (14)–(17), we derive that

$$\begin{aligned}&\|\widehat{H}(u_n) - \widehat{H}(u_{n-1}) - \rho(F(v_n, w_n) - F(v_{n-1}, w_{n-1}))\| \\ &\leq \sqrt[q]{\tau^q - q\rho(\sigma + \delta)\varsigma^q + c_q\rho^q(1 + \frac{1}{n})^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q}\|u_n - u_{n-1}\|.\end{aligned} \tag{18}$$

Since the mapping G is α-Lipschitz continuous, it follows that

$$\|G(u_n) - G(u_{n-1})\| \leq \alpha\|u_n - u_{n-1}\|. \tag{19}$$

Combining (12), (13), (18) and (19), we obtain

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \frac{1}{l + \rho k}(\sqrt[q]{\tau^q - q\rho(\sigma + \delta)\varsigma^q + c_q\rho^q(1 + \frac{1}{n})^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q} \\ &\quad + \alpha\rho)\|u_n - u_{n-1}\| + \frac{1}{l + \rho k}\|e_n - e_{n-1}\| \\ &= \varphi_n\|u_n - u_{n-1}\| + \frac{1}{l + \rho k}\|e_n - e_{n-1}\|,\end{aligned} \tag{20}$$

where for each $n \in \mathbb{N}$,

$$\varphi_n = \frac{1}{l+\rho k}\Big(\sqrt[q]{\tau^q - q\rho(\sigma+\delta)\varsigma^q + c_q\rho^q\big(1+\frac{1}{n}\big)^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q} + \alpha\rho\Big).$$

Let us put

$$\varphi = \frac{1}{l+\rho k}\Big(\sqrt[q]{\tau^q - q\rho(\sigma+\delta)\varsigma^q + c_q\rho^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q} + \alpha\rho\Big).$$

Then, in the light of the assumptions, we infer that $\varphi_n \to \varphi$, as $n \to \infty$. By virtue of (10) we know that $\varphi \in (0,1)$. Therefore, there exists $n_0 \in \mathbb{N}$ and $\hat{\varphi} \in (\varphi, 1)$ such that $\varphi_n \leq \hat{\varphi}$ for all $n \geq n_0$. Thereby, using (20), we derive that for all $n > n_0$,

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \hat{\varphi}\|u_n - u_{n-1}\| + \frac{1}{l+\rho k}\|e_n - e_{n-1}\| \\
&\leq \hat{\varphi}[\hat{\varphi}\|u_{n-1} - u_{n-2}\| + \frac{1}{l+\rho k}\|e_{n-1} - e_{n-2}\|] + \frac{1}{l+\rho k}\|e_n - e_{n-1}\| \\
&= \hat{\varphi}^2\|u_{n-1} - u_{n-2}\| + \frac{\hat{\varphi}}{l+\rho k}\|e_{n-1} - e_{n-2}\| + \frac{1}{l+\rho k}\|e_n - e_{n-1}\| \\
&\leq \ldots \\
&\leq \hat{\varphi}^{n-n_0}\|u_{n_0+1} - u_{n_0}\| + \sum_{i=1}^{n-n_0}\frac{\hat{\varphi}^{i-1}}{l+\rho k}\|e_{n-(i-1)} - e_{n-i}\|.
\end{aligned} \tag{21}$$

Making use of (21), we deduce that for all $m \geq n > n_0$,

$$\begin{aligned}
\|u_m - u_n\| &\leq \sum_{j=n}^{m-1}\|u_{j+1} - u_j\| \leq \sum_{j=n}^{m-1}\hat{\varphi}^{j-n_0}\|u_{n_0+1} - u_{n_0}\| \\
&\quad + \sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}\frac{\hat{\varphi}^{i-1}}{l+\rho k}\|e_{j-(i-1)} - e_{j-i}\| \\
&= \sum_{j=n}^{m-1}\hat{\varphi}^{j-n_0}\|u_{n_0+1} - u_{n_0}\| + \sum_{j=n}^{m-1}\hat{\varphi}^j\Big[\sum_{i=1}^{j-n_0}\frac{1}{l+\rho k}\frac{\|e_{j-(i-1)} - e_{j-i}\|}{\hat{\varphi}^{j-(i-1)}}\Big].
\end{aligned} \tag{22}$$

Taking into account that $\hat{\varphi} \in (0,1)$, making use of (22) we deduce that for any $m \geq n > n_0$, $\|u_m - u_n\| \to 0$, as $n \to \infty$, and so $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in E. The completeness of E implies the existence of a point $u \in E$ such that $u_n \to u$, as $n \to \infty$. Now thanks to the fact that the mappings S and T are l_1, l_2-D-Lipschitz continuous, respectively, by using (8), we infer that $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are also Cauchy sequences in E and so $v_n \to v$ and $w_n \to w$ for some $v, w \in E$, as $n \to \infty$. In the meanwhile, since $v_n \in S(u_n)$ for each $n \geq 0$, it yields

$$\begin{aligned}
d(v, S(u)) &= \inf\{\|v - s\| : s \in S(u)\} \\
&\leq \|v - v_n\| + d(v_n, S(u)) \\
&\leq \|v - v_n\| + D(S(u_n), S(u)) \\
&\leq \|v - v_n\| + l_1\|u_n - u\|.
\end{aligned}$$

The right-hand side of the above inequality approaches zero as $n \to \infty$. In view of the closedness of the set $S(u)$, we conclude that $v \in S(u)$. By an argument analogous to the previous one, one can show that $w \in T(u)$. At the same time, by Theorem 2.16 and the assumptions, we obtain

$$\|R^{\widehat{H}}_{\rho,\widehat{M}}(z_n) - R^{\widehat{H}}_{\rho,\widehat{M}}(z)\| \leq \frac{1}{l+\rho k}\|z_n - z\|. \tag{23}$$

Owing to the facts that the mappings $\widehat{H}, F$ and G are continuous, $u_n \to u$, $v_n \to v$, $w_n \to w$, $e_n \to 0$, as $n \to \infty$, applying (8) it can be easily seen that

$$z_n \to z = H(u) - \rho G(u) - \rho F(v,w) + \rho\lambda, \text{ as } n \to \infty.$$

This fact ensures that the right-hand side of (23) tends to zero, as $n \to \infty$ and so $R^{\widehat{H}}_{\rho,\widehat{M}}(z_n) \to R^{\widehat{H}}_{\rho,\widehat{M}}(z)$, as $n \to \infty$. From (8) it follows that $u = R^{\widehat{H}}_{\rho,\widehat{M}}(z)$. Now, by virtue of (7) and Lemma 3.1, we conclude that (u,v,w) is a solution of the SVIP (4). The proof is finished. □

As a direct consequence of the above theorem, we have the following corollary.

Corollary 3.10. *Assume that $E, F, \widehat{H}, S, T, \widehat{M}$ and λ are the same as in problem* (5) *such that $\widehat{H}$ is a l-strongly accretive, τ-Lipschitz continuous and ς-expansive mapping, and $\widehat{M}$ is a $\widehat{H}$-k-strongly accretive mapping. Let the mapping F be ϵ_1, ϵ_2-Lipschitz continuous in the first and second arguments, respectively, σ-strongly accretive with respect to S and $\widehat{H}$ in the first argument, and δ-strongly accretive with respect to T and $\widehat{H}$ in the second argument. Suppose further that the mappings S and T are l_1, l_2-D-Lipschitz continuous, respectively. If there exits a constant $\rho > 0$ such that*

$$\sqrt[q]{\tau^q - q\rho(\sigma+\delta)\varsigma^q + c_q\rho^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q} < l + k\rho, \tag{24}$$

where c_q is a constant guaranteed by Lemma 2.3, and for the case when q is an even natural number, in addition to (24), (11) *holds, then the iterative sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ guaranteed by Algorithm 3.5 converge strongly to u, v and w, respectively, and (u,v,w) is a solution to problem* (5).

4. $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings and some comments

In this section, our attention is paid to the concept of $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping introduced in [15]. The results relating to it appeared in [15] are investigated and analyzed and some remarks regarding them are pointed out. We also show that all the results given in [15] can be derived making use of our conclusions presented in Sections 2 and 3.

Definition 4.1. [15, Definition 2.6] *Let $H : (E\times E)\times(E\times E) \to E$, and $A, B, C, D : E \to E$ be the single-valued mappings. Then*

(i) *$H((A,.),(C,.))$ is said to be (μ_1,γ_1)-strongly mixed cocoercive regarding (A,C) with $\mu_1,\gamma_1 > 0$ if*

$$\langle H((Ax,u),(Cx,u)) - H((Ay,u),(Cy,u)), J_q(x-y)\rangle \geq \mu_1\|Ax-Ay\|^q + \gamma_1\|x-y\|^q,$$
$$\forall x,y,u \in E;$$

(ii) *$H((.,B),(.,D))$ is said to be (μ_2,γ_2)-relaxed mixed cocoercive regarding (B,D) with $\mu_2,\gamma_2 > 0$ if*

$$\langle H((u,Bx),(u,Dx)) - H((u,By),(u,Dy)), J_q(x-y)\rangle \geq -\mu_2\|Bx-By\|^q + \gamma_2\|x-y\|^q,$$
$$\forall x,y,u \in E;$$

(iii) *$H((A,B),(C,D))$ is said to be symmetric mixed cocoercive regarding (A,C) and (B,D) if $H((A,.),(C,.))$ is (μ_1,γ_1)-strongly mixed cocoercive regarding (A,C) and $H((.,B),(.,D))$ is (μ_2,γ_2)-relaxed mixed cocoercive regarding (B,D);*

(iv) *$H((A,B),(C,D))$ is said to be τ-mixed Lipschitz continuous regarding A, B, C and D with $\tau > 0$ if*

$$\|H((Ax,Bx),(Cx,Dx)) - H((Ay,By),(Cy,Dy))\| \leq \tau\|x-y\|, \quad \forall x,y \in E.$$

Proposition 4.2. *Let $A, B, C, D : E \to E$ and $H : (E\times E)\times(E\times E) \to E$ be the mappings and let $\widehat{H} : E \to E$ be a mapping defined by $\widehat{H}(x) := H((Ax,Bx),(Cx,Dx))$ for all $x \in E$. Then the following assertions hold:*

(i) *If $H((A,B),(C,D))$ is symmetric mixed cocoercive regarding (A,C) and (B,D), A is α_1-expansive and B is β_1-Lipschitz continuous, then $\widehat{H}$ is $(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-strongly accretive (resp., accretive and $-(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-relaxed accretive) provided that $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 > 0$ (resp., $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 = 0$ and $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 < 0$);*

(ii) *If $H((A,B),(C,D))$ is τ-mixed Lipschitz continuous regarding A, B, C and D, then $\widehat{H}$ is τ-Lipschitz continuous.*

Proof. (i) In the light of the assumptions mentioned in part (i), for all $x, y \in E$, it yields

$$\begin{aligned}\langle \widehat{H}(x) - \widehat{H}(y), J_q(x-y)\rangle &= \langle H((Ax,Bx),(Cx,Dx)) - H((Ay,By),(Cy,Dy)), J_q(x-y)\rangle \\ &= \langle H((Ax,Bx),(Cx,Dx)) - H((Ay,Bx),(Cy,Dx)), J_q(x-y)\rangle \\ &\quad + \langle H((Ay,Bx),(Cy,Dx)) - H((Ay,By),(Cy,Dy)), J_q(x-y)\rangle \\ &\geq \mu_1\|Ax - Ay\|^q + \gamma_1\|x-y\|^q - \mu_2\|Bx - By\|^q + \gamma_2\|x-y\|^q \\ &\geq (\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)\|x-y\|^q.\end{aligned}$$

If $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 > 0$ (resp., $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 = 0$ and $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 < 0$) then the preceding inequality implies that $\widehat{H}$ is a $(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-strongly accretive (resp., accretive and $-(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-relaxed accretive) mapping.

(ii) Taking into account that the mapping $(H(A,B),(C,D))$ is τ-mixed Lipschitz continuous regarding A, B, C and D, for all $x, y \in E$, it follows that

$$\|\widehat{H}(x) - \widehat{H}(y)\| = \|H((Ax,Bx),(Cx,Dx)) - H((Ay,By),(Cy,Dy))\| \leq \tau\|x-y\|,$$

i.e., $\widehat{H}$ is τ-Lipschitz continuous. This completes the proof. □

Let us emphasize that thanks to Proposition 4.2(i), every symmetric mixed cocoercive mapping $H : (E\times E)\times(E\times E) \to E$ with respect to the mappings $A, B, C, D : E \to E$, in which A is α_1-expansive, B is β_1-Lipschitz continuous, and $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 > 0$ (resp., $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 = 0$ and $\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 < 0$) is actually a univariate $(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-strongly accretive (resp., accretive and $-(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-relaxed accretive) mapping and is not a new one. In the meanwhile, the notion of mixed Lipschitz continuity of the mapping $H : (E\times E)\times(E\times E) \to E$ with respect to the mappings A, B, C and D appeared in Definition 4.1(iv) is exactly the same concept of Lipschitz continuity of a univariate mapping $\widehat{H} = H((A,B),(C,D)) : E \to E$ given in Definition 2.4(iv) and is not a new one.

Definition 4.3. [15, Definition 2.7] *For given mappings $f, g : E \to E$ and $M : E \times E \multimap E$,*

(i) *$M(f,.)$ is said to be α-strongly accretive regarding f with $\alpha > 0$, if*

$$\langle u - v, J_q(x-y)\rangle \geq \alpha\|x-y\|^q, \quad \forall x, y, w \in E, u \in M(f(x), w), v \in M(f(y), w);$$

(ii) *$M(., g)$ is said to be β-relaxed accretive regarding g with $\beta > 0$, if*

$$\langle u - v, J_q(x-y)\rangle \geq -\beta\|x-y\|^q, \quad \forall x, y, w \in E, u \in M(w, g(x)), v \in M(w, g(y));$$

(iii) *$M(.,.)$ is said to be $\alpha\beta$-symmetric accretive regarding f and g if $M(f,.)$ is α-strongly accretive regarding f and $M(., g)$ is β-relaxed accretive regarding g with $\alpha \geq \beta$ and $\alpha = \beta$ if and only if $x = y$.*

Proposition 4.4. *Suppose that $f, g : E \to E$ are two single-valued mappings and $M : E \times E \multimap E$ is a set-valued mapping. Assume further that the set-valued mapping $\widehat{M} : E \multimap E$ is defined by $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$. If $M(.,.)$ is $\alpha\beta$-symmetric accretive with respect to f and g, then $\widehat{M}$ is $(\alpha - \beta)$-strongly accretive (resp., accretive) provided that $\alpha > \beta$ (resp., $\alpha = \beta$).*

Proof. Since $M(.,.)$ is $\alpha\beta$-symmetric accretive with respect to f and g, in the light of Definition 4.3(iii), $M(f,.)$ is α-strongly accretive with respect to f and $M(.,g)$ is β-relaxed accretive with respect to g. Then, for all $x, y \in E$, $u \in \widehat{M}(x)$ and $v \in \widehat{M}(y)$, we deduce that

$$\begin{aligned}\langle u - v, J_q(x-y)\rangle &= \langle u - w + w - v, J_q(x-y)\rangle \\ &= \langle u - w, J_q(x-y)\rangle + \langle w - v, J_q(x-y)\rangle \\ &\geq \alpha\|x-y\|^q - \beta\|x-y\|^q \\ &= (\alpha-\beta)\|x-y\|^q,\end{aligned}$$

for all $w \in M(f(y), g(x))$. For the case when $\alpha > \beta$ (resp., $\alpha = \beta$), the last inequality ensures that the mapping $\widehat{M}$ is $(\alpha-\beta)$-strongly accretive (resp., accretive). The proof is completed. □

It is very essential to note that based on Proposition 4.4, every symmetric accretive mapping is actually a strongly accretive or accretive mapping. In fact, the notion of $\alpha\beta$-symmetric accretive mapping given in Definition 4.3(iii) (that is, [15, Definition 2.7(vii)]) is exactly the same concept of $r = (\alpha-\beta)$-strongly accretive (resp., accretive) mapping presented in part (ii) (resp., part (i)) of Definition 2.5 provided that $\alpha > \beta$ (resp., $\alpha = \beta$), and is not a new one.

For given single-valued mappings $H : (E \times E) \times (E \times E) \to E$, $A, B, C, D, f, g : E \to E$ and a set-valued mapping $M : E \times E \multimap E$, Gupta et al. [15] considered the following assumptions throughout sections 3 and 4 of [15].

Assumption (a_1): Let H be symmetric mixed cocoercive regarding (A, C) and (B, D).

Assumption (a_2): Let A be α_1-expansive and B be β_1-Lipschitz continuous.

As an extension of notions of $H(.,.)$-accretive mapping [38], B-monotone operator [26], generalized $H(.,.)$-accretive mapping [21], $H((.,.),(.,.))$-mixed cocoercive operators [20] and several another notions of accretive and monotone operators existing in the literature, recently Gupta et al. [15] introduced and studied the concept of $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping as follows.

Definition 4.5. [15, Definition 3.1] *Let Assumption (a_1) holds. M is said to be $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive regarding (A, C), (B, D) and (f, g) if*

(i) *M is $\alpha\beta$-symmetric accretive regarding f and g;*
(ii) *$(H((.,.),(.,.)) + \rho M(f, g))(E) = E$, for all $\rho > 0$.*

Remark 4.6. It should be pointed out that in view of the above-mentioned discussions, Definition 4.5 coincides exactly with Definition 2.6. In fact, by defining $\widehat{M} : E \multimap E$ as $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$, thanks to the fact that M is an $\alpha\beta$-symmetric accretive with respect to the mappings f and g, Proposition 4.4 implies that $\widehat{M}$ is $(\alpha-\beta)$-strongly accretive (resp., accretive) when $\alpha > \beta$ (resp., $\alpha = \beta$) and so $\widehat{M}$ is an accretive mapping. Now, by defining the mapping $\widehat{H} : E \to E$ as $\widehat{H}(x) := H((Ax, Bx), (Cx, Dx))$ for all $x \in E$ and in the light of the fact that M is an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping with respect to (A, C), (B, D) and (f, g), in virtue of Definition 4.5 we have

$$(\widehat{H} + \rho\widehat{M})(E) = (H((A, B), (C, D)) + \rho M(f, g))(E) = E,$$

for every $\rho > 0$. Hence, according to Definition 2.6, $\widehat{M}$ is a $\widehat{H}$-accretive mapping. Thereby, for the case when $\alpha > \beta$ (resp., $\alpha = \beta$), the class of $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mappings coincides exactly with the class of $\widehat{H}$-$(\alpha-\beta)$-strongly accretive (resp., $\widehat{H}$-accretive) mappings. In other words, the notion of $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping is actually the same concept of $\widehat{H}$-$(\alpha-\beta)$-strongly accretive (resp., $\widehat{H}$-accretive) mapping when $\alpha > \beta$ (resp., $\alpha = \beta$), and is not a new one.

Example 4.7. Let $q = 2$ and $E = \mathbb{R}^2$ with usual inner product defined by

$$\langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_1 + x_2y_2.$$

Let $A, B, C, D : \mathbb{R}^2 \to \mathbb{R}^2$ be defined, respectively, by

$$Ax = \begin{pmatrix} 4x_1 \\ 4x_2 \end{pmatrix} = 4x, Bx = \begin{pmatrix} -3x_1 \\ -3x_2 \end{pmatrix} = -3x,$$
$$Cx = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 2x, Dx = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Suppose that $H : (\mathbb{R}^2 \times \mathbb{R}^2) \times (\mathbb{R}^2 \times \mathbb{R}^2) \to \mathbb{R}^2$ is defined by

$$H((x, y), (z, w)) = x + y + z + w, \quad \forall x, y, z, w \in \mathbb{R}^2.$$

Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x) = \begin{pmatrix} 5x_1 - \frac{2}{3}x_2 \\ \frac{2}{3}x_1 + 5x_2 \end{pmatrix}, g(x) = \begin{pmatrix} \frac{7}{4}x_1 + \frac{3}{4}x_2 \\ -\frac{3}{4}x_1 + \frac{7}{4}x_2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,$$

respectively. In addition, let $M : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $M(x, y) = x - y$ for all $x, y \in \mathbb{R}^2$. Taking into account that for all $x, y, u \in \mathbb{R}^2$,

$$\begin{aligned}
\langle H((Ax, u), (Cx, u)) - H((Ay, u), (Cy, u)), J_2(x - y)\rangle &= \langle Ax + Cx - Ay - Cy, x - y\rangle \\
&= \langle Ax - Ay, x - y\rangle + \langle Cx - Cy, x - y\rangle \\
&= \langle 4x - 4y, x - y\rangle + \langle 2x - 2y, x - y\rangle \\
&= 4\|x - y\|^2 + 2\|x - y\|^2 \\
&= \frac{1}{4}\|4x - 4y\|^2 + 2\|x - y\|^2 \\
&= \frac{1}{4}\|Ax - Ay\|^2 + 2\|x - y\|^2
\end{aligned}$$

and

$$\begin{aligned}
\langle H((u, Bx), (u, Dx)) - H((u, By), (u, Dy)), J_2(x - y)\rangle &= \langle Bx + Dx - By - Dy, x - y\rangle \\
&= \langle Bx - By, x - y\rangle + \langle Dx - Dy, x - y\rangle \\
&= \langle -3x + 3y, x - y\rangle + \langle x - y, x - y\rangle \\
&= -3\|x - y\|^2 + \|x - y\|^2 \\
&= -\frac{1}{3}\|3x - 3y\|^2 + \|x - y\|^2 \\
&= -\frac{1}{3}\|Bx - By\|^2 + \|x - y\|^2,
\end{aligned}$$

the authors [15] concluded that $H((A, .), (C, .))$ is $(\mu_1, \gamma_1) = (\frac{1}{4}, 2)$-strongly mixed cocoercive with respect to (A, C) and $H((., B), (., D))$ is $(\mu_2, \gamma_2) = (\frac{1}{3}, 1)$-relaxed mixed cocoercive with respect to (B, D), and so $H((A, B), (C, D))$ is symmetric mixed cocoercive with respect to (A, C) and (B, D). Thanks to the fact that for all $x, y \in \mathbb{R}^2$,

$$\begin{aligned}
&\|H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy))\| \\
&= \|Ax + Bx + Cx + Dx - Ay - By - Cy - Dy\| = 4\|x - y\|,
\end{aligned}$$

they deduced that H is $\tau = 4$-mixed Lipschitz continuous with respect to A, B, C and D.

By virtue of the facts that for all $x, y, w \in \mathbb{R}^2$, $u \in M(f(x), w)$ and $v \in M(f(y), w)$,

$$\begin{aligned}
\langle u - v, J_2(x - y)\rangle &= \langle M(f(x), w) - M(f(y), w), x - y\rangle \\
&= \langle f(x) - w - f(y) + w, x - y\rangle \\
&= \langle f(x) - f(y), x - y\rangle \\
&= \left\langle \begin{pmatrix} 5(x_1 - y_1) - \frac{2}{3}(x_2 - y_2) \\ \frac{2}{3}(x_1 - y_1) + 5(x_2 - y_2) \end{pmatrix}, \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\rangle \\
&= 5(x_1 - y_1)^2 + 5(x_2 - y_2)^2 \\
&= 5\|x - y\|^2
\end{aligned}$$

and for all $u \in M(w, g(x))$ and $v \in M(w, g(y))$,

$$\begin{aligned}
\langle u - v, J_2(x - y)\rangle &= \langle M(w, g(x)) - M(w, g(y)), x - y\rangle \\
&= \langle w - g(x) - w + g(y), x - y\rangle \\
&= -\langle g(x) - g(y), x - y\rangle \\
&= \left\langle \begin{pmatrix} -\frac{7}{4}(x_1 - y_1) - \frac{3}{4}(x_2 - y_2) \\ \frac{3}{4}(x_1 - y_1) - \frac{7}{4}(x_2 - y_2) \end{pmatrix}, \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\rangle \\
&= -\frac{7}{4}(x_1 - y_1)^2 - \frac{7}{4}(x_2 - y_2)^2 = -\frac{7}{4}\|x - y\|^2,
\end{aligned}$$

Gupta et al. [15] deduced that $M(f, .)$ is α-strongly accretive regarding f with $\alpha = 5$ and $M(., g)$ is β-relaxed accretive regarding g with $\beta = \frac{7}{4}$, and so $M(., .)$ is $\alpha\beta$-symmetric accretive regarding f and g. Since for every real constant $\rho > 0$, $[H((A, B), (C, D)) + \rho M(f, g)](\mathbb{R}^2) = \mathbb{R}^2$, they concluded that M is an $\alpha\beta$-$H((., .), (., .))$-mixed accretive with respect to $(A, B), (C, D)$ and (f, g). Let us define the mappings $\widehat{H}, \widehat{M} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\widehat{H}(x) := H((Ax, Bx), (Cx, Dx))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$. Then, for all $x \in E$, we have $\widehat{H}(x) = Ax + Bx + Cx + Dx = 4x$ and

$$\widehat{M}(x) = f(x) - g(x) = \begin{pmatrix} 5x_1 - \frac{2}{3}x_2 \\ \frac{2}{3}x_1 + 5x_2 \end{pmatrix} - \begin{pmatrix} \frac{7}{4}x_1 + \frac{3}{4}x_2 \\ -\frac{3}{4}x_1 + \frac{7}{4}x_2 \end{pmatrix} = \begin{pmatrix} \frac{13}{4}x_1 - \frac{17}{12}x_2 \\ \frac{17}{12}x_1 + \frac{13}{4}x_2 \end{pmatrix}.$$

Since for all $x, y \in \mathbb{R}^2$,

$$\langle Ax - Ay, J_2(x - y)\rangle = \langle 4x - 4y, x - y\rangle = 4\|x - y\|^2$$

and

$$\|Bx - By\| = \| - 3x + 3y\| = 3\|x - y\|,$$

it follows that A is 4-expansive and B is 3-Lipschitz continuous. Now taking $\alpha_1 = 4$ and $\beta_1 = 3$, in the light of the fact that $\mu_1\alpha_1^2 - \mu_2\beta_1^2 = 1 > 0$, from Proposition 4.2(i) it is expected that the mapping $\widehat{H}$ to be $(\mu_1\alpha_1^2 - \mu_2\beta_1^2) = 1$-strongly accretive. At the same time, invoking Proposition 4.2(ii), we expect that $\widehat{H}$ to be $\tau = 4$-Lipschitz continuous. The facts that for all $x, y \in \mathbb{R}^2$,

$$\langle \widehat{H}(x) - \widehat{H}(y), J_2(x - y)\rangle = \langle 4x - 4y, x - y\rangle = 4\|x - y\|^2 \geq \|x - y\|^2$$

and

$$\|\widehat{H}(x) - \widehat{H}(y)\| = 4\|x - y\|$$

confirm our expectations. Moreover, owing to the fact that $\alpha > \beta$, from Proposition 4.4 it is expected that the mapping $\widehat{M}$ to be $\alpha - \beta = \frac{13}{4}$-strongly accretive. The fact that for all $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, $u \in \widehat{M}(x)$

and $v \in \widehat{M}(y)$,

$$\begin{aligned}\langle u - v, J_2(x - y)\rangle &= \langle \widehat{M}(x) - \widehat{M}(y), x - y\rangle \\ &= \left\langle \begin{pmatrix} \frac{13}{4}(x_1 - y_1) - \frac{17}{12}(x_2 - y_2) \\ \frac{17}{12}(x_1 - y_1) + \frac{13}{4}(x_2 - y_2) \end{pmatrix}, \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\rangle \\ &= \frac{13}{4}(x_1 - y_1)^2 + \frac{13}{4}(x_2 - y_2)^2 \\ &= \frac{13}{4}\|x - y\|^2\end{aligned}$$

confirms the expectation. In virtue of the above-mentioned arguments, we infer that our observations are compatible with our derived assertions in Propositions 4.2 and 4.4.

Proposition 4.8. [15, Proposition 3.4] *Let M be an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping regarding (A, C), (B, D) and (f, g). Let Assumptions (a_1) and (a_2) hold with $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$. If the following inequality $\langle u - v, J_q(x - y)\rangle \geq 0$ satisfied for all $(y, v) \in \mathrm{Graph}(M(f, g))$, then $(x, u) \in \mathrm{Graph}(M(f, g))$, where $\mathrm{Graph}(M(f, g)) = \{(x, u) \in E \times E : u \in M(f(x), g(x))\}$.*

Proof. Assume that the mappings $\widehat{H} : E \to E$ and $\widehat{M} : E \multimap E$ are defined by $\widehat{H}(x) := H((Ax, Bx), (Cx, Dx))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$, respectively. Taking into account that $\alpha_1 > \beta_1$, $\mu_1 > \mu_2$ and $\gamma_1, \gamma_2 > 0$, from Proposition 4.2(i) it follows that the mapping $\widehat{H}$ is $(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-strongly accretive and so it is a strictly accretive mapping. Since $\alpha > \beta$, Proposition 4.4 ensures that $\widehat{M}$ is $(\alpha - \beta)$-strongly accretive and as it was pointed out in Remark 4.6, $\widehat{M}$ is a $\widehat{H}$-accretive mapping. We now note that all the conditions of Lemma 2.10 are satisfied and so Lemma 2.10 implies that $(x, u) \in \mathrm{Graph}(\widehat{M}) = \mathrm{Graph}(M(f, g))$. This gives the desired result. □

It is significant to emphasize that there are some small mistakes in the context of Proposition 3.4 of [15]. In fact, in the context of the mentioned proposition, (u, x) and (v, y) must be replaced by (x, u) and (y, v), respectively, as we have done in the context of Proposition 4.8.

With the aim of defining the proximal-point mapping associated with an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping, Gupta et al. [15] gave the following conclusion in which the required conditions for the mapping $(H(A, B), (C, D)) + \rho M(f, g))^{-1}$ to be single-valued for every $\rho > 0$ are stated.

Theorem 4.9. [15, Theorem 3.5] *Let M be an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping regarding (A, C), (B, D) and (f, g). If Assumptions (a_1) and (a_2) hold with $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$, then the mapping $(H((A, B), (C, D)) + \rho M(f, g))^{-1}$ is single-valued.*

Proof. Define the mappings $\widehat{H} : E \to E$ and $\widehat{M} : E \multimap E$ for all $x \in E$, respectively, by $\widehat{H}(x) := H((Ax, Bx), (Cx, Dx))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$. The same argument used in the proof of Proposition 4.8 shows that $\widehat{H}$ is $(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-strongly accretive and so it is a strictly accretive mapping, and $\widehat{M}$ is a $\widehat{H}$-accretive mapping. Then, all the conditions of Corollary 2.13 are satisfied and so according to Corollary 2.13 the mapping $(\widehat{H} + \rho\widehat{M})^{-1} = (H((A, B), (C, D)) + \rho M(f, g))^{-1}$ is single-valued for every constant $\rho > 0$. The proof is finished. □

Based on Theorem 4.9 (that is, [15, Theorem 3.5]), Gupta et al. [15] defined the proximal-point mapping $R^{H((.,.),(.,.))}_{\rho,M(.,.)}$ associated with an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping M and an arbitrary real constant $\rho > 0$ as follows.

Definition 4.10. [15, Definition 3.6] *Let M be an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping regarding (A, C), (B, D) and (f, g). If Assumptions (a_1) and (a_2) hold with $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$, then the proximal-point mapping $R^{H((.,.),(.,.))}_{\rho,M(.,.)} : E \to E$ is defined by*

$$R^{H((.,.),(.,.))}_{\rho,M(.,.)}(u) = (H((A, B), (C, D)) + \rho M(f, g))^{-1}(u), \quad \forall u \in E.$$

It is worth noting that by defining the mappings $\widehat{H} : E \to E$ and $\widehat{M} : E \multimap E$ as $\widehat{H}(x) := H((Ax, Bx), (Cx, Dx))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$, thanks to the assumptions of Definition 4.10, Propositions 4.2 and 4.4 imply that $\widehat{H}$ is a strictly accretive mapping and $\widehat{M}$ is a $\widehat{H}$-accretive mapping. Then, in the light of Definition 2.15, for any real constant $\rho > 0$, the proximal-point mapping $R^{\widehat{H}}_{\rho,\widehat{M}} = R^{H((.,.),(.,.))}_{\rho,M(.,.)} : E \to E$ associated with a $\widehat{H}$-accretive mapping $\widehat{M}$ ($\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping M) is defined by

$$R^{H((.,.),(.,.))}_{\rho,M(.,.)}(u) = R^{\widehat{H}}_{\rho,\widehat{M}}(u) = (\widehat{H} + \rho\widehat{M})^{-1}(u) = (H((A, B), (C, D)) + \rho M(f, g))^{-1}(u), \quad \forall u \in E.$$

In view of the above-mentioned arguments, it is very essential to note that the concept of the proximal-point mapping $R^{H((.,.),(.,.))}_{\rho,M(.,.)}$ associated with an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping M and an arbitrary real constant $\rho > 0$ given in Definition 4.10 is actually the same notion of the proximal-point mapping $R^{\widehat{H}}_{\rho,\widehat{M}}$ associated with $\widehat{H}$-accretive mapping $\widehat{M}$ and real constant $\rho > 0$ given in Definition 2.15 and is not a new one.

In order to prove the Lipschitz continuity of the proximal-point mapping $R^{H((.,.),(.,.))}_{\rho,M(.,.)}$ and to compute an estimate of its Lipschitz constant, the authors closed section 3 in [15] with the following assertion.

Theorem 4.11. [15, Theorem 3.7] *Let $M : E \times E \multimap E$ be an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping with respect to $(A, C), (B, D)$ and (f, g). If Assumptions (a_1) and (a_2) hold with $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$, then the proximal-point mapping $R^{H((.,.),(.,.))}_{\rho,M(.,.)} : E \to E$ is $\frac{1}{l+\rho k}$-Lipschitz continuous, that is,*

$$\|R^{H((.,.),(.,.))}_{\rho,M(.,.)}(u) - R^{H((.,.),(.,.))}_{\rho,M(.,.)}(v)\| \leq \frac{1}{l + \rho k}\|u - v\|, \quad \forall u, v \in E,$$

where $l = \mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2$ and $k = \alpha - \beta$.

Proof. Let us define the mappings $\widehat{H} : E \to E$ and $\widehat{M} : E \multimap E$ by $\widehat{H}(x) := H((Ax, Bx), (Cx, Dx))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$, respectively. By using an argument similar to that in the proof of Proposition 4.8, it follows that the mapping $\widehat{H}$ is $(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-strongly accretive and $\widehat{M}$ is a $\widehat{H}$-$(\alpha - \beta)$-strongly accretive mapping. Now, taking $l = \mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2$ and $k = \alpha - \beta$, Theorem 2.16 implies that the proximal-point mapping $R^{\widehat{H}}_{\rho,\widehat{M}} = R^{H((.,.),(.,.))}_{\rho,M(.,.)} : E \to E$ is $\frac{1}{l+\rho k} = \frac{1}{\mu_1\alpha_1^q-\mu_2\beta_1^q+\gamma_1+\gamma_2+\rho(\alpha-\beta)}$-Lipschitz continuous, i.e.,

$$\begin{aligned}
\|R^{H((.,.),(.,.))}_{\rho,M(.,.)}(u) - R^{H((.,.),(.,.))}_{\rho,M(.,.)}(v)\| &= \|R^{\widehat{H}}_{\rho,\widehat{M}}(u) - R^{\widehat{H}}_{\rho,\widehat{M}}(v)\| \\
&\leq \frac{1}{l + \rho k}\|u - v\| \\
&= \frac{1}{\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2 + \rho(\alpha - \beta)}\|u - v\|, \quad \forall u, v \in E.
\end{aligned}$$

This completes the proof. □

Let $S, T : E \multimap CB(E)$ be the set-valued mappings, and let $A, B, C, D, f, g : E \to E$, $F : E \times E \to E$ and $H : (E \times E) \times (E \times E) \to E$ be single-valued mappings. Suppose that the set-valued mapping $M : E \times E \multimap E$ is an $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping regarding $(A, C), (B, D)$ and (f, g). Recently, Gupta et al. [15] considered and studied the following generalized set-valued variational inclusion:

For given $\lambda \in E$, find $u \in E$, $v \in S(u)$ and $w \in T(u)$ such that

$$\lambda \in F(v, w) + M(f(u), g(u)). \tag{25}$$

Using the notion of the proximal-point mapping $R^{H((.,.),(.,.))}_{\rho,M(.,.)}$, they presented a characterization of a solution of problem (25) as follows.

Lemma 4.12. [15, Lemma 4.1] *Let $A, B, C, D, f, g, F, S, T, H, M$ and λ be the same as in problem* (25) *and let Assumptions* (a_1) *and* (a_2) *hold such that $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$. Then $u \in E$, $v \in S(u)$ and $w \in T(u)$ is a solution of problem* (25)*([15, problem 8]) if and only if $u \in E$, $v \in S(u)$ and $w \in T(u)$ satisfy the following relation:*

$$u = R^{H((.,.),(.,.))}_{\rho,M(.,.)}[H((Au, Bu),(Cu, Du)) - \rho F(v, w) + \rho\lambda],$$

where $\lambda > 0$ is a constant.

Proof. Defining the mappings $\widehat{H} : E \to E$ and $\widehat{M} : E \multimap E$, respectively, by $\widehat{H}(x) := H((Ax, Bx),(Cx, Dx))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$, with the help of the assumptions and Propositions 4.2 and 4.4, by a same way as the proof of Proposition 4.8, we conclude that $\widehat{H}$ is a strictly accretive mapping and $\widehat{M}$ is a $\widehat{H}$-accretive mapping. Then problem (25) becomes actually the same problem (5). We now infer that all the conditions of Lemma 3.2 are satisfied and so in the light of Lemma 3.2, $(u, v, w) \in E \times S(u) \times T(u)$ is a solution of problem (5) (and so it is a solution of problem (25)) if and only if (u, v, w) satisfies the relation

$$u = R^{\widehat{H}}_{\rho,\widehat{M}}[\widehat{H}(u) - \rho F(v, w) + \rho\lambda] = R^{H((.,.),(.,.))}_{\rho,M(.,.)}[H((Au, Bu),(Cu, Du)) - \rho F(v, w) + \rho\lambda],$$

where $\rho > 0$ is a constant and $R^{H((.,.),(.,.))}_{\rho,M(.,.)} = R^{\widehat{H}}_{\rho,\widehat{M}} = (\widehat{H} + \rho\widehat{M})^{-1} = (H((A, B),(C, D)) + \rho M(f, g))^{-1}$. □

With the help of Lemma 4.12 and Nadler's technique [28], Gupta et al. [15] suggested an iterative algorithm for finding an approximate solution of problem (25) as follows.

Algorithm 4.13. [15, Algorithm 4.3] *Let $A, B, C, D, f, g, F, S, T, H, M$ and λ be the same as in problem* (25) *and let Assumptions* (a_1) *and* (a_2) *hold such that $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$. For any given $z_0 \in E$, we can choose $u_0 \in E$ such that the sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ satisfy*

$$\begin{cases} u_n = R^{H((.,.),(.,.))}_{\rho,M(.,.)}(z_n), \\ v_n \in S(u_n) : \|v_n - v_{n+1}\| \le (1 + \frac{1}{n+1})D(S(u_n), S(u_{n+1})), \\ w_n \in T(u_n) : \|w_n - w_{n+1}\| \le (1 + \frac{1}{n+1})D(T(u_n), T(u_{n+1})), \\ z_{n+1} = H((Au_n, Bu_n),(Cu_n, Du_n)) - \rho F(v_n, w_n) + \rho\lambda + e_n, \\ \sum\limits_{j=1}^{\infty} \|e_j - e_{j-1}\|\varpi^{-j} < \infty, \quad \forall \varpi \in (0, 1), \lim\limits_{n\to\infty} e_n = 0, \end{cases}$$

where $\rho > 0$ is a constant, $\lambda \in E$ is any given element and $\{e_n\}_{n=0}^{\infty} \subset E$ is an error to take into account a possible inexact computation of the proximal-point mapping point for all $n \ge 0$, and $D(.,.)$ is the Huasdorff metric on $CB(E)$.

Remark 4.14. It is important to emphasize that

(i) thanks to Theorem 4.9 and Definition 4.10, the conditions mentioned in Assumptions (a_1) and (a_2) with $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$ are essential for the proximal-point mapping $R^{H((.,.),(.,.))}_{\rho,M(.,.)}$ to be single-valued and must not be dropped from the contexts of Lemma 4.1 and Algorithm 4.3 of [15]. Hence, the mentioned conditions must be added to the contexts of Lemma 4.1 and Algorithm 4.3 of [15], as it is done in Lemma 4.12 and Algorithm 4.13;

(ii) contrary to the claim in [15], the characterization of the solution for problem (25) involving $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping M given in Lemma 4.12 is exactly the same characterization of the solution for problem (5) involving $\widehat{H}$-accretive mapping $\widehat{M}$ presented in Lemma 3.2, and is not a new one;

(iii) by defining the mappings $\widehat{H} : E \to E$ and $\widehat{M} : E \multimap E$, respectively, as $\widehat{H}(x) := H((Ax, Bx),(Cx, Dx))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$, by the same argument used in Proposition 4.8, and applying the assumptions and Propositions 4.2 and 4.4 we conclude that $\widehat{H}$ is a strictly accretive mapping and $\widehat{M}$ is a $\widehat{H}$-accretive mapping. Thereby, Algorithm 4.13 (that is, [15, Algorithm 4.3]) is actually the same Algorithm 3.5 and is not a new one.

Under some suitable conditions, Gupta et al. [15] proved the strong convergence of the sequence $\{(u_n, v_n, w_n)\}_{n=0}^{\infty}$ generated by Algorithm 4.13 to a solution of problem (25), and in support of this assertion which is the most important result derived in [15], the authors presented Example 4.7 in [15].

Theorem 4.15. [15, Theorem 4.6] *Let us consider problem* (25) *and assume that*

(i) *S and T are l_1- and l_2-D-Lipschitz continuous, respectively;*
(ii) *$H((A,B),(C,D))$ is τ-mixed Lipschitz continuous regarding A, B, C and D;*
(iii) *F is σ-strongly accretive regarding S and $H((A,B),(C,D))$ in the first component and δ-strongly accretive regarding T and $H((A,B),(C,D))$ in the second component;*
(iv) *F is ϵ_1, ϵ_2-Lipschitz continuous in the first and second components, respectively;*
(v) $0 < \sqrt[q]{\tau^q + c_q\rho^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q - \rho q(\sigma+\delta)\tau^q} < l + \rho k$,

where $l = \mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2$, $k = \alpha - \beta$, $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2, \rho > 0$. Then, problem 25 (*that is, [15, problem 8]*) *has a solution (u, v, w), where $u \in E$, $v \in S(u)$ and $w \in T(u)$, and the iterative sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ generated by Algorithm 4.13 (that is, [15, Algorithm 4.3]) converge strongly to u, v and w, respectively.*

Here it is to be noted that by a careful reading the proof of Theorem 4.6 in [15], we found that there is a fatal error in it. In fact, under the conditions of σ-strong accretivity of F regarding S and $H((A,B),(C,D))$ in the first component, δ-strong accretivity of F regarding T and $H((A,B),(C,D))$ in the second component, and τ-mixed Lipschitz continuity of $H((A,B),(C,D))$ regarding A, B, C and D, the authors [15] derived relation (20) in it as follows:

$$\begin{aligned}
&\langle F(v_n, w_n) - F(v_{n-1}, w_{n-1}), J_q(H((Au_n, Bu_n),(Cu_n, Du_n)) \\
&\quad - H((Au_{n-1}, Bu_{n-1}),(Cu_{n-1}, Du_{n-1})))\rangle \\
&\leq \langle F(v_n, w_n) - F(v_{n-1}, w_n), J_q(H((Au_n, Bu_n),(Cu_n, Du_n)) \\
&\quad - H((Au_{n-1}, Bu_{n-1}),(Cu_{n-1}, Du_{n-1})))\rangle \\
&\quad + \langle F(v_{n-1}, w_n) - F(v_{n-1}, w_{n-1}), J_q(H((Au_n, Bu_n),(Cu_n, Du_n)) \\
&\quad - H((Au_{n-1}, Bu_{n-1}),(Cu_{n-1}, Du_{n-1})))\rangle \\
&\leq (\sigma+\delta)\|H(Au_n, Bu_n),(Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}),(Cu_{n-1}, Du_{n-1}))\|^q \\
&\leq (\sigma+\delta)\tau^q\|u_n - u_{n-1}\|^q.
\end{aligned} \tag{26}$$

But, by using (ii) and (iii) mentioned in the context of Theorem 4.15, what we obtain is not (26) (that is, [15, relation (20)]). In fact, by (iii) it yields

$$\begin{aligned}
&\langle F(v_n, w_n) - F(v_{n-1}, w_{n-1}), J_q(H((Au_n, Bu_n),(Cu_n, Du_n)) \\
&\quad - H((Au_{n-1}, Bu_{n-1}),(Cu_{n-1}, Du_{n-1})))\rangle \\
&\leq \langle F(v_n, w_n) - F(v_{n-1}, w_n), J_q(H((Au_n, Bu_n),(Cu_n, Du_n)) \\
&\quad - H((Au_{n-1}, Bu_{n-1}),(Cu_{n-1}, Du_{n-1})))\rangle \\
&\quad + \langle F(v_{n-1}, w_n) - F(v_{n-1}, w_{n-1}), J_q(H((Au_n, Bu_n),(Cu_n, Du_n)) \\
&\quad - H((Au_{n-1}, Bu_{n-1}),(Cu_{n-1}, Du_{n-1})))\rangle \\
&\geq (\sigma+\delta)\|H(Au_n, Bu_n),(Cu_n, Du_n)) - H((Au_{n-1}, Bu_{n-1}),(Cu_{n-1}, Du_{n-1}))\|^q.
\end{aligned} \tag{27}$$

It can be easily observed that (ii) and (27) do not imply (26). In order to resolve this problem, the mapping $H((A,B),(C,D))$, in addition to the conditions mentioned in the context of Theorem 4.15 must be satisfied in an additional condition. In fact, it is sufficient that the mapping $H((A,B),(C,D))$ is expansive. In the light of the above-mentioned discussion, the assumptions appeared in the context of Theorem 4.6 of [15] do not ensure the assertion.

In the following a correct version of [15, Theorem 4.6] is given and then with the help of our main result derived in Section 3, an easy proof for it is provided. For this aim, we need to present the following concept.

Definition 4.16. *For given mappings $A, B, C, D : E \to E$, a mapping $H : (E \times E) \times (E \times E) \to E$ is said to be ς-expansive with respect to A, B, C and D if there exists a constant $\varsigma > 0$ such that*

$$\|H((Ax, Bx), (Cx, Dx)) - H((Ay, By), (Cy, Dy))\| \geq \varsigma\|x - y\|, \quad \forall x, y \in E.$$

Clearly, it is expansive with respect to A, B, C and D if and only if $\varsigma = 1$.

Theorem 4.17. *Let $A, B, C, D, f, g, F, S, T, H, M$ and λ be the same as in problem* (25) *and let Assumptions* (a_1) *and* (a_2) *hold such that $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$. Suppose further that*

(i) *S and T are l_1- and l_2-D-Lipschitz continuous, respectively;*
(ii) *$H((A, B), (C, D))$ is ς-expansive and τ-mixed Lipschitz continuous with respect to A, B, C and D;*
(iii) *F is σ-strongly accretive with respect to S and $H((A, B), (C, D))$ in the first argument and δ-strongly accretive with respect to T and $H((A, B), (C, D))$ in the second argument, respectively;*
(iv) *F is ϵ_1, ϵ_2-Lipschitz continuous in the first and second arguments, respectively;*
(v) *there exists a constant $\rho > 0$ such that*

$$\sqrt[q]{\tau^q - q\rho(\sigma + \delta)\varsigma^q + c_q\rho^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q} < l + \rho k,$$

where $l = \mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2$, $k = \alpha - \beta$, $\alpha > \beta$, $\mu_1 > \mu_2$, $\alpha_1 > \beta_1$ and $\gamma_1, \gamma_2 > 0$, c_q is a constant guaranteed by Lemma 2.3, and for the case when q is an even natural number, in addition to the above inequality, the constant ρ satisfies the following condition:

$$q\rho(\sigma + \delta)\varsigma^q < \tau^q + c_q\rho^q(\epsilon_1 l_1 + \epsilon_2 l_2)^q.$$

Then, the iterative sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ generated by Algorithm 4.13 converge strongly to u, v and w, respectively, and (u, v, w) is a solution of problem (25).

Proof. Define the mappings $\widehat{H} : E \to E$ and $\widehat{M} : E \multimap E$ as $\widehat{H}(x) := H((Ax, Bx), (Cx, Dx))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in E$, respectively. In the light of the assumptions, and applying Propositions 4.2 and 4.4 it follows that $\widehat{H}$ is a $(\mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2)$-strongly accretive and τ-Lipschitz continuous mapping, and $\widehat{M}$ is a $\widehat{H}$-$(\alpha - \beta)$-strongly accretive mapping. At the same time, $\widehat{H}$ is a ς-expansive mapping, and problem (25) and Algorithm 4.13 become actually the same problem (5) and Algorithm 3.5, respectively. Now, taking $l = \mu_1\alpha_1^q - \mu_2\beta_1^q + \gamma_1 + \gamma_2$ and $k = \alpha - \beta$, we note that all the conditions of Corollary 3.10 are satisfied and so the assertion follows from Corollary 3.10 immediately. □

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