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On the convergence of Ishikawa iterates defined by nonlinear quasi-contractions

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Abstract. In this study, we establish the convergence of Ishikawa iterates defined by nonlinear quasi-contractive mappings on TVS-cone metric space. Further, our results generalize many existing results in the literature.

1. Introduction

In last 15 years generalizations of Banach contraction mapping principle on cone metric spaces and their applications, have important role in nonlinear analysis. For references and historical notes see [1, 2].

Lj. Ćirić [7] first introduced the notion of quasi-contraction and proved the fixed point theorem for this class of mappings. The result of Ćirić was extended to cone metric spaces by D. Ilić and V. Rakočević [14] and Z. Kadelburg, S. Radenović and V. Rakočević [18]. In first of this papers authors considered cone metric spaces with normal cone. In second paper normality condition was omitted but authors in proof existence of fixed point result use strong assumptions that coefficient of quasi-contractivity is less then $\frac{1}{2}$. The fixed point theorem of Ćirić was extended to nonlinear quasi-contractions on metric spaces by A. A. Ivanov [16], J. Danes [10] and I. Aranđelović, M. Rajović and V. Kilibarda [3]. Fixed point theorem for nonlinear quasi-contractions defined on cone metric spaces was presented in [2].

In modern nonlinear analysis and its applications theory of Ishikawa iterates (see [15, 19]) has important role because it can be applied in many cases in which classical methods (convergence of Picard or Mann's iterations) is not useful. In many papers Ishikawa iteration sequence was used as tool to obtain approximative fixed points of nonexpansive and pseudo-contractive mappings defined on Hilbert and Banach spaces.

First result on the convergence of Ishikawa iterates defined by quasi-contractions was obtained by L. Qihou [27] for Hilbert spaces. This result is generalized in L. Qihou [22] for non-compact subset of a Hilbert

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space, C. E. Chidume [5] for Banach spaces L^p and ℓ^p $(2 < p)$, C. E. Chidume [6] for Banach spaces L^p and Xu [28] for arbitrary Banach space. B. E. Rhoades [23] proved first result on the convergence of Ishikawa iterates defined by nonlinear quasi-contractions. In [8] Lj. B. Ćirić proved that results of Xu and Rhoades is uncorrect. He proved general results for nonlinear quasi-contractions defined on Takahashi convex metric space [8, 9]. I. Aranđelović, M. Rajović and V. Kilibarda [4] obtained further generalization of Ćirić's results.

Recently V.-S. Du [11] introduced the notion of TVS-cone metric space. Fixed point result for nonlinear quasi-contractions defined on this class of spaces was proved by I. Aranđelović, D. Kečkić [2].

In this paper, we present further generalizations of Ćirić's results [8, 9] and main result of [4] to nonlinear quasi-contractive mappings defined on TVS-cone metric space.

2. Preliminary Notes

In 1928, K. Menger [20] introduce the notion of convexity in metric space.

Definition 2.1. *Let (X, d) be a metric space, such that for each $x, y \in X$ with $x \neq y$ there exists $z \in X \backslash \{x, y\}$ such that $d(x, z) + d(z, y) = d(x, y)$. Then X is called a convex metric space.*

Menger's definition includes many geometric properties of Euclidian convex sets, but it is restrictive in fixed point applications.

In 1970, W. Takahashi [26] introduced the new concept of convexity in metric space and generalized some important fixed point theorems previously proved for Banach spaces.

Definition 2.2. *Let (X, d) be a metric space and $I = [0, 1]$ the closed unit interval. A Takahashi convex structure on X is a function $W : X \times X \times I \to X$ which has the property that for every $x, y \in X$ and $\lambda \in I$*

$$d(z, W(x, y, \lambda) \leq \lambda d(z, x) + (1 - \lambda)d(z, y)$$

for every $z \in X$.

If (X, d) is equipped with a Takahashi convex structure W, then (X, d, W) is called a Takahashi convex metric space, or a metric space of hyperbolic type.

The term "metric space of hyperbolic type"is introduced by W.A. Kirk [19] and today is widely used.

Let (X, d) be a Takahashi convex metric space and

$$m(x, y) = W(x, y, \frac{1}{2}),$$

for any $x, y \in X$. From definition follows

$$d(m(x, z), m(y, z)) \leq \frac{1}{2} d(x, y).$$

Let $m(x, y)$ be a midpoint straight line segment, which links ponts x and y. Then in hyperbolic geometry we have strict inequality, while in parabolic geometry it is equality. This implies that "metric space of non-elliptic typečould be more appropriate term than "metric space of hyperbolic type".

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be open unite disc in complex plane, $\overline{U}$ its closure, and $\theta, \theta : U^2 \to U$ be mappings defined by

$$\theta(z_1, z_2) = |\frac{z_1 - z_2}{1 + \overline{z_2} z_1}|,$$

$$\rho(z_1, z_2) = \ln \frac{1 + \theta(z_1, z_2)}{1 - \theta(z_1, z_2)}.$$

Then ρ is hyperbolic metric in Poincaré model of Lobachevsky plane defined on U (see [13]) and (U, ρ, W_h) is metric space of hyperbolic type, where $W_h(z_1, z_2, \lambda)$ is uniqe point on straight line segment (in sense of Lobachevsky), which links ponts z_1 and z_2 such that $\rho(z_1, W_h(z_1, z_2, \lambda)) = \lambda\rho(z_1, z_2)$ and $\rho(z_2, W_h(z_1, z_2, \lambda)) =$

$(1 - \lambda)\rho(z_1, z_2)$. So, we get that function W_h defined Takahashi convex structure on $[0, \frac{1}{2}] \cap U$. Hence, $([0, \frac{1}{2}], \rho, W_h)$ is Takahashi convex metric space.

Let X be a nonempty set and $f : X \to X$ arbitrary mapping. $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that a sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) defined is the orbit of f at point x_0.

Let (α_n) and (β_n) be two sequences of real numbers. Let (X, d) be a Takahashi convex metric space and W Takahashi convex structure on X. Then for arbitrary $x_0 \in X$ sequence of Ishikawa iterates of $(f, (\alpha_n), (\beta_n))$ at point x_0 is defined by:

$$\begin{cases} y_n = W(f(x_n), x_n, \beta_n) \\ x_{n+1} = W(f(y_n), x_n, \alpha_n). \end{cases}$$

Let E be a linear topological space. By Θ we denote the zero element of E. A subset P of E is called a cone if:

1) P is closed, nonempty and $P \neq \{\Theta\}$;

2) $a, b \in \mathbf{R}$, $a, b > 0$, and $x, y \in P$ imply $ax + by \in P$;

3) $P \cap (-P) = \{\Theta\}$.

Given a cone $P \subseteq E$, we define partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ and $x \neq y$, whereas $x \ll y$ will stand for $y - x \in \mathrm{int}P$ (interior of P).

Let E be a linear topological space and let $P \subseteq E$ be a cone. We say that P is a solid cone if and only if $\mathrm{int}P \neq \emptyset$.

In the following we always suppose that E is a locally convex Hausdorff topological vector space, P is a solid cone in E such that $\leq$ is partial ordering on E with respect to P. By I we denote identity operator on E i.e. $I(x) = x$ for each $x \in E$.

Let $e \in \mathrm{int}P$. The nonlinear scalarization function ξ_e defined by

$$\xi_e(y) = \inf\{t \in \mathbf{R} : y \in te - P\},$$

for all $y \in Y$, has the following properties (see [11]):

1) ξ_e is positively homogeneous and continuous on Y;

2) if $y_1 \in y_2 + P$, then $\xi_e(y_2) \leq \xi_e(y_1)$;

3) ξ_e is sub-additive on Y.

In [2] the next statement was proved.

Lemma 2.3. *(Aranđelović, Kečkić [2]) Let $e \in \mathrm{int}P$, and let $\xi_e(x) = \inf\{t \in \mathbf{R} : te - x \in P\}$ be its scalarization function.*

a) Let $\Theta \ll x \ll \lambda e$, for some real $\lambda > 0$. Then $0 \leq \xi_e(x), -\xi_e(-x) < \lambda$.

b) The function $E \ni x \mapsto \|x\| = \max\{|\xi_e(x)|, |\xi_e(-x)|\}$ is a seminorm on a locally convex topological space considered over real field.

Definition 2.4. *Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:*

1) $\Theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \Theta$ if and only if $x = y$;

2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a TVS-cone metric on X and (X, d) is called a TVS-cone metric space.

Cone metric spaces in Huang-Zhang sense [12] are included in our definition since every Banach space is locally convex Hausdorff topological vector space.

The following definitions were introduced in [12]. Let (X, d) be a solid TVS-cone metric space, $x \in X$ and (x_n) a sequence in X. Then

1) (x_n) TVS-cone converges to x if for every $c \in \mathrm{int}P$ there exists a positive integer N such that for all $n \geq N$ $d(x_n, x) \ll c$, we denote this by $\lim x_n = x$ or $x_n \to x$;

2) (x_n) is a TVS-cone Cauchy sequences if for every $c \in \mathrm{int}P$ there exists a positive integer N such that for all $m, n \geq N$ $d(x_m, x_n) \ll c$;

3) (X, d) is a TVS-cone complete cone metric space if every Cauchy sequence is convergent.

Let (X, d) be a TVS-cone metric space and $a \in \mathrm{int}P$. In [11] W.-S. Du proved that for any $e \in \mathrm{int}P$:

1) Function $d_e : X \times X \to [0, +\infty)$ defined by $d_e = \xi_e \circ d$ is a metric;

2) If (x_n) TVS-cone converges to x, then $\lim d_e(x_n, x) = 0$;

3) If (x_n) is a TVS-cone Cauchy sequences then (x_n) is a Cauchy sequences (in usual sense) in (X, d_e);

4) If (X, d) is a TVS-cone complete cone metric space, then (X, d_e) is complete metric space.

In [2] the next statement was proved.

Lemma 2.5. *Let E be a TVS space, let $P \subseteq E$ be a cone. A sequence $x_n \in E$ converges to $x \in E$ in TVS metric d, if and only if $x_n \to x$ in all metrics d_e, where e runs through* $\mathrm{int}(P)$*. Also, x_n is a TVS-cone Cauchy sequence if and only if x_n is a Cauchy sequence in all metrics d_e.*

By Φ we denote the set of all real functions $\varphi : [0, \infty) \to [0, \infty)$ which have the following properties:

(a) $\varphi(0) = 0$;

(b) $\varphi(x) < x$ for all $x > 0$;

(c) $\lim_{x\to\infty}(x - \varphi(x)) = \infty$.

Define

$$\Phi_1 = \{\varphi \in \Phi : \varphi \text{ is monotone nondecreasing and } \overline{\lim}_{t\to r+}\varphi(t) < r \text{ for any } r > 0\}$$

and

$$\Phi_2 = \{\varphi \in \Phi : \overline{\lim}_{t\to r}\varphi(t) < r \text{ for any } r > 0\}.$$

Let X be a nonempty set and $f : X \to X$ an arbitrary mapping. The element $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that the sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) is the orbit of f at point x_0.

In [4] authors presented the following fixed point theorem which generalized earlier results obtained by L. Qihou [22, 27], C. E. Chidume [5, 6] and Ćirić [8, 9].

Theorem 2.6. *Let (X, d, W) be a Takahashi convex complete metric space, $f : X \to X$ and (α_n), (β_n) two sequences of real numbers such that $0 < \alpha_n, \beta_n < 1$ for all n and $\sum \alpha_n = \infty$. If there exist $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \in \Phi_2$ such that*

$$d(f(x), f(y)) \le \max\{\varphi_1(d(x, y)), \varphi_2(d(x, f(x))), \varphi_3(d(y, f(y))), \varphi_4(d(x, f(y))), \varphi_5(d(f(x), y))\},$$

for any $x, y \in X$, then for arbitrary $x_0 \in X$ sequence of Ishikawa iterates of $(f, (\alpha_n), (\beta_n))$ at point x_0 is defined by:

$$\begin{cases} y_n = W(f(x_n), x_n, \beta_n) \\ x_{n+1} = W(f(y_n), x_n, \alpha_n). \end{cases}$$

converge to unique fixed point of f.

Remark. In [4] condition $\varphi_1, \ldots, \varphi_5 \in \Phi_1 \cup \Phi_2$ be used. But, if φ is nondecreasing, then $\overline{\lim}_{t\to r-}\varphi(t) \le \varphi(r) < r$. So, $\Phi_1 \subseteq \Phi_2$. Indeed, Thus, the condition can be simplified to $\varphi_1, \ldots, \varphi_5 \in \Phi_2$.

3. Main Results

Definition 3.1. *Let (X, d) be a TVS cone metric space and $I = [0, 1]$ the closed unit interval. A Takahashi convex structure on X is a function $W : X \times X \times I \to X$ which has the property that for every $x, y \in X$ and $\lambda \in I$*

$$d(z, W(x, y, \lambda)) \le \lambda d(z, x) + (1 - \lambda)d(z, y)$$

for every $z \in X$.

If (X, d) is equipped with a Takahashi convex structure W, then (X, d, W) is called a Takahashi convex TVS cone metric space.

Let (α_n) and (β_n) be two sequences of real numbers. Let (X, d) be a Takahashi convex metric space and W Takahashi convex structure on X. Then for arbitrary $x_0 \in X$ sequence of Ishikawa iterates of $(f, (\alpha_n), (\beta_n))$ at point x_0 is defined by:

$$\begin{cases} y_n = W(f(x_n), x_n, \beta_n) \\ x_{n+1} = W(f(y_n), x_n, \alpha_n). \end{cases}$$

By Ψ we denote the set of all functions $\psi : P \to P$ which have the following properties:
(a) $\psi(\Theta) = \Theta$;
(b) $(I - \psi)(\mathrm{int}(P)) \subseteq \mathrm{int}P$;
(c) $\lim_{t\to\infty}(\|tx\| - \|\psi(tx)\|) = \infty$, for any $x \in P \setminus \{\Theta\}$, where $\|.\|$ is an arbitrary semi-norm on E;

For a function $\psi \in \Psi$ we say that $\psi \in \Psi_2$ if for any $x \in \mathrm{int}P$ and for each $(x_n) \subseteq \mathrm{int}P$ such that $x_n \to x$, there exists a positive integer n_0 such that $n > n_0$ implies $\psi(x_n) \ll (1 - \varepsilon)x$, here ε does not depend on the choice of the sequence (x_n).

Now we shall prove our main result, which is a generalization of earlier results presented in [4, 6, 9].

Theorem 3.2. *Let (X, d, W) be a Takahashi convex complete TVS - cone metric space, $f : X \to X$ and (α_n), (β_n) two sequences of real numbers such that $0 < \alpha_n, \beta_n < 1$ for all n and $\sum \alpha_n = \infty$. If there exist $\psi_1, \psi_2, \varphi_3, \psi_4, \psi_5 \in \Psi_2$ such that for any $x, y \in X$, there exists*

$$u \in \{\psi_1(d(x, y)), \psi_2(d(x, f(x))), \psi_3(d(y, f(y))), \psi_4(d(x, f(y))), \psi_5(d(f(x), y))\} \tag{3.1}$$

such that

$$d(f(x), f(y)) \leq u. \tag{3.2}$$

Then for arbitrary $x_0 \in X$ sequence of Ishikawa iterates of $(f, (\alpha_n), (\beta_n))$ at point x_0 is defined by:

$$\begin{cases} y_n = W(f(x_n), x_n, \beta_n) \\ x_{n+1} = W(f(y_n), x_n, \alpha_n). \end{cases}$$

converge, in the TVS metric, to unique fixed point of f.

Proof. Let $e \in \mathrm{int}P$, $k \in \{1, \ldots, 5\}$ and $\varphi_k : [0, \infty) \to [0, \infty)$ defined by

$$\varphi_k = \xi_e(\psi_k(te)).$$

Then
(a) $\varphi_k(0) = 0$ since $\xi_e(\Theta) = 0$ and $\psi_k(\Theta) = \Theta$;
(b) By Lemma 2.3, taking into account $\psi_k(te) \ll te$, we have $\varphi_k(t) = \|\psi_k(te)\| < t$.
(c) $\lim_{t\to\infty}(t - \varphi_k(t)) = \lim_{t\to\infty}(\|te\| - \|\psi_k(te)\|) = \infty$.
Therefore, $\varphi_k \in \Phi$.
(d) If $\psi_k \in \Psi_2$ then $\varphi_k \in \Phi_2$.

Let $t \in (0, +\infty)$ and $(t_n) \to t$. From $\psi_k \in \Psi_2$ and $t_n e \to te$ it follows that there exists a positive integer n_0 such that $n > n_0$ implies $\psi_k(t_n e) \leq (1 - \varepsilon)t_n e \ll (1 - \varepsilon/2)t_n e$. By Lemma 2.3, it follows that $\varphi_k(t_n) = \|\psi_k(t_n)\| < (1 - \varepsilon/2)t_n$, for $n > n_0$. Hence $\overline{\lim}_{s\to t}\varphi(s) \leq (1 - \varepsilon/2)t < t$ because t_n and t are arbitrary, which implies that $\varphi_k \in \Phi_2$.

Now, (X, d_e) is a complete metric space and $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \in \Phi_2$. From 1) and 2) it follows that

$$\xi_e(d(f(x), f(y)) \leq \xi_e(u) \leq \|u\|$$

for some

$$u \in \{\psi_1(d(x, y)), \psi_2(d(x, f(x))), \psi_3(d(y, f(y))), \psi_4(d(x, f(y))), \psi_5(d(f(x), y))\}$$

which implies

$$d_e(f(x), f(y)) \leq \max\{\varphi_1(d_e(x, y)), \varphi_2(d_e(x, f(x))), \varphi_3(d_e(y, f(y))), \varphi_4(d_e(x, f(y))), \varphi_5(d_e(f(x), y))\},$$

for any $x, y \in X$. The statement follows from Theorem 2.6 and 2.5. □

Example 3.3. *Let $X = [0, \frac{1}{2}]$, $E = C_{\mathbb{R}}[0,1]$ equipped with the strongest locally convex topology and $P = \{g \in E : g(t) \geq 0, t \in [0,1]\}$. Then E is a locally convex not normable space by exercise 7 from [25], and P is a non-normal solid cone, by Theorem 2.3. from [17]. Also, $d : X^2 \to P$, defined by*

$$d(x,y) = \frac{\rho(x,y) + |y-x|}{2} e^t$$

is a TVS-cone metric on E. Further function $W : X \times X \times [0,1] \to X$ defined by

$$W(x,y,\lambda) = \frac{W_h(x,y,\lambda) + \lambda x + (1-\lambda)y}{2}$$

is Takakahashi convex structure on X, because for any $z \in X$

$$\begin{aligned} W(x,y,\lambda) &\leq \frac{\lambda\rho(x,z) + (1-\lambda)\rho(y,z) + \lambda|x-z| + (1-\lambda)|y-z|}{2} \\ &= \lambda\frac{\rho(x,z) + |x-z|}{2} + (1-\lambda)\frac{\rho(y,z) + |y-z|}{2}. \end{aligned}$$

Define $f : X \to X$ and $\psi_i : P \to P$ $i = 1,5$ be mappings defned by:

$$f(x) = \begin{cases} x - \frac{x^2}{2} & (0 \leq x < 1) \\ \frac{7}{8} & x = 1 \end{cases},$$

$$\psi_1(p)|_t = \psi_2(p)|_t = \psi_3(p)|_t = \begin{cases} (p(t)e^{-t} - \frac{p(t)^2 e^{-2t}}{2})e^t & (0 \leq p(t) \leq e^t) \\ e^t & (p(t) > e^t) \end{cases}.$$

$$\psi_4(p)|_t = \psi_5(p)|_t = \begin{cases} 0 & (0 \leq p(t) < \frac{e^t}{2}) \\ \frac{15}{16}p(t) & (\frac{e^t}{2} \geq p(t)) \end{cases}.$$

Then existence and uniqueness of fixed points, and convergence of arbitrary sequences of Ishikawa iterates follows from Theorem 3.2. The results of a recent work [4–6, 8, 27] are not applicable in this case.

Definition 3.4. *If $A : E \to E$ is an one to one function such that $A(P) = P$, $(I - A)$ is one to one and $(I - A)(P) = P$ then we say that A is contractive operator.*

Next corollary include results presented in [5, 6, 27].

Corollary 3.5. *Let (X, d) be a Takahashi convex complete TVS - cone metric space, $f : X \to X$ and (α_n), (β_n) two sequences of real numbers such that $0 < \alpha_n, \beta_n < 1$ for all n and $\sum \alpha_n = \infty$. If there exists contractive bounded linear operators A_1, A_2, A_3, A_4, A_5 such that for any $x, y \in X$ there exists*

$$u \in \{A_1(d(x,y)), A_2(d(x,f(x))), A_3(d(y,f(y))), A_4(d(x,f(y))), A_5(d(f(x),y))\}$$

such that $d(f(x), f(y)) \leq u$. Then f has the unique fixed point $y \in X$ and for each $x \in X$ the sequence of Ishikawa iterates defined by f at x converge to y in the TVS metric.

Proof. Let $k \in \{1, \ldots, 5\}$. Then $A_k(\Theta) = \Theta$ since A_k is linear. From $(I - A)(P) = P$ by Open mapping theorem (see [24]) it follows that $(I - A)(\mathrm{int}P) \subseteq \mathrm{int}P$. So $A(x) \ll x$ for any $x \in \mathrm{int}P$. Let $x \in \mathrm{int}P$, $(x_n) \subseteq \mathrm{int}P$ and $x_n \to x$. Then $\lim A(x_n) \ll x$ since A is continuous.

$$\lim_{t\to\infty}(\|tx\| - \|A_k(tx)\|) = (\|x - A_k(x)\|)\lim_{t\to\infty} t = \infty,$$

for any $x \in P \setminus \{\Theta\}$", where $\|.\|$ is an arbitrary real semi-norm on E.

Hence $A \in \Psi_2$. The statement follows from Theorem 3.2. □

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