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The doubly metric dimension of corona product graphs

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Abstract. The doubly metric dimension of a connected graph G is the minimum cardinality of doubly resolving sets in it. It is well known that deciding the doubly metric dimension of G is NP-complete. The corona product $G \odot H$ of two vertex-disjoint graphs G and H is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H, then joining the ith vertex of G to every vertex in the ith copy of H. In this paper some formulae on the doubly metric dimension of corona product $G \odot H$ of graphs G and H are established in terms of the order of G with the adjacency dimension of H and the doubly metric dimension of $K_1 \odot H$, respectively. We determine both sharp upper and lower bounds on doubly metric dimension of corona product graphs with disconnected and connected coronas involved, respectively, and characterize the corresponding extremal graphs. We also characterize all graphs G of diameter two with doubly metric dimension two. Furthermore, the exact values are obtained for the doubly metric dimensions of corona product graphs, being the corona either a path or a cycle.

1. Introduction

Nowadays, both the metric dimension and doubly metric dimension of graphs are highly attracting the attention of many researchers. The concept of metric dimension of graphs was independently introduced by Harary *et al.* [10] and Slater [26]. Since then some related results to this topic are published in [2, 4, 5, 7, 8, 14–17, 27, 29]. Cáceres *et al.* [4] introduced the definition of doubly resolving set in order to determine bounds on the metric dimension of Cartesian product of graphs. For some other results on the doubly resolving set, please see [6, 12, 21–23]. Readers are referred to recent survey [20] for more information and background on many of these variants.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$ where $|V(G)|$ will be denoted by n_G in the following. A graph G with $n_G = 1$ is said to be *trivial*. We denote by P_n, C_n, N_n, K_n, $K_{s,n-s}$ the path, the cycle, the empty graph, the complete graph and the complete bipartite graph of order n, respectively. We also denote $\overline{G}$ as the complement of G. The *distance* $d_G(u, v)$ between a pair of vertices u and v is the length of a shortest path between u and v. The *diameter* of G is $diam(G) = max\{d_G(u, v) : u, v \in V(G)\}$. For any vertex $u \in V(G)$, we denote by $N_G(u)$ the set of *neighbors* of u, whose cardinality is just the *degree* of u, written as $deg_G(u)$ in the following. Two vertices $u, v \in V(G)$ are *twins* if $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$. A set U

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of G is a *twin-set* if any two vertices in it are twins in G. If the graph G is clear from the context, we will drop the subscript G from these notations. A *universal vertex* is the vertex adjacent to all other vertices. A *pendant* vertex u is a vertex with degree $deg_G(u) = 1$. All the pendant vertices of graph G form a set $L(G)$ with cardinality $\ell(G)$. A set of vertices $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G) \setminus S$ has a neighbor in S.

For $x, y \in V(G)$, x and y are *resolved* by $v \in V(G)$ if $d(v, x) \neq d(v, y)$, and they are *doubly resolved* by $u, v \in V(G)$ if $d(x, u) - d(y, u) \neq d(x, v) - d(y, v)$. A set W is a *resolving set* of G if each pair of vertices of G is resolved by some vertex in W. The minimum cardinality of resolving sets of G, denoted by $\beta(G)$, is the *metric dimension* of G. A set $W \subseteq V(G)$ is a *doubly resolving set* (a DR, for short) of G, and W *doubly resolves* G, if each pair of vertices of G is doubly resolved by some pair of vertices in W. The *doubly metric dimension* of G, denoted by $\psi(G)$, is the minimum cardinality of DR sets of G. The *representation* of $v \in V(G)$ on an ordered set $W = \{u_1, u_2, \ldots, u_m\}$ is the vector $r(v|W) = (d(v, u_1), \ldots, d(v, u_m))$. Let $\overrightarrow{c}$ be an m-dimensional vector where each component is a constant c. A set $W \subseteq V(G)$ is a *resolving set* of a graph G if $r(u|W) \neq r(v|W)$ for any two distinct vertices $u, v \in V(G)$, and W is a DR set of G if $r(u|W) - r(v|W) \neq \overrightarrow{c}$ for any constant c. Jannesari *et al.* [13] introduced the concept of adjacency dimension to study the metric dimension of lexicographic product of graphs. A set $W \subseteq V(G)$ is an *adjacency generator* (an AG, for short) of G if there is a vertex $u_i \in W$ such that $|N(u_i) \bigcap \{u, v\}| = 1$ for $u, v \in V(G) \setminus W$. The *adjacency dimension* of G, denoted by $\mu(G)$, is the minimum cardinality of adjacency generators of G, where the minimum AG is called an *adjacency basis* of G. Clearly, $\beta(G) \geq 1$, $\psi(G) \geq 2$, $\mu(G) \geq 1$, with $\beta(G) \leq \psi(G)$ and $\beta(G) \leq \mu(G)$ for any connected graph G.

The *corona product* $G \odot H$ of two vertex-disjoint graphs G and H is defined as the graph obtained from G and H by taking one copy of G and n_G copies of H, then joining the ith vertex of G to every vertex in the ith copy of H. The *join graph* $G \vee H$ is obtained from G and H by adding an edge between each vertex of G and each vertex of H. Moreover, $W_n = K_1 \odot C_n$ and $F_n = K_1 \odot P_n$ are the wheel graph and the fan graph, respectively. The universal vertex in $K_1 \odot H$ is called the *center* of $K_1 \odot H$. From the structure of $G \odot H$, G is called the *basis* and H is called the *corona* in it. Let G be a nontrivial connected graph with $V(G) = \{u_1, \ldots, u_{n_G}\}$, H be a graph with p nontrivial connected components and q isolated vertices. Then $H = N_{n_H}$ for $p = 0$ or $H = (\bigcup_{r=1}^{p} T_r) \bigcup N_q$ for $p \geq 1$ where T_r is a connected component of order $n_r > 1$ in H. Let $H_i = (V_i, E_i)$ be the ith copy of H with $H_i = (\bigcup_{r=1}^{p} T_r^i) \bigcup N_q^i$ for $p \geq 1$. In Section 2, 3 and 4 (unless otherwise stated), G is a nontrivial connected graph and H is a (non necessarily connected) nontrivial graph with $p \geq 1$ nontrivial connected components and $q \geq 0$ isolated vertices. Moreover, the case $p = 1$ and $q = 0$ correspond to the case when H is connected.

Kratica *et al.* [19] proved that deciding the doubly metric dimension of an arbitrary graph is NP-complete. The problem of the corona product graphs has been studied in [1, 9, 18, 24, 25, 28], while in this paper we focus on $\psi(G \odot H)$. In Section 2, we provide some basic results. In Section 3, we research the doubly metric dimension of $G \odot H$ with disconnected corona and determine both sharp upper and lower bounds on $\psi(G \odot H)$. In Section 4, we study the doubly metric dimension of $G \odot H$ with connected corona and determine the exact values of $\psi(G \odot H)$ for $H \in \{P_{n_H}, C_{n_H}\}$.

2. Preliminaries

In this section, we list or prove some basic results.

Lemma 2.1. ([4, 12]) Let G be a connected graph of order $n \geq 3$. Then $\psi(G) \leq n - 1$ with equality if and only if $G \in \{K_n, K_{1,n-1}, K_{2,n-2}, K_2 \vee N_{n-2}\}$.

Lemma 2.2. ([5]) Let G be a nontrivial connected graph of order n. Then
(*i*) $\beta(G) = n - 1$ if and only if $G \cong K_n$.
(*ii*) $\beta(G) = n - 2$ if and only if $G \in \{K_{s,n-s}, K_s \vee N_{n-s}, K_s \vee (K_1 \bigcup K_{n-s-1})\}$ for $n \geq 4$, $s \geq 1$ and $n - s \geq 2$.

Lemma 2.3. ([11]) Let G be a nontrivial connected graph with twins x and y. Then $d(x, u) = d(y, u)$ for $u \in V(G) \setminus \{x, y\}$.

Lemma 2.4. *Let G be a nontrivial connected graph with a twin-set U and S be a DR set of G. Then S contains at least $|U| - 1$ vertices of U.*

Proof. Assume, to the contrary, that there are two vertices $u, v \in U \setminus S$. We have $d(u, s) = d(v, s)$ for $s \in S$ by Lemma 2.3. This is a contradiction to the assumption that S is a DR set of G. □

Lemma 2.5. *Let G be a connected graph with $\ell \geq 2$ universal vertices. If S is an AG of G, then S contains at least $\ell - 1$ universal vertices of G.*

Proof. If there are two universal vertices $u, v \in V(G) \setminus S$, then $|N(s) \bigcap \{u, v\}| = 2$ for $s \in S$. This is a contradiction to the definition of AG. □

Proposition 2.6. ([13]) Let G be a connected graph with diameter 2. Then $\beta(G) = \mu(G)$.

Lemma 2.7. ([9]) Let G be a nontrivial connected graph and H be a nontrivial graph. Then $\beta(G \odot H) = n_G \mu(H)$. Moreover, $\beta(G \odot K_{s,n-s}) = (n-2)n_G$ for $n - s \geq s \geq 2$.

Lemma 2.8. ([28]) Let G be a nontrivial connected graph and H be a nontrivial graph. If $x, y \in V_i$, then $d_{G \odot H}(x, w) = d_{G \odot H}(y, w)$ for $w \in V(G \odot H) \setminus V_i$.

Lemma 2.9. ([28]) Let G and H be two vertex-disjoint nontrivial connected graphs, respectively. Then $\beta(G \odot H) \geq n_G \beta(H)$.

Lemma 2.10. *Let G be a connected graph with $\delta(G) = 1$. If S is a DR set of G, then $L(G) \subseteq S$.*

Proof. To the contrary, assume that there is a vertex $u_i \in L(G) \setminus S$. The vertex u_i is adjacent to u_j in G, we have $d(u_i, s) = d(u_j, s) + 1$ for $s \in V(G) \setminus \{u_i\}$. That is, $d(u_i, s) - d(u_j, s) = d(u_i, t) - d(u_j, t) = 1$ for $s, t \in V(G) \setminus \{u_i\}$. These two vertices u_i and u_j are not doubly resolved by S, a contradiction. Hence, $L(G) \subseteq S$. □

Lemma 2.11. *Let G be a nontrivial connected graph and H be a graph.*
(*i*) *$d_{G \odot H}(u, w) = d_{G \odot H}(v, w)$ for $u, v \in V(T^i_r)$ and $w \in V(G \odot H) \setminus V(T^i_r)$.*
(*ii*) *If S is a DR set of $G \odot H$, then $V(N^i_q) \subseteq S$ and $S \bigcap V(T^i_r) \neq \emptyset$.*
(*iii*) *If $S \subseteq V(G \odot H)$ with $S \bigcap V_k \neq \emptyset$ where $V_k = V(H_k)$ for $1 \leq k \leq n_G$, then S doubly resolves two distinct vertices $u \in V_i \bigcup \{u_i\}$ and $v \in V_j \bigcup \{u_j\}$ in $G \odot H$.*
(*iv*) *If S is a minimum DR set of $G \odot H$, then $S \bigcap V(G) = \emptyset$.*

Proof. (*i*) Clearly, $d_{G \odot H}(u, w) = 1 + d_{G \odot H}(u_i, w) = d_{G \odot H}(v, w)$ for $u_i \in V(G)$ and $w \in V(G \odot H) \setminus V(T^i_r)$.

(*ii*) We have $V(N^i_q) \subseteq S$ by Lemma 2.10. We next prove $S \bigcap V(T^i_r) \neq \emptyset$. Assume, to the contrary, that $S \bigcap V(T^i_r) = \emptyset$. Since $n_r \geq 2$, there are two vertices $u, v \in V(T^i_r) \setminus S$. By Lemma 2.8 and (*i*), we can directly obtain $d_{G \odot H}(u, s) = d_{G \odot H}(v, s)$ for $s \in S$, a contradiction. Hence, $S \bigcap V(T^i_r) \neq \emptyset$.

(*iii*) For $u \in V_i$ and $v \in V_j$, we have $d_{G \odot H}(u, s) \leq 2 < 3 \leq d_{G \odot H}(v, s)$ and $d_{G \odot H}(u, t) \geq 3 > 2 \geq d_{G \odot H}(v, t)$ for $s \in S \bigcap V_i$ and $t \in S \bigcap V_j$. Then, S doubly resolves these two vertices u and v. Similarly, two vertices $s \in S \bigcap V_i$ and $t \in S \bigcap V_j$ doubly resolve u and v for $u = u_i$ and $v \in V_j \bigcup \{u_j\}$. By symmetry, the result holds if $u \in V_i$ and $v = u_j$.

(*iv*) Let $S_i = S \bigcap V_i$ and $W = \bigcup_{i=1}^{n_G} S_i$. Then $S_i \neq \emptyset$ by (*ii*). To the contrary, assume that $S \bigcap V(G) \neq \emptyset$. Then, $S \setminus W \neq \emptyset$. Our aim is to show that W doubly resolves $u, v \in V(G \odot H)$ with the following two cases: $u \in V_i \bigcup \{u_i\}$ and $v \in V_j \bigcup \{u_j\}$; $u, v \in V_i \bigcup \{u_i\}$. By (*iii*), we only consider the case $u, v \in V_i \bigcup \{u_i\}$.

Suppose that $u, v \in V_i$. By Lemma 2.8, there exists a vertex $s \in S_i$ satisfying $d_{G \odot H}(u, s) \neq d_{G \odot H}(v, s)$. Then, two vertices $s \in S_i$ and $t \in S_j$ doubly resolve u, v.

Suppose that $u = u_i$ and $v \in V_i$. Clearly, $d_{G \odot H}(u, t) - d_{G \odot H}(v, t) = -1$ for $t \in S_j$. If $v \in S_i$, then v and $t \in S_j$ doubly resolve u and v. If $v \notin S_i$, then $d_{G \odot H}(v, s) \in \{1, 2\}$ and $d_{G \odot H}(u, s) = 1$ for $s \in S_i$. Assume first that there exists a vertex $s \in S_i$ satisfying $d_{G \odot H}(v, s) = 1$, we derive that $s \in S_i$ and $t \in S_j$ doubly resolve u and v. Now assume that $d_{G \odot H}(v, s) = 2$ for $s \in S_i$. Let $r(v|S) = (2, \ldots, 2, \ell_1, \ldots, \ell_{|S|-|S_i|})$ with $\ell_k = d_{G \odot H}(v, t)$ for $1 \leq k \leq |S| - |S_i|$ and $t \in S \setminus S_i$. Then $r(u|S) = (1, \ldots, 1, \ell_1 - 1, \ldots, \ell_{|S|-|S_i|} - 1)$ and $r(v|S) - r(u|S) = \overrightarrow{1}$, which is a contradiction. Thus, W is a DR set of $G \odot H$, contradicting the minimality of S. □

3. $G \odot H$ with disconnected corona

In this section, we provide some formulae for $\psi(G\odot H)$ in terms of n_G with $\mu(H)$ and $\psi(K_1\odot H)$, respectively. We also determine both upper and lower bounds on $\psi(G \odot H)$ and characterize the corresponding extremal graphs. In this section (unless otherwise stated), G is a nontrivial connected graph and H is a (non necessarily connected) nontrivial graph with $p \geq 1$ nontrivial connected components and $q \geq 0$ isolated vertices. Moreover, the case $p = 1$ and $q = 0$ correspond to the case when H is connected.

3.1. General results

We first show the closed formulae for $\psi(G \odot H)$ in terms of n_G and $\mu(H)$.

Proposition 3.1. ([9]) Let G be a nontrivial graph. If S is an AG of G, then at most one vertex of G is not dominated by S.

Theorem 3.2. *Let G, H be two nontrivial graphs and let G be connected. Then*

$$\psi(G \odot H) = \begin{cases} n_G\mu(H) & \text{if there is an adjacency basis of } H \text{ as a dominating set;} \\ n_G(\mu(H) + 1) & \text{otherwise.} \end{cases}$$

Proof. Suppose that there is an adjacency basis of H that is a dominating set. We have $\psi(G\odot H) \geq \beta(G\odot H) = n_G\mu(H)$ by Lemma 2.7. Next, we show $\psi(G \odot H) \leq n_G\mu(H)$. Let W_i be the adjacency basis of H_i that is a dominating set. Our aim is to show that $S = \bigcup_{i=1}^{n_G} W_i$ doubly resolves $u, v \in V(G \odot H)$. There are two cases: $u \in V_i \bigcup\{u_i\}$ and $v \in V_j \bigcup\{u_j\}$; $u, v \in V_i \bigcup\{u_i\}$. We only need to consider the case $u, v \in V_i \bigcup\{u_i\}$ by Lemma 2.11 (*iii*). For $u = u_i$ and $v \in V_i$, we have $d_{G\odot H}(v, t) - d_{G\odot H}(u, t) = 1$ for $t \in W_j$ and $d_{G\odot H}(v, s) - d_{G\odot H}(u, s) = 0$ for some $s \in W_i$ as W_i is a dominating set. For $u, v \in V_i$, we have $|N_{H_i}(s) \bigcap\{u, v\}| = 1$ for some $s \in W_i$. Hence, $d_{G\odot H}(u, s) - d_{G\odot H}(v, s) \neq d_{G\odot H}(u, t) - d_{G\odot H}(v, t)$ for $s \in W_i$ and $t \in W_j$. Therefore, S is a DR set of $G \odot H$, implying $\psi(G \odot H) = n_G\mu(H)$.

Suppose that any adjacency basis of H is not a dominating set. We first show $\psi(G \odot H) \geq n_G(\mu(H) + 1)$. Let X be a minimum DR set of $G \odot H$ and $X_i = X \bigcap V_i$. By Lemma 2.11 (*iv*), $X \bigcap V(G) = \emptyset$. For $u, v \in V_i \setminus X_i$, there is a vertex $s \in X_i$ such that $d_{G\odot H}(u, s) \neq d_{G\odot H}(v, s)$ by Lemma 2.8. Hence, $d_{G\odot H}(u, s) \in \{1, 2\}$ and $d_{G\odot H}(v, s) \in \{1, 2\}$ for $s \in X_i$, that is, $|N_{H_i}(s) \bigcap\{u, v\}| = 1$. Thus, X_i is an AG of H_i.

If X_i is not a dominating set of H_i, then there is a vertex $w \in V_i$ such that $d_{G\odot H}(w, s) - d_{G\odot H}(u_i, s) = 1$ for $u_i \in V(G)$ and $s \in X$, which contradicts that X doubly resolves $G \odot H$. Thus, X_i is a dominating set of H_i. From the above argument, X_i is an AG of cardinality greater than $\mu(H_i)$, that is, $\mu(H_i) < |X_i|$. Hence, $\psi(G \odot H) = |X| \geq \sum_{i=1}^{n_G} |X_i| \geq \sum_{i=1}^{n_G}(\mu(H_i) + 1) = n_G(\mu(H) + 1)$.

Next, we show $\psi(G \odot H) \leq n_G(\mu(H) + 1)$. Let W_i be an adjacency basis of H_i that is not a dominating set. Then, by Proposition 3.1, there is exactly one vertex $w \in V_i$ which is not dominated by W_i. Set $T_i = W_i \bigcup\{w\}$ and $S = \bigcup_{i=1}^{n_G} T_i$. We claim that S doubly resolves $u, v \in V(G \odot H)$ for $u, v \in V_i \bigcup\{u_i\}$. For $u, v \in V_i$, we have $d_{G\odot H}(u, s) - d_{G\odot H}(v, s) \neq d_{G\odot H}(u, t) - d_{G\odot H}(v, t)$ for $t \in W_j$ and $s \in W_i$ as $|N_{H_i}(s) \bigcap\{u, v\}| = 1$. For $u \in V_i$ and $v = u_i$, we have $d_{G\odot H}(u, s) = d_{G\odot H}(v, s)$ for some $s \in T_i$ as T_i is a dominating set of H_i and $d_{G\odot H}(u, t) \neq d_{G\odot H}(v, t)$ for $t \in T_j$. By Lemma 2.11 (*iii*), S doubly resolves $u, v \in V(G \odot H)$ and $\psi(G \odot H) \leq n_G(\mu(H) + 1)$, ending the proof. □

Corollary 3.3. *Let G and H be two vertex-disjoint connected graphs of order $n_G \geq 2$ and $n_H \geq 3$, respectively. If H contains at least two universal vertices, then $\psi(G \odot H) = n_G\mu(H) = n_G\beta(H)$.*

Proof. Certainly, each minimum AG of H is a dominating set and $\psi(G \odot H) = n_G\mu(H) = n_G\beta(H)$ by Lemma 2.5, Proposition 2.6 and Theorem 3.2. □

Clearly, $n - 2$ leaves of $K_{1,n-1}$ form the unique minimum AG of $K_{1,n-1}$ which is not a dominating set. We obtain the following result.

Corollary 3.4. *Let G be a nontrivial connected graph and $K_{1,n-1}$ be a graph of order $n \geq 3$. Then $\psi(G \odot H) = n_G(\mu(K_{1,n-1}) + 1)$.*

Next, closed formulae for $\psi(G \odot H)$ in terms of n_G and $\psi(K_1 \odot H)$ are given.

Theorem 3.5. *Let G, H be two nontrivial graphs and let G be connected. Then*

$$\psi(G \odot H) = \begin{cases} n_G(\psi(K_1 \odot H) - 1) & \text{a minimum DR set of } K_1 \odot H \text{ contains its center;} \\ n_G\psi(K_1 \odot H) & \text{otherwise.} \end{cases}$$

Proof. Suppose that the center of $K_1 \odot H$ belongs to some minimum DR set of $K_1 \odot H$. We claim $\psi(G \odot H) \leq n_G(\psi(K_1 \odot H) - 1)$. Let W_i be the minimum DR set of $K_1 \odot H_i$ with the vertex $u_i \in W_i$ where $V(K_1) = \{u_i\}$, $T_i = W_i \setminus \{u_i\}$ and $S = \sum_{i=1}^{n_G} T_i$. We claim that S doubly resolves $u, v \in V(G \odot H)$ for $u, v \in V_i \bigcup \{u_i\}$. For $u, v \in V_i$, we assert that $d_{G\odot H}(u, t) = d_{G\odot H}(v, t)$ for $t \in T_j$ and $d_{G\odot H}(u, s) \neq d_{G\odot H}(v, s)$ for some $s \in T_i$. For $u \in V_i$ and $v = u_i$, there is a vertex $s \in T_i$ such that $d_{G\odot H}(u, s) = d_{K_1\odot H_i}(u, s) \leq 1$, otherwise contradicts that W_i doubly resolves $K_1 \odot H_i$. Hence, $s \in T_i$ and $t \in T_j$ doubly resolve u and v. Thus, S doubly resolves $u, v \in V(G \odot H)$ by Lemma 2.11 (*iii*) and $\psi(G \odot H) \leq n_G(\psi(K_1 \odot H) - 1)$.

Now we prove that $\psi(G \odot H) \geq n_G(\psi(K_1 \odot H) - 1)$. Let X be a minimum DR set of $G \odot H$ and $X_i = X \bigcap V_i$. Next, we prove that $X_i \bigcup \{u_i\}$ is a DR set of $K_1 \odot H_i$. For $u, v \in V_i$, there is a vertex $s \in X_i$ such that $d_{K_1\odot H_i}(u, s) - d_{K_1\odot H_i}(v, s) \neq d_{K_1\odot H_i}(u, u_i) - d_{K_1\odot H_i}(v, u_i)$. For $u \in V_i$ and $v = u_i$, there is a vertex $s \in X_i$ such that $d_{K_1\odot H_i}(u, s) - d_{K_1\odot H_i}(v, s) \neq 1$. Thus, $s \in X_i$ and u_i doubly resolve u and v. Hence, $\psi(K_1 \odot H_i) \leq |X_i| + 1$ and $\psi(G \odot H) = |X| \geq \sum_{i=1}^{n_G} |X_i| \geq \sum_{i=1}^{n_G}(\psi(K_1 \odot H_i) - 1) = n_G(\psi(K_1 \odot H) - 1)$.

Suppose that any minimum DR set of $K_1 \odot H$ does not contain its center. Let W_i be a minimum DR set of $K_1 \odot H_i$ and $S' = \sum_{i=1}^{n_G} W_i$. We check that S' doubly resolves $u, v \in V(G \odot H)$ for $u, v \in V_i \bigcup \{u_i\}$. Clearly, u, v are doubly resolved by W_i as W_i is a DR set of $K_1 \odot H_i$ where $V(K_1) = \{u_i\}$. Hence, S' doubly resolves $G \odot H$ by Lemma 2.11 (*iii*) and $\psi(G \odot H) \leq n_G\psi(K_1 \odot H)$.

We now show that $\psi(G \odot H) \geq n_G\psi(K_1 \odot H)$. Let T be a minimum DR set of $G \odot H$ and $T_i = T \bigcap V_i$. We claim that $T_i \bigcup \{u_i\}$ doubly resolves $K_1 \odot H_i$. For $u, v \in V_i$, by Lemma 2.8, we have $d_{K_1\odot H_i}(u, u_i) = d_{K_1\odot H_i}(v, u_i)$ and $d_{K_1\odot H_i}(u, s) \neq d_{K_1\odot H_i}(v, s)$ for some $s \in T_i$. For $u \in V_i$ and $v = u_i$, we assert that $d_{K_1\odot H_i}(u, u_i) - d_{K_1\odot H_i}(v, u_i) = 1$ and $d_{K_1\odot H_i}(u, s) - d_{K_1\odot H_i}(v, s) \neq 1$ for some $s \in T_i$. Hence, $T_i \bigcup \{u_i\}$ doubly resolves $K_1 \odot H$. Since any minimum DR set of $K_1 \odot H_i$ does not contain the vertex of K_1, $\psi(K_1 \odot H_i) < |T_i| + 1$. Therefore, $\psi(G \odot H) = |T| \geq \sum_{i=1}^{n_G} |T_i| \geq \sum_{i=1}^{n_G} \psi(K_1 \odot H_i) = n_G\psi(K_1 \odot H)$, ending the proof. □

Since all vertices of N_{n_H} form the unique minimum DR set of $K_1 \odot N_{n_H}$ and there are two universal vertices in $K_1 \odot K_{1,n_H-1}$, we obtain the following result.

Corollary 3.6. *Let G be a nontrivial connected graph and* $H \in \{N_{n_H}, K_{1,n_H-1}\}$. *Then* $\psi(G \odot N_{n_H}) = n_G\psi(K_1 \odot N_{n_H})$ *and* $\psi(G \odot K_{1,n_H-1}) = n_G(\psi(K_1 \odot K_{1,n_H-1}) - 1)$.

Since $1 \leq \mu(H) \leq n_H - 1$ and $2 \leq \psi(H) \leq n_H - 1$ for a graph H, from Theorems 3.2 and 3.5 the following bounds on $\psi(G \odot H)$ are derived.

Corollary 3.7. *Let G be a nontrivial connected graph and H be a nontrivial graph. Then* $n_G \leq \psi(G \odot H) \leq n_G n_H$.

3.2. Upper and lower bounds on $\psi(G \odot H)$

In this subsection, we establish more precise upper and lower bounds on $\psi(G \odot H)$ and characterize the corresponding extremal graphs. Before presenting the main result, some lemmas are proven. Note that p and q denote the number of nontrivial connected components and of isolated vertices of H, respectively, and also that H is disconnected, that is, that $p + q \geq 2$, when necessary.

Lemma 3.8. *Let G, H be two nontrivial graphs and let G be connected. If there is a nontrivial connected component* T_r *in H with* $\psi(T_r) \leq n_r - 2$, *then* $\psi(G \odot H) \leq n_G(n_H - p - 1)$.

Proof. Note that there is a nontrivial connected component T_r in H and $p \geq 1$. Let B_i consist of all isolated vertices of H_i, A_i consist of all but one vertex in each of the $p - 1$ nontrivial connected components of $H_i \setminus T_r^i$ and C_i be the DR set of T_r^i with $n_r - 2$ vertices. Set $x, y \in V(T_r^i) \setminus C_i$. The aim is to show that $S = \bigcup_{i=1}^{n_G}(A_i \bigcup B_i \bigcup C_i)$ doubly resolves $u, v \in V(G \odot H)$ for $u, v \in V_i \bigcup \{u_i\}$.

Suppose that $u = u_i$ and $v \in V_i$. Then, we have $d_{G\odot H}(u,t) - d_{G\odot H}(v,t) = -1$ for $t \in S \bigcap V_j$. If $v \notin V(T_r^i)$, then $d_{G\odot H}(u,v) - d_{G\odot H}(v,v) \neq d_{G\odot H}(u,t) - d_{G\odot H}(v,t)$ for $v \in S$ or $d_{G\odot H}(u,s) - d_{G\odot H}(v,s) \neq d_{G\odot H}(u,t) - d_{G\odot H}(v,t)$ for $v \notin S$ and $s \in S \bigcap N_{H_i}(v)$. If $v \in V(T_r^i) \bigcap C_i$, then v and $t \in S \bigcap V_j$ doubly resolve u and v. Next, we consider $v \in \{x, y\}$. Assume, w.l.o.g, that $v = x$. If $N_{T_r^i}(x) = \{y\}$, then x is a pendant vertex of T_r^i. Using Lemma 2.10, we can derive $x \in C_i$, which yields a contradiction. Therefore, there is a vertex $s \in N_{T_r^i}(x) \setminus \{y\}$ such that $d_{T_r^i}(x,s) = 1 = d_{G\odot H}(v,s)$ and $d_{G\odot H}(u,s) = 1$. This implies that $s \in N_{T_r^i}(x) \setminus \{y\}$ and $t \in S \bigcap V_j$ doubly resolve u and v.

Suppose that $u, v \in V_i$. $d_{G\odot H}(u,t) - d_{G\odot H}(v,t) = 0$ for $t \in S \bigcap V_j$ by Lemma 2.8. If $u, v \in S \bigcap V_i$, then u and v are doubly resolved by themselves. If $u \in S \bigcap V_i$ and $v \notin S \bigcap V_i$, then u and $t \in S \bigcap V_j$ doubly resolve u and v. If $u, v \notin S \bigcap V_i$, then $u, v \in \{x, y\}$ or u, v belong to two different nontrivial connected components of H_i. For $u, v \in \{x, y\}$, w.l.o.g., assume that there exists a vertex $s \in (N_{T_r^i}(u) \setminus \{v\}) \setminus (N_{T_r^i}(v) \setminus \{u\})$ satisfying $d_{T_r^i}(u,s) = 1 = d_{G\odot H}(u,s)$ and $d_{G\odot H}(v,s) = 2$. Thus, $s \in (N_{T_r^i}(u) \setminus \{v\}) \setminus (N_{T_r^i}(v) \setminus \{u\})$ and $t \in S \bigcap V_j$ doubly resolve u and v. We now consider that u, v belong to two different connected components of H_i. Assume, w.l.o.g., that $u \in V(T_k^i)$ and $k \neq r$. We assert that $s \in S \bigcap N_{T_k^i}(u)$ and $t \in S \bigcap V_j$ doubly resolve u and v.

Based on the above cases and Lemma 2.11 (*iii*), S doubly resolves $u, v \in V(G \odot H)$ and $\psi(G \odot H) \leq n_G(n_H - p - 1)$. □

Lemma 3.9. *Let G, H be two nontrivial graphs and let G be connected. If there is a connected component $T_r \in \{K_{2,n_r-2}, K_2 \vee N_{n_r-2}\}$ of order $n_r \geq 4$, then $\psi(G \odot H) \leq n_G(n_H - p - 1)$.*

Proof. Let $T_r \cong K_{2,n_r-2}$ with $V(T_r^i) = \{x_1^i, x_2^i, y_1^i, y_2^i, \dots, y_{n_r-2}^i\}$ and $E(T_r^i) = \{x_m^i y_t^i : 1 \leq m \leq 2, 1 \leq t \leq n_r - 2\}$. Set A_i consist of all but one vertex in each of the $p-1$ nontrivial connected components of $H_i \setminus T_r^i$, B_i be the set of isolated vertices of H_i and $C_i = \{x_2^i, y_2^i, \dots, y_{n_r-2}^i\}$ be the set of vertices of T_r^i. Our aim is to show that $S = \bigcup_{i=1}^{n_G}(A_i \bigcup B_i \bigcup C_i)$ doubly resolves $u, v \in V(G \odot H)$ for $u, v \in V_i \bigcup \{u_i\}$.

Suppose that $u = u_i$ and $v \in V_i$. Then, $d_{G\odot H}(u,t) - d_{G\odot H}(v,t) = -1$ for $t \in S \bigcap V_j$. If $v \in S \bigcap V_i$, then v and $t \in S \bigcap V_j$ doubly resolve u and v. If $v \notin S \bigcap V_i$, then $d_{G\odot H}(v,s) - d_{G\odot H}(u,s) = 0$ for $s \in N_{H_i}(v) \bigcap S$. Hence, $s \in N_{H_i}(v) \bigcap S$ and $t \in S \bigcap V_j$ doubly resolve u and v.

Suppose that $u, v \in V_i$. If $u, v \in S \bigcap V_i$, then u, v are doubly resolved by themselves. If $u \notin S \bigcap V_i$ and $v \in S \bigcap V_i$, then v and $t \in S \bigcap V_j$ doubly resolve u and v. If $u, v \notin S \bigcap V_i$, then $s \in (N_{H_i}(u) \setminus N_{H_i}(v)) \bigcap S$ and $t \in S \bigcap V_j$ doubly resolve u and v. There always exists the vertex $s \in (N_{H_i}(u) \setminus N_{H_i}(v)) \bigcap S$ due to the selection of S.

Consequently, S doubly resolves $u, v \in V(G \odot H)$ by Lemma 2.11 (*iii*) and $\psi(G \odot H) \leq n_G(n_H - p - 1)$. Analogously, we can also derive $\psi(G \odot H) \leq n_G(n_H - p - 1)$ for $T_r \cong K_2 \vee N_{n_r-2}$ and thus we omit the proof. □

Lemma 3.10. *Let G, H be two nontrivial graphs and let G be connected. If there is a connected component T_r of order $n_r \geq 3$, then $\psi(G \odot H) \geq n_G(p + q + 1)$.*

Proof. Let S be a minimum DR set of $G \odot H$. By Lemma 2.11 (*iv*), we obtain $S \bigcap V(G) = \emptyset$ and $|S \bigcap V(T_r^i)| \geq 1$. We just need to prove that $|S \bigcap V(T_r^i)| \geq 2$ for $n_r \geq 3$. Assume, to the contrary, that $S \bigcap V(T_r^i) = \{s\}$. There are two vertices $u, v \in V(T_r^i) \setminus S$ as $n_r \geq 3$. If $d_{G\odot H}(u,s) = d_{G\odot H}(v,s)$, then we have $r(u|S) = r(v|S)$ by Lemma 2.11 (*i*), implying that these two vertices u and v are not doubly resolved by S, a contradiction. If $d_{G\odot H}(u,s) \neq d_{G\odot H}(v,s)$, then either $d_{G\odot H}(u,s) = 2$ or $d_{G\odot H}(v,s) = 2$. W.l.o.g., assume that $d_{G\odot H}(u,s) = 2$. It is clear that $d_{G\odot H}(u_i,s) = 1$ and $d_{G\odot H}(u,t) = d_{G\odot H}(u_i,t) + 1$ for $u_i \in V(G)$ and $t \in S \setminus V(T_r^i)$, that is, $r(u|S) - r(u_i|S) = \overrightarrow{1}$, a contradiction. Thus, $|S \bigcap V(T_r^i)| \geq 2$ and $|S| \geq n_G(p + q + 1)$ by Lemma 2.11 (*ii*). The result follows. □

In the following we show both a sharp upper bound and a lower bound on $\psi(G \odot H)$.

Theorem 3.11. *Let G, H be two nontrivial graphs and let G be connected. Then*

$$n_G(p + q) \leq \psi(G \odot H) \leq n_G(n_H - p)$$

with left equality if and only if $H = pK_2 \bigcup N_q$ and right equality if and only if $H = N_{n_H}$ for $p = 0$ or $H = (\bigcup_{r=1}^{p} T_r) \bigcup N_q$ for $p \geq 1$, where $T_r \in \{K_{n_r}, K_{1,n_r-1}\}$ for $1 \leq r \leq p$.

Proof. We first show the upper bound. Let A_i consist of all but one vertex in each of the p nontrivial connected components of H_i, B_i consist of all isolated vertices of H_i. Suppose that $p \geq 1$. The goal is to show that $S = \bigcup_{i=1}^{n_G}(A_i \bigcup B_i)$ doubly resolves $u, v \in V(G \odot H)$ for $u, v \in V_i \bigcup \{u_i\}$.

Suppose that $u = u_i$ and $v \in V_i$. There exists a vertex $t \in S \bigcap V_j$ such that $d_{G\odot H}(v,t) = d_{G\odot H}(u,t) + 1$. If $v \in S \bigcap V_i$, then v and $t \in S \bigcap V_j$ doubly resolve u and v. If $v \notin S \bigcap V_i$, then v belongs to a nontrivial connected component of H_i. It is routine to obtain that $d_{G\odot H}(v,s) = 1$ and $d_{G\odot H}(u,s) = 1$ for $s \in S \bigcap N_{H_i}(v)$. Then, $s \in S \bigcap N_{H_i}(v)$ and $t \in S \bigcap V_j$ doubly resolve u and v.

Suppose that $u, v \in V_i$. If $u, v \in S \bigcap V_i$, then u, v are doubly resolved by themselves. If $u, v \notin S \bigcap V_i$, then u and v belong to two different nontrivial connected components of H_i. We get that $s \in S \bigcap N_{H_i}(u)$ and $t \in S \bigcap N_{H_i}(v)$ doubly resolve u and v. If $u \in S \bigcap V_i$ and $v \notin S \bigcap V_i$, then $d_{G\odot H}(u,u) - d_{G\odot H}(v,u) < 0$. Certainly, these two vertices $u \in S \bigcap V_i$ and $t \in S \bigcap V_j$ doubly resolve u and v.

The above cases and Lemma 2.11 (*iii*) show that S doubly resolves $u, v \in V(G\odot H)$ and $\psi(G\odot H) \leq n_G(n_H - p)$ for $p \geq 1$.

In the following we show the extremal graphs. By Lemmas 2.1, 3.8 and 3.9, we need to prove $\psi(G \odot H) = n_G(n_H - p)$ for $H = (\bigcup_{r=1}^{p} T_r) \bigcup N_q$, where $T_r \in \{K_{n_r}, K_{1,n_r-1}\}$. It suffices to prove $\psi(G \odot H) \geq n_G(n_H - p)$. Let S be a minimum DR set of $G \odot H$. Then we have $S \bigcap V(G) = \emptyset$ by Lemma 2.11 (*iv*). It suffices to show $|S_i| = |S \bigcap V_i| \geq n_H - p$. To the contrary, assume that $|S_i| \leq n_H - p - 1$. As $V(N_q^i) \subseteq S_i$ and $S_i \bigcap V(T_r^i) \neq \emptyset$ by Lemma 2.11 (*ii*), we just need to consider that there are two vertices $u, v \in V(T_r^i) \setminus S_i$. Suppose first that $T_r^i \cong K_{n_r}$. We have $r(u|S) = r(v|S)$, which leads to a contradiction. Secondly we suppose that $T_r^i \cong K_{1,n_r-1}$. If u is the universal vertex of T_r^i, then $d_{G\odot H}(v,s) = 2$ for $s \in S_i$. Since $d_{G\odot H}(u_i,s) = 1$ for $s \in S_i$, we conclude $r(u_i|S) - r(v|S) = \overrightarrow{-1}$, a contradiction. If neither u nor v is the universal vertex of T_r^i, then we can directly get $r(u|S) = r(v|S)$, a contradiction. Hence, $|S_i| \geq n_H - p$ and $\psi(G \odot H) = n_G(n_H - p)$.

Suppose that $p = 0$. It is clear that $A_i = \emptyset$, and we can also verify that $S = \bigcup_{i=1}^{n_G} B_i$ doubly resolves $u, v \in V(G \odot H)$. We have $\psi(G \odot N_{n_H}) \geq n_G n_H$ by Lemma 2.10, and so $\psi(G \odot N_{n_H}) = n_G n_H$. Note that $\psi(G \odot H) \leq n_G(n_H - p)$ for $p \geq 1$. Then, $\psi(G \odot H) \leq n_G n_H$ with equality if only if $H = N_{n_H}$.

Next, we show the lower bound. We assert $\psi(G \odot H) \geq n_G(p + q)$ by Lemma 2.11 (*ii*). We now prove that $\psi(G \odot H) = n_G(p + q)$ for $H = pK_2 \bigcup N_q$. As $n_H - p = p + q$, we obtain $\psi(G \odot H) \leq n_G(n_H - p) = n_G(p + q)$ and $\psi(G \odot H) = n_G(p + q)$. If there is a connected component of order at least 3 in H, then we get $\psi(G \odot H) \geq n_G(p + q + 1)$ by Lemma 3.10. Thus, $\psi(G \odot H) = n_G(p + q)$ if only if $H = pK_2 \bigcup N_q$. This completes the proof. □

Since $\psi(G \odot H) \geq \beta(G \odot H)$, we get the following result by Lemmas 2.2, 2.9, 3.9 and Theorem 3.11.

Remark 3.12. *Let G be a nontrivial connected graph. Then*
(*i*) $\psi(G \odot H) = n_G(n_H - 1)$ *for* $H \in \{K_{n_H}, K_{1,n_H-1}\}$, *where* $n_H \geq 2$.
(*ii*) $\psi(G \odot H) = n_G(n_H - 2)$ *for* $H \in \{K_{2,n_H-2}, K_2 \vee N_{n_H-2}\}$, *where* $n_H \geq 4$.

4. $G \odot H$ with connected corona

In this section, we characterize all graphs G of diameter 2 with $\psi(G) = 2$. A sharp lower bound on $G \odot H$ with $n_H \geq 3$ is also studied. Moreover, we give the exact values of $\psi(W_n)$, $\psi(F_n)$, and $\psi(G \odot H)$ with $H \in \{P_n, C_n\}$.

Lemma 4.1. *Let G be a graph of order* $n \geq 6$ *and* $diam(G) = 2$. *Then* $\psi(G) \geq 3$.

Proof. Assume, to the contrary, that $\psi(G) = 2$. Set $W = \{u, v\}$ be a DR set of G. There are 3^2 vectors: $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$. Since W is a DR set of G, there are at most 5 different representations such that $r(x|W) - r(y|W) \neq \overrightarrow{c}$ for $x, y \in V(G)$, that is, $n \leq 5$, a contradiction. Thus, $\psi(G) \geq 3$. □

Proposition 4.2. *Let G be a connected graph with* $diam(G) = 2$. *Then* $\psi(G) \geq 2$ *with equality if and only if* $G \in \{P_3, (K_2 \bigcup K_1) \vee K_1, (K_2 \bigcup K_2) \vee K_1, F_4, \overline{P_5}, C_5\}$.

Proof. We know $\psi(G) \geq 2$ by definition. For characterizing the graphs with $\psi(G) = 2$, it suffices to consider the graph G of order $n \leq 5$ by Lemma 4.1. Through the computer, we find that there are 1, 4 and 14 non-isomorphic connected graphs of order 3, 4 and 5 with $diam(G) = 2$, respectively. Since $n \leq 5$, we can easily obtain $\psi(G) = 2$ if and only if $G \in \{P_3, (K_2 \bigcup K_1) \vee K_1, (K_2 \bigcup K_2) \vee K_1, F_4, \overline{P_5}, C_5\}$. □

Theorem 4.3. *Let G be a nontrivial connected graph and H be a graph of order $n_H \geq 5$ and $diam(H) = 2$. If a minimum DR set of $K_1 \odot H$ contains its center, then $\psi(G \odot H) = n_G \psi(H)$.*

Proof. Let W be a minimum DR set of $K_1 \odot H$ that contains the vertex u of K_1. Set $W' = W \setminus \{u\}$. We claim that W' is a DR set of H. Otherwise, there are two vertices $x, y \in V(H)$ such that $r(x|W') - r(y|W') = \overrightarrow{c}$ where $c \in \{0, \pm1, \pm2\}$. Since $n_H \geq 5$, we have $|W| \geq 3$ by Lemma 4.1 and $c \in \{0, \pm1\}$. For $c = 0$, we have $r(x|W') = r(y|W')$ and $r(x|W) = r(y|W)$ as $d_{K_1 \odot H}(x, u) = d_{K_1 \odot H}(y, u)$ which contradicts that W doubly resolves $K_1 \odot H$. For $c = 1$, we have $r(x|W') = \overrightarrow{2}$ and $r(y|W') = \overrightarrow{1}$ as $|W'| \geq 2$. Thus, $r(x|W) - r(u|W) = \overrightarrow{1}$ as $r(u|W') = \overrightarrow{1}$, a contradiction. Hence, W' doubly resolves H.

Let S be a minimum DR set of H with $|S| < |W'|$. If there is a vertex $x \in V(H)$ with representation $r(x|S) = \overrightarrow{2}$, then we show that $D = S \bigcup \{x\}$ doubly resolves $K_1 \odot H$. Certainly, D doubly resolves $a, b \in V(H)$ as $d_H(a, b) = d_{K_1 \odot H}(a, b)$. We next consider $a \in V(H)$ and $b = u$. Since $r(x|S) = \overrightarrow{2}$, there is no vertex in $V(H) \setminus \{x\}$ with representation $r(a|S) = \overrightarrow{1}$. Then, $r(u|D) - r(a|D) \neq \overrightarrow{c}$ for $a \in V(H) \setminus \{x\}$ and $r(u|D) - r(x|D) \neq \overrightarrow{c}$ as $x \in D$. If no vertex in H has representation $\overrightarrow{2}$ with respect to S, then we prove that $D = S \bigcup \{u\}$ doubly resolves $K_1 \odot H$. Set $r(u|D) = (1, \ldots, 1, 0)$. Then $r(x|D) - r(u|D) = \overrightarrow{c}$ if and only if $r(x|D) = (2, \ldots, 2, 1)$ for $x \in V(H)$. As no vertex in H has representation $\overrightarrow{2}$ with respect to S, $r(x|D) - r(u|D) \neq \overrightarrow{c}$ and D doubly resolves $K_1 \odot H$. That is, $|S| + 1 < |W'| + 1 = |W| = \psi(K_1 \odot H)$, a contradiction. Hence, W' is a minimum DR set of H and $\psi(K_1 \odot H) - 1 = \psi(H)$. Thus, $\psi(G \odot H) = n_G \psi(H)$ by Theorem 3.5. □

From Lemma 2.10 and Theorem 3.11, the following result is obtained as n_G pendant vertices of $G \odot K_1$ form a DR set of $G \odot K_1$.

Corollary 4.4. *Let G and H be two connected graphs of order $n_G \geq 2$ and $n_H \geq 1$, respectively. Then $\psi(G \odot H) = n_G$ for $n_H = 1$, $n_G \leq \psi(G \odot H) \leq n_G(n_H - 1)$ for $n_H \geq 2$ with left equality if and only if $H = K_2$ and right equality if and only if $H \in \{K_{n_H}, K_{1,n_H-1}\}$.*

Lemma 4.5. *Let G and H be two connected graphs of order $n_G \geq 2$ and $n_H \geq 6$, respectively. Then $\psi(G \odot H) \geq 3n_G$.*

Proof. Let S be a minimum DR set of $G \odot H$. We have $S \bigcap V(G) = \emptyset$ and $|S \bigcap V_i| \geq 2$ by Lemma 2.11 (*iv*) and the proof of Lemma 3.10. Next, we show $|S \bigcap V_i| \geq 3$ for $n_H \geq 6$. To the contrary, assume that $S_i = S \bigcap V_i$ and $|S_i| = 2$. Let $W = V_i \setminus S_i$. Then we have $|W| = n_H - 2 \geq 4$. Note that $d_{G \odot H}(a, s) \in \{1, 2\}$ for $a \in W$ and $s \in S_i$. There are at most 2^2 different representations of vertices in W since $d_{G \odot H}(a, t) = d_{G \odot H}(b, t)$ for $a, b \in W$ and $t \in S \setminus S_i$. If there is a vertex $b \in W$ with $r(b|S_i) = (2, 2)$, then $r(b|S) - r(u_i|S) = \overrightarrow{1}$ as $r(u_i|S_i) = (1, 1)$ and $d_{G \odot H}(b, t) - d_{G \odot H}(u_i, t) = 1$ for $t \in S \setminus S_i$, a contradiction. Thus, we have $|W| \leq 2^2 - 1 = 3$, this contradicts the fact that $|W| \geq 4$. Hence, $|S_i| \geq 3$ and $\psi(G \odot H) \geq 3n_G$. □

Let $\mathcal{A} = \{C_n, P_n, W_4, F_4\}$ for $3 \leq n \leq 5$ and $\mathcal{B}$ consist of all graphs in Fig. 1.

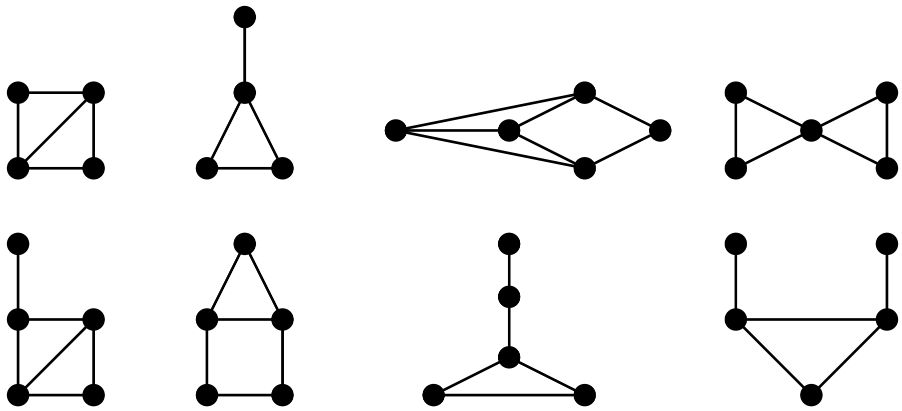

Figure 1: The set $\mathcal{B}$ of graphs

Theorem 4.6. *Let G and H be two connected graphs of order $n_G \geq 2$ and $n_H \geq 3$, respectively. Then $\psi(G \odot H) \geq 2n_G$ with equality if and only if $H \in \mathcal{A} \bigcup \mathcal{B}$.*

Proof. By Lemma 3.10, we have $\psi(G \odot H) \geq 2n_G$. For characterizing the graphs with $\psi(G \odot H) = 2n_G$, it suffices to consider the graph H of order $3 \leq n_H \leq 5$ by Lemma 4.5. Through the computer, we find that there are 2, 6 and 21 non-isomorphic connected graphs of order 3, 4 and 5, respectively. Moreover, it is routine to verify that $\psi(G \odot H) = 2n_G$ if and only if $H \in \mathcal{A} \bigcup \mathcal{B}$. Hence, the result follows. □

Next, we determine the exact value of $\psi(G \odot H)$ in the following where $H \in \{P_n, C_n\}$. The metric dimension of wheel graphs was determined by Buczkowski *et al.* [2], and Cáceres *et al.* [3] showed $\beta(F_n) = \beta(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$. We can check that $\psi(W_3) = \psi(W_4) = \psi(W_5) = \psi(F_3) = \psi(F_5) = 3$ and $\psi(F_4) = 2$. Next, we only need to consider $n \geq 6$.

Lemma 4.7. *Let u be the universal vertex of W_n or F_n, where $n \geq 6$. If S is a minimum DR set of W_n or F_n, then $u \notin S$.*

Proof. We first consider W_n for $n \geq 6$. Assume, to the contrary, that $u \in S$. Then, $D = S \setminus \{u\}$ is not a DR set of W_n. There are two vertices $x, y \in V(W_n)$ such that $r(x|D) - r(y|D) = \overrightarrow{c}$. We have $|D| \geq 2$ and $c \in \{0, \pm 1\}$ since $n \geq 6$ and $diam(W_n) = 2$.

Suppose that $x \neq u$ and $y \neq u$. First we have $r(x|D) - r(y|D) \neq \overrightarrow{0}$. Otherwise $r(x|S) - r(y|S) = \overrightarrow{0}$ as a contradiction to the fact that S is a doubly resolving set of W_n. Assume, w.l.o.g., that $r(x|D) - r(y|D) = \overrightarrow{1}$. Obviously, $x \notin D$ and $d(y, s) \leq 1$ for $s \in D$. For $y \in D$, we may assume that $r(y|D) = (0, 1, 1)$ or $r(y|D) = (0, 1)$. If $r(y|D) = (0, 1, 1)$, then we obtain $r(x|D) = (1, 2, 2)$ and $x \in N_{C_n}(y)$. It is clear that $N_{C_n}(y) \subseteq D$, we get the contradiction. If $r(y|D) = (0, 1)$, then $r(x|D) = (1, 2)$. There is a vertex $z \in N_{C_n}(x) \setminus \{y\}$ satisfying $r(z|D) = (2, 2)$. We can easily acquire $r(z|S) - r(u|S) = \overrightarrow{1}$, which leads to a contradiction. For $y \notin D$, we have $r(y|D) = \overrightarrow{1}$ and $r(x|D) = \overrightarrow{2}$. Then, $r(x|D) - r(u|D) = \overrightarrow{1}$ as $r(u|D) = \overrightarrow{1}$. Therefore, $r(x|S) - r(u|S) = \overrightarrow{1}$ is impossible since S is a DR set of W_n.

Suppose that $x = u$ and $y \neq u$. We have $r(x|D) = \overrightarrow{1}$. If $r(x|D) - r(y|D) = \overrightarrow{0}$, then $r(y|D) = (1, 1)$ and $|S| = 3$. As $n \geq 6$, there always exists a vertex $u_i \in V(C_n)$ satisfying $r(u_i|D) = (2, 2)$, which implies that $r(u_i|S) - r(u|S) = \overrightarrow{1}$, a contradiction. If $r(x|D) - r(y|D) = \overrightarrow{1}$, then $r(y|D) = \overrightarrow{0}$, it is impossible. If $r(x|D) - r(y|D) = \overrightarrow{-1}$, then $r(y|D) = \overrightarrow{2}$. Hence, $r(y|S) - r(x|S) = \overrightarrow{1}$ is a contradiction.

Therefore, $u \notin S$. Analogously, we can obtain $u \notin S$ in F_n and thus we omit the proof. □

Let $\{u_i, u_j\} \subseteq S \subseteq V(C_n)$ and Q be a u_i, u_j-path in C_n. If Q contains only two vertices of S, then u_i and u_j are *neighboring vertices* in S and the set of internal vertices of Q is a *gap* of S. Two gaps of S are *neighboring gaps* if they are determined by a vertex in S and its two neighboring vertices in S. Certainly, if $|S| = t$, then there are t gaps of S in C_n and the gaps of S can be empty in this definition.

Lemma 4.8. *Let W_n be a wheel graph of $n \geq 6$ and $S \subseteq V(C_n)$. Then S is a DR set of W_n if and only if the following two conditions hold:*

(*i*) *Each gap of S consists of at most two vertices.*

(*ii*) *The neighboring gaps of a gap with two vertices consist of at most one vertex.*

Proof. Let $V(W_n) = \{u, u_1, u_2, \ldots, u_n\}$ and $E(W_n) = \{uu_r, u_r u_{r+1} : 1 \leq r \leq n\}$. We first prove the necessity. Let S be a DR set of W_n. To the contrary, assume that the gap $\{u_i, \ldots, u_{i+k}\}$ of S consists of at least three vertices. It is easy to obtain $d(u_{i+1}, s) - d(u, s) = 1$ for $s \in S$, this leads to a contradiction. Assume that there is a neighboring gap $\{u_{i-1}, u_i\}$ of a gap $\{u_{i+2}, u_{i+3}\}$. These two gaps are determined by $u_{i+1}, u_{i-2}, u_{i+4} \in S$. Then, $d(u_i, u_{i+1}) - d(u_{i+2}, u_{i+1}) = 0$ and $d(u_i, s) - d(u_{i+2}, s) = 0$ for $s \in S \setminus \{u_{i+1}\}$, a contradiction.

Now we consider the sufficiency. Let $S \subseteq V(C_n)$ and S satisfy (*i*) and (*ii*). Our aim is to show that S is a DR set of W_n. For $u, u_j \in V(W_n)$, if $u_j \in S$, then there always exists a vertex $s \in S \setminus \{u_j\}$ such that $d(u, u_j) - d(u_j, u_j) \neq d(u, s) - d(u_j, s)$. If $u_j \notin S$, then u_j belongs to a gap with size 1 or a gap with size 2. Let $u_{j-1}, u_j, u_{j+1} \in V(C_n)$ and $u_{j-1}, u_{j+1} \in S$. As $n \geq 6$, $d(u, s) - d(u_j, s) \neq d(u, u_{j-1}) - d(u_j, u_{j-1})$ for some $s \in S \setminus \{u_{j-1}, u_{j+1}\}$. Let $u_{j-1}, u_j, u_{j+1}, u_{j+2} \in V(C_n)$ and $u_{j-1}, u_{j+2} \in S$. Then $d(u, u_{j-1}) - d(u_j, u_{j-1}) \neq d(u, u_{j+2}) - d(u_j, u_{j+2})$.

For $u_i, u_j \in V(C_n)$, if $u_i, u_j \in S$, then u_i and u_j are doubly resolved by u_i and u_j. If $u_i \in S$ and $u_j \notin S$, then $d(u_i, u_i) - d(u_j, u_i) \leq -1$ and $d(u_i, s) - d(u_j, s) \geq 0$ for some $s \in S \setminus \{u_i\}$. The vertex s always exists because the set S satisfies (*i*) and (*ii*). We analyze $u_i, u_j \notin S$ by the following four cases.

Case 1. u_i, u_j belong to a same gap of S.

The gap is determined by $u_{i'}, u_{j'} \in S$, where $u_{i'} u_i, u_{j'} u_j \in E(W_n)$. We have $d(u_i, u_{i'}) - d(u_j, u_{i'}) = -1 \neq 1 = d(u_i, u_{j'}) - d(u_j, u_{j'})$.

Case 2. u_i belongs to a gap R with size 1, u_j belongs to a gap R^* with size 1.

Suppose that R and R^* are neighboring gaps. There are five consecutive vertices $u_{i-1}, u_i, u_{i+1}, u_j, u_{j+1} \in V(C_n)$ and $u_{i-1}, u_{i+1}, u_{j+1} \in S$. We have $d(u_i, u_{i+1}) - d(u_j, u_{i+1}) \neq d(u_i u_{i-1}) - d(u_j, u_{i-1})$. Suppose that R and R^* are not neighboring gaps. Let $u_{i-1}, u_i, u_{i+1}, u_{j-1}, u_j, u_{j+1} \in V(C_n)$ and $u_{i-1}, u_{i+1}, u_{j-1}, u_{j+1} \in S$. We obtain $d(u_i, u_{i-1}) - d(u_j, u_{i-1}) \neq d(u_i, u_{j-1}) - d(u_j, u_{j-1})$.

Case 3. u_i belongs to a gap R with size 1, u_j belongs to a gap R^* with size 2.

Let $u_{i-1}, u_i, u_{i+1}, u_{j-1}, u_j, u_{j+1}, u_{j+2} \in V(C_n)$ and $u_{i-1}, u_{i+1}, u_{j-1}, u_{j+2} \in S$. Suppose that R and R^* are neighboring gaps. Assume, w.l.o.g., that $u_{i+1} = u_{j-1}$. We have $d(u_i, u_{i+1}) - d(u_j, u_{i+1}) \neq d(u_i, u_{i-1}) - d(u_j, u_{i-1})$. Suppose that R and R^* are not neighboring gaps. It is evident to find that $d(u_i, u_{i+1}) - d(u_j, u_{i+1}) \neq d(u_i, u_{j-1}) - d(u_j, u_{j-1})$.

Case 4. u_i belongs to a gap R with size 2, u_j belongs to a gap R^* with size 2.

These two gaps R and R^* are not neighboring gaps since S satisfies (*ii*). Let $u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{j-1}, u_j, u_{j+1}, u_{j+2} \in V(C_n)$ and $u_{i-1}, u_{i+2}, u_{j-1}, u_{j+2} \in S$. We have $d(u_i, u_{i-1}) - d(u_j, u_{i-1}) \neq d(u_i, u_{j-1}) - d(u_j, u_{j-1})$.

As mentioned above, the set S is a DR set of W_n. □

Proposition 4.9. $\psi(W_n) = \lceil \frac{2n}{5} \rceil$ *for* $n \geq 6$.

Proof. Let $V(W_n) = \{u, u_1, u_2, \ldots, u_n\}$ and $E(W_n) = \{uu_r, u_r u_{r+1} : 1 \leq r \leq n\}$. We first show $\psi(W_n) \leq \lceil \frac{2n}{5} \rceil$ by dividing into the following five cases.

Suppose that $n \equiv 0 \ (mod\ 5)$. Let $n = 5r$ with $r \geq 2$ and $\lceil \frac{2n}{5} \rceil = 2r$. Then we construct $S = \{u_{5i+1}, u_{5i+4} : 0 \leq i \leq r-1\}$, where $|S| = 2r$ and S satisfies (*i*) and (*ii*). Therefore, S doubly resolves W_n by Lemma 4.8.

Suppose that $n \equiv 1 \ (mod\ 5)$. Let $n = 5r + 1$ with $r \geq 1$ and $\lceil \frac{2n}{5} \rceil = 2r + 1$. Set $S = \{u_{5i+1}, u_{5i+4} : 0 \leq i \leq r-1\} \bigcup \{u_{5r}\}$, where $|S| = 2r + 1$ and S satisfies (*i*) and (*ii*). Hence, S doubly resolves W_n by Lemma 4.8.

Suppose that $n \equiv 2 \ (mod\ 5)$. Let $n = 5r + 2$ with $r \geq 1$ and $\lceil \frac{2n}{5} \rceil = 2r + 1$. We can construct $S = \{u_{5i+1}, u_{5i+3} : 0 \leq i \leq r-1\} \bigcup \{u_{5r+1}\}$, where $|S| = 2r + 1$ and S satisfies (*i*) and (*ii*). Hence, S doubly resolves W_n by Lemma 4.8.

Suppose that $n \equiv 3 \ (mod\ 5)$. Let $n = 5r + 3$ with $r \geq 1$ and $\lceil \frac{2n}{5} \rceil = 2r + 2$. Then we construct $S = \{u_{5i+1}, u_{5i+3} : 0 \leq i \leq r\}$, where $|S| = 2r + 2$ and S satisfies (*i*) and (*ii*). Hence, S doubly resolves W_n by Lemma 4.8.

Suppose that $n \equiv 4 \ (mod\ 5)$. Let $n = 5r + 4$ with $r \geq 1$ and $\lceil \frac{2n}{5} \rceil = 2r + 2$. Let $S = \{u_{5i+1}, u_{5i+3} : 0 \leq i \leq r\}$, where $|S| = 2r + 2$ and S satisfies (*i*) and (*ii*). Then S doubly resolves W_n by Lemma 4.8.

As stated above, $\psi(W_n) \leq \lceil \frac{2n}{5} \rceil$. Let S be a minimum DR set of W_n. We show $\psi(W_n) \geq \lceil \frac{2n}{5} \rceil$ in the following.

Suppose that $|S| = 2t$. There are at most $2t$ gaps in C_n. By Lemma 4.8, there are at most t gaps with two vertices. Thus, the number of vertices in the gaps of S is at most $3t$. We have $n - 2t \leq 3t$, and so $|S| = 2t \geq \lceil \frac{2n}{5} \rceil$.

Suppose that $|S| = 2t + 1$. There are at most $2t + 1$ gaps in C_n. At most t gaps consist of two vertices by Lemma 4.8. Hence, the number of vertices in the gaps of S is at most $3t + 1$. Then, $n - 2t - 1 \leq 3t + 1$ and $|S| = 2t + 1 \geq \lceil \frac{2n}{5} \rceil$. □

Proposition 4.10. *$\psi(F_n) = \lceil \frac{2n}{5} \rceil$ for $n \geq 6$.*

Proof. By the proof of Proposition 4.9, we have constructed the DR set S of W_n that can produce a gap of S with two vertices in C_n. Deleting the edge between these two vertices of the gap does not change the distance between any vertices with elements of S. Note that F_n is obtained by deleting an edge of C_n in W_n. It is simple to get that the set S is also a DR set of F_n. Therefore, $\psi(F_n) \leq \lceil \frac{2n}{5} \rceil$.

Let $D \subseteq V(W_n)$ consist of at most $\lceil \frac{2n}{5} \rceil - 1$ vertices. It is clear that D is not a DR set of W_n. Let $V(W_n) = \{u, u_1, u_2, \ldots, u_n\}$ and $E(W_n) = \{uu_r, u_r u_{r+1} : 1 \leq r \leq n\}$. We only need to consider $D \subseteq V(C_n)$ by Lemma 4.7. Next, according to Lemma 4.8, the following two cases are distinguished.

Suppose that there is a gap of D with at least three vertices. Let $\{u_{i-1}, u_i, \ldots, u_{i+k}\}$ be the gap. Then $r(u_i|D) = \overrightarrow{2}$. Deleting any edge of $E(C_n)$ in W_n can not change the representation of u_i. Thus, D is not a DR set of F_n as $r(u|D) = \overrightarrow{1}$.

Suppose that there are two neighboring gaps of D with two vertices. Let $\{u_i, u_{i+1}\}$ and $\{u_{i+3}, u_{i+4}\}$ be the two neighboring gaps, which are determined by $u_{i-1}, u_{i+2}, u_{i+5} \in D$. If we delete any edge of $E(C_n)$ in W_n, then it is not difficult to check that u_{i+1} and u_{i+3}, u_{i+1} and u, or u_{i+3} and u are not doubly resolved by D.

From above, we have $\psi(F_n) \geq \lceil \frac{2n}{5} \rceil$ and $\psi(F_n) = \lceil \frac{2n}{5} \rceil$ for $n \geq 6$. □

Propositions 4.9 and 4.10 can be extended as follows.

Theorem 4.11. *Let G be a nontrivial connected graph and H be a path or cycle. Then $\psi(G \odot H) = n_G \lceil \frac{2n_H}{5} \rceil$.*

Proof. By Corollary 4.4, $\psi(G \odot H) = n_G \lceil \frac{2n_H}{5} \rceil$ holds for $n_H \leq 2$. Moreover, $\psi(G \odot H) = n_G \lceil \frac{2n_H}{5} \rceil$ for $3 \leq n_H \leq 5$ follows from Theorem 4.6. Combining Theorem 3.5, Propositions 4.9 and 4.10, we have $\psi(G \odot H) = n_G \lceil \frac{2n_H}{5} \rceil$ for $n_H \geq 6$, ending the proof. □

Note that $diam(K_1 \odot H) \leq 2$ for any graph H. From Lemma 2.1 and Proposition 4.2, the following result holds.

Corollary 4.12. *Let $G = K_1 \odot H$ be a graph of order $n \geq 3$. Then $2 \leq \psi(K_1 \odot H) \leq n - 1$ with left equality if and only if $H \in \{N_2, K_2 \bigcup K_1, K_2 \bigcup K_2, P_4\}$ and right equality if and only if $H \in \{K_{n-1}, N_{n-1}, K_{1,n-2}\}$.*

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