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On Π−Nekrasov matrices

Dunja Arsić[a], Maja Nedović[a]

[a]*Department for Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21000 Novi Sad, Serbia*

Abstract. In this paper, we consider Π−Nekrasov matrices, a generalization of $\{P_1, P_2\}$−Nekrasov matrices obtained by introducing the set $\Pi = \{P_1, P_2, ..., P_m\}$ of m simultaneous permutations of rows and columns of the given matrix. For point-wise and block Π−Nekrasov matrices we give infinity norm bounds for the inverse. For Π−Nekrasov B−matrices, obtained through a special rank one perturbation, we present main results on infinity norm bounds for the inverse and error bounds for linear complementarity problems. Numerical examples illustrate the benefits of new bounds.

1. Introduction

In the paper by Varah, see [28], upper bound for the infinity norm of the inverse for strictly diagonally dominant matrices was obtained. In recent years, Varah bound was modified in many different ways, in order to obtain infinity norm bounds of the inverse for different subclasses of H−matrices, see [7, 9, 17, 18, 21, 22].
The most direct application of these results lies in determination of upper bounds for the condition number of a matrix:

$$\kappa(A) = \|A\|\|A^{-1}\|.$$

Also, many researchers have used results on bounding the norm of the inverse for bounding errors in linear complementarity problems, see [11, 12, 14, 25]. In this paper we expand these results to other matrix classes, such as Π−Nekrasov, block Π−Nekrasov and Π−Nekrasov B−matrices.

The starting point in the following considerations is the well-known class of Nekrasov matrices. Unlike the class of strictly diagonally dominant matrices, Nekrasov class is not closed under simultaneous permutations of rows and columns. Therefore, different authors have considered modifications of Nekrasov condition involving permutation matrices. One way to define a generalization of Nekrasov class is to consider so-called Gudkov matrices, matrices that can be transformed to Nekrasov matrices through a similarity permutation, see [16]. Another modification of Nekrasov condition was given in the form of $\{P_1, P_2\}$−Nekrasov matrices, see [8]. For a matrix to be $\{P_1, P_2\}$−Nekrasov, the permutation P_1 should repair some of the rows in the original matrix (transforming these rows to Nekrasov-dominant rows), while P_2 should repair the remaining rows. The most common choice for P_1 and P_2 consists of identical and counteridentical permutation.

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Email addresses: dunjaarsic@uns.ac.rs (Dunja Arsić), maja.nedovic@uns.ac.rs (Maja Nedović)

Conditions that involve permutation matrices, especially similarity permutations of rows and columns, have interesting interpretations in applications. For instance, in relation with the adjacency graph, a similarity permutation can be viewed as relabeling the vertices of the graph without changing the edges, see [24]. Changing the order in which the nodes are approached in a specific algorithm can, in some cases, significantly affect the performance of the algorithm. In this paper we consider involving additional permutations in order to improve results on infinity norm bounds for the inverse and error bounds for linear complementarity problems. We present norm bounds for the inverse both in point-wise and the block case of $\Pi-$Nekrasov matrices.

There are matrices that, in the point-wise case, don't belong to the class of $H-$matrices at all, but, for some choices of partition of the index set, do belong to block $\Pi-$Nekrasov matrices. For these matrices, well-known bounds for the norm of the inverse in the point-wise case cannot be applied, but block case bounds can be applied. This is true for matrices with zero diagonal entries, that often appear in mathematical models in ecology or population biology. Namely, when modelling ecological systems, self-interactions of included populations are often considered to be zero, see [15]. Therefore, the corresponding matrices have all the diagonal entries equal to zero. These matrices are called hollow matrices and they don't belong to the class of $H-$matrices in the usual point-wise sense. Therefore, when dealing with hollow matrices, none of the results and tools developed for $H-$matrices can be applied. In some of these cases we can use the block approach.

The main results of the paper refer to the class of $\Pi-$Nekrasov $B-$matrices. This class is a generalization of $\Pi-$Nekrasov class (in the real case) through a rank one perturbation. It is the subclass of the class of $P-$matrices, real square matrices with all principal minors positive. For these matrices, we obtained infinity norm bound for the inverse and error bound for linear complementarity problems. Numerical examples show that new error bounds can give tighter results in some cases compared to already known error bounds for $S-$Nekrasov and $B-S-$Nekrasov matrices. Moreover, new bounds work in some cases where bounds developed for $S-$Nekrasov and $B-S-$Nekrasov matrices cannot be applied, for any choice of the partition of the index set into S and $\overline{S}$.

The paper is organized as follows. In the remainder of Section 1 we recall preliminaries on Nekrasov and $\{P_1, P_2\}-$Nekrasov matrices together with well-known infinity norm bounds for the inverse of these matrices. In Section 2, we consider $\Pi-$Nekrasov matrices and block generalizations of $\Pi-$Nekrasov matrices and define improved norm bounds compared to bounds given in [22]. In Section 3, the main results are presented. We consider the class of $\Pi-$Nekrasov $B-$matrices and define infinity norm bounds for the inverse and error bounds for linear complementarity problems for matrices in this class. Section 4 consists of numerical examples that illustrate the effectiveness of new bounds, comparisons to already known results and concluding remarks.

It is well-known that a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called strictly diagonally dominant (SDD) matrix if, for each $i \in N$, it holds that

$$|a_{ii}| > r_i(A) = \sum_{k \in N, k \neq i} |a_{ik}|,$$

or, in the form of vectors, $d(A) > r(A)$, where $r(A) = [r_1(A), ..., r_n(A)]^T$ is the vector of deleted row sums, and the vector of moduli of diagonal entries is given by $d(A) = [|a_{11}|, ..., |a_{nn}|]^T$. A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an $H-$matrix if its comparison matrix $\langle A \rangle = [m_{ij}]$ defined by

$$\langle A \rangle = [m_{ij}] \in \mathbb{C}^{n,n}, \qquad m_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j \end{cases}$$

is an $M-$matrix, i.e., $\langle A \rangle^{-1} \geq 0$, see [1]. For some subclasses of $H-$matrices we know how to construct a corresponding diagonal scaling matrix that transforms the given $H-$matrix to an SDD matrix, see [10]. This can be used further in obtaining eigenvalue localization, investigation of Schur complement properties, see [27, 29], or in determining error bounds for linear complementarity problems, see [14]. In [1] it is pointed out that for any nonsingular $H-$matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $|A^{-1}| \leq \langle A \rangle^{-1}$.

Nekrasov matrices, see [16, 20], are defined by condition

$$|a_{ii}| > h_i(A), \quad \text{for all } \; i \in N,$$

or, in vector form, $d(A) > h(A)$, where the sums $h_i(A)$, $i \in N$ are defined recursively by

$$h_1(A) = r_1(A), \quad h_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \;\; i = 2, 3, \dots n,$$

and $h(A) = [h_1(A), ..., h_n(A)]^T$. For a given matrix A, $A = D - L - U$ represents the standard splitting of A into its diagonal (D), strictly lower $(-L)$ and strictly upper $(-U)$ triangular parts.

As the *SDD* class is closed under simultaneous permutations of rows and columns, while the class of Nekrasov matrices is not, the following generalization of Nekrasov condition was considered.

For a permutation matrix P, of order $n \geq 2$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a $P-$Nekrasov matrix if P^TAP is a Nekrasov matrix. In other words, if

$$|(P^TAP)_{ii}| > h_i(P^TAP), \quad \text{for all } \; i \in N,$$

or, in vector form, $d(P^TAP) > h(P^TAP)$. The union of all $P-$Nekrasov classes by all corresponding permutation matrices P is known as Gudkov class, see [16, 26].

Suppose that for the given matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ and two given permutation matrices P_1 and P_2,

$$d(A) > \min\left\{h^{P_1}(A), h^{P_2}(A)\right\},$$

where

$$h^{P_k}(A) = P_k h(P_k^T A P_k), \;\; k = 1, 2.$$

We call such a matrix $\{P_1, P_2\}-$Nekrasov matrix, see [8].

Examples show, see [8], that it is possible that, for the set of two given permutation matrices $\{P_1, P_2\}$, the given matrix A is neither P_1-Nekrasov nor P_2-Nekrasov, but A does belong to $\{P_1, P_2\}-$Nekrasov class.

With permutations involved, Nekrasov row sums can be expressed as follows.

Lemma 1.1 ([8]). *Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, with $a_{ii} \neq 0$ for all $i \in N$, and given a permutation matrix $P \in \mathbb{C}^{n,n}$, then*

$$h_i^P(A) = |a_{ii}|\Big(P(|D_P| - |L_P|)^{-1}|U_P|e\Big)_i,$$

where $e \in \mathbb{C}^n$ is the vector with all components equal to 1 and D_P is diagonal, $-L_P$ strictly lower and $-U_P$ strictly upper triangular part of the matrix P^TAP, i.e., $P^TAP = D_P - L_P - U_P$ is the standard splitting of the matrix P^TAP.

Let us now recall well-known results on bounding the norm of the inverse for *SDD*, Nekrasov and $\{P_1, P_2\}-$Nekrasov matrices.

Theorem 1.2 ([28]). *Given an SDD matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, the following bound applies:*

$$\|A^{-1}\|_\infty \leq \frac{1}{\min\limits_{i \in N}(|a_{ii}| - r_i(A))}.$$

Theorem 1.3 ([18]). *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a Nekrasov matrix. Then*

$$\|A^{-1}\|_\infty \leq \max_{i \in N} \frac{z_i(A)}{|a_{ii}| - h_i(A)},$$

where $z_1(A) = 1$ and $z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1$, $i = 2, 3, \dots n$.

Norm bounds for $\{P_1, P_2\}-$Nekrasov matrices were presented in [8]. In [30] the improved norm bound for this type of matrices is given as follows.

Theorem 1.4 ([30]). *Suppose that, for a given set of permutation matrices $\{P_1, P_2\}$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, is a $\{P_1, P_2\}-$Nekrasov matrix. Then,*

$$\|A^{-1}\|_\infty \leq \max_{i \in N} \frac{\frac{z_i^{P_{k_i}}(A)}{|a_{ii}|}}{1 - \min\left\{\frac{h_i^{P_1}(A)}{|a_{ii}|}, \frac{h_i^{P_2}(A)}{|a_{ii}|}\right\}},$$

where $z_1(A) = 1$, $z_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1$, $i = 2, 3, \ldots n$, the corresponding vector is $z(A) = [z_1(A), ..., z_n(A)]^T$, $z^P(A) = Pz(P^T AP)$, and for the given $i \in N$ the corresponding index $k_i \in \{1, 2\}$ is chosen in such a way that

$$\min\left\{h_i^{P_1}(A), h_i^{P_2}(A)\right\} = h_i^{P_{k_i}}(A).$$

In the recent paper [17], bounds for the norm of the inverse for $P-$Nekrasov matrices and $\{P_1, P_2\}-$Nekrasov matrices were further improved in the following way.

Theorem 1.5 ([17]). *Let $P \in \mathbb{R}^{n,n}$ be a permutation matrix of order $n \geq 2$ and let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a $P-$Nekrasov matrix. Then*

$$\|A^{-1}\|_\infty \leq \max_{i \in N} \frac{\{G^P e\}_i}{\{G^P \langle A \rangle e\}_i} = \max_{i \in N} \frac{e_i^T G^P e}{e_i^T G^P \langle A \rangle e},$$

where the matrix $G^P = G^P(A)$ is defined in the following way

$$G^P = P(|D_P| - |L_P|)^{-1} P^T.$$

Theorem 1.6 ([17]). *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, be a $\{P_1, P_2\}-$Nekrasov matrix, where $P_1, P_2 \in \mathbb{R}^{n,n}$ are some permutation matrices. Then*

$$\|A^{-1}\|_\infty \leq \max_{i \in N} \frac{\{Ge\}_i}{\{G \langle A \rangle e\}_i},$$

where the matrix $G = G(A, P_1, P_2)$ is defined in the following way. The $i-$th row of the matrix G coincides with the $i-$th row of the matrix G^{P_1} if either the $i-$th row of the matrix $G^{P_2} \langle A \rangle$ is not strictly diagonally dominant or

$$\frac{e_i^T G^{P_1} e}{e_i^T G^{P_1} \langle A \rangle e} \leq \frac{e_i^T G^{P_2} e}{e_i^T G^{P_2} \langle A \rangle e}.$$

Similarly, the $i-$th row of the matrix G coincides with the $i-$th row of the matrix G^{P_2} if either the $i-$th row of the matrix $G^{P_1} \langle A \rangle$ is not strictly diagonally dominant or

$$\frac{e_i^T G^{P_2} e}{e_i^T G^{P_2} \langle A \rangle e} \leq \frac{e_i^T G^{P_1} e}{e_i^T G^{P_1} \langle A \rangle e}.$$

Applying the previous result to $P-$Nekrasov case, the bound for the norm of the inverse for $P-$Nekrasov matrices was further improved.

Theorem 1.7 ([17]). *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, and let $P_1, P_2 \in \mathbb{R}^{n,n}$ be permutation matrices. Assume that A is a P_1-Nekrasov matrix. Then, A is a $\{P_1, P_2\}-$Nekrasov matrix and*

$$\|A^{-1}\|_\infty \leq \max_{i \in N} \frac{\{Ge\}_i}{\{G \langle A \rangle e\}_i} \leq \max_{i \in N} \frac{\{G^{P_1} e\}_i}{\{G^{P_1} \langle A \rangle e\}_i},$$

where the matrix $G = G(A, P_1, P_2)$ is defined as in Theorem 1.6 and the matrix $G^{P_1} = G^{P_1}(A)$ is defined as in Theorem 1.5.

Applying these results, in the following section we give norm bounds for the inverse of $\Pi-$Nekrasov matrices in the point-wise and block case. The bounds presented in the next section improve the bounds given in [22].

2. Infinity norm bounds for the inverse of Π−Nekrasov and block Π−Nekrasov matrices

In [8], it is noted that $\{P_1, P_2\}$−Nekrasov condition could be generalized by involving more than two permutation matrices, as follows.

Definition 2.1. *Given a set of m permutation matrices $\Pi = \{P_k\}_{k=1}^m$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is a Π−Nekrasov matrix if*

$$d(A) > \min_{k=1,\dots,m} h^{P_k}(A).$$

We could also formulate this generalization in a slightly different manner. Assume that for a given matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ it holds that for each $i \in N$ there exists a permutation matrix P_i such that $|a_{ii}| > h_i^{P_i}(A)$. In that way, we obtain more general condition by changing the order of quantifiers in Gudkov condition (in the same fashion as CKV-type condition is obtained from CKV condition, see [5, 19]). Matrices satisfying this condition belong to nonsingular H−matrices. This can be proved following the arguments given in [17].

Lemma 2.2. *Given a set of permutation matrices $\Pi = \{P_1, P_2, \dots, P_m\}$, $P_1, P_2, \dots, P_m \in \mathbb{R}^{n,n}$, $n \geq 2$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a Π−Nekrasov matrix if and only if*

$$\max_{j=1,2,\dots,m} \{e_i^T G^{P_j} \langle A \rangle e\} > 0, \quad i = 1, 2, \dots, n.$$

Theorem 2.3. *Given a set of m permutation matrices $\Pi = \{P_1, P_2, \dots, P_m\}$, $P_1, P_2, \dots, P_m \in \mathbb{R}^{n,n}$, $n \geq 2$, any Π−Nekrasov matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a nonsingular H−matrix.*

Proof: From Lemma 2.2 it follows that a matrix $G = G(A, \Pi)$ can be composed of the rows of the matrices G^{P_j}, $j = 1, 2, \dots, m$ such that

$$B := G\langle A \rangle$$

is an *SDD* M−matrix. More precisely, the i−th row of the matrix G, $1 \leq i \leq n$ coincides with the i−th row of G^{P_k}, $k \in \{1, 2, \dots, m\}$, if

$$\frac{e_i^T G^{P_k} e}{e_i^T G^{P_k} \langle A \rangle e} = \min_{j \in L} \frac{e_i^T G^{P_j} e}{e_i^T G^{P_j} \langle A \rangle e},$$

where L is a subset of $\{1, 2, \dots, m\}$ and the set of indices such that for every $j \in L$ the i−th row of the matrix $G^{P_j} \langle A \rangle$ is strictly diagonally dominant.

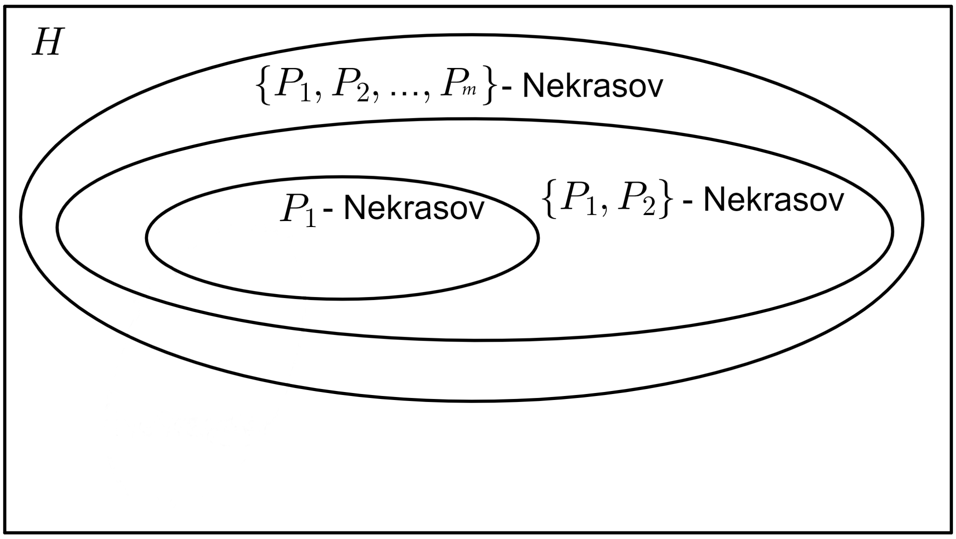
Since the *SDD* matrix B is nonsingular, so are the matrices G and $\langle A \rangle$. Furthermore, it follows that

$$\langle A \rangle^{-1} = B^{-1} G.$$

Since both matrices B^{-1} and G are nonnegative, $\langle A \rangle$ is monotone, whence it is a nonsingular M−matrix. This proves that A is a nonsingular H−matrix.

□

For the given set of permutation matrices $\Pi = \{P_1, P_2, \dots, P_m\}$, the following relations between the corresponding matrix classes hold.



Example in Section 4 shows that there are matrices that don't belong to any of the classes $\{P_1, P_2\}-$Nekrasov, $\{P_1, P_3\}-$Nekrasov, $\{P_2, P_3\}-$Nekrasov, but do belong to a class of $\{P_1, P_2, P_3\}-$Nekrasov matrices. Note that in cases when the identical permutation is included in Π, Nekrasov class is a subclass of $\Pi-$Nekrasov class.

According to [17], the following bound for the infinity norm of the inverse holds.

Theorem 2.4. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n},\ n \geq 2,$ be a $\Pi-$Nekrasov matrix, where Π is the set of m permutation matrices $P_1, P_2, ..., P_m \in \mathbb{R}^{n,n}$. Then*

$$\|A^{-1}\|_\infty \leq \max_{i\in N} \frac{\{Ge\}_i}{\{G\langle A\rangle e\}_i}, \tag{1}$$

where the matrix $G = G(A, \Pi)$ is defined in the following way. The $i-$th row of the matrix G coincides with the $i-$th row of the matrix G^{P_k}, $k \in \{1, 2, ..., m\}$ if

$$\frac{e_i^T G^{P_k} e}{e_i^T G^{P_k}\langle A\rangle e} = \min_{j\in L} \frac{e_i^T G^{P_j} e}{e_i^T G^{P_j}\langle A\rangle e},$$

where L is a subset of $\{1, 2, ..., m\}$ and the set of indices such that for every $j \in L$ the $i-$th row of the matrix $G^{P_j}\langle A\rangle$ is strictly diagonally dominant.

Proof: By Theorem 2.3, A is a nonsingular $H-$matrix and the matrix $B = [b_{ij}] = G\langle A\rangle$ is an SDD $M-$matrix. G is also nonsingular and nonnegative.

In [17], it is proved that

$$\|A^{-1}\|_\infty \leq \|\langle A\rangle^{-1}\|_\infty \leq \|(\Delta^{-1}B)^{-1}\|_\infty,$$

where the diagonal matrix $\Delta = diag(\delta_1, ..., \delta_n)$ is defined by the relation $\Delta e = Ge$.

Since both B and $\Delta^{-1}B$ are SDD matrices, by applying to $\Delta^{-1}B$ the classical bound for the inverse of an SDD matrix, we immediately obtain the inequality

$$\|(\Delta^{-1}B)^{-1}\|_\infty \leq \max_{1\leq i\leq n} \frac{\{Ge\}_i}{|b_{ii}| - r_i(B)}.$$

In order to complete the proof, it remains to observe that for the $M-$matrix B we have

$$|b_{ii}| - r_i(B) = \{G\langle A\rangle e\}_i, \quad i = 1, ..., n.$$

□

Notice that if Π is the set of permutation matrices $P_1, P_2, ..., P_m \in \mathbb{R}^{n,n}$ and if A is a P_1-Nekrasov matrix, then A is also a $\{P_1, P_2\}-$Nekrasov matrix and a $\Pi-$Nekrasov matrix and

$$\|A^{-1}\|_\infty \leq \max_{i\in N} \frac{\{G(A,\Pi)e\}_i}{\{G(A,\Pi)\langle A\rangle e\}_i} \leq \max_{i\in N} \frac{\{G(A,P_1,P_2)e\}_i}{\{G(A,P_1,P_2)\langle A\rangle e\}_i} \leq \max_{i\in N} \frac{\{G^{P_1}e\}_i}{\{G^{P_1}\langle A\rangle e\}_i}. \tag{2}$$

In the same manner, bounds for the block case can be further improved by introducing more than two permutations. Improvements stated in previous remarks are illustrated with numerical examples in Section 4.

Block generalizations of the class of $H-$matrices were considered in [23].

For a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and a partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N, one can present A in the block form as $[A_{ij}]_{l\times l}$. For rectangular blocks, $\|A_{ij}\|_\infty$ is defined as follows:

$$\|A_{ij}\|_\infty = \sup_{x\in W_j,\, x\neq 0} \frac{\|A_{ij}x\|_\infty}{\|x\|_\infty} = \sup_{\|x\|_\infty=1} \|A_{ij}x\|_\infty.$$

Also, denote

$$(\|A_{ii}^{-1}\|_\infty)^{-1} = \inf_{x\in W_i,\, x\neq 0} \frac{\|A_{ii}x\|_\infty}{\|x\|_\infty}, \quad i \in L,$$

where the last quantity is zero if A_{ii} is singular.

Now, we consider two different ways of introducing the $l\times l$ comparison matrix for a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and a partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N.

The comparison matrix of type I is denoted by $\rangle A\langle^\pi = [p_{ij}]$, where

$$p_{ii} = (\|A_{ii}^{-1}\|_\infty)^{-1}, \quad p_{ij} = -\|A_{ij}\|_\infty, \quad i, j \in L, \quad i \neq j.$$

The comparison matrix of type II is denoted by $\langle A\rangle^\pi = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & i = j \textit{ and } \det A_{ii} \neq 0, \\ -\|A_{ii}^{-1}A_{ij}\|_\infty, & i \neq j \textit{ and } \det A_{ii} \neq 0, \\ 0, & \textit{otherwise}. \end{cases}$$

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$ and a given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N we say that A is a block $\pi - H-$matrix of type I (type II) if $\rangle A\langle^\pi$ is an $H-$matrix ($\langle A\rangle^\pi$ is an $H-$matrix).

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N and for a given permutation matrix P of the index set L, we say that A is a block $\pi - P-$Nekrasov matrix of type I (type II) if $\rangle A\langle^\pi$ is a $P-$Nekrasov matrix ($\langle A\rangle^\pi$ is a $P-$Nekrasov matrix).

For a given $A = [a_{ij}] \in \mathbb{C}^{n,n}$, given partition $\pi = \{p_j\}_{j=0}^{l}$ of the index set N and for the given set of permutation matrices $\Pi = \{P_1, P_2, ..., P_m\}$ of the index set L, we say that A is a block $\pi - \Pi-$Nekrasov matrix of type I (type II) if $\rangle A\langle^\pi$ is $\Pi-$Nekrasov matrix ($\langle A\rangle^\pi$ is $\Pi-$Nekrasov matrix).

In [6] the following results can be found.

Theorem 2.5 ([6]). *If $A = [A_{ij}]_{n\times n}$ is a block $\pi - H-$matrix of type I and $\rangle A\langle^\pi$ is its comparison matrix of type I, then*

$$\|A^{-1}\|_\infty \leq \|(\rangle A\langle^\pi)^{-1}\|_\infty.$$

Theorem 2.6 ([6]). *If $A = [A_{ij}]_{n\times n}$ is a block $\pi - H-$matrix of type II and $\langle A\rangle^\pi$ is its comparison matrix of type II, then*

$$\|A^{-1}\|_\infty \leq \max_{i\in L} \|A_{ii}^{-1}\|_\infty \|(\langle A\rangle^\pi)^{-1}\|_\infty.$$

Recently, in [22], upper bounds for the infinity norm of the inverse for block generalizations of $\{P_1, P_2\}-$Nekrasov matrices of type I and type II were presented. Now we present upper bounds for the infinity norm of the inverse for block $\Pi-$Nekrasov matrices.

Theorem 2.7. *Suppose that, for a given partition* $\pi = \{p_j\}_{j=0}^{l}$ *of the index set* N *and for the given set of permutation matrices* $\Pi = \{P_1, P_2, ..., P_m\}$ *of the index set* L, *a matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *is a block* $\pi - \Pi-$*Nekrasov matrix of type I. Then,*

$$\|A^{-1}\|_\infty \leq \max_{i\in L} \frac{\{Ge\}_i}{\{G\rangle A\langle^\pi e\}_i},$$

where the matrix $G = G(\rangle A\langle^\pi, \Pi)$ *is defined in the same manner as in Theorem 2.4.*

Proof: From Theorem 2.5 and the fact that a block $\pi-\Pi-$Nekrasov matrix of type I is a block $\pi-H-$matrix of type I, we know that

$$\|A^{-1}\|_\infty \leq \|(\rangle A\langle^\pi)^{-1}\|_\infty.$$

From the definition of block $\pi - \Pi-$Nekrasov matrix of type I, we know that the comparison matrix $\rangle A\langle^\pi$ is a $\Pi-$Nekrasov matrix. Therefore, we can apply the upper bound for the infinity norm of the inverse given in Theorem 2.4 to the matrix $\rangle A\langle^\pi$ and obtain

$$\|A^{-1}\|_\infty \leq \|(\rangle A\langle^\pi)^{-1}\|_\infty \leq \max_{i\in L} \frac{\{Ge\}_i}{\{G\rangle A\langle^\pi e\}_i},$$

where the matrix G is defined as $G = G(\rangle A\langle^\pi, \Pi)$ in the same manner as in Theorem 2.4, only considering the matrix $\rangle A\langle^\pi$ instead of the matrix A.

□

Theorem 2.8. *Suppose that, for a given partition* $\pi = \{p_j\}_{j=0}^{l}$ *of the index set* N *and for the given set of permutation matrices* $\Pi = \{P_1, P_2, ..., P_m\}$ *of the index set* L, *a matrix* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *is a block* $\pi - \Pi-$*Nekrasov matrix of type II. Then,*

$$\|A^{-1}\|_\infty \leq \max_{i\in L} \|A_{ii}^{-1}\|_\infty \max_{i\in L} \frac{\{Ge\}_i}{\{G\langle A\rangle^\pi e\}_i},$$

where the matrix G *is defined as* $G = G(\langle A\rangle^\pi, \Pi)$ *in the same manner as in Theorem 2.4, only considering the matrix* $\langle A\rangle^\pi$ *instead of the matrix* A.

Proof: In a similar manner as previous, the proof follows from Theorem 2.6, the fact that every block $\pi - \Pi-$Nekrasov matrix of type II is also a block $\pi - H-$matrix of type II, the definition of block $\pi - \Pi-$Nekrasov matrix of type II and Theorem 2.4.

□

Theorem 2.9. *Let* $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, *let* $\pi = \{p_j\}_{j=0}^{l}$ *be a partition of the index set* N *and let* $\Pi = \{P_1, P_2, ..., P_m\} \in \mathbb{R}^{l,l}$ *be a set of permutation matrices of the index set* L. *Assume that* A *is a block* $\pi - P_1-$*Nekrasov matrix of type I. Then,* A *is a block* $\pi - \Pi-$*Nekrasov matrix of type I and*

$$\|A^{-1}\|_\infty \leq \max_{i\in L} \frac{\{Ge\}_i}{\{G\rangle A\langle^\pi e\}_i} \leq \max_{i\in L} \frac{\{G^{P_1}e\}_i}{\{G^{P_1}\rangle A\langle^\pi e\}_i},$$

where the matrix $G = G(\rangle A\langle^\pi, \Pi)$ *is defined in the same manner as in Theorem 2.4 and the matrix* $G^{P_1} = G^{P_1}(\rangle A\langle^\pi)$ *is defined as in Theorem 1.5.*

Proof: From Theorem 2.5 and the fact that a block $\pi - P_1-$Nekrasov matrix of type I is a block $\pi - H-$matrix of type I, we know that

$$\|A^{-1}\|_\infty \leq \|(\rangle A\langle^\pi)^{-1}\|_\infty.$$

From the definition of block $\pi - P_1-$Nekrasov matrix of type I, we know that the comparison matrix $\rangle A\langle^\pi$ is a P_1-Nekrasov matrix. Therefore, we can apply (2) to the matrix $\rangle A\langle^\pi$ and obtain

$$\|A^{-1}\|_\infty \leq \|(\rangle A\langle^\pi)^{-1}\|_\infty \leq \max_{i\in L} \frac{\{Ge\}_i}{\{G(\rangle A\langle^\pi\rangle e\}_i} \leq \max_{i\in L} \frac{\{G^{P_1}e\}_i}{\{G^{P_1}(\rangle A\langle^\pi\rangle e\}_i},$$

where the matrix G is defined as $G = G(\rangle A\langle^\pi, \Pi)$ in the same manner as in Theorem 2.4, only considering the matrix $\rangle A\langle^\pi$ instead of the matrix A.

□

Theorem 2.10. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, let $\pi = \{p_j\}_{j=0}^{l}$ be a partition of the index set N and let $\Pi = \{P_1, P_2, ..., P_m\}$, $P_1, P_2, ..., P_m \in \mathbb{R}^{l,l}$ be a set of permutation matrices of the index set L. Assume that A is a block $\pi - P_1-$Nekrasov matrix of type II. Then, A is a block $\pi - \Pi-$Nekrasov matrix of type II and*

$$\|A^{-1}\|_\infty \leq \max_{i\in L} \|A_{ii}^{-1}\|_\infty \max_{i\in L} \frac{\{Ge\}_i}{\{G\langle A\rangle^\pi e\}_i} \leq \max_{i\in L} \|A_{ii}^{-1}\|_\infty \max_{i\in L} \frac{\{G^{P_1}e\}_i}{\{G^{P_1}\langle A\rangle^\pi e\}_i},$$

where the matrix $G = G(\langle A\rangle^\pi, \Pi)$ is defined in the same manner as in Theorem 2.4 and the matrix $G^{P_1} = G^{P_1}(\langle A\rangle^\pi)$ is defined as in Theorem 1.5.

Proof: In a similar manner as previous, the proof follows from Theorem 2.6, the fact that every block $\pi - P_1-$Nekrasov matrix of type II is also a block $\pi - H-$matrix of type II, the definition of block $\pi - P_1-$Nekrasov matrix of type II and (2).

□

Remark 2.11. *According to Theorem 10 from [17], bounds for point-wise case given in Theorem 1.6 generally improve the bounds presented in [30] via more suitable criteria for choosing the i−th row in $G = G(A, P_1, P_2)$ in cases when the i−th row is SDD in both $G^{P_1}\langle A\rangle$ and $G^{P_2}\langle A\rangle$. Therefore, our bounds for the block case given in Theorem 2.7 and Theorem 2.8 generally improve the corresponding bounds in [22]. Further improvement in estimation of the norm bound is obtained by introducing additional permutation matrices.*

Previous results and remarks for the block case will be illustrated with numerical examples in Section 4.

3. Main results on LCP for Π−Nekrasov B−matrices

The linear complementarity problem (LCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Ax + q \geq 0, \quad (Ax + q)^T x = 0,$$

or to show that such vector does not exist. Here, $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and $q \in \mathbb{R}^n$. Many mathematical problems can be described in LCP form. It is well known that LCP(A, q) has a unique solution for any $q \in \mathbb{R}^n$ if and only if a matrix A is a $P-$matrix, a real square matrix with all its principal minors positive, see [4]. An $H-$matrix with positive diagonal entries is a $P-$matrix. In defining an upper error bound for LCP(A, q) where A is a $P-$matrix, see [2], the following fact can be a starting point

$$\|x - x^*\|_\infty \leq \max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \|r(x)\|_\infty.$$

Here, x^* is a solution of the LCP(A, q), $r(x) = \min(x, Ax + q)$, $D = diag(d_1, ..., d_n)$, with $0 \leq d_i \leq 1$ for each $i \in N$, and the min operator denotes the componentwise minimum of two vectors. Obviously, the upper bound for the infinity norm of the inverse matrix of $I - D + DA$ plays an important role in determining LCP error bound.

If A is a certain structure matrix, more results on LCP(A, q) can be found in [11, 12, 14, 25].

Definition 3.1. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be writen in the form*

$$A = B^{+} + C, \tag{3}$$

where

$$B^{+} = [b_{ij}] = \begin{bmatrix} a_{11} - r_1^{+} & \dots & a_{1n} - r_1^{+} \\ \vdots & \ddots & \vdots \\ a_{n1} - r_n^{+} & \dots & a_{nn} - r_n^{+} \end{bmatrix}, \quad C = [c_{ij}] = \begin{bmatrix} r_1^{+} & \dots & r_1^{+} \\ \vdots & \ddots & \vdots \\ r_n^{+} & \dots & r_n^{+} \end{bmatrix}$$

and $r_i^{+} := \max\{0, a_{ij} | j \neq i\}$. For the given set of permutation matrices $\Pi = \{P_1, P_2, ..., P_m\}$, A is called a $\Pi-$Nekrasov $B-$matrix if B^{+} is a $\Pi-$Nekrasov matrix with positive diagonal entries.

Theorem 3.2. *Given any diagonal matrix $D = diag(d_1, d_2, ..., d_n)$ with $d_i \in [0, 1]$ for all $i \in N$ and given a set of permutation matrices $\Pi = \{P_1, P_2, ..., P_m\}$, then if A is a $\Pi-$Nekrasov $B-$matrix then $I - D + DA$ is a $\Pi-$Nekrasov $B-$matrix, where I is the identity matrix.*

Proof: Let A be a $\Pi-$Nekrasov $B-$matrix. Then B^{+} is a $\Pi-$Nekrasov matrix with positive diagonal entries and so is the matrix $I - D + DB^{+}$, according to [22]. Now, observe the matrix $\bar{A} = I - D + DA$ whose entries are

$$\bar{a}_{ij} = \begin{cases} 1 - d_i + d_i a_{ii}, & i = j \\ d_i a_{ij}, & i \neq j \end{cases},$$

where $\bar{r}_i^{+} = \max\{0, \bar{a}_{ij} | j \neq i\} = \max\{0, d_i a_{ij} | j \neq i\} = d_i \max\{0, a_{ij} | j \neq i\} = d_i r_i^{+}$.
It follows that

$$\bar{A} = I - D + DA = I - D + D(B^{+} + C) = (I - D + DB^{+}) + DC = \bar{B}^{+} + \bar{C},$$

and since $\bar{B}^{+} = I - D + DB^{+}$ is a $\Pi-$Nekrasov matrix with positive diagonal entries, we conclude that $\bar{A} = I - D + DA = \bar{B}^{+} + \bar{C}$ is a $\Pi-$Nekrasov $B-$matrix.

□

Lemma 3.3 ([13]). *Suppose $P = (p_1, p_2, ..., , p_n)^T e$, where $e = (1, 1, ..., 1)$ and $p_i \geq 0$ for all $i \in N$. Then $\|(I + P)^{-1}\|_\infty \leq n - 1$.*

Theorem 3.4. *Suppose that, for a given set of permutation matrices $\Pi = \{P_1, P_2, ..., P_m\}$, a matrix $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $n \geq 2$ is a $\Pi-$Nekrasov $B-$matrix. Then*

$$\|A^{-1}\|_\infty \leq (n-1) \max_{i \in N} \frac{\frac{z_i^{P_{k_i}}(B^{+})}{|b_{ii}|}}{1 - \min\limits_{1 \leq j \leq m} \{\frac{h_i^{P_j}(B^{+})}{|b_{ii}|}\}},$$

where the matrix $B^{+} = [b_{ij}]$ is defined in (3), $z_1(B^{+}) = 1$, $z_i(B^{+}) = \sum_{j=1}^{i-1} |b_{ij}| \frac{z_j(B^{+})}{|b_{jj}|} + 1$, $i = 2, 3, ..., n$, the corresponding vector is $z(B^{+}) = [z_1(B^{+}), ..., z_n(B^{+})]^T$, $z^P(B^{+}) = Pz(P^T B^{+} P)$ and for the given $i \in N$ the corresponding index $k_i \in \{1, 2, ..., m\}$ is chosen in such a way that $\min\limits_{1 \leq j \leq m} \{h_i^{P_j}(B^{+})\} = h_i^{P_{k_i}}(B^{+})$.

Proof: Since A is a $\Pi-$Nekrasov $B-$matrix, B^{+} is a $\Pi-$Nekrasov matrix with positive diagonal entries and also a $Z-$matrix. Thus, B^{+} is an $M-$matrix and $(B^{+})^{-1}$ is nonnegative. Hence, from $A = B^{+} + C$ we have

$$A^{-1} = (B^{+}(I + (B^{+})^{-1}C))^{-1} = (I + (B^{+})^{-1}C)^{-1}(B^{+})^{-1},$$

which implies that

$$\|A^{-1}\|_\infty \leq \|(I + (B^{+})^{-1}C)^{-1}\|_\infty \|(B^{+})^{-1}\|_\infty.$$

Note that $C = (r_1^+, ..., r_n^+)^T e$ is a nonnegative matrix. Therefore, $(B^+)^{-1}C$ can be written as $(p_1, ..., p_n)^T e$ where $p_i \geq 0$ for all $i \in N$. By Lemma 3.3 we get

$$\|(I + (B^+)^{-1}C)^{-1}\|_\infty \leq n - 1.$$

Since B^+ is a $\Pi-$Nekrasov matrix, from Theorem 1.4 it follows

$$\|A^{-1}\|_\infty \leq (n-1) \max_{i\in N} \frac{\frac{z_i^{P_{k_i}}(B^+)}{|b_{ii}|}}{1 - \min\limits_{1\leq j\leq m}\{\frac{h_i^{P_j}(B^+)}{|b_{ii}|}\}},$$

where $k_i \in \{1, 2, ..., m\}$ is chosen in such a way that $\min\limits_{1\leq j\leq m}\{h_i^{P_j}(B^+)\} = h_i^{P_{k_i}}(B^+)$.

□

Now we give an upper bound of $\max\limits_{d_i\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty$ when A is a $\Pi-$Nekrasov $B-$matrix.

Theorem 3.5. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a $\Pi-$Nekrasov B-matrix with positive diagonal entries for a given set of permutation matrices $\Pi = \{P_1, P_2, ..., P_m\}$ and let $D = diag(d_1, ..., d_n)$ with $0 \leq d_i \leq 1$ for each $i \in N$. Then*

$$\max_{d_i\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq (n-1) \max_{i\in N} \frac{\max\limits_{1\leq j\leq m}\{\theta_i^{P_j}(B^+)\}}{1 - \min\limits_{1\leq j\leq m}\{\frac{h_i^{P_j}(B^+)}{b_{ii}}\}}, \tag{4}$$

where the matrix $B^+ = [b_{ij}]$ is defined in (3)*,*

$$\begin{aligned} \theta_1(A) &= \frac{1}{\min\{|a_{11}|, 1\}}, \\ \theta_i(A) &= \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|}\theta_j(A) + \frac{1}{\min\{|a_{ii}|, 1\}}, \quad i = 2, 3, ..., n, \end{aligned} \tag{5}$$

the corresponding vector is $\theta(A) = [\theta_1(A), ..., \theta_n(A)]^T$ and $\theta^P(A) = P\theta(P^T AP)$.

Proof: With a slightly different notation, in [11], it is proved that for a $\{P_1, P_2\}-$Nekrasov matrix A with positive diagonal entries, for each $i \in N$, it holds

$$\frac{z_i(P^T(I - D + DA)P)}{(P^T(I - D + DA)P)_{ii}} \leq \theta_i(P^T AP),$$

where $P \in \{P_1, P_2\}$. Therefore, it follows that

$$\frac{z_i^P(I - D + DB^+)}{|1 - d_i + d_i b_{ii}|} = \frac{(Pz(P^T(I - D + DB^+)P))_i}{1 - d_i + d_i b_{ii}} \leq \theta_i^P(B^+) \leq \max_{1\leq j\leq m}\{\theta_i^{P_j}(B^+)\}.$$

Since, from [12, 22]

$$\frac{h_i^P(I - D + DB^+)}{|1 - d_i + d_i b_{ii}|} \leq \frac{h_i^P(B^+)}{|b_{ii}|},$$

for $P \in \Pi$, we first apply the result of Theorem 3.4, with k_i chosen as above, and with $\bar{B}^+ = I - D + DB^+ = [\bar{b}_{ij}]$, we obtain

$$\begin{aligned}
\max_{d_i\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty &\le (n-1)\max_{i\in N} \frac{\frac{z_i^{P_{k_i}}(\bar{B}^+)}{|\bar{b}_{ii}|}}{1-\min\limits_{1\le j\le m}\{\frac{h_i^{P_j}(\bar{B}^+)}{|\bar{b}_{ii}|}\}} \\
&= (n-1)\max_{i\in N} \frac{\frac{z_i^{P_{k_i}}(I-D+DB^+)}{|1-d_i+d_ib_{ii}|}}{1-\min\limits_{1\le j\le m}\{\frac{h_i^{P_j}(I-D+DB^+)}{|1-d_i+d_ib_{ii}|}\}} \\
&\le (n-1)\max_{i\in N} \frac{\max\limits_{1\le j\le m}\{\theta_i^{P_j}(B^+)\}}{1-\min\limits_{1\le j\le m}\{\frac{h_i^{P_j}(B^+)}{b_{ii}}\}}.
\end{aligned}$$

□

Now, we can apply our result in measuring the sensitivity of the solution of the linear complementarity problem of a $P-$matrix. In [3], a constant for a $P-$matrix A was introduced:

$$\beta_p(A) = \max_{d_i\in[0,1]^n} \|(I-D+DA)^{-1}D\|_p,$$

where $\|\cdot\|_p$ is a matrix norm induced by a vector norm for $p \ge 1$. The constant $\beta_p(A)$ is considered to be an indicator of perturbation sensitivity and has been used in error analysis of the LCP.

For $p=\infty$, according to the fact that

$$\beta_\infty(A) \le \max_{d_i\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty \cdot \max_{d_i\in[0,1]^n} \|D\|_\infty$$

and Theorem 3.5, we obtain the following perturbation bound for linear complementarity problem of $\Pi-$Nekrasov $B-$matrices.

Corollary 3.6. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a $\Pi-$Nekrasov $B-$matrix, $\Pi = \{P_1, P_2, ..., P_m\}$, with positive diagonal entries. Then*

$$\beta_\infty(A) \le (n-1)\max_{i\in N} \frac{\max\limits_{1\le j\le m}\{\theta_i^{P_j}(B^+)\}}{1-\min\limits_{1\le j\le m}\{\frac{h_i^{P_j}(B^+)}{b_{ii}}\}},$$

where the matrix $B^+ = [b_{ij}]$ is defined in (3), $\theta_i(A)$ for $i \in N$ is defined in (5), the corresponding vector is $\theta(A) = [\theta_1(A), ..., \theta_n(A)]^T$ and $\theta^P(A) = P\theta(P^TAP)$.

4. Numerical examples and concluding remarks

With the following examples we illustrate the results and improve some already known bounds.

Example 4.1. *Matrices with all diagonal entries equal to zero are called hollow matrices. These matrices often appear in practical applications. For instance, hollow matrices play a role in modelling ecological systems consisting of several populations with no self interactions, see [15]. Matrices that appear in applications often have large dimensions, but also a certain block structure. Therefore, it is suitable to use block approach in order to collect information on the original, large matrix through analyzing a comparison matrix of a smaller format.*

Consider the following matrix.

$$A_1=\begin{bmatrix}
0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1\\
2 & 0 & -2 & 1 & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0\\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0\\
1 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-3}\\
-1 & 1 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 10 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
10^{-3} & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-3}\\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-3}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 10^{-3}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-3} & 10^{-3} & 2 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1.5 & 0 & 0 & 0\\
10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 10^{-3}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & -0.5 & 2 & 0 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0\\
0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\\
0 & 0 & 0.5 & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 0 & 0 & 10^{-3} & 2 & 0 & 2\\
0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{bmatrix}.$$

Matrix A_1 was considered in [22]. Since all the diagonal entries are equal to zero, A_1 does not belong to the class of $H-$matrices in the point-wise sense and, therefore, neither of the point-wise bounds for the norm of the inverse can be applied. However, for the partition π into 3 by 3 blocks, and for P_1 being identical and P_2 counteridentical permutation of order 7, this matrix does belong to $\pi-\{P_1,P_2\}-$Nekrasov matrices of type II. In [22], for identical and counteridentical permutation of order 7, the obtained bound for the norm of the inverse is 10.5111, while the exact value is equal to 2.86631.

If, in addition to identical permutation P_1 and counteridentical permutation P_2, we also consider the permutation

$$P_3=\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

it follows that matrix $\langle A_1\rangle^{\pi}$ is a $\Pi-$Nekrasov matrix, where $\Pi=\{P_1,P_2,P_3\}$ and by applying our bound from Theorem 2.8 we have improved the result from [22] by obtaining 9.630199.

Therefore, involving additional permutation matrices can lead, in some cases, to tighter norm estimation.

Example 4.2. *Consider the following matrix*

$$A_2=\begin{bmatrix}
1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2}\\
\frac{1}{5} & 1 & \frac{2}{5} & \frac{2}{5}\\
-1 & 0 & 1 & -\frac{1}{6}\\
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix}.$$

Matrix A_2 can be written $A_2=B^{+}+C$ as in (3)*, where*

$$B^{+}=\begin{bmatrix}
\frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0\\
-\frac{1}{5} & \frac{3}{5} & 0 & 0\\
-1 & 0 & 1 & -\frac{1}{6}\\
-\frac{1}{4} & 0 & 0 & \frac{1}{2}
\end{bmatrix},\quad C=\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\
\frac{2}{5} & \frac{2}{5} & \frac{2}{5} & \frac{2}{5}\\
0 & 0 & 0 & 0\\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.$$

Since

$$h_1(B^+) = \frac{1}{3}, \quad h_2(B^+) = \frac{2}{15}, \quad h_3(B^+) = \frac{5}{6}, \quad h_4(B^+) = \frac{1}{6},$$

it follows that B^+ is a Nekrasov matrix and hence A_2 is $\{P_1, P_2\}-$Nekrasov $B-$matrix for $P_1 = I$ and any permutation P_2. In [12] the same matrix is observed and an LCP bound for $B - S-$Nekrasov matrices is applied and obtained bound is 25.3636. As mentioned in [12], the bound from Theorem 2 in [13] cannot be used. Here, we apply our bound (4) *and for $P_1 = I$ and $P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ improve the bound from [12] by obtaining*

$$\max_{d_i \in [0,1]^n} \|(I - D + DA_2)^{-1}\|_\infty \leq 23.7321.$$

Example 4.3. *Consider the following matrix*

$$A_3 = \begin{bmatrix} 5.6 & -5.6 & 0 & 0 \\ -1.3 & 4.1 & -3 & 0.1 \\ 0 & -2.2 & 7 & -2.9 \\ 0.4 & 0 & 0 & 3.4 \end{bmatrix}.$$

Matrix A_3 can be written $A_3 = B^+ + C$ as in (3), *where*

$$B^+ = \begin{bmatrix} 5.6 & -5.6 & 0 & 0 \\ -1.4 & 4 & -3.1 & 0 \\ 0 & -2.2 & 7 & -2.9 \\ 0 & -0.4 & -0.4 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}.$$

By computations, matrix A_3 is neither $S-$Nekrasov nor $B-S-$Nekrasov matrix for any S and, therefore, the LCP bound from [12] cannot be applied. However, matrix A_3 is a $\{P_1, P_2\}-$Nekrasov $B-$matrix for identical and counteridentical permutations P_1 and P_2 and by (4) *we obtain* $\max_{d_i \in [0,1]^n} \|(I - D + DA_3)^{-1}\|_\infty \leq 28.9524$.

Example 4.4. *Consider the following matrix*

$$A_4 = \begin{bmatrix} 1 & -0.2 & -0.2 & -0.2 & -0.2 & -0.2 & -0.2 \\ -0.1 & 1 & -0.2 & 0 & 0 & -0.3 & 0 \\ -0.2 & 0 & 1 & -0.3 & -0.1 & 0 & 0 \\ -0.2 & -0.2 & -0.2 & 1 & -0.2 & -0.2 & -0.2 \\ 0 & -0.2 & -0.1 & 0 & 1 & -0.2 & 0 \\ -0.1 & -0.2 & 0 & 0 & -0.3 & 1 & 0 \\ -0.2 & -0.2 & -0.2 & -0.2 & -0.2 & -0.2 & 1 \end{bmatrix}.$$

For $P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$

it can be easily shown that matrix A_4 is neither P_1-Nekrasov, P_2-Nekrasov, P_3-Nekrasov nor $\{P_1, P_2\}-$Nekrasov, $\{P_1, P_3\}-$Nekrasov, $\{P_2, P_3\}-$Nekrasov matrix, but A_4 does belong to $\Pi-$Nekrasov class, where $\Pi = \{P_1, P_2, P_3\}$. Therefore, neither of the well-known bounds for the inverse for $P-$Nekrasov and $\{P_1, P_2\}-$Nekrasov matrices can be applied. However, we can apply the new bound (1) *and obtain 18.61349 while the exact value is 6.2492.*

As we can see from these examples and the previous considerations, the benefits of the presented results are the following. First, the matrix classes we investigated cover some matrices that are not $H-$matrices (in the point-wise sense). Therefore, already known bounds for different subclasses of $H-$matrices cannot be applied to these matrices. Second, infinity norm bounds given for the block $\Pi-$Nekrasov matrices are tighter, or at least as tight as the bounds in [22]. The main results, error bounds for LCP involving $\Pi-$Nekrasov $B-$matrices, are tighter, in some cases, than already known error bounds for $S-$Nekrasov and $B-S-$Nekrasov matrices. Furthermore, our error bounds work in some cases where already known bounds for $S-$Nekrasov and $B-S-$Nekrasov matrices cannot be applied at all. For future work, it would be interesting to further investigate the criteria for choosing suitable permutation matrices in some more specific problems in applications. Also, it would be interesting to consider other block generalizations of these matrix classes, as well as different modifications and generalizations of linear complementarity problems.

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