Filomat 37:13 (2023), 4287–4295
https://doi.org/10.2298/FIL2313287L

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A double phase equidiffusive logistic equation

Zhenhai Liu[a,b,*], **Nikolaos S. Papageorgiou**[c]

[a] *Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, P.R. China.*
[b] *Guangxi Key Laboratory of Universities Optimization Control and Engineering Calculation, Guangxi Minzu University, Nanning, Guangxi, 530006, P.R. China*
[c] *Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece*

Abstract. We consider a (p, q)-equation with unbalanced growth (double phase problem) and a logistic reaction of the equidiffusive type. We show the existence and uniqueness of a positive solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following double phase Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = \theta(z)u(z)^{q-1} - f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < q < p < N, u > 0. \end{cases} \tag{1}$$

Given $a \in L^\infty(\Omega) \setminus \{0\}$ with $a(z) \geq 0$ for a.a. $z \in \Omega$ and $r \in (1, \infty)$, by Δ_r^a we denote the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2}Du).$$

If $a \equiv 1$, then we have the usual r-Laplacian denoted by Δ_r. In (1) the equation is driven by the sum of two such operators with different exponents. Such an operator is not homogeneous. We do not assume that the weight function $a(\cdot)$ is bounded away from zero (that is, we do not have that $\operatorname{essinf}_\Omega a > 0$). This means that the integrand

$$\eta(z, t) = a(z)t^p + t^q \quad \forall z \in \Omega, \ \forall t \geq 0$$

of the energy functional corresponding to the differential operator of (1),exhibits unbalanced growth, that is, we have

$$t^q \leq \eta(z, t) \leq c_0[t^p + t^q] \quad \text{for a.a.} z \in \Omega, \ \text{all } t \geq 0, \ \text{some } c_0 > 0.$$

Such functionals were first considered by Marcellini [10] and Zhikov [20] in the context of problems of the calculus of variations and of nonlinear elasticity theory. The hardening properties of strongly anisotropic

2020 *Mathematics Subject Classification*. Primary 35J25; Secondary 35J92
Keywords. Unbalanced growth,generalized Orlicz spaces, modular function, equidiffusive logistic reaction, positive solutions
Received: 04 July 2022; Accepted: 31 July 2022
Communicated by Calogero Vetro
Research supported by NNSF of China Grant No. 12071413 and the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement No. 823731 CONMECH.
* Corresponding author: Zhenhai Liu
Email addresses: zhhliu@hotmail.com (Zhenhai Liu), npapg@math.ntua.gr (Nikolaos S. Papageorgiou)

materials vary by the point. This leads to mathematical models described by operators as the one in problem (1). Note that the ellipticity of integrand $\eta(z,t)$ varies and depends on the point of the space. The weight function $a(\cdot)$ regulates the mixture between different materials with power hardening of rates p and q respectively. To deal with such problems, we need to move beyond the usual functional framework of the standard Lebesgue and Sobolev spaces and use generalized Orlicz-Sobolev spaces. For such unbalanced growth problems, there is no global (that is, up to the boundary) regularity theory. Only some local results, which can be found in Baroni-Colombo-Mingione [1]. Marcellini [11] and the references therein. Surveys of the recent advances on this topic can be found in Mingione-Rădulescu [12], Papageorgiou [13] and Rădulescu [19]. This lack of a global regularity theory, eliminates many powerful tools which are readily available when dealing with problems which have balanced growth.

The reaction (source) of (1) is logistic and it is of equidiffusive type since $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to +\infty$. Logistic equations are important among others in mathematical biology, since they describe the steady state of the evolution of biological populations in the presence of variable rates of reproduction and of mortality (see Gurtin-Mac Camy [6]). Recently there have been some existence and multiplicity results for unbalanced double phase problems. Indicatively, we mention the works of Gasiński-Winkert [4], Ge-Lv-Lu [5], Liu-Dai [8], Liu-Papageorgiou [9], Papageorgiou-Pudelko-Rădulescu [14],Papageorgiou-Rădulescu-Repovš [16]. None of these works deals with logistic equations.

2. Hypotheses and Mathematical Background

As we already mentioned in the Introduction, the study of unbalanced double phase problems, requires the use of generalized Orlicz spaces. For a comprehensive introduction to the theory of these spaces, we refer to the book of Harjulehto-Hästo [7].

In what follows by $\widehat{\lambda}_1(q)$ we denote the principal eigenvalue of the Dirichlet q-Laplacian. We know (see Gasiński-Papageorgiou [3]) that

$\widehat{\lambda}_1(q) > 0$ and it is simple and isolated;

$\widehat{\lambda}_1(q) = \inf\left[\frac{\|Du\|_q^q}{\|u\|_q^q} : u \in W_0^{1,q}, u \neq 0\right]$. This infimum is realized on the corresponding one dimensional eigenspace which is included in $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ and the elements of this eigenspace have constant sign.

We mention that $C_0^1(\bar{\Omega})$ is an ordered Banach space with positive cone $C_+ = \{v \in C_0^1(\bar{\Omega}) : v(z) \geq 0$ for all $z \in \bar{\Omega}\}$. This cone has a nonempty interior given by

$$\mathrm{int}C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ and $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$. By $\widehat{u}_1(q)$ we denote the positive, L^q-normalized (that is, $\|\widehat{u}_1(q)\|_q = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(q) > 0$. As a consequence of the nonlinear regularity theory and of the nonlinear maximum principle, we have $\widehat{u}_1(q) \in \mathrm{int}C_+$.

By $C^{0,1}(\bar{\Omega})$, we denote the Banach space of Lipschitz continuous functions defined on $\bar{\Omega}$ with value in $\mathbb{R}$.

Our hypotheses on the coefficients $a(\cdot)$ and $\theta(\cdot)$ are:

$\underline{H_0}$: $a \in C^{0,1}(\bar{\Omega})\backslash\{0\}, a(z) \geq 0$ for all $z \in \bar{\Omega}, 1 < q < p < N$ with $\frac{p}{q} < 1 + \frac{1}{N}$ and $\theta \in L^\infty(\Omega)$ and satisfies $\theta(z) \geq \widehat{\lambda}_1(q)$ for a.a. $z \in \Omega$, and the inequality is strict on a set of positive Lebesgue measure.

Remark 2.1. *The conditions on the weight function $a(\cdot)$ and the exponents $\{p,q\}$, guarantee boundness of the solutions of (1) (see [4]), the absence of the Lavrentiev phenomenon (see [11, 12]), the validity of the Poincare inequality in the appropriate Orlicz-Sobolev space (see [7],p.138) and lead to useful embeddings of the relavant spaces (since $p < q^* = \frac{Nq}{N-q}$).*

Recall that the integrand corresponding to the energy functional of the differential operator, is the function

$$\eta(z,t) = a(z)t^p + t^q,$$

which is a Caratheodory function (that is, $z \to \eta(z,t)$ ia measurable and $t \to \eta(z,t)$ is continuous). Let $L^0(\Omega)$ be the space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual, we identify two such functions which differ only on a Lebesgue-null set. The generalized Orlicz space $L^\eta(\Omega)$ is defined by

$$L^\eta(\Omega) = \{u \in L^0(\Omega) : \rho_\eta(u) < \infty\},$$

where $\rho_\eta(\cdot)$ is the modular function defined by

$$\rho_\eta(u) = \int_\Omega \eta(z, |u|)\, dz.$$

We equip $L^\eta(\Omega)$ with the so-called "Luxemburg norm" given by

$$\|u\|_\theta = \inf\left\{\lambda > 0 : \rho_\eta\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Then $L^\eta(\Omega)$ becomes a Banach space which is separable and uniformly convex (since $\eta(z,\cdot)$ is a uniformly convex function). We know that a uniformly convex Banach space is reflexive (Milman-Pettis theorem, see Papageorgrou-Winkert [18], p.225).

Using $L^\eta(\Omega)$ we can define the corresponding generalized Orlicz-Sobolev space $W^{1,\eta}(\Omega)$ by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) : |Du| \in L^\eta(\Omega)\}$$

with Du being the weak gradient of u. We equip $W^{1,\eta}(\Omega)$ with the normal

$$\|u\|_{1,\eta} = \|u\|_\eta + \|Du\|_\eta \quad \text{for all } u \in W^{1,\eta}(\Omega),$$

where $\|Du\|_\eta = \||Du|\|\eta$. Also, we set

$$W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}.$$

For this space, the Poincare inequality holds, that is, there exists $\widehat{c} > 0$ such that $\|u\|_\eta \leq \widehat{c}\|Du\|_\eta$ for all $u \in W_0^{1,\eta}(\Omega)$. So, on $W_0^{1,\eta}(\Omega)$ we can use the equivalent norm

$$\|u\| = \|Du\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

The spaces $W^{1,\eta}(\Omega)$, $W_0^{1,\eta}(\Omega)$ are separable Banach spaces which are uniformly convex (hence reflexive).

The norm $\|\cdot\|$ and the modular function are closely related.

Proposition 2.2. (a) $\|u\| = \lambda \Leftrightarrow \rho_\eta(\frac{Du}{\lambda}) \leq 1$;
(b) $\|u\| < 1(resp. = 1, > 1) \Leftrightarrow \rho_\eta(Du) < 1(resp. = 1, > 1)$;
(c) $\|u\| < 1 \Rightarrow \|u\|^p \leq \rho_\eta(Du) \leq \|u\|^q$;
(d) $\|u\| > 1 \Rightarrow \|u\|^q \leq \rho_\eta(Du) \leq \|u\|^p$;
(e) $\|u\| \to 0(resp. \to +\infty) \Leftrightarrow \rho_\eta(Du) \to 0(resp. \to +\infty)$.

We have the following useful embeddings among the spaces introduced above.

Proposition 2.3. $\underline{(a)}L^\eta(\Omega) \hookrightarrow L^s(\Omega), W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,s}(\Omega)$ *continuously for all* $1 \leq s \leq q$;
$\underline{(b)}W_0^{1,\eta}(\Omega) \hookrightarrow L^s(\Omega)$ *continuously (resp.,compactly), if* $1 \leq s \leq q^*$ *(resp., if* $1 \leq s < q^*$*);*
$\underline{\overline{(c)}}L^q(\Omega) \hookrightarrow L^\eta(\Omega)$ *continuously.*

Let $V : W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_\Omega [a(z)|Du|^{p-2} + |Du|^{q-2}](Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,\eta}(\Omega).$$

The operator is continuous and strictly monotone (thus maximal monotone too) and coercive.

The Lebesgue space $L^\infty(\Omega)$ is an ordered Banach space for the pointwise order, with positive (order) cone

$$L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega) : 0 \le u(z) \quad \text{for a.a. } z \in \Omega\}.$$

This cone has a nonempty interior given by

$$\text{int} L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega)_+ : 0 < \operatorname*{ess\,inf}_\Omega u\}.$$

If $u \in L^0(\Omega)$, then $u^\pm = \max\{\pm u(z), 0\}$ for all $z \in \Omega$. We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,\eta}(\Omega)$, then $u^\pm \in W_0^{1,\eta}(\Omega)$.

Now, we introduce our hypotheses on the perturbation $f(z, x)$:
(H_1) : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $f(z, 0) = 0$ for *a.a.* $z \in \Omega$ and
(i) $|f(z, x)| \le \widehat{a}(z)[1 + x^{r-1}]$ for *a.a.* $z \in \Omega$, all $x \ge 0$, with $\widehat{a} \in L^\infty(\Omega)_+$, $p \le r < q^*$;
(ii) $\lim_{x\to+\infty} \dfrac{f(z, x)}{x^{p-1}} = +\infty$ uniformly for a.a. $z \in \Omega$ and for *a.a.* $z \in \Omega$, $x \to \dfrac{f(z, x)}{x^{q-1}}$ is strictly increasing on $\mathring{\mathbb{R}}_+ = (0, +\infty)$;
(iii) $\lim_{x\to0^+} \dfrac{f(z, x)}{x^{q-1}} = 0$ uniformly for a.a. $z \in \Omega$.

Remark 2.4. *Since we look for positive solutions and the above hypotheses concern the positive semiaxis* $\mathbb{R}_+ = [0, +\infty)$, *without any loss of generality, we may assume that* $f(z, x) = 0$ *for a.a.* $z \in \Omega$, *all* $x \le 0$. *We point out we do not assume that* $f \ge 0$. *So,* f *may be sign-changing. On account of hypothesis* $H_1(\underline{iii})$, *we see that asymptotically as* $x \to 0^+$ *the quotient* $\dfrac{\theta(z)x^{q-1} - f(z, x)}{x^{q-1}}$ *partially interacts with the principal eigenvalue of* $(-\Delta_q, W_c^{1,\eta}(\Omega))$ *(so we have nonuniform nonresonance).*

Example 2.5. *The following functions satisfy hypotheses* H_1. *For the sake of simplicity, we drop the z-dependence.*

$$f_1(x) = (x^+)^{r-1} \quad \text{with } p < r < q^*,$$

$$f_2(x) = (x^+)^{p-1} \ln(x^+) - (x^+)^{\tau-1} \quad \text{with } 1 < \tau < q.$$

Note that $f_1(\cdot)$ *is part of the classical equidiffusive reaction* $x \to x^{q-1} - x^{r-1}$ *for all* $x \ge 0$. *On the other hand,* $f_2(\cdot)$ *is sign changing.*

In the sequel, we will use another modular function $\rho_a(\cdot)$ defined by

$$\rho_\alpha(Du) = \int_\Omega a(z)|Du|^p dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

This function is continuous, convex, thus weakly lower semicontinuous too.

Let $\varphi : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\rho_\alpha(Du) + \frac{1}{q}\|Du\|_q^q + \int_\Omega F(z, u^+)dz - \frac{1}{q}\int_\Omega \theta(z)(u^+)^q dz \text{ for all } u \in W_0^{1,\eta}(\Omega),$$

with $F(z, x) = \int_0^x f(z, s)ds$. Evidently, $\varphi \in C^1(W_0^{1,\eta}(\Omega))$.

3. Positive Solution

First, we show the existence of a positive solution for problem (1).

Proposition 3.1. *If hypotheses H_0, H_1 hold, then problem* (1) *has at least one positive solution $u_0 \in W_0^{1,\eta}(\Omega) \bigcap L^\infty(\Omega)$ and for every $K \subseteq \Omega$ compact $0 \le c_K \le u_0(z)$ for a.a. $z \in K$*

Proof. On account of hypotheses $H_1(i),(iii)$, given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z,x) \ge -\frac{\varepsilon}{q}|x|^q - c_\varepsilon |x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{2}$$

If $u \in W_0^{1,\eta}(\Omega)$ with $\|u\| \ge 1$, then we have

$$\begin{aligned} \varphi(u) \quad &\ge \quad \frac{1}{p}\|u\|^q - c_1\|u\|_r^q + c_\varepsilon \|u\|_r^r \text{ for some } c_1 > 0 \\ &\qquad \text{(see Proposition 2.2,(2) and hypothesis } H_0) \\ &\ge \quad \frac{1}{p}\|u\|^q + [c_\varepsilon \|u\|_r^{r-q} - c_1]\|u\|_r^q \quad \text{(since } q < r). \end{aligned}$$

Since $q < p < r$, it follows that $\varphi(\cdot)$ is coercive. Moreover, Using Proposition 2.3, we have that $\varphi(\cdot)$ is sequentially weakly lower semicontinuous.

By the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,\eta}(\Omega)$ such that

$$\varphi(u_0) = \inf\{\varphi(u) : u \in W_0^{1,\eta}(\Omega)\}. \tag{3}$$

On account of hypothesis $H_1(iii)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \le \frac{\varepsilon}{q}|x|^q \text{ for a.a. } z \in \Omega, \text{ all } |x| \le \delta. \tag{4}$$

From Section 2, we know that the principal eigenfunction $\widehat{u_1} = \widehat{u_1}(q) \in \text{int}C_+$. So, we can find $t \in (0,1)$ small such that

$$0 \le t\widehat{u_1}(z) \le 0 \quad \text{for all } z \in \overline{\Omega}. \tag{5}$$

Then from (4) and (5), we have

$$\begin{aligned} \varphi(t\widehat{u_1}) \quad &\le \quad \frac{t^p}{p}\rho_a(D\widehat{u_1}) + \frac{t^q}{q}[\|D\widehat{u_1}\|_q^q - \int_\Omega \theta(z)\widehat{u}_1^q dz + \epsilon] \text{ (recall that } \|\widehat{u_1}\|_q = 1) \\ &= \quad \frac{t^p}{p}\rho_a(D\widehat{u_1}) + \frac{t^q}{q}[\int_\Omega (\widehat{\lambda}_1(q) - \theta(z))\widehat{u}_1^q dz + \epsilon]. \end{aligned} \tag{6}$$

Since $\widehat{u_1} \in intC_+$ and using the properties of $\theta(\cdot)$ (see hypotheses H_0), we have

$$\beta = \int_\Omega [\theta(z) - \widehat{\lambda}_1(q)]\widehat{u}_1^q dz > 0.$$

So, choosing $\epsilon \in (0,\beta)$, from (6) it follows that

$$\varphi(t\widehat{u_1}) \le c_2 t^p - c_3 t^q \quad \text{for some } c_2, c_3 > 0.$$

We know that $q < p$. So, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} \varphi(t\widehat{u_1}) \quad &< \quad 0, \\ &\Rightarrow \quad \varphi(u_0) < 0 = \varphi(0) \text{ (see (3))} \\ &\Rightarrow \quad u_0 \ne 0. \end{aligned}$$

From (3) we have

$$\begin{aligned} & \langle \varphi'(u_0), h)\rangle = 0 \quad \text{for all } h \in W_0^{1,\eta}(\Omega), \\ \Rightarrow \quad & \langle V(u_0), h\rangle = \int_\Omega \theta(z)(u_0^+)^{q-1} h dz - \int_\Omega f(z, u_0^+) h dz \quad \text{for all } \ h \in W_0^{1,\eta}(\Omega). \end{aligned}$$

Choosing $h = -u_0^- \in W_0^{1,\eta}(\Omega)$, we obtain

$$\begin{aligned} & \rho_a(Du_0^-) \le 0, \\ \Rightarrow \quad & u_0 \ge 0, \ u_0 \ne 0 \quad \text{(see Proposition 1).} \end{aligned}$$

Therefore $u_0 \in W_0^{1,\eta}(\Omega) \setminus \{0\}$ is a positive solution of problem (1). Theorem 3.1 of Gasiński-Winkert [4], implies that $u_0 \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$.

Hypotheses $H_1(i), (iii)$ imply that given $\epsilon > 0$, we can find $\widehat{c_\epsilon} > 0$ such that

$$f(z, x) \le \epsilon x^{q-1} + \widehat{c_\epsilon} x^{r-1} \quad \text{for a.a. } z \in \Omega, \ \text{ all } x \ge 0. \tag{7}$$

Therefore we have

$$\begin{aligned} -\Delta_p^a u_0 - \Delta_q u_0 &= \theta(z) u_0^{q-1} - f(z, u_0) \\ &\ge [\theta(z) - \epsilon] u_0^{q-1} - \widehat{c_\epsilon} \|u_0\|_\infty^{r-p} u_0^{p-1} \ \text{ in } \Omega \text{ (recall } p < r). \end{aligned}$$

Choosing $\epsilon \in (0, \widehat{\lambda}_1(q))$, we infer that for some $c_4 > 0$, we have

$$-\Delta_p^a u_0 - \Delta_q u_0 + c_4 u_0^{p-1} \ge 0 \quad \text{in } \Omega.$$

Then Proposition 2.4 of Papageorgiou-Vetro-Vetro [17], implies that for all $K \subseteq \Omega$ compact we have

$$0 < c_K \le u_0(z) \ \text{ for a.a. } z \in K.$$

□

Remark 3.2. *Evidently we have that* $u_0(z) > 0$ *for a.a.* $z \in \Omega$.

Next we show that this positive solution of (1) is in fact unique.

Proposition 3.3. *If hypotheses* H_0, H_1 *hold, then the positive solution of* (1) *is unique.*

Proof. From Proposition 3.1, we already have a positive solution $u_0 \in W_0^{1,\eta}(\Omega) \bigcap L^\infty(\Omega)$. Let v_0 be another positive solution of (1). Again, we have $v_0 \in W_0^{1,\eta}(\Omega) \bigcap L^\infty(\Omega)$. For $\varepsilon > 0$, we set $u_0^\varepsilon = u_0 + \varepsilon$ and $v_0^\varepsilon = v_0 + \varepsilon$. We have

$$u_0^\varepsilon, \ v_0^\varepsilon \in W^{1,\eta}(\Omega) \text{ and } u_0^\varepsilon, \ v_0^\varepsilon \in intL^\infty(\Omega)_+.$$

Invoking Proposition 4.1.22, p.274, of Papageorgiou-Rădulescu-Repovš [15], we can say that

$$\frac{u_0^\varepsilon}{v_0^\varepsilon} \in L^\infty(\Omega), \ \frac{v_0^\varepsilon}{u_0^\varepsilon} \in L^\infty(\Omega). \tag{8}$$

Consider the integral functional $j : L^1(\Omega) \to \bar{R} = R \cup \{+\infty\}$
defined by

$$j(u) = \begin{cases} \frac{1}{p} \rho_a(D_u^{\frac{1}{q}}) + \frac{1}{q} \|D_u^{\frac{1}{q}}\|_q^q & \text{if } u \ge 0, \ u^{\frac{1}{q}} \in W^{1,\eta}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $dom j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$). Consider the integranal $\widehat{\eta}(z,t)$ defined by

$$\widehat{\eta}(z,t) = \frac{a(z)}{p}t^p + \frac{1}{q}t^q \textit{ for all } z \in \Omega, \textit{ all } t \geq 0.$$

We see that for every $z \in \Omega$,

- $t \to \widehat{\eta}(z,t)$ is strictly increasing on $\mathbb{R}_+ = [0,\infty)$;
- $t \to \widehat{\eta}(z,t^{\frac{1}{q}})$ is convex (recall $q < \rho$).

We set $\widehat{H}(z,y) = \widehat{\eta}(z,|y|)$ for all $z \in \Omega$, all $y \in \mathbb{R}^N$. Evidently for all $z \in \Omega$, $\widehat{H}(z,\cdot)$ is convex.
Let $u_1,\ u_2 \in dom j$ and set $v = [tu_1 + (1-t)u_2]^{\frac{1}{q}},\ t \in [0,1]$. From Diaz-Saa[2](see Lemma 1), we have

$$\begin{aligned}
|Dv| &\leq [t|Du_1^{\frac{1}{q}}|^q + (1-t)|Du_2^{\frac{1}{q}}|^q]^{\frac{1}{q}} \\
\Rightarrow \widehat{\eta}(z,|Dv|) &\leq \widehat{\eta}(z,[t|Du_1^{\frac{1}{q}}|^q + (1-t)|Du_2^{\frac{1}{q}}|^q]^{\frac{1}{q}}) \text{ (since } \widehat{\eta}(z,\cdot) \text{ is increasing)} \\
&\leq t\widehat{\eta}(z,|Du_1^{\frac{1}{q}}|) + (1-t)\widehat{\eta}(z,|Du_2^{\frac{1}{q}}|) \text{ (since } t \to \widehat{\eta}(z,t^{\frac{1}{q}}) \text{ is convex)}, \\
&\Rightarrow \widehat{H}(z,Dv) \leq t\widehat{H}(z,Du_1^{\frac{1}{q}}) + (1-t)\widehat{H}(z,Du_2^{\frac{1}{q}}) \\
&\Rightarrow j(\cdot) \text{ is convex.}
\end{aligned}$$

Let $h = (u_0^\varepsilon)^q - (v_0^\varepsilon)^q \in W_0^{1,\eta}(\Omega)$. On account of (8) for $|t| < 1$ small we have

$$(u_0^\varepsilon)^q + th \in dom j \textit{ and } (v_0^\varepsilon)^q + th \in dom j.$$

This and the convexity of $j(\cdot)$ imply that the directional derivatives of $j(\cdot)$ at $(u_0^\varepsilon)^q$ and at $(v_0^\varepsilon)^q$ in the direction h exist and using the chain rule and Green's identity, we have

$$\begin{aligned}
j'((u_0^\varepsilon)^q)(h) =& \frac{1}{q}\int_\Omega \frac{-\Delta_0^\alpha u_0 - \Delta_q u_0}{(u_0^\varepsilon)^{q-1}} h dz \\
=& \frac{1}{q}\int_\Omega \frac{\theta(z)u_0^{q-1} - f(z,u_0)}{(u_0^\varepsilon)^{q-1}} h dz,
\end{aligned}$$

$$\begin{aligned}
j'((v_0^\varepsilon)^q)(h) =& \frac{1}{q}\int_\Omega \frac{-\Delta_0^a v_0 - \Delta_q v_0}{(v_0^\varepsilon)^{q-1}} h dz \\
=& \frac{1}{q}\int_\Omega \frac{\theta(z)v_0^{q-1} - f(z,v_0)}{(v_0^\varepsilon)^{q-1}} h dz.
\end{aligned}$$

The convexity of $j(\cdot)$, implies that the directional derivative $j^{'}(\cdot)$ is monotone. Therefore, we have

$$\begin{aligned}
0 \leq& \int_\Omega \theta(z)\left[\frac{u_0^{q-1}}{(u_0^\varepsilon)^{q-1}} - \frac{v_0^{q-1}}{(v_0^\varepsilon)^{q-1}}\right]((u_0^\varepsilon)^q - (v_0^\varepsilon)^q)dz \\
&+ \int_\Omega \left[\frac{f(z,v_0)}{(v_0^\varepsilon)^{q-1}} - \frac{f(z,u_0)}{(u_0^\varepsilon)^{q-1}}\right]((u_0^\varepsilon)^q - (v_0^\varepsilon)^q)dz. \qquad (9)
\end{aligned}$$

Note that

$$\left|\frac{u_0^{q-1}}{(u_0^\varepsilon)^{q-1}} - \frac{v_0^{q-1}}{(v_0^\varepsilon)^{q-1}}\right| \leq 2,\ u_0^\varepsilon,\ v_0^\varepsilon \in L^\infty(\Omega). \qquad (10)$$

Therefore using the dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0^+}\int_\Omega \theta(z)\left[\frac{u_0^{q-1}}{(u_0^\varepsilon)^{q-1}}-\frac{v_0^{q-1}}{(v_0^\varepsilon)^{q-1}}\right]((u_0^\varepsilon)^q-(v_0^\varepsilon)^q)dz=0. \tag{11}$$

Also we have

$$\left|\frac{f(z,v_0)}{(v_0^\varepsilon)^{q-1}}-\frac{f(z,u_0)}{(u_0^\varepsilon)^{q-1}}\right|\le \frac{f(z,v_0)}{v_0^{q-1}}+\frac{f(z,u_0)}{u_0^{q-1}}.$$

Hypotheses $H_1(i)$, (iii) implies that there exists $c_5>0$ such that

$$f(z,x)\le |x|^{q-1}+c_5|x|^{r-1} \text{ for a.a. } z\in\Omega,\ \text{all } x\in R. \tag{12}$$

Using (12) we have

$$\frac{f(z,v_0)}{v_0^{q-1}}+\frac{f(z,u_0)}{u_0^{q-1}}\le a+c_5[v_0^{r-q}+u_0^{r-q}]\le c_6 \text{ for some } c_6>0 \text{ (see (10)).}$$

Hence using once again the dominated convergence theorem, we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_\Omega &\left[\frac{f(z,v_0)}{(v_0^\varepsilon)^{q-1}}-\frac{f(z,u_0)}{(u_0^\varepsilon)^{q-1}}\right]((u_0^\varepsilon)^q-(v_0^\varepsilon)^q)dz\\ &=\int_\Omega\left[\frac{f(z,v_0)}{v_0^{q-1}}-\frac{f(z,u_0)}{u_0^{q-1}}\right](u_0^q-v_0^q)dz.\end{aligned} \tag{13}$$

If in (9) we let $\varepsilon\to 0^+$, then using (11) and (13), we obtain

$$0\le\int_\Omega\left[\frac{f(z,v_0)}{v_0^{q-1}}-\frac{f(z,u_0)}{u_0^{q-1}}\right](u_0^q-v_0^q)dz. \tag{14}$$

The strict monotonicity of $x\to\dfrac{f(z,x)}{x^{q-1}}$ on $\mathring{\mathbb{R}}_+=(0,+\infty)$ (see hypotheses $H,(ii)$) and (14) imply that $u_0=v_0$. The proves the uniqueness of the positive solution $u_0=W_0^{1,\eta}(\Omega)\cap L^\infty(\Omega)$. □

So, finally we can state the following existence and uniqueness theorem for the Dirichlet equidiffusive logistic problem.

Theorem 3.4. *If hypotheses H_0,H_1 hold, then problem* (1) *has a unique positive solution $u_0\in W_0^{1,\eta}(\Omega)\cap L^\infty(\Omega)$ and for every $K\subseteq\Omega$ compact, we have*

$$0<c_K\le u_0(z)\ for\ a.a. z\in K.$$

References

[1] P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase. Calc. Var. 57(2019),Art.62,48pp

[2] J.I. Diaz, J.E. Saa, Existence et unicité de solutions positives pour certaines equations elliptiques quasilineaires. CRAS Paris, t.305(1987),521-524.

[3] L. Gasiński, N.S. Papageorgiou, Nonlinear Analysis, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[4] L. Gasiński, P. Winkert, Constant sign solutions for double phase problems with superlinear nonlinearity. Nonlin. Anal. 195 (2020) 111739, 9pp.

[5] B. Ge, D. Lv, J. Lu, Multiple solutions for a class of double phase problems without the Ambrosetti-Rabinowitz condition. Nonlin. Anal. 188(2019) 294-315.

[6] M.E. Gurtin, R.C. Mac Camy, On the difusion of biological populations, Math. Biosci. 33 (1977) 35-49.
[7] P. Harjulehto, P. Hästo, Orlicz Spaces and Generalized Orlicz Spaces. Springer, Cham. 2019.
[8] W. Liu, G. Dai, Existence and multiplicity results for double phase problems. J. Differential Equ. 265(2018), 4311-4334.
[9] Z. Liu, N.S Papageorgiou, Double phase Dirichlet problems with unilateral constraints. J. Differential Equ. 316(2022), 249-269.
[10] P. Marcellini, Reguarity of minimizers of integrals of the calculus of the variations with nonstandard growth coditions. Arch. Rational Mech. Anal.105(1989),267-284.
[11] P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes. J.Math.Anal.Appl. 501(2021),124408, 32pp.
[12] G. Mingione, V.D. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity. J.Math.Anal.Appl. 501(2021) 125197, 41pp.
[13] N.S. Papageorgiou, Double phase problems: a survey of some recent results. Opuscula Math. 42(2)(2022),257-278.
[14] N.S. Papageorgiou,V.D. Rădulescu, D Repovš, Nonautonomous (p,q)-equations with unbalanced growth. Math. Annalen, doi:10.1007/s00208-022-02381-1.
[15] N.S. Papageorgiou,V.D. Rădulescu, D Repovš, Nonlinear Analysis-Theory and Methods. Springer, Cham, 2019.
[16] N.S. Papageorgiou,V.D. Rădulescu, D Repovš, Existence and multiplicity of solutions for double-phase Robin problems. Bull. London Math. Soc. 52(2020) 546-560.
[17] N.S. Papageorgiou, C. Vetro, F. Vetro, Multiple solutions for parametric double phase Dirichlet problems. Comm. Contemp. Math. 23(2021)2050006,18pp.
[18] N.S. Papageorgiou, P. Winkert, Applied Nonlinear Functional Analysis, De Gruyter, Berlin, 2018.
[19] V.D. Rădulescu, Isotropic and anisotropic double phase problems: old and new. Opuscula Math. 39(2)(2019),259-280.
[20] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity. Math. USSR-Izv. 29(1987), 33-66.