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Pointwise semi-slant Riemannian (PSSR) maps from almost Hermitian manifolds

Yılmaz Gündüzalp[a], Mehmet Akif Akyol[b]

[a]*Department of Mathematics, Dicle University, 21280, Diyarbakır, Türkiye*
[b]*Department of Mathematics, Faculty of Art and Sciences, Bingöl University, 12000, Bingöl, Türkiye*

Abstract. In this paper, as a generalization of pointwise slant submanifolds [B-Y. Chen and O. J. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds*, Turk J Math 36, (2012), 630-640.], pointwise slant submersions [J.W.Lee and B. Ṣahin, *Pointwise slant submersions*, Bulletin of the Korean Mathematical Sosiety, 51(4), (2014), 115-1126.] and pointwise slant Riemannian maps [Y. Gündüzalp and M. A. Akyol, *Pointwise slant Riemannian maps from Kaehler manifolds*, Journal of Geometry and Physics, 179, (2002), 104589.], we introduce pointwise semi-slant Riemannian maps (briefly, **PSSR** maps) from almost Hermitian manifolds to Riemannian manifolds, present examples and characterizations. We also investigate the harmonicity of such maps. Moreover, we give Chen-Ricci inequality for a **PSSR** map. Finally, we study some curvature relations in complex space forms, involving Casorati curvatures for **PSSR** maps.

1. Introduction

In differential geometry, it is good to use some types of maps in order to compare objects (in particular, manifolds). The theory of smooth maps between Riemannian manifolds plays a preeminent role in differential geometry and also in physics. The main smooth maps are isometric immersions, Riemannian submersions and Riemannian maps. These kinds of maps have many applications, including in supergravity and superstring theories, Yang-Mills theory, Kaluza-Klein theory, geometric modeling, computer vision, medical imaging, cartography and sustainability science ([9–11, 29–33, 55–57]).

Slant submanifolds were introduced by B. Y. Chen [13] as a generalization of almost complex submanifolds and totally real submanifolds of an almost Hermitian manifold in 1990. After that, N. Papaghiuc [34] defined the notion of semi-slant submanifolds of an almost Hermitian manifold as a generalization of CR-submanifolds and slant submanifolds of an almost Hermitian manifold.

In [12], Casorati introduced Casorati curvature which is a very natural concept of regular surfaces in the three-dimensional Euclidean space. The curvature is obtained by the normalized sum of the squared principal curvatures of the surface. After that, many geometers published some optimal inequalities involving Casorati curvatures in ([6], [7], [24], [25], [49], [50], [59], [60]).

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Email addresses: `ygunduzalp@dicle.edu.tr` (Yılmaz Gündüzalp), `mehmetakifakyol@bingol.edu.tr` (Mehmet Akif Akyol)

The main extrinsic (the squared mean curvature) and main intrinsic invariants (the scalar curvature and the Ricci curvature) of a submanifold in a real space form was established by B. Y. Chen in [14] (see also [15]). For the inequalities, see: ([5, 27, 28, 48, 51, 52]).

In 1998, pointwise slant submanifolds were introduced by F. Etayo [17] as a natural generalization of slant submanifolds. In 2012, the notion were investigated by B. Y. Chen and O. J. Garay [16]. The notion of pointwise semi-slant submanifolds were intoduced by B. Şahin in [47].

Dual to slant submanifolds, B. Şahin introduced in [44] the notion of slant submersions as a generalization of anti-inviariant and holomorphic submersions in [45]. The notion of semi-slant Riemannian submersions were introduced by K. S. Park and R. Prasad in 2013 [36]. As a natural generalization of slant submersions. pointwise slant submersions were introduced by J. W. Lee and B. Şahin in [26].

A. Fischer [23] introduced Riemannian maps as a generalization of isometric immersions and Riemannian submersions. After that, many geometers published many papers related to Riemannian maps in ([2], [3], [4], [19],[20], [37], [38], [39], [40], [41], [43], [46], [53], [54]). Moreover, B. Şahin defined slant Riemannian maps in [42]. As a generalization of slant Riemannian maps and semi-slant submersions, semi-slant Riemannian maps were introduced by [35]. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For Riemannian maps and their applications in spacetime geometry (see: [1], [18]).

In 2022, as a more general class of Riemannian maps including slant submanifolds, slant submersions, slant Riemannian maps, pointwise slant submanifolds, pointwise slant submersions, pointwise slant Riemannian maps were introduced by the authors [20] as follows.

Definition 1.1. Let φ be a Riemannian map from an almost Hermitian manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}). If at each given $q \in B_1$, the Wirtinger angle $\theta(U_1)$ between J_1U_1 and the space $(ker\varphi_*)_q$ is independent of the choice of the non-zero vector field $U_1 \in (ker\varphi_*)$, then we say that φ is a pointwise slant Riemannian map. In this case, the angle θ can be regarded as a function on B_1, which is called the slant function of the pointwise slant Riemannian map.

In the present paper, as a more generalization of the above mentioned notions, we will define the notions of **PSSR** maps from almost Hermitian manifolds onto Riemannian manifolds and investigate the geometry of the total space and the base space.

We organize the paper as follows. In Sec. 2 we deal with some necessary notions and recall some basic notions. In Sec. 3 we introduce the definition of **PSSR** maps from almost Hermitian manifolds onto Riemannian manifolds, giving many examples and investigate the geometry of foliations which are arisen from the definition of a **PSSR** map. We also investigate the harmonicity of such maps and find necessary and sufficient conditions for **PSSR** maps to be totally geodesic. In Sec. 4, we give Chen-Ricci inequality for a **PSSR** map. In Sec. 5, we study some curvature relations in complex space form, involving Casorati curvatures for **PSSR** maps.

2. Preliminaries

In this section, we review some basic concepts and results on geometric structures for Riemannian maps.

Let (B_1, g_{B_1}, J_1) be an almost Hermitian manifold. This means that B_1 admits a tensor field J_1 of type $(1,1)$ on B_1 such that

$$J_1^2 = -I, \quad g_{B_1}(J_1Y_1, J_1Y_2) = g_{B_1}(Y_1, Y_2), \quad Y_1, Y_2 \in \Gamma(TB_1). \tag{1}$$

An almost Hermitian manifold B_1 is called Kaehler manifold [58] if

$$(\nabla_{Y_1} J_1)Y_2 = 0, \quad Y_1, Y_2 \in \Gamma(TB_1), \tag{2}$$

where ∇ denotes the Riemannian connection of the metric g_{B_1} on B_1.

Let (B_1, g_{B_1}) and (B_2, g_{B_2}) be Riemannian manifolds and $\varphi : (B_1, g_{B_1}) \to (B_2, g_{B_2})$ is a differentiable map. Then the differential φ_* of φ can be viewed a section of the bundle $Hom(TB_1, \varphi^{-1}TB_2) \to B_1$, where $\varphi^{-1}TB_2$

is the pullback bundle which has fibres $(\varphi^{-1}TB_2)_q = T_{\varphi(q)}B_2$, $q \in B_1$. $Hom(TB_1, \varphi^{-1}TB_2)$ has a connection ∇ induced from the Levi-Civita connection ∇^{B_1} and the pullback connection. The second fundamental form of φ is given by [8]

$$(\nabla\varphi_*)(Y_1, Y_2) = \nabla^{\varphi}_{Y_1}\varphi_* Y_2 - \varphi_*(\nabla^{B_1}_{Y_1} Y_2) \tag{3}$$

for $Y_1, Y_2 \in \Gamma(TB_1)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric. Remind that φ is said to be harmonic if we get the tension field $\tau(\varphi) = trace(\nabla\varphi_*) = 0$ and we call the map a totally geodesic map if $(\nabla\varphi_*)(Y_1, Y_2) = 0$. On the other hand, it is shown in [41] that $(\nabla\varphi_*)(Y_1, Y_2)$ has no components in $Im\varphi_*$, provided that $Y_1, Y_2 \in \Gamma((ker\varphi_*)^{\perp})$. More precisely,

$$(\nabla\varphi_*)(Y_1, Y_2) \in \Gamma((range\varphi_*)^{\perp}), \ \ \forall Y_1, Y_2 \in \Gamma((ker\varphi_*)^{\perp}), \tag{4}$$

here $(range\varphi_*)^{\perp}$ is the subbundle of $\varphi^{-1}(TB_2)$ with fibre $\Gamma(\varphi_*(T_qB_1)^{\perp})$, $q \in B_1$.

Let φ be a Riemannian map from a Riemannian manifold (B_1, g_{B_1}) to a Riemannian manifold (B_2, g_{B_2}). Then, we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{T}_{Y_1} Y_2 = h\nabla_{vY_1} vY_2 + v\nabla_{vY_1} hY_2 \tag{5}$$

and

$$\mathcal{A}_{Y_1} Y_2 = v\nabla_{hY_1} hY_2 + h\nabla_{hY_1} vY_2 \tag{6}$$

for every $Y_1, Y_2 \in \Gamma(TB_1)$, where ∇ is the Levi-Civita connection of g_{B_1}. In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $Y_1 \in \Gamma(TB_1)$, $\mathcal{T}_{Y_1}$ and $\mathcal{A}_{Y_1}$ are skew-symmetric operators reversing the horizontal and the vertical distributions. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_{U_1} U_2 = \mathcal{T}_{U_2} U_1, \ \ \mathcal{A}_{Y_1} Y_2 = -\mathcal{A}_{Y_2} Y_1, \tag{7}$$

for any $\forall U_1, U_2 \in \Gamma(ker\varphi_*), \forall Y_1, Y_2 \in \Gamma(ker\varphi_*)^{\perp}$. Using (5) and (6), we obtain

$$\nabla_{U_1} U_2 = \mathcal{T}_{U_1} U_2 + \hat{\nabla}_{U_1} U_2; \tag{8}$$

$$\nabla_{U_1} Y_1 = \mathcal{T}_{U_1} Y_1 + h\nabla_{U_1} Y_1; \tag{9}$$

$$\nabla_{Y_1} U_1 = \mathcal{A}_{Y_1} U_1 + v\nabla_{Y_1} U_1; \tag{10}$$

$$\nabla_{Y_1} Y_2 = \mathcal{A}_{Y_1} Y_2 + h\nabla_{Y_1} Y_2, \tag{11}$$

for any $Y_1, Y_2 \in \Gamma((ker\varphi_*)^{\perp})$, $U_1, U_2 \in \Gamma(ker\varphi_*)$, here $\hat{\nabla}_{U_1} U_2 = v\nabla_{U_1} U_2$.

3. PSSR maps from Kaehler manifolds

Definition 3.1. *Let (B_1, g_{B_1}, J_1) be an almost Hermitian manifold and (B_2, g_{B_2}) be a Riemannian manifold. Then we say that a Riemannian map $\varphi : B_1 \to B_2$ is a pointwise semi-slant Riemannian map (briefly,* **PSSR** *map) if there exists a pair of orthogonal distributions D^{θ} and $\mathrm{D}^{\top}$ on $ker\varphi_*$ such that*

1. *The space $ker\varphi_*$ admits the orthogonal direct decomposition $\mathrm{D}^{\theta} \oplus \mathrm{D}^{\top}$.*
2. *The distribution $\mathrm{D}^{\top}$ is invariant.*
3. *The distribution D^{θ} is pointwise slant with semi-slant function θ.*

In this case, the angle θ can be regarded as a function on B_1, which is called the semi-slant function of the **PSSR** *map.*

Now, we present some examples for proper **PSSR** maps. Let J_1 be an almost complex structure on $\mathbb{R}^8$ as follows:

$$J_1(y_1, ..., y_8) = (y_2, -y_1, ..., y_8, -y_7).$$

Example 3.1. Define a map $\varphi : \mathbb{R}^8 \to \mathbb{R}^6$ by

$$\varphi(y_1, ..., y_8) = (y_1 \cos x - y_3 \sin x, y_2 \sin y - y_4 \cos y, y_5, y_6, \pi, e),$$

where $x, y : \mathbb{R}^8 \to \mathbb{R}$ are real valued functions. Then the map φ is a **PSSR** map such that

$$\mathrm{D}^\top =< \frac{\partial}{\partial y_7}, \frac{\partial}{\partial y_8} > \ and\, \mathrm{D}^\theta =< \sin x \frac{\partial}{\partial y_1} + \cos x \frac{\partial}{\partial y_3}, \cos y \frac{\partial}{\partial y_2} + \sin y \frac{\partial}{\partial y_4} >$$

with the semi-slant function θ with $\cos\theta = \sin(x + y)$.

Example 3.2. Let $(\mathbb{R}^8, g_{\mathbb{R}^8})$ be the Euclid space. Consider $\{J_1, J_2\}$ a pair of almost complex structures on $\mathbb{R}^8$ satisfying $J_1 J_2 = -J_2 J_1$, here

$$J_1(a_1, ..., a_8) = (-a_3, -a_4, a_1, a_2, -a_7, -a_8, a_5, a_6)$$

and

$$J_2(a_1, ..., a_8) = (-a_2, a_1, a_4, -a_3, -a_6, a_5, a_8, -a_7).$$

For any real-valued function $\lambda : \mathbb{R}^8 \to \mathbb{R}$, we define new almost complex structure J_λ on $\mathbb{R}^8$ by $J_\lambda = (\cos\lambda)J_1 + (\sin\lambda)J_2$.
Then, $\mathbb{R}^8_\lambda = (\mathbb{R}^8, J_\lambda, g_{\mathbb{R}^8})$ is an almost Hermitian manifold.
Consider a Riemannian map $\varphi : \mathbb{R}^8_\lambda \to \mathbb{R}^8$ by

$$\varphi(y_1, ..., y_8) = (0, 0, 0, 0, 0, y_6, 0, y_8).$$

Then we obtain

$$(ker\varphi_*)^\perp =< X_1 = \frac{\partial}{\partial y_6}, X_2 = \frac{\partial}{\partial y_8} >$$

and

$$ker\varphi_* =< V_1 = \frac{\partial}{\partial y_5}, V_2 = \frac{\partial}{\partial y_7}, V_3 = \frac{\partial}{\partial y_1}, V_4 = \frac{\partial}{\partial y_2}, V_5 = \frac{\partial}{\partial y_3}, V_6 = \frac{\partial}{\partial y_4} > .$$

On the other hand, we get $J_\lambda(V_3) = \sin\lambda V_4 + \cos\lambda V_5$, $J_\lambda(V_4) = -\sin\lambda V_3 + \cos\lambda V_6$, $J_\lambda(V_5) = -\cos\lambda V_3 - \sin\lambda V_6$, $J_\lambda(V_6) = -\cos\lambda V_4 + \sin\lambda V_5$. Thus, φ is a **PSSR** map with the semi-slant function $\theta = \lambda$ with $\mathrm{D}^\theta =< V_1, V_2 >$ and $\mathrm{D}^\top =< V_3, V_4, V_5, V_6 >$.

Let $\varphi : (B_1, g_{B_1}, J_1) \to (B_2, g_{B_2})$ be a **PSSR** map. Then, for $V \in \Gamma(ker\varphi_*)$, we can write

$$J_1 V = Q_1 V + Q_2 V, \tag{12}$$

here $Q_1 V \in \Gamma(\mathrm{D}^\top)$ and $Q_2 V \in \Gamma(\mathrm{D}^\theta)$. For $V \in \Gamma(ker\varphi_*)$

$$J_1 V = \zeta V + \eta V, \tag{13}$$

here $\zeta V \in \Gamma(ker\varphi_*)$ and $\eta V \in \Gamma(ker\varphi_*)^\perp$. Also, for $Y \in \Gamma(ker\varphi_*)^\perp$, we get

$$J_1 Y = \bar{\zeta} Y + \bar{\eta} Y, \tag{14}$$

here $\bar{\zeta} Y \in \Gamma(ker\varphi_*)$ and $\bar{\eta} Y \in \Gamma(ker\varphi_*)^\perp$. For $Z \in \varphi^{-1} T B_2$, we have

$$Z = \bar{Q}_1 Z + \bar{Q}_2 Z, \tag{15}$$

here $\bar{Q}_1 Z \in \Gamma(range\varphi_*)$ and $\bar{Q}_2 Z \in \Gamma(range\varphi_*)^\perp$. Then,

$$(ker\varphi_*)^\perp = \eta D^\theta \oplus \nu, \tag{16}$$

here ν is the orthogonal complement of ηD^θ in $\Gamma(ker\varphi_*)^\perp$ and is invariant under J_1.

In addition to,

$$\zeta D^\top = D^\top,\ \eta D^\top = 0,\ \zeta D^\theta \subset D^\theta,\ \bar{\zeta}((ker\varphi_*)^\perp) = D^\theta$$
$$\zeta^2 + \bar{\zeta}\eta = -I,\ \bar{\eta}^2 + \eta\bar{\zeta} = -I,\ \eta\zeta + \bar{\eta}\eta = 0,\ \bar{\zeta}\bar{\eta} + \zeta\bar{\zeta} = 0.$$

Also, if (B_1, g_{B_1}, J_1) is Kaehler, for $V_1, V_2 \in \Gamma(ker\varphi_*)$, then it is easy to get

$$(\nabla_{V_1}\eta)V_2 = \bar{\eta}T_{V_1}V_2 - T_{V_1}\zeta V_2 \tag{17}$$
$$(\nabla_{U_1}\zeta)V_2 = \bar{\zeta}T_{V_1}V_2 - T_{V_1}\eta V_2, \tag{18}$$

here ∇ is the Levi-Civita connection on B_1 and define

$$(\nabla_{V_1}\eta)V_2 = h\nabla_{V_1}\eta V_2 - \eta\hat{\nabla}_{V_1}V_2 \tag{19}$$
$$(\nabla_{V_1}\zeta)V_2 = \hat{\nabla}_{V_1}\zeta V_2 - \zeta\hat{\nabla}_{V_1}V_2. \tag{20}$$

Theorem 3.3. *Let φ be a* **PSSR** *map from an almost Hermitian manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}) with semi-slant function θ. Then, we have*

$$\zeta^2 V_1 = -(\cos^2\theta)V_1,\ \ V_1 \in \Gamma(D^\theta). \tag{21}$$

Proof. Since,

$$\cos\theta = \frac{g_{B_1}(J_1V_1, \zeta V_1)}{|J_1V_1||\zeta V_1|} = -\frac{g_{B_1}(V_1, \zeta^2 V_1)}{|V_1||\zeta V_1|}$$

and $\cos\theta == \frac{|\zeta V_1|}{|J_1V_1|}$, for $V_1 \in \Gamma(D^\theta)$ we obtain

$$\cos^2\theta = -\frac{g_{B_1}(V_1, \zeta^2 V_1)}{|V_1|^2}.$$

Hence,

$$\zeta^2 V_1 = -(\cos^2\theta)V_1.$$

Also, conversely, it can be directly verified.

Using (1), (13) and (21), for $V_1, V_2 \in \Gamma(D^\theta)$ we have

$$g_{B_1}(\zeta V_1, \zeta V_2) = \cos^2\theta g_{B_1}(V_1, V_2) \tag{22}$$
$$g_{B_1}(\eta V_1, \eta V_2) = \sin^2\theta g_{B_1}(V_1, V_2). \tag{23}$$

When semi-slant function θ, locally we can write an orthonormal frame $\{X_1, J_1X_1, ..., X_k, J_1X_k, \bar{X}_1, \sec\theta\zeta\bar{X}_1, csc\theta\eta\bar{X}_1, ..., \bar{X}_s, \sec\theta\zeta\bar{X}_s, csc\theta\eta\bar{X}_s, \hat{X}_1, J_1\hat{X}_1, ..., \hat{X}_t, J_1\hat{X}_t\}$ of TB_1 such that $\{X_1, J_1X_1, ..., X_k, J_1X_k\}$ is an orthonormal frame of $D^\top$, $\{\bar{X}_1, \sec\theta\zeta\bar{X}_1, ..., \bar{X}_s, \sec\theta\zeta\bar{X}_s\}$ an orthonormal frame of D^θ, $\{csc\theta\eta\bar{X}_1, ..., csc\theta\eta\bar{X}_s\}$ an orthonormal frame of ηD^θ, and $\{\hat{X}_1, J_1\hat{X}_1, ..., \hat{X}_t, J_1\hat{X}_t\}$ an orthonormal frame of ν.

Lemma 3.4. *Let φ be a* **PSSR** *map from a Kaehler manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}) with semi-slant function θ. If η is parallel, then $V_1 \in \Gamma(D^\theta)$ we get:*

$$T_{\zeta V_1}\zeta V_1 = -\cos^2\theta T_{V_1}V_1. \tag{24}$$

Proof. Assume that η is parallel. Then, for $V_1, V_2 \in \Gamma(D^\theta)$ we obtain

$$T_{V_1}\zeta V_2 = \bar{\eta}T_{V_1}V_2.$$

By replacing V_1 and V_2 we get

$$T_{V_2}\zeta V_1 = \bar{\eta} T_{V_2} V_1.$$

So,

$$T_{V_2}\zeta V_1 = T_{V_1}\zeta V_2.$$

If we write ζV_1 instead of V_2 and using (21), then the proof is completed.

The proof of the following Theorem is the same with Theorem 49 in [45].

Theorem 3.5. *Let φ be a* **PSSR** *map from a Kaehler manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}). Then, the distribution $\mathrm{D}^{\top}$ is integrable if and only if for $V_1, V_2 \in \Gamma(\mathrm{D}^{\top})$ we get*

$$\eta(\hat{\nabla}_{V_1} V_2 - \hat{\nabla}_{V_2} V_1) = 0.$$

Theorem 3.6. *Let φ be a* **PSSR** *map from a Kaehler manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}) such that $\mathrm{D}^{\top}$ is integrable. Then, φ is harmonic if and only if $trace(\nabla\varphi_*) = 0$ on D^{θ} and $\bar{H} = 0$, here $\bar{H} = 0$ denotes the mean curvature vector field of $\Gamma(range\varphi_*)$.*

Proof. Using (4), we get $trace(\nabla\varphi_*)|_{(ker\varphi_*)} \in \Gamma(range\varphi_*)$ and $trace(\nabla\varphi_*)|_{(ker\varphi_*)^{\perp}} \in \Gamma(range\varphi_*)^{\perp}$ so that

$$trace(\nabla\varphi_*) = 0 \Leftrightarrow trace(\nabla\varphi_*)|_{(ker\varphi_*)} = 0,\ trace(\nabla\varphi_*)|_{(ker\varphi_*)^{\perp}} = 0.$$

Since $\mathrm{D}^{\top}$ is invariant under J_1, locally we can write an orthonormal frame $\{X_1, J_1X_1, ..., X_p, J_1X_p\}$ of $\mathrm{D}^{\top}$. Using the integrability of $\mathrm{D}^{\top}$,

$$\begin{aligned}(\nabla\varphi_*)(J_1X_j, J_1X_j) &= -\varphi_*\nabla_{J_1X_j}J_1X_j = -\varphi_*J_1(\nabla_{X_j}J_1X_j + [J_1X_j, X_j]) \\ &= \varphi_*\nabla_{X_j}X_j = -(\nabla\varphi_*)(X_j, X_j),\ 1 \leq j \leq p.\end{aligned}$$

So,

$$trace(\nabla\varphi_*)|_{(ker\varphi_*)} = 0 \Leftrightarrow trace(\nabla\varphi_*)|_{\mathrm{D}^{\theta}} = 0.$$

Furthermore, it is easy to get that

$$trace(\nabla\varphi_*)|_{(ker\varphi_*)^{\perp}} = k\bar{H},\ k = dim(ker\varphi_*)^{\perp}$$

so that

$$trace(\nabla\varphi_*)|_{(ker\varphi_*)^{\perp}} = 0 \Leftrightarrow \bar{H} = 0.$$

Thus, we have the result.

Using (24), we have:

Corollary 3.7. *Let φ be a* **PSSR** *map from a Kaehler manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}) such that $\mathrm{D}^{\top}$ is integrable and the semi-slant function θ. Assume that the tensor η is parallel. Then, φ is harmonic if and only if $\bar{H} = 0$.*

Theorem 3.8. *Let φ be a* **PSSR** *map from a Kaehler manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}) with the semi-slant function θ. Then, $\mathrm{D}^{\top}$ defines a totally geodesic foliation on B_1 if and only if*

$$g_{B_1}(\hat{\nabla}_{U_1}J_1U_2, \bar{\zeta}Y) = g_{B_2}((\nabla\varphi_*)(U_1, J_1U_2), \varphi_*\bar{\eta}Y) \tag{25}$$

and

$$-\cos^2\theta g_{B_1}(\hat{\nabla}_{U_1}U_2, U_3) = g_{B_2}((\nabla\varphi_*)(U_1, U_2), \varphi_*\eta\zeta U_3) + g_{B_1}(\mathrm{T}_{U_1}J_1U_2, \eta U_3) \tag{26}$$

for $U_1, U_2 \in \Gamma(\mathrm{D}^{\top})$, $U_3 \in \Gamma(\mathrm{D}^{\theta})$ and $Y \in \Gamma(ker\varphi_*)^{\perp}$.

Proof. For any $U_1, U_2 \in \Gamma(\mathrm{D}^{\top})$ and $Y \in \Gamma(ker\varphi_*)^{\perp}$, we get:

$$\begin{aligned} g_{B_1}(\nabla_{U_1} U_2, Y) &= g_{B_1}(\nabla_{U_1} J_1 U_2, J_1 Y) \\ &= g_{B_1}(\nabla_{U_1} J_1 U_2, \bar{\zeta} Y) + g_{B_1}(\nabla_{U_1} J_1 U_2, \bar{\eta} Y) \\ &= g_{B_1}(\hat{\nabla}_{U_1} J_1 U_2, \bar{\zeta} Y) - g_{B_2}((\nabla \varphi_*)(U_1, J_1 U_2), \varphi_* \bar{\eta} Y). \end{aligned} \tag{27}$$

Moreover, for all $U_1, U_2 \in \Gamma(\mathrm{D}^{\top})$ and $U_3 \in \Gamma(\mathrm{D}^{\theta})$ we obtain:

$$\begin{aligned} g_{B_1}(\nabla_{U_1} U_2, U_3) &= g_{B_1}(\nabla_{U_1} J_1 U_2, J_1 U_3) \\ &= \cos^2 \theta g_{B_1}(\hat{\nabla}_{U_1} U_2, U_3) + g_{B_2}((\nabla \varphi_*)(U_1, U_2), \varphi_* \eta \zeta U_3) \\ &+ g_{B_1}(\mathrm{T}_{U_1} J_1 U_2, \eta U_3). \end{aligned} \tag{28}$$

So the proof comes from (27) and (28).

Theorem 3.9. *Let φ be a* **PSSR** *map from a Kaehler manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}) with the semi-slant function θ. Then, D^{θ} defines a totally geodesic foliation on B_1 if and only if*

$$\begin{aligned} \sin^2 \theta g_{B_1}([U_1, Y], U_2) &= \sin 2\theta Y(\theta) g_{B_1}(U_1, U_2) \\ &+ g_{B_1}(\mathrm{A}_Y \eta \zeta U_1, U_2) - g_{B_1}(\mathrm{A}_Y \eta U_1, \zeta U_2) \\ &- g_{B_2}(\varphi_* h \nabla_Y \eta U_1, \varphi_* \eta U_2) \end{aligned} \tag{29}$$

and

$$g_{B_1}(\mathrm{T}_{U_1} \eta \zeta U_2, U_3) = g_{B_1}(\mathrm{T}_{U_1} \eta U_2, J_1 U_3) \tag{30}$$

for $U_1, U_2 \in \Gamma(\mathrm{D}^{\theta})$, $U_3 \in \Gamma(\mathrm{D}^{\top})$ and $Y \in \Gamma(ker\varphi_*)^{\perp}$.

Proof. For any $U_1, U_2 \in \Gamma(\mathrm{D}^{\theta})$ and $Y \in \Gamma(ker\varphi_*)^{\perp}$, we obtain:

$$\begin{aligned} g_{B_1}(\nabla_{U_1} U_2, Y) &= -g_{B_1}([U_1, Y], U_2) - g_{B_1}(\nabla_Y \zeta U_1, J_1 U_2) \\ &- g_{B_1}(\nabla_Y \eta U_1, J_1 U_2). \end{aligned}$$

From (21), we have

$$\begin{aligned} \sin^2 \theta g_{B_1}(\nabla_{U_1} U_2, Y) &= -\sin^2 \theta g_{B_1}([U_1, Y], U_2) + \sin 2\theta Y(\theta) g_{B_1}(U_1, U_2) \\ &+ g_{B_1}(\mathrm{A}_Y \eta \zeta U_1, U_2) - g_{B_1}(\mathrm{A}_Y \eta U_1, \zeta U_2) \\ &- g_{B_2}(\varphi_* h \nabla_Y \eta U_1, \varphi_* \eta U_2). \end{aligned} \tag{31}$$

Additionally, for all $U_1, U_2 \in \Gamma(\mathrm{D}^{\theta})$ and $U_3 \in \Gamma(\mathrm{D}^{\top})$, we get:

$$g_{B_1}(\nabla_{U_1} U_2, U_3) = g_{B_1}(\nabla_{U_1} J_1 U_2, J_1 U_3).$$

By using (9),(13) and (21), we arrive at

$$\sin^2 \theta g_{B_1}(\nabla_{U_1} U_2, U_3) = -g_{B_1}(\mathrm{T}_{U_1} \eta \zeta U_2, U_3) + g_{B_1}(\mathrm{T}_{U_1} \eta U_2, J_1 U_3). \tag{32}$$

Thus, the proof comes from (31) and (32).

Theorem 3.10. *Let φ be a* **PSSR** *map from a Kaehler manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}) with the semi-slant function θ. Then, the space $ker\varphi_*$ defines a totally geodesic foliation on B_1 if and only if*

$$\begin{aligned} g_{B_1}(\mathrm{A}_Y J_1 Q_2 U_1, J_1 U_2) &= -\sin^2 \theta g_{B_1}([U_1, Y], U_2) + \cos^2 \theta g_{B_1}(v \nabla_Y Q_2 U_1, U_2) \\ &+ \sin 2\theta Y(\theta) g_{B_1}(Q_1 U_1, U_2) + g_{B_1}(\mathrm{A}_Y \eta \zeta Q_1 U_1, U_2) \\ &- g_{B_1}(\mathrm{A}_Y \eta Q_1 U_1, J_1 U_2) - g_{B_1}(h \nabla_Y \eta Q_1 U_1, J_1 U_2) \\ &- g_{B_1}(v \nabla_Y J_1 Q_2 U_1, J_1 U_2). \end{aligned} \tag{33}$$

for $U_1, U_2 \in \Gamma(ker\varphi_*)$ *and* $Y \in \Gamma(ker\varphi_*)^\perp$.

Proof. For any $U_1, U_2 \in \Gamma(ker\varphi_*)$ and $Y \in (ker\varphi_*)^\perp$, from (1),(2) and (12), we get

$$\begin{aligned} g_{B_1}(\nabla_{U_1}U_2, Y) &= -g_{B_1}([U_1, Y], U_2) - g_{B_1}(\nabla_Y J_1 Q_1 U_1, J_1 U_2) \\ &- g_{B_1}(\nabla_Y J_1 Q_2 U_1, J_1 U_2). \end{aligned}$$

By using (13) and (21), we have

$$\begin{aligned} g_{B_1}(\nabla_{U_1}U_2, Y) &= -g_{B_1}([U_1, Y], U_2) - \cos^2\theta g_{B_1}(\nabla_Y Q_1 U_1, U_2) \\ &+ \sin 2\theta Y(\theta) g_{B_1}(Q_1 U_1, U_2) + g_{B_1}(\nabla_Y \eta\zeta Q_1 U_1, U_2) \\ &- g_{B_1}(\nabla_Y \eta Q_1 U_1, J_1 U_2) - g_{B_1}(\nabla_Y J_1 Q_2 U_1, J_1 U_2). \end{aligned}$$

By using (10) and (11), we obtain

$$\begin{aligned} \sin^2\theta g_{B_1}(\nabla_{U_1}U_2, Y) &= -\sin^2\theta g_{B_1}([U_1, Y], U_2) + \cos^2\theta g_{B_1}(v\nabla_Y Q_2 U_1, U_2) \\ &+ \sin 2\theta Y(\theta) g_{B_1}(Q_1 U_1, U_2) + g_{B_1}(\mathrm{A}_Y \eta\zeta Q_1 U_1, U_2) \\ &- g_{B_1}(\mathrm{A}_Y \eta Q_1 U_1, J_1 U_2) - g_{B_1}(h\nabla_Y \eta Q_1 U_1, J_1 U_2) \\ &- g_{B_1}(\mathrm{A}_Y J_1 Q_2 U_1, J_1 U_2) - g_{B_1}(v\nabla_Y J_1 Q_2 U_1, J_1 U_2). \end{aligned}$$

Theorem 3.11. *Let φ be a* **PSSR** *map from a Kaehler manifold (B_1, g_{B_1}, J_1) to a Riemannian manifold (B_2, g_{B_2}) with the semi-slant function θ. Then, the space $(ker\varphi_*)^\perp$ defines a totally geodesic foliation on B_1 if and only if*

$$v\nabla_{U_1}\bar{\zeta}U_2 = -\mathrm{A}_{U_1}\bar{\eta}U_2 \tag{34}$$

and

$$\begin{aligned} g_{B_2}(\nabla_{U_1}\varphi_*(\eta V_2), \varphi_*\bar{\eta}U_2) &= g_{B_2}(\nabla_{U_1}\varphi_*(\eta\zeta V_2), \varphi_* U_2) \\ &- g_{B_1}(\mathrm{A}_{U_1}\eta V_2, \bar{\zeta}U_2). \end{aligned} \tag{35}$$

for $V_2 \in \Gamma(\mathrm{D}^\theta)$, $V_1 \in \Gamma(\mathrm{D}^\top)$ and $U_1, U_2 \in \Gamma(ker\varphi_*)^\perp$.

Proof. For any $V_1 \in \Gamma(\mathrm{D}^\top)$ and $U_1, U_2 \in \Gamma(ker\varphi_*)^\perp$, from (1), (2), (10), (11) and (14), we get

$$g_{B_1}(\nabla_{U_1}U_2, V_1) = g_{B_1}(v\nabla_{U_1}\bar{\zeta}U_2 + \mathrm{A}_{U_1}\bar{\eta}U_2, J_1 V_1). \tag{36}$$

Also, for $V_2 \in \Gamma(\mathrm{D}^\theta)$, by using (21) we have

$$\begin{aligned} \sin^2\theta g_{B_1}(\nabla_{U_1}U_2, V_2) &= g_{B_1}(h\nabla_{U_1}\eta\zeta V_2, U_2) - g_{B_1}(h\nabla_{U_1}\eta V_2, \bar{\eta}U_2) \\ &- g_{B_1}(\mathrm{A}_{U_1}\eta V_2, \bar{\zeta}U_2). \end{aligned}$$

By using (3)and (4) we obtain

$$\begin{aligned} \sin^2\theta g_{B_1}(\nabla_{U_1}U_2, V_2) &= g_{B_2}(\nabla_{U_1}\varphi_*(\eta\zeta V_2), \varphi_* U_2) - g_{B_2}(\nabla_{U_1}\varphi_*(\eta V_2), \varphi_*\bar{\eta}U_2) \\ &- g_{B_1}(\mathrm{A}_{U_1}\eta V_2, \bar{\zeta}U_2). \end{aligned} \tag{37}$$

So, (36) and (37) complete proof.

4. Chen-Ricci inequality

Let (B_1, g_{B_1}, J_1) be a Kaehler manifold. The Riemannian-Christoffel curvature tensor of a complex space form $B_1(\nu)$ of constant holomorphic sectional curvature ν satisfies

$$\begin{aligned} R_{\mathcal{B}_1}(Y_1, Y_2, \mathcal{Y}_3, \mathcal{Y}_4) &= \frac{\nu}{4}\{g_{\mathcal{B}_1}(Y_1, \mathcal{Y}_4) g_{\mathcal{B}_1}(Y_2, \mathcal{Y}_3) - g_{\mathcal{B}_1}(Y_1, \mathcal{Y}_3) g_{\mathcal{B}_1}(Y_2, \mathcal{Y}_4) \\ &+ g_{\mathcal{B}_1}(Y_1, J_1\mathcal{Y}_3) g_{\mathcal{B}_1}(J_2 Y_2, \mathcal{Y}_4) - g_{\mathcal{B}_1}(Y_2, J_1\mathcal{Y}_3) g_{\mathcal{B}_1}(J_1 Y_1, \mathcal{Y}_4) \\ &+ 2g_{\mathcal{B}_1}(Y_1, J_1 Y_2) g_{\mathcal{B}_1}(J_1\mathcal{Y}_3, \mathcal{Y}_4)\} \end{aligned} \tag{38}$$

for all vector fields $Y_1, Y_2, Y_3, Y_4 \in \Gamma(TB_1)$([58]).

Let $(B_1^{b_1}(\nu), g_{B_1}, J_1)$ be a complex space form, (B_2, g_{B_2}) a Riemannian manifold and $\varphi : B_1(\nu) \to B_2$ be a **PSSR** map with $(range\varphi_*)^\perp = \{0\}$ and $dim(ker\varphi_*) = p = 2k_1 + 2k_2$. For every $q \in B_1$, we consider $\{X_1, X_2 = J_1X_1, ..., X_{2k_1-1}, X_{2k_1} = J_1X_{2k_1-1}, X_{2k_1+1}, X_{2k_1+2} = \sec\theta\zeta X_{2k_1+1} ..., X_{2k_1+2k_2-1}, X_p = \sec\theta\zeta X_{2k_1+2k_2-1}\}$ and $\{X_{p+1}, X_{p+2}, ..., X_{b_1}\}$ two orthonormal bases of $(ker\varphi_*)$ and $(ker\varphi_*)^\perp$, respectively. One can get easily,

$$g^2_{B_1}(J_1X_k, X_{k+1}) = \begin{cases} 1, & for\ k \in \{1, 2, ..., 2k_1 - 1\}; \\ \cos^2\theta, & for\ k \in \{2k_1 + 1, ..., 2k_1 + 2k_2 - 1\}. \end{cases}$$

Then

$$\sum_{k,s=1}^{p} g^2_{B_1}(J_1X_k, X_{k+1}) = 2(k_1 + k_2\cos^2\theta). \tag{39}$$

Let's denote $\mathcal{T}^\alpha_{ks}$ by

$$\mathcal{T}^\alpha_{ks} = g_{B_1}(\mathcal{T}_{X_k}X_s, X_\alpha) \tag{40}$$

where $1 \leq k, s \leq p$ and $p + 1 \leq \alpha \leq b_1$.

Now, for $ker\varphi_*$ using (1.27) of [22] and (38) , since φ is a **PSSR** map with $(range\varphi_*)^\perp = \{0\}$ then, for each unit vector $F_1 \in \Gamma(ker\varphi_*)$ we arrive at

$$\begin{aligned} Ric^{ker\varphi_*}(F_1) &= \frac{\nu}{4}[p + 2 + 3\cos^2\theta] \\ &- pg_1(\mathcal{T}_{F_1}F_1, \mathcal{H}) + \sum_{k=1}^{p} g_1(\mathcal{T}_{X_k}F_1, \mathcal{T}_{F_1}X_k), \end{aligned} \tag{41}$$

here $\mathcal{H}$ is the mean curvature vector field of the fiber.

From here, we get:

Theorem 4.1. *Let $\varphi : (B_1(\nu), g_{B_1}) \to (B_2, g_{B_2})$ be a* **PSSR** *map with $(range\varphi_*)^\perp = \{0\}$. Then, we have*

$$Ric^{ker\varphi_*}(F_1) \geq \frac{\nu}{4}[p + 2 + 3\cos^2\theta] - pg_{B_1}(\mathcal{T}_{F_1}F_1, \mathcal{H}). \tag{42}$$

For a unit vertical vector $F_1 \in \Gamma(ker\varphi_*)$, the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

By polarization, using (41), we obtain:

Theorem 4.2. *Let $\varphi : (B_1(\nu), g_{B_1}) \to (B_2, g_{B_2})$ be a* **PSSR** *map with $(range\varphi_*)^\perp = \{0\}$. Then, the Ricci tensor $S^{ker\varphi_*}$ on $ker\varphi_*$ satisfies*

$$S^{ker\varphi_*}(F_1, F_2) \geq \frac{\nu}{4}[p + 2 + 3\cos^2\theta]g_{B_1}(F_1, F_2) - pg_{B_1}(\mathcal{T}_{F_1}F_2, \mathcal{H}). \tag{43}$$

For $F_1, F_2 \in \Gamma(ker\varphi_*)$, the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Similarly, for $ker\varphi_*$ using (1.27) of [22] and (38), we obtain

$$\begin{aligned} 2\rho^{ker\varphi_*} &= \frac{\nu}{4}[p(p-1) + 6(k_1 + k_2\cos^2\theta)] \\ &- p^2\|\mathcal{H}\|^2 + \sum_{k,s=1}^{p} g_1(\mathcal{T}_{X_k}X_s, \mathcal{T}_{X_k}X_s), \end{aligned} \tag{44}$$

here $\rho^{ker\varphi_*} = \sum_{1\leq k<s\leq p} R^{ker\varphi_*}(X_k, X_s, X_s, X_k)$.
Therefore, we can state the following result.

Theorem 4.3. *Let $\varphi : (B_1(\nu), g_{B_1}) \to (B_2, g_{B_2})$ be a* **PSSR** *map with $(range\varphi_*)^\perp = \{0\}$. Then, we have*

$$2\rho^{ker\varphi_*} \geq \frac{\nu}{4}[p(p-1) + 6(k_1 + k_2\cos^2\theta)] - p^2\|\mathcal{H}\|^2 \tag{45}$$

the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Now, we give Chen-Ricci inequality on $ker\varphi_*$ for a **PSSR** map with $(range\varphi_*)^\perp = \{0\}$.

By using (7), (40) and (44), we arrive at

$$\begin{aligned} 2\rho^{ker\varphi_*} &= \frac{\nu}{4}[p(p-1) + 6(k_1 + k_2\cos^2\theta)] \\ &- p^2\|\mathcal{H}\|^2 + \sum_{\alpha=p+1}^{b_1}\sum_{k,s=1}^{p}(\mathcal{T}^\alpha_{ks})^2. \end{aligned} \tag{46}$$

From [21], we know that

$$\begin{aligned} \sum_{\alpha=p+1}^{b_1}\sum_{k,s=1}^{p}(\mathcal{T}^\alpha_{ks})^2 &= \frac{1}{2}p^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{\alpha=p+1}^{b_1}\left[\mathcal{T}^\alpha_{11} - \mathcal{T}^\alpha_{22} - \dots - \mathcal{T}^\alpha_{pp}\right]^2 + 2\sum_{\alpha=p+1}^{b_1}\sum_{s=2}^{p}(\mathcal{T}^\alpha_{1s})^2 \\ &\quad -2\sum_{\alpha=p+1}^{b_1}\sum_{2\leq k<s\leq p}\left[\mathcal{T}^\alpha_{kk}\mathcal{T}^\alpha_{ss} - \left(\mathcal{T}^\alpha_{ks}\right)^2\right]. \end{aligned} \tag{47}$$

If we put (47) in (46), we obtain

$$\begin{aligned} 2\rho^{ker\varphi_*} &= \frac{\nu}{4}[p(p-1) + 6(k_1 + k_2\cos^2\theta)] \\ &- \frac{1}{2}p^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{\alpha=p+1}^{b_1}\left[\mathcal{T}^\alpha_{11} - \mathcal{T}^\alpha_{22} - \dots - \mathcal{T}^\alpha_{pp}\right]^2 \\ &+2\sum_{\alpha=p+1}^{b_1}\sum_{s=2}^{p}(\mathcal{T}^\alpha_{1s})^2 - 2\sum_{\alpha=p+1}^{b_1}\sum_{2\leq k<s\leq p}\left[\mathcal{T}^\alpha_{kk}\mathcal{T}^\alpha_{ss} - \left(\mathcal{T}^\alpha_{ks}\right)^2\right]. \end{aligned}$$

From here, we get

$$\begin{aligned} &2\rho^{ker\varphi_*} \geq \frac{\nu}{4}[p(p-1) + 6(k_1 + k_2\cos^2\theta)] \\ &-\frac{1}{2}p^2\|\mathcal{H}\|^2 - 2\sum_{\alpha=p+1}^{b_1}\sum_{2\leq k<s\leq p}\left[\mathcal{T}^\alpha_{kk}\mathcal{T}^\alpha_{ss} - \left(\mathcal{T}^\alpha_{ks}\right)^2\right]. \end{aligned} \tag{48}$$

On the other hand, using (1.27) of [22], taking $U = W = X_k$, $V = F = X_s$ and from (40), we have

$$\begin{aligned} 2\sum_{2\leq k<s\leq p} R^{B_1}(X_k, X_s, X_s, X_k) &= 2\sum_{2\leq k<s\leq p} R^{ker\varphi_*}(X_k, X_s, X_s, X_k) \\ &+2\sum_{\alpha=p+1}^{b_1}\sum_{2\leq k<s\leq p}\left[\mathcal{T}^\alpha_{kk}\mathcal{T}^\alpha_{ss} - \left(\mathcal{T}^\alpha_{ks}\right)^2\right]. \end{aligned}$$

From the last equality, (48) can be written as

$$2\rho^{ker\varphi_*} \geq \frac{\nu}{4}[p(p-1)+6(k_1+k_2\cos^2\theta)] - \frac{1}{2}p^2\|\mathcal{H}\|^2$$
$$+2\sum_{2\leq k<s\leq p} R^{ker\varphi_*}(X_k,X_s,X_s,X_k) - 2\sum_{2\leq k<s\leq p} R^{B_1}(X_k,X_s,X_s,X_k). \tag{49}$$

Also, using the equality

$$2\rho^{ker\varphi_*} = 2\sum_{2\leq k<s\leq p} R^{ker\varphi_*}(X_k,X_s,X_s,X_k) + 2\sum_{s=1}^{p} R^{ker\varphi_*}(X_1,X_s,X_s,X_1).$$

If we put the last equality in (49), then we have

$$2Ric^{ker\varphi_*}(X_1) \geq \frac{\nu}{4}[p(p-1)+6(k_1+k_2\cos^2\theta)]$$
$$-\frac{1}{2}p^2\|\mathcal{H}\|^2 - 2\sum_{2\leq k<s\leq p} R^{B_1}(X_k,X_s,X_s,X_k).$$

Since B_1 is a complex space form, curvature tensor R^{B_1} of B_1 provides equation (38), therefore we acquire

$$Ric^{ker\varphi_*}(X_1) \geq \frac{\nu}{4}(p-1) + \frac{3\nu}{4}(1+\cos^2\theta) - \frac{1}{4}p^2\|\mathcal{H}\|^2.$$

Thus, we can give the following result:

Theorem 4.4. *Let $\varphi : B_1(\nu) \to B_2$ be a* **PSSR** *map from a complex space form $(B_1(\nu), g_{B_1})$ onto a Riemannian manifold (B_2, g_{B_2}) with $(range\varphi_*)^\perp = \{0\}$. Then we have*

$$Ric^{ker\varphi_*}(X_1) \geq \frac{\nu}{4}(p-1) + \frac{3\nu}{4}(1+\cos^2\theta) - \frac{1}{4}p^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only if

$$\mathcal{T}^\alpha_{11} = \mathcal{T}^\alpha_{22} + ... + \mathcal{T}^\alpha_{pp}$$
$$\mathcal{T}^\alpha_{1s} = 0, s = 2, ..., p.$$

Corollary 4.5. *Let $\varphi : B_1(\nu) \to B_2$ be a* **PSSR** *map from a complex space form $(B_1(\nu), g_{B_1})$ onto a Riemannian manifold (B_2, g_{B_2}) with $(range\varphi_*)^\perp = \{0\}$ and the semi-slant function $\theta = \frac{\pi}{2}$. Then we get*

$$Ric^{ker\varphi_*}(X_1) \geq \frac{\nu}{4}(p+2) - \frac{1}{4}p^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only if

$$\mathcal{T}^\alpha_{11} = \mathcal{T}^\alpha_{22} + ... + \mathcal{T}^\alpha_{pp}$$
$$\mathcal{T}^\alpha_{1s} = 0, s = 2, ..., p.$$

Corollary 4.6. *Let $\varphi : B_1(\nu) \to B_2$ be a* **PSSR** *map from a complex space form $(B_1(\nu), g_{B_1})$ onto a Riemannian manifold (B_2, g_{B_2}) with $(range\varphi_*)^\perp = \{0\}$ and the semi-slant function $\theta = 0$. Then we obtain*

$$Ric^{ker\varphi_*}(X_1) \geq \frac{\nu}{4}(p+5) - \frac{1}{4}p^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only if

$$\mathcal{T}^\alpha_{11} = \mathcal{T}^\alpha_{22} + ... + \mathcal{T}^\alpha_{pp}$$
$$\mathcal{T}^\alpha_{1s} = 0, s = 2, ..., p.$$

5. Casorati curvatures

The following lemma plays a key role in the proof of our theorem:

Lemma 5.1. *Let $W = \{(y_1, y_2, ..., y_m) \in R^m : y_1 + y_2 + ... + y_m = z\}$ be a hyperplane of R^m, and $g : R^m \to R$ a quadratic form given by*

$$g(y_1, y_2, ..., y_m) = c\Sigma_{k=1}^{m-1}(y_k)^2 + d(y_m)^2 - 2\Sigma_{1\leq k<s\leq m} y_k y_s,\ c > 0,\ d > 0.$$

Then the constrained extremum problem $min_{(y_1,y_2,...,y_m)\in W}\, g$ has the following solution:

$$y_1 = y_2 = ... = y_{m-1} = \frac{z}{c+1},\ y_m = \frac{z}{d+1} = \frac{z(m-1)}{(c+1)d} = (c-m+2)\frac{z}{c+1},$$

provided that $d = \frac{m-1}{c-m+2}$[49].

Let φ be a **PSSR** map from a complex space form $(B_1^{b_1}(\nu), J_1, g_{B_1})$ to a Riemannian manifold $(B_2^{b_2}, g_{B_2})$ with $(range\varphi_*)^\perp = \{0\}$. Suppose $\{X_1, ..., X_p\}$ is an orthonormal basis of the vertical space $ker\varphi_{*q}$, for $q \in B_1$, and $\{X_{p+1}, ..., X_{b_1}\}$ be an orthonormal basis of the horizontal space $(ker\varphi_{*q})^\perp$.
We defined the scalar curvature $\tau^{ker\varphi_*}$ on the vertical space $ker\varphi_{*q}$ by

$$\tau^{ker\varphi_*} = \Sigma_{k,s=1}^{p} g_{B_1}(R^{ker\varphi_*}(X_k, X_s)X_s, X_k)$$

and the normalized scalar curvature $\kappa^{ker\varphi_*}$ of $ker\varphi_{*q}$ as

$$\kappa^{ker\varphi_*} = \frac{2\tau^{ker\varphi_*}}{p(p-1)}.$$

Then, we can write

$$\begin{aligned} T_{ks}^{\beta} &= g_{\mathcal{B}_1}(T(X_k, X_s), X_\beta),\ k, s = 1, ..., p,\ \beta = p+1, ..., b_2, \\ \|T\|^2 &= \Sigma_{k,s=1}^{p} g_{\mathcal{B}_1}(T(X_k, X_s), T(X_k, X_s)), \\ traceT &= \Sigma_{k=1}^{p} T(X_k, X_k),\ \|traceT\|^2 = g_{\mathcal{B}_1}(traceT, traceT) \end{aligned}$$

and the squared norm of T over the manifold B_1, denoted by $C^{ker\varphi_*}$, is called the vertical Casorati curvatures of the vertical space $(ker\varphi_*)_q$. Thus, we get

$$C^{ker\varphi_*} = \frac{1}{p}\|T\|^2 = \frac{1}{p}\Sigma_{\beta=p+1}^{b_1}\Sigma_{k,s=1}^{p}(T_{ks}^{\beta})^2.$$

Now, assume that $L^{ker\varphi_*}$ is a $t-$dimensional subspace $(ker\varphi_*)_q$, $2 \leq t$ and let $\{X_1, X_2, ..., X_t\}$ be an orthonormal basis of $L^{ker\varphi_*}$. Then the Casorati curvature $C^{ker\varphi_*}(L^{ker\varphi_*})$ of $L^{ker\varphi_*}$ defined as

$$C^{ker\varphi_*}(L^{ker\varphi_*}) = \frac{1}{t}\|T\|^2 = \frac{1}{t}\Sigma_{\beta=p+1}^{b_1}\Sigma_{k,s=1}^{t}(T_{ks}^{\beta})^2.$$

The normalized $\sigma^{ker\varphi_*}-$ Casorati curvatures $\sigma_C^{ker\varphi_*}(p-1)$ and $\bar{\sigma}_C^{ker\varphi_*}(p-1)$ of $ker\varphi_*)_q$ are given by
$[\sigma_C^{ker\varphi_*}(p-1)]_q = \frac{1}{2}C_q^{ker\varphi_*} + \frac{p+1}{2p} inf\{C^{ker\varphi_*}(L^{ker\varphi_*}) : L^{ker\varphi_*}$ a hyperplane of $(ker\varphi_*)_q\}$, and
$[\bar{\sigma}_C^{ker\varphi_*}(p-1)]_q = 2C_q^{ker\varphi_*} - \frac{2p-1}{2p} inf\{C^{ker\varphi_*}(L^{ker\varphi_*}) : L^{ker\varphi_*}$ a hyperplane of $(ker\varphi_*)_q\}$.

Theorem 5.2. *Let φ be a* **PSSR** *map from a complex space form $(B_1^{b_1}(\nu), J_1, g_{B_1})$ to a Riemannian manifold $(B_2^{b_2}, g_{B_2})$ with $(range\varphi_*)^\perp = \{0\}$ and $3 \leq p$. Then the normalized $\sigma-$ Casorati curvatures $\sigma_C^{ker\varphi_*}$ and $\bar{\sigma}_C^{ker\varphi_*}$ on $(ker\varphi_*)_q$ satisfy*

$$(i)\ \kappa^{ker\varphi_*} \leq \sigma_C^{ker\varphi_*}(p-1) + \frac{\nu}{4} + \frac{3\nu}{2p(p-1)}(k_1 + k_2\cos^2\theta), \tag{50}$$

$$(ii)\ \kappa^{ker\varphi_*} \leq \bar{\sigma}_C^{ker\varphi_*}(p-1) + \frac{\nu}{4} + \frac{3\nu}{2p(p-1)}(k_1 + k_2 \cos^2\theta). \tag{51}$$

Furthermore, the equality case holds in any inequalities at a point $q \in B_1$ if and only if with respect to suitable orthonormal basis $\{X_1, ..., X_p\}$ on $\Gamma(ker\varphi_*)_q$ and $\{X_{p+1}, ..., X_{b_1}\}$ on $\Gamma((ker\varphi_*)_q)^\perp)$, the components of T satisfy

$$T^\beta_{11} = T^\beta_{22} = ... = T^\beta_{p-1p-1} = \frac{1}{2}T^\beta_{pp},\ \ \beta \in \{p+1, p+2, ..., b_1\},$$

$$T^\beta_{ks} = 0,\ \ k, s \in \{1, , ..., p\}(k \neq s),\ \ \beta \in \{p+1, p+2, ..., b_1\}.$$

Proof. Using (1.27) of [22] and (38) we have

$$\begin{aligned} 2\tau^{ker\varphi_*} &= \frac{\nu}{4}(p^2-p) + \frac{3\nu}{2}(k_1 + k_2\cos^2\theta) \\ &- pC^{ker\varphi_*} + \|traceT\|^2. \end{aligned} \tag{52}$$

Now we define a function $Q^{ker\varphi_*}$ associated with the following quadratic polynomial with respect to the components of T :

$$\begin{aligned} Q^{ker\varphi_*} &= \frac{1}{2}[(p^2-p)C^{ker\varphi_*} + (p^2-1)C^{ker\varphi_*}(\mathrm{L}^{ker\varphi_*})] \\ &- 2\tau^{ker\varphi_*} + \frac{\nu}{4}(p^2-p) + \frac{3\nu}{2}(k_1 + k_2\cos^2\theta). \end{aligned}$$

Without loos of generality, by supposing that the hyperplane $\mathrm{L}^{ker\varphi_*}$ is spanned by $\{X_1, ..., X_{p-1}\}$, one can produce

$$\begin{aligned} Q^{ker\varphi_*} &= \Sigma^{b_1}_{\beta=p+1}\Sigma^{p-1}_{k=1}[p(T^\beta_{kk})^2 + (p+1)(T^\beta_{kp})^2] \\ &+ \Sigma^{b_1}_{\beta=p+1}[2(p+1)\Sigma^{p-1}_{1=k<s}(T^\beta_{ks})^2 \\ &- 2\Sigma^{p}_{1=k<s}T^\beta_{kk}T^\beta_{ss} + \frac{p-1}{2}(T^\beta_{pp})^2]. \end{aligned} \tag{53}$$

Using (53), we obtain the critical points

$$T^c = (T^{p+1}_{11}, T^{p+1}_{12}, ..., T^{p+1}_{pp}, ..., T^{b_1}_{11}, ..., T^{b_1}_{pp})$$

of $Q^{ker\varphi_*}$ are solutions of the next system of equations:

$$\begin{cases} \frac{\partial Q^{ker\varphi_*}}{\partial T^\beta_{kk}} = 2(r+1)T^\beta_{kk} - 2\Sigma^p_{t=1}T^\beta_{tt} = & 0 \\ \frac{\partial Q^{ker\varphi_*}}{\partial T^\beta_{pp}} = (r-1)T^\beta_{pp} - 2\Sigma^{p-1}_{t=1}T^\beta_{tt} = & 0 \\ \frac{\partial Q^{ker\varphi_*}}{\partial T^\beta_{ks}} = 4(r+1)T^\beta_{ks} = & 0 \\ \frac{\partial Q^{ker\varphi_*}}{\partial T^\beta_{kp}} = 2(r+1)T^\beta_{kp} = & 0, \end{cases} \tag{54}$$

here $k, s \in \{1, 2, ..., p-1\}$, $k \neq s$ and $\beta \in \{p+1, ..., b_1\}$. Frankly (54) is a system consisting only in linear homogeneous equations and it is easy to checky that every solution T^c has $T^\beta_{ks} = 0$ for $k \neq s$, and the determinant corresponding to the first two series of linear homogeneous equations in (54) has vanishes. Furthermore, the Hessian matrix of $Q^{ker\varphi_*}$ is defined as

$$\mathcal{H}(Q^{ker\varphi_*}) = \begin{pmatrix} \mathcal{H}_1 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 \\ 0 & 0 & \mathcal{H}_3 \end{pmatrix},$$

here

$$\mathcal{H}_1 = \begin{pmatrix} 2p & -2 & \dots & -2 & -2 \\ -2 & 2p & \dots & -2 & -2 \\ \dots & \dots & \dots & \dots & \dots \\ -2 & -2 & \dots & 2p & -2 \\ -2 & -2 & \dots & -2 & p-1 \end{pmatrix},$$

denotes the zero matrix of suitable dimensions and the matrices $\mathcal{H}_2$, $\mathcal{H}_3$ are ones having the following diagonal forms

$$\begin{aligned} \mathcal{H}_2 &= diag(4(p+1), 4(p+1), ..., 4(p+1)), \\ \mathcal{H}_3 &= diag(2(p+1), 2(p+1), ..., 2(p+1)). \end{aligned}$$

Then a standard computation shows that the eigenvalues of $\mathcal{H}(Q^{ker\varphi_*})$ are

$$\xi_{11} = 0,\ \xi_{22} = p+3,\ \xi_{33} = ... = \xi_{pp} = 2(p+1),\ \xi_{ks} = 4(p+1),$$
$$\xi_{kb_1} = 2(p+1),\ \forall k, s \in \{1, 2, ..., p-1\},\ k \neq s.$$

Also it follows that $Q^{ker\varphi_*}$ is parabolic and achieves a global minimum value $Q^{ker\varphi_*}(c)$ for T^c- the solution of (54). However we obtain $Q^{ker\varphi_*}(c) = 0$ and we get $Q^{ker\varphi_*} \geq 0$. Thus,

$$\begin{aligned} 2\tau^{ker\varphi_*} \leq & \frac{1}{2}[(p^2-p)C^{ker\varphi_*} + (p^2-1)C^{ker\varphi_*}(\mathrm{L}^{ker\varphi_*})] \\ & + \frac{\nu}{4}(p^2-p) + \frac{3\nu}{2}(k_1 + k_2\cos^2\theta) \end{aligned} \tag{55}$$

and using (55) we obtain

$$\begin{aligned} \kappa^{ker\varphi_*} \leq & [\frac{1}{2}C^{ker\varphi_*} + \frac{p+1}{2p}C^{ker\varphi_*}(\mathrm{L}^{ker\varphi_*})] \\ & + \frac{\nu}{4} + \frac{3\nu}{2p(p-1)}(k_1 + k_2\cos^2\theta) \end{aligned} \tag{56}$$

for all hyperplane $\mathrm{L}^{ker\varphi_*}$ of B_1. Now, taking the infimum in (56) over every hyperplane $\mathrm{L}^{ker\varphi_*}$, we get (i)

$$\begin{aligned} & \kappa^{ker\varphi_*} \leq \sigma_C^{ker\varphi_*}(p-1) + \frac{\nu}{4} \\ & + \frac{3\nu}{2p(p-1)}(k_1 + k_2\cos^2\theta) \end{aligned} \tag{57}$$

Besides, simply we can check that the equality sign holds in the (57) if and only if

$$T_{ks}^{\beta} = 0,\ \forall k, s \in \{1, 2, ..., p\},\ k \neq s,\ \beta \in \{p+1, ..., b_1\},$$

and

$$T_{pp}^{\beta} = 2T_{11}^{\beta} = ... = 2T_{p-1\,p-1}^{\beta},\ \forall k, s \in \{p+1, p+2, ..., b_1\}.$$

In a similar way we have (ii).

Using the Theorem 5.2, we obtain the following results:

Corollary 5.3. *Let φ be a* **PSSR** *map from a complex space form $(B_1^{b_1}(\nu), J_1, g_{B_1})$ to a Riemannian manifold $(B_2^{b_2}, g_{B_2})$ with $(range\varphi_*)^{\perp} = \{0\}$, the semi-slant function $\theta = \frac{\pi}{2}$ and $3 \leq p$. Then the normalized $\sigma-$ Casorati curvatures $\sigma_C^{ker\varphi_*}$ and $\bar{\sigma}_C^{ker\varphi_*}$ on $(ker\varphi_*)_q$ satisfy*

$$(i)\ \kappa^{ker\varphi_*} \leq \sigma_C^{ker\varphi_*}(p-1) + \frac{\nu}{4} + \frac{3k_1\nu}{2p(p-1)}, \tag{58}$$

$$(ii)\ \kappa^{ker\varphi_*} \le \bar{\sigma}_C^{ker\varphi_*}(p-1) + \frac{\nu}{4} + \frac{3k_1\nu}{2p(p-1)}. \tag{59}$$

Furthermore, the equality case holds in any inequalities at a point $q \in B_1$ if and only if with respect to suitable orthonormal basis $\{X_1, ..., X_p\}$ on $\Gamma(ker\varphi_*)_q$ and $\{X_{p+1}, ..., X_{b_1}\}$ on $\Gamma((ker\varphi_*)_q)^\perp)$, the components of T satisfy

$$T^\beta_{11} = T^\beta_{22} = ... = T^\beta_{p-1p-1} = \frac{1}{2}T^\beta_{pp},\ \ \beta \in \{p+1, p+2, ..., b_1\},$$

$$T^\beta_{ks} = 0,\ \ k, s \in \{1, , ..., p\}(k \ne s),\ \ \beta \in \{p+1, p+2, ..., b_1\}.$$

Corollary 5.4. *Let φ be a* **PSSR** *map with $(range\varphi_*)^\perp = \{0\}$ from a complex space form $(B_1^{b_1}(\nu), g_{B_1})$ to a Riemannian manifold $(B_2^{b_2}, g_{B_2})$ with the semi-slant function $\theta = 0$ and $3 \le p$. Then the normalized $\sigma-$ Casorati curvatures $\sigma_C^{ker\varphi_*}$ and $\bar{\sigma}_C^{ker\varphi_*}$ on $(ker\varphi_*)_q$ satisfy*

$$(i)\ \kappa^{ker\varphi_*} \le \sigma_C^{ker\varphi_*}(p-1) + \frac{(p+2)\nu}{4(p-1)},$$

$$(ii)\ \kappa^{ker\varphi_*} \le \bar{\sigma}_C^{ker\varphi_*}(p-1) + \frac{(p+2)\nu}{4(p-1)}.$$

Furthermore, the equality case holds in any inequalities at a point $q \in B_1$ if and only if with respect to suitable orthonormal basis $\{X_1, ..., X_p\}$ on $\Gamma(ker\varphi_*)_q$ and $\{X_{p+1}, ..., X_{b_1}\}$ on $\Gamma((ker\varphi_*)_q)^\perp)$, the components of T satisfy

$$T^\beta_{11} = T^\beta_{22} = ... = T^\beta_{p-1p-1} = \frac{1}{2}T^\beta_{pp},\ \ \beta \in \{p+1, p+2, ..., b_1\},$$

$$T^\beta_{ks} = 0,\ \ k, s \in \{1, , ..., p\}(k \ne s),\ \ \beta \in \{p+1, p+2, ..., b_1\}.$$

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