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On torse-forming vector fields and their applications in submanifold theory

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Abstract. The present article utilises a property of torse-forming vector fields to deduce some criteria for invariant submanifolds of Riemannian manifolds to be totally geodesic. Certain features of submanifolds of Riemannian manifolds as η-Ricci Bourguignon soliton have been developed.

1. Introduction

Geometry of a manifold is represented by the vector fields defined on the manifold. There are several well known vector fields such as concircular vector fields, concurrent vector fields, geodesic vector fields etc. For details we refer [3, 4, 11, 12, 24, 25, 27, 30, 31, 34]. Torse-forming vector fields(in short TFV fields) are generalization of such vector fields. TFV fields demand special attention due to their applications not only in relativity and cosmology but in theory of submanifolds also. In this short article, we intend to highlight some applications of TFV fields in submanifold theory.

The theory of submanifolds has been substantially developed by Chen [6–10]. Among all submanifolds of a Riemannian manifold totally geodesic submanifolds are nicer than others due to their interesting geodesic inheritance properties. A geodesic on the ambient manifold remains geodesic in totally geodesic submanifolds. A lot of works have been done to determine conditions for a submanifold of an almost contact manifold to be totally geodesic [18–22, 28]. Almost contact manifolds provide sometimes beautiful properties due to the existence of Reeb vector fields and structure tensors on such manifolds. But the drawback of such a structure is that it cannot be defined on even dimensional manifolds. So natural question arises that what conditions make submanifolds of any Riemannian manifold to be totally geodesic? To answer such a question, we study submanifolds of Riemannian manifolds admitting TFV fields [15, 33].

Totally umbilical submanifolds [6, 16] form another important class of submanifolds of Riemannian manifolds. Natural intuition urges us to find the relation between totally umbilical submanifolds and totally geodesic submanifolds admitting TFV fields.

On the other hand, nowadays, the literature associated with submanifold theory and solitons of several geometric flows is being nurtured by several geometers [3, 4, 21]. In this connection it should be mentioned

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that the theory of Ricci flow was coined by Hamilton [14] and was successfully applied by Perelman [17] to solve the famous Poincare conjecture. J. P. Bourguignon [2] introduced Bourguignon flow and Catino [5] gave the concept of Ricci Bourguignon flow and its soliton. The present authors also have studied soliton of such a flow in [23]. A soliton is a special type of solution of such a flow. The theory of Ricci soliton has been first used by Sharma [24] in the context of contact geometry. Very recently Blaga and Özgur [3, 4] have worked on η-Ricci Bourguignon solitons associated with concurrent vector fields and geometry of submanifolds. The present paper is organized as follows:

After the introductory section, we add some basics about TFV fields and submanifold theory in Section 2. In Section 3, we establish some necessary and sufficient conditions for a submanifold of a Riemannian manifold with a TFV field to be totally geodesic. The last section is devoted to study submanifolds as Ricci Bourguignon solitons and η-Ricci Bourguignon solitons admitting TFV fields.

2. Preliminaries

A vector field θ on a Riemannian manifold $\bar{M}$ is termed as a TFV field [15, 33] if it agrees with the equation

$$\bar{\nabla}_E\theta = lE + m(E)\theta \tag{1}$$

for any vector field E on $\bar{M}$, where l is a smooth function defined on $\bar{M}$ and m is a 1-form and $\bar{\nabla}$ representing the Levi-Civita connection on $\bar{M}$. Again if $m(\theta) = 0$, the vector field is called torqued vector field [7–10]. When the 1-form is identically zero on the manifold, the vector field θ is designated as concircular [32]. If $l = 1$ and $m = 0$, then the vector field θ is called concurrent [13, 29]. If $l = 0$ and $m \neq 0$, the vector field is recurrent.

As a consequence of (1) we have

$$\begin{aligned} \bar{R}(E,F)\theta &= (El)F - (Fl)E \\ &+ (\nabla_E m)F - (\nabla_F m)E \\ &+ m(F)lE - m(E)lF, \end{aligned} \tag{2}$$

where $\bar{R}$ is Riemann curvature of $\bar{M}$ and E, F are arbitrary vector fields of $\bar{M}$.

Let us now add some results on TFV fields.

Proposition 2.1. *Let θ be a TFV field. If there is a unit vector field ψ which is orthogonal to θ and $£_\theta\psi = 0$, then θ is a recurrent vector field.*

Proof. Let θ be a TFV field. Then using (1)

$$\begin{aligned} (£_\theta g)(E,F) &= g(lE + m(E)\theta, F) + g(lF + m(F)\theta, E) \\ &= 2lg(E,F) + m(E)g(\theta,F) + m(F)g(\theta,E). \end{aligned} \tag{3}$$

Let ψ be a unit vector field orthogonal to θ, then $(£_\theta g)(\psi,\psi) = 2l$. Which gives $g(£_\theta\psi,\psi) - l = 0$. For $£_\theta\psi = 0$, $l = 0$. Consequently, θ is recurrent. □

Proposition 2.2. *Let θ be a TFV field on an n-dimensional Riemannian manifold with $m(\theta) = 0$. Then θ is concircular if and only if the Ricci tensor S agrees with $S(\theta,\theta) \geq \frac{1}{4}||d\alpha||^2 - \frac{n-1}{2n}\theta(\mathrm{Tr}£_\theta g)$, where α is any one form dual to θ.*

Proof. Let θ be a TFV field on an n-dimensional Riemannian manifold with $m(\theta) = 0$, then from (1) $\tilde{\nabla}_\theta\theta = l\theta$. Hence the vector field is a geodesic vector field. Now from [11], we know that a geodesic vector field θ is concircular if and only if $S(\theta,\theta) \geq \frac{1}{4}||d\alpha||^2 - \frac{n-1}{2n}\theta(\mathrm{Tr}£_\theta g)$. Hence the proof follows. □

Proposition 2.3 *Let θ be a TFV field on an n-dimensional compact and connected Riemannian manifold such that $m(\theta) = 0$ and $\Delta\theta = -\lambda\theta$, for a positive constant λ. Then the manifold is isometric to an n-sphere $S^n(\lambda)$ if and only if the energy $E(\theta)$ satisfies*

$$E(\theta) \leq \frac{1}{8n\lambda}\int_M (\mathrm{Tr}(£_\theta g))^2.$$

Proof. Suppose θ is a TFV field with $m(\theta) = 0$. Then θ is a geodesic vector field. Hence the proof follows by Theorem 2 of the paper [11]. □

Proposition 2.4. *A TFV field on a Riemannian manifold is not torse-forming after scale changing transformation unless the transformation is identity transformation.*

Proof. Define a scale changing transformation by $\bar{\theta} = a\theta$, $\bar{m} = am$, $\bar{l} = al$, $\bar{g} = ag$ for a real number a. If ∇ is Levi-Civita connection of g and $\bar{\nabla}$ is Levi-Civita connection of $\bar{g}$, then $\bar{\nabla} = a\nabla$. Now

$$\begin{aligned}\bar{\nabla}_E\bar{\theta} &= \bar{\nabla}_E(ag)\\ &= a^2(lE + m(E)\theta). \end{aligned}\tag{4}$$

If, after the transformation the TFV field is to remain torse-forming, then

$$\begin{aligned}\bar{\nabla}_E\bar{\theta} &= \bar{l}E + \bar{m}(E)\bar{\theta}\\ &= alE + a^2m(E)\theta. \end{aligned}\tag{5}$$

From (4) and (5), we have $a = 1$. Hence, the proof follows. □

Suppose $f : (M, g) \to (\bar{M}, \bar{g})$ be an isometric immersion of a d-dimensional Riemannian manifold (M, g) into n-dimensional Riemannian manifold $(\bar{M}, \bar{g})$. By ∇ and $\bar{\nabla}$, we represent the Levi-Civita connections of M and $\bar{M}$ respectively. TM and $T^{\perp}M$ will denote the tangent and normal bundle of M. Then for any $E, F \in TM$ the second fundamental form (in brief SFF) of the immersion ρ is given by the formula [6]

$$\rho(E, F) = \bar{\nabla}_E F - \nabla_E F. \tag{6}$$

By the following equations, we define the covariant derivative $\bar{\nabla}\rho$ and the second covariant derivative $\bar{\nabla}^2\rho$ of the SFF ρ [6] respectively.

$$(\bar{\nabla}_E\rho)(F, Z) = \nabla^{\perp}_E\rho(F, Z) - \rho(\nabla_E F, Z) - \rho(F, \nabla_E Z). \tag{7}$$

$$\begin{aligned}(\bar{\nabla}^2\rho)(Z, W; E, F) &= \nabla^{\perp}_E((\bar{\nabla}_F\rho)(Z, W)) - (\bar{\nabla}_F\rho)(\nabla_E Z, W)\\ &- (\bar{\nabla}_E\rho)(Z, \nabla_F W) - (\bar{\nabla}_{\nabla_E F}\rho)(Z, W), \end{aligned}\tag{8}$$

where $\nabla^{\perp}$ is the Levi-Civita connection in the normal bundle. If θ is tangent to the submanifold, from (1) and (6), we get

$$\nabla_E\theta = lE + m(E)\theta, \tag{9}$$

$$\rho(E, \theta) = 0 \tag{10}$$

for any tangent vector field E of the submanifold.

3. Conditions for a submanifold of a Riemannian manifold with TFV field to be totally geodesic

In this section, we would like to study submanifolds which admit TFV fields and who's SFF satisfy some recurrent and parallel conditions. For details about recurrent and parallel SFF we refer [1, 18, 28].

Lemma 3.1. *If a submanifold of a Riemannian manifold possesses a TFV field and if the SFF is parallel, then the submanifold is totally geodesic. The converse is trivial.*

Proof. Let the SFF of the submanifold be parallel. Then

$$(\bar{\nabla}_W \rho)(E, F) = 0.$$

In view of (7), we get from above

$$\nabla^{\perp}_W \rho(E, F) - \rho(\nabla_W E, F) - \rho(E, \nabla_W F) = 0.$$

Putting $E = \theta$ in the above equation and using (10), we get

$$\rho(\nabla_W \theta, F) = 0.$$

By virtue of (9) and (10), the above equation gives $\rho(E, F) = 0$, provided $l \neq 0$. Thus we get the proof. □

Lemma 3.2. *If a submanifold of a Riemannian manifold possesses a TFV field and if the SFF is recurrent, then the submanifold is totally geodesic. The converse is trivially true.*

Proof. Let the SFF of the submanifold is recurrent. Then

$$(\bar{\nabla}_W \rho)(E, F) = A(W)\rho(E, F),$$

where A is a 1-form defined on the submanifold. Then, as before, using (7), (9) and (10) we infer that $\rho(E, F) = 0$. This completes the proof. □

Lemma 3.3. *If a submanifold of a Riemannian manifold possesses a TFV field and if the SFF is* 2*-recurrent, then the submanifold is totally geodesic. The converse is trivial.*

Proof. Suppose the SFF of the submanifold is 2-recurrent. Then

$$(\bar{\nabla}_W \bar{\nabla}_U \rho)(E, F) = B(W, U)\rho(E, F),$$

where B is certain 2-form defined on the manifold. Simplifying the above equation we get

$$\nabla^{\perp}_W((\bar{\nabla}_U \rho)(E, F)) - (\bar{\nabla}_U \rho)(\nabla_W E, F) - (\bar{\nabla}_U \rho)(E, \nabla_W F) - (\bar{\nabla}_{\nabla_W U} \rho)(E, F) = B(W, U)\rho(E, F).$$

Putting $E = F = \theta$ and using (7), (8), (9) and (10) in the above equation we have after simplification $\rho(U, W) = 0$, provided $l \neq 0$. The above situation leads us to conclude the proof. □

Lemma 3.4. *If a submanifold of a Riemannian manifold possesses a TFV field and if the submanifold has parallel third fundamental form, then the submanifold is totally geodesic. The converse is trivial.*

Proof. Suppose the submanifold has parallel third fundamental form. Then

$$(\bar{\nabla}_W \bar{\nabla}_U \rho)(E, F) = 0.$$

Proceeding as before, we have the proof. □

Lemma 3.5. *If a submanifold of a Riemannian manifold possesses a TFV field and if the SFF is semi-parallel, then the submanifold is totally geodesic, provided the* 1*-form associated with the vector field is invariant by the action of the vector field. The converse is trivial.*

Proof. Suppose the SFF is semi-parallel. Then

$$\bar{R}(U, V)\rho(E, F) = 0.$$

The above equation gives

$$R^{\perp}(U, V)\rho(E, F) - \rho(R(U, V)E, F) - \rho(E, R(U, V)F) = 0,$$

where $R^{\perp}$ and R are the normal and tangential parts of $\bar{R}$. Putting $E = V = \theta$ and using (10) in the above equation, we obtain

$$\rho(R(U, \theta)\theta, F) = 0.$$

By virtue of (2), the above equation gives

$$dl(\theta)\rho(U, F) + \rho((\nabla_{\theta} m)U, F) + lm(\theta)\rho(U, F) = 0.$$

If $(\nabla_{\theta} m)U = 0$, the above equation gives $\rho(U, F) = 0$. Hence the proof is complete. □

Remark 3.1. *If the vector field is concurrent vector field the above lemma takes the form: If a submanifold of a Riemannian manifold admits a concurrent vector field and if the SFF is semi-parallel, then the submanifold is totally geodesic. The converse is trivially true.*

Lemma 3.6. *If a submanifold of a Riemannian manifold admits a TFV field and if the submanifold is totally umbilical, then the submanifold is totally geodesic. The converse is trivially true.*

Proof. Suppose the submanifold is totally umbilical [6, 16]. Then

$$\rho(E, F) = g(E, F)H, \tag{11}$$

where H is the mean curvature vector field. Since the dimension of the submanifold is d and θ is tangential to the submanifold, we can write

$$\theta = c_1 e_1 + c_2 e_2 + ... + c_d e_d,$$

where $\{e_1, e_2, ..., e_d\}$ is a basis of the tangent bundle of the submanifold and $c_1, c_2, ..., c_d$ are scalars. Putting $F = \theta$ in (11), and using (10), we have

$$(c_1 g(E, e_1) + ... + c_d g(E, e_d))H = 0.$$

Consequently, $H = 0$. Using this fact in (11), we infer that $\rho(E, F) = 0$. Hence the submanifold is totally geodesic. Thus the proof follows. □

By virtue of the above lemmas, we conclude the following

Theorem 3.1. If a Riemannian manifold has a submanifold with a TFV field, then the bellow stated criteria regarding the submanifold are equivalent.

- It is totally geodesic.
- It is totally umbilical.
- It has parallel SFF.
- It has recurrent SFF.
- It has 2-recurrent SFF.
- It has parallel third fundamental form.
- It has semi-parallel SFF with the property that the TFV field keeps the 1-form associated with it invariant.

Now we frame an example of a Riemannian manifold with TFV field and its submanifold for illustration of the deduced properties.

Example 3.1. Choose $\bar{M} = \{(u_1, u_2, u_3, t) : t \neq 0\}$ as an ambient manifold of dimension 4; (u_1, u_2, u_3, t) being coordinates in $\mathbb{R}^4$. The vector fields

$$e_1 = e^{-t}\frac{\partial}{\partial u_1}, e_2 = e^{-t}\frac{\partial}{\partial u_2}, e_3 = e^{-t}\frac{\partial}{\partial u_3}, e_4 = e^{-t}\frac{\partial}{\partial t}$$

are obviously linearly independent. Define $\bar{g}(e_i, e_j) = 1$ whenever $i = j$, and 0 in other cases. Here i and j varies from 1 to 4.

Let $\bar{\nabla}$ is the Levi-Civita connection of the metric g. Then one obtains

$$[e_i, e_4] = e^{-t}e_i \quad i = 1, 2, 3, \quad [e_i, e_j] = 0, \quad 1 \leq i, j \leq 3.$$

By Koszul formula,

$$\begin{array}{lll} \bar{\nabla}_{e_1}e_1 = -e^{-t}e_4, & \bar{\nabla}_{e_2}e_2 = -e^{-t}e_4, & \bar{\nabla}_{e_3}e_3 = -e^{-t}e_4, \\ \bar{\nabla}_{e_4}e_4 = 0, & \bar{\nabla}_{e_4}e_i = 0, & \bar{\nabla}_{e_i}e_4 = e^{-t}e_i \end{array}$$

for $1 \leq i \leq 3$, $\bar{\nabla}_{e_i}e_j = 0$, otherwise. If we define the 1-form m by $m(E) = -e^{-t}g(E, e_4)$, then from above data, we see that e_4 is a TFV field with $l = e^{-t}$.

Cconsider the isometric immersion $f : M \to \bar{M}$ revealed by

$$f(u_1, u_2, t) = (u_1, 0, u_2, t).$$

We observe that $M = \{(u_1, u_2, t) \in \mathbb{R}^3 : t \neq 0\}$, where (u_1, u_2, t) are the coordinates of $\mathbb{R}^3$ is a submanifold of $\bar{M}$. We choose the vector fields

$$e_1 = e^{-t}\frac{\partial}{\partial x_1}, e_2 = e^{-t}\frac{\partial}{\partial x_3}, e_4 = e^{-t}\frac{\partial}{\partial t},$$

which are linearly independent at each point of M. Here $\{e_1, e_2, e_4\}$ is an orthonormal frame of M under the restriction of $\bar{g}$ on the submanifold.

By Koszul formula,

$$\begin{array}{ll} \nabla_{e_1}e_1 = -e^{-t}e_4, & \nabla_{e_2}e_2 = -e^{-t}e_4, \\ \nabla_{e_4}e_4 = 0, & \nabla_{e_4}e_i = 0, \end{array}$$

and $\nabla_{e_i}e_4 = e^{-t}e_i$ for $1 \leq i \leq 2$, $\bar{\nabla}_{e_i}e_j = 0$, otherwise.

Now, we easily see that

$$\begin{array}{lll} \rho(e_1, e_1) = 0, & \rho(e_1, e_2) = 0, & \rho(e_1, e_3) = 0, \\ \rho(e_2, e_1) = 0, & \rho(e_2, e_2) = 0, & \rho(e_2, e_3) = 0, \\ \rho(e_3, e_1) = 0, & \rho(e_3, e_2) = 0, & \rho(e_3, e_3) = 0. \end{array}$$

Hence the submanifold is totally geodesic and totally umbilical. It agrees with Theorem 3.1.

4. Ricci-Bourguignon solitons and η-Ricci Bourguignon solitons as submanifolds

Theorem 4.1. *If a submanifold of a Riemannian manifold with TFV field is a Ricci Bourguignon $(g, \theta, \alpha, \beta)$ with $\beta \neq 0$ and its potential vector field is TFV field θ given by $\nabla_E\theta = lE + m(E)\theta$, with $(\nabla_{e_i}m)\theta = (\nabla_\theta m)e_i$, where $\{e_i\}, i = 1, .., d$, is an orthonormal basis of the tangent space at each point of the manifold, then the scalar curvature of the manifold is constant if and only if l is constant.*

Proof. A Ricci Bourguignon soliton is a self similar solution of the Ricci Bourguignon flow. The data $(g, \theta, \alpha, \beta)$ is said to be Ricci Bourguignon soliton if

$$\frac{1}{2}\pounds_\theta g + S = (\alpha + \beta r)g, \tag{12}$$

where S is the Ricci tensor and r is the scalar curvature and α, β are real numbers and $\pounds_\theta$ denotes Lie derivative with respect to θ. Let M with linear connection ∇ be submanifold of the Riemannian manifold $\bar{M}$ with linear connection $\bar{\nabla}$. If the submanifold is Ricci Bourguignon soliton from (12)

$$S(E, F) = (\alpha + \beta r)g(E, F) - \frac{1}{2}(g(\nabla_E\theta, F) + g(E, \nabla_F\theta)). \tag{13}$$

Contracting the above equation

$$r = n(\alpha + \beta r) - \mathrm{div}\theta,$$

where divθ is divergence of θ. Now, from (2)

$$\begin{aligned}\bar{R}(E,F)\theta &= (El)F-(El)E\\ &+ (\nabla_E m)F-(\nabla_F m)E\\ &+ m(F)lE-m(E)lF.\end{aligned}$$

Contracting E, we get from above

$$S(F,\theta)=(1-d)Fl+\sum_{i=1}^{d}\Big(g\big((\nabla_{e_i}m)F,e_i\big)-g\big((\nabla_F m)e_i,e_i\big)\Big)+(d-1)m(F)l. \tag{14}$$

Comparing (13) and (14)

$$\begin{aligned}(\alpha+\beta r)g(F,\theta)-\frac{1}{2}\Big(g(\nabla_\theta,\theta)+g(F,\nabla_\theta\theta)\Big) &= (1-d)Fl+\sum_{i=1}^{d}\Big(g\big((\nabla_{e_i}m)F,e_i\big)-g\big((\nabla_F m)e_i,e_i\big)\Big)\\ &+ (d-1)m(F)l.\end{aligned}$$

Putting $F=\theta$

$$\begin{aligned}(\alpha+\beta r)-g(l\theta+m(\theta)\theta,\theta) &= (1-d)\theta l+\sum_{i=1}^{d}\Big(g\big((\nabla_{e_i}m)\theta,e_i\big)-g\big((\nabla_\theta m)e_i,e_i\big)\Big)\\ &+ (d-1)m(\theta)l.\end{aligned}$$

If $(\nabla_{e_i}m)\theta=(\nabla_\theta m)e_i$, we have

$$r=\frac{l+d-\alpha}{\beta}. \tag{15}$$

□

Corollary 4.1. *If a submanifold of a Riemannian manifold with TFV field is a Ricci Bourguignon soliton and its potential vector field is TFV field θ given by $\nabla_E\theta=lE+m(E)\theta$, with $(\nabla_{e_i}m)\theta=(\nabla_\theta m)e_i$, where $\{e_i\}$, $i=1,..,d$, is an orthonormal basis of the tangent space at each point of the manifold, then the submanifold is a manifold of zero scalar curvature if and only if the dimension of the manifold is $\alpha-l$.*

If a Riemannian manifold admits a vector field θ, a 1-form η and a Riemannian metric g, then the set of data $(g,\theta,\alpha,\beta,\gamma)$ is called a η-Ricci Bourguignon soliton if

$$\frac{1}{2}\pounds_\theta g+S=(\alpha+\beta r)g+\gamma\eta\otimes\eta, \tag{16}$$

where S is the Ricci tensor and r is the scalar curvature and α,β,γ are real numbers. For details, see [3, 27]. In this section we study some properties of submanifolds as η-Ricci Bourguignon solitons whose potential vector field θ is torse-forming.

Theorem 4.2. *If a submanifold of a Riemannian manifold admits a TFV field θ, then it is η-Ricci Bourguignon soliton if and only if its Ricci tensor is given by $S(E,F)=(\alpha+\beta r-l)g(E,F)+\gamma\eta(E)\eta(F)+\frac{1}{2}m(E)g(F,\theta)+\frac{1}{2}m(F)g(F,\theta)$.*

Proof. Let M be a submanifold of a Riemannian manifold $\bar{M}$. Let θ be a TFV field tangent to M. For a vector field E, tangent to M, we get

$$\bar{\nabla}_E\theta=\nabla_E\theta+\rho(E,\theta).$$

By (1), the above equation gives

$$lE+m(E)\theta=\nabla_E\theta+\rho(E,\theta).$$

Comparing tangential components, we have

$$\nabla_E\theta=lE+m(E)\theta.$$

It is to be seen that

$$\begin{aligned}\frac{1}{2}(\pounds_\theta g)(E,F) &= [g(\nabla_E\theta,F)+g(\nabla_F\theta,E)]\\ &= \frac{1}{2}[g(lE+m(E)\theta,F)+g(lF+m(E)\theta,E)]\\ &= lg(E,F)+\frac{1}{2}m(E)g(\theta,F)+\frac{1}{2}m(F)g(\theta)E. \end{aligned} \tag{17}$$

From (16) and (17), we infer

$$S(E,F)=(\alpha+\beta r-l)g(E,F)+\gamma\eta(E)\eta(F)+\frac{1}{2}m(E)g(F,\theta)+\frac{1}{2}m(F)g(F,\theta). \tag{18}$$

The converse is obvious. □

Corollary 4.2. *If a submanifold of a Riemannian manifold, with a TFV field θ and a 1-form m dual to θ, admits η-Ricci Bourguignon soliton, then it is a generalized quasi-Einstein manifold.*

Proof. The proof is immediate consequence of the above theorem. □

Corollary 4.3. *If M is a d-dimensional submanifold of a Riemannian manifold such that the submanifold admits a TFV field θ, given by $\nabla_E\theta = lE+\eta(E)\theta$ and η is dual to θ and if it also admits an η-Ricci Bourguignon soliton $(g,\theta,\alpha,\beta,\gamma)$,* then $\beta=\frac{1}{d}$ if and only if $(\alpha-l)d+\gamma+1$.

Proof. If $m=\eta$, contracting the equation (18) with respect to an orthonormal basis $\{e_1,e_2,....,\theta\}$, one obtains

$$r=(\alpha+\beta r-l)d+(\gamma+1). \tag{19}$$

Thus the proof follows. □

Corollary 4.4. *If M is a d-dimensional submanifold of a Riemannian manifold such that the submanifold admits a TFV field θ, given by $\nabla_E\theta = lE+\eta(E)\theta$, where η is dual to θ and if it also admits an η-Ricci Bourguignon soliton $(g,\theta,\alpha,\beta,\gamma)$, then the scalar curvature of the manifold vanishes if $\alpha=l$, $\beta d\neq 1$.*

Proof. The proof follows from (19). □

Theorem 4.3. *If a d-dimensional ($d\neq 1$) submanifold of a Riemannian manifold admits a TFV field θ given by $\nabla_E\theta = lE+\eta(E)\theta$, where η is dual to θ and $\nabla\eta=0$, and if the submanifold admits η-Ricci Bourguignon soliton, them* grad$l$ is also a TFV field, provided $\alpha+\beta r+\gamma+1-ld\neq 0$.

Proof. Taking $m=\eta$, from (18), we infer that

$$S(E,\theta)=(\alpha+\beta r-l+\gamma+1)\eta(E) \tag{20}$$

If $\nabla_E\theta = lE+\eta(E)\theta$, we have

$$\begin{aligned} R(E,F)\theta &= (El)F-(Fl)E\\ &+ (\nabla_E\eta)F-(\nabla_F\eta)E\\ &+ \eta(F)lE-\eta(E)lF. \end{aligned}$$

If $\nabla\eta=0$, contracting the above equation

$$S(E,\theta)=(1-d)Fl-l(1-d)\eta(F). \tag{21}$$

Comparison of (20) and (21) yields

$$Fl=\frac{\alpha+\beta r+\gamma+1-ld}{1-d}\eta(F).$$

Since by our assumption η is dual to θ and $Fl = g(\text{grad}l, F)$, the consequence is

$$\text{grad}l = \frac{\alpha + \beta r + \gamma + 1 - ld}{1 - d}\theta.$$

Thus, grad$l$ and θ are collinear. So, if θ is torse-forming, grad$l$ is so. □

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