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Minimal cusco maps and the topology of uniform convergence on compacta

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Abstract. Minimal cusco maps have applications in functional analysis, in optimization, in the study of weak Asplund spaces, in the study of differentiability of functions, etc. It is important to know their topological properties. Let X be a Hausdorff topological space, $MC(X)$ be the space of minimal cusco maps with values in $\mathbb{R}$ and τ_{UC} be the topology of uniform convergence on compacta. We study complete metrizability and cardinal invariants of $(MC(X), \tau_{UC})$. We prove that for two nondiscrete locally compact second countable spaces X and Y, $(MC(X), \tau_{UC})$ and $(MC(Y), \tau_{UC})$ are homeomorphic and they are homeomorphic to the space $C(I^{\mathfrak{c}})$ of continuous real-valued functions on $I^{\mathfrak{c}}$ with the topology of uniform convergence.

1. Introduction

Minimal cusco maps are very important tool in functional analysis (see [1, 3, 4, 20, 28, 32]), in optimization [7], in the study of weak Asplund spaces [30], etc.

As for topologies on spaces of set-valued maps, there are mainly two approaches in the literature - function space topologies [13, 20–22] and hyperspace topologies [20, 23], in which set-valued maps are identified with their graphs and are considered as elements of a hyperspace.

In our paper we study the topology of uniform convergence on compacta τ_{UC} on the space $MC(X)$ of all minimal cusco maps from a topological space X to $\mathbb{R}$, the space of real numbers equipped with the usual Euclidean metric. If X is locally compact, then $(MC(X), \tau_{UC})$ is a locally convex topological vector space [24]. If X is hemicompact, then $(MC(X), \tau_{UC})$ is metrizable [21]. We will prove that if X is a locally compact hemicompact space, then $(MC(X), \tau_{UC})$ is completely metrizable. It is known [33] that if two infinite-dimensional completely metrizable locally convex topological vector spaces have the same density, then they are homeomorphic.

We will study density and other cardinal invariants of $(MC(X), \tau_{UC})$. If $(MC(X), \tau_{UC})$ is metrizable, then all cardinal invariants, including density, weight and cellularity coincide on $(MC(X), \tau_{UC})$. We find further conditions on X under which the cardinal invariants coincide on $(MC(X), \tau_{UC})$ as well as characterizations of some cardinal invariants of $(MC(X), \tau_{UC})$.

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The topology τ_{UC} of uniform convergence on compacta on $MU(X, Y)$, the space of all minimal usco maps from a topological space X to a metric space Y, was studied in many papers [18, 21, 22, 24, 25]. A nice generalization of the Arzela-Ascoli Theorem from continuous functions to minimal usco/cusco maps into metric spaces was proved in [18]. In papers [13] and [22] the topology τ_{UC} of uniform convergence on compacta on the space of densely continuous forms was studied. There is a connection between minimal usco maps and densely continuous forms.

2. Preliminaries

In what follows let X, Y be Hausdorff topological spaces, $\mathbb{N}$ be the set of positive integers and $\mathbb{R}$ be the space of real numbers with the usual metric. The symbols $\overline{A}$ and $IntA$ will stand for the closure and interior of the set A in a topological space.

A set-valued map, or a multifunction, from X to Y is a function that assigns to each element of X a subset of Y. Following [9] the term map is reserved for a set-valued map. If F is a map from X to Y, then its graph is the set $\{(x, y) \in X \times Y : y \in F(x)\}$. In our paper we will identify maps with their graphs.

Notice that if $f : X \to Y$ is a single-valued function, we will use the symbol f also for the graph of f.

A map $F : X \to Y$ is upper semicontinuous at a point $x \in X$ if for every open set V containing $F(x)$, there exists an open set U such that $x \in U$ and

$$F(U) = \bigcup \{F(u) : u \in U\} \subset V.$$

F is upper semicontinuous if it is upper semicontinuous at each point of X. Following Christensen [6] we say, that a map F is usco if it is upper semicontinuous and takes nonempty compact values. A map F from a topological space X to a linear topological space Y is cusco if it is usco and $F(x)$ is convex for every $x \in X$.

Finally, a map F from a topological space X to a topological (linear topological space) Y is said to be minimal usco (minimal cusco) if it is a minimal element in the family of all usco (cusco) maps (with the domain X and the range Y); that is, if it is usco (cusco) and does not contain properly any other usco (cusco) map from X into Y. Using the Kuratowski-Zorn principle we can guarantee that every usco (cusco) map from X to Y contains a minimal usco (cusco) map from X to Y (see [5, 9]).

In papers [14, 16] we can find interesting characterizations of minimal usco and minimal cusco maps using quasicontinuous and subcontinuous selections, which will be also useful for our analysis.

A function $f : X \to Y$ is quasicontinuous at $x \in X$ [15, 31] if for every neighborhood V of $f(x)$ and every neighbourhood U of x there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

Let A be a dense subset of a topological space X and Y be a topological space. Let $f : A \to Y$ be a function. We say that f is densely defined. A densely defined function f is a densely defined quasicontinuous selection of a set-valued map F, if $f(x) \in F(x)$ for every $x \in dom f$, the domain of f and $f : dom f \to Y$ is quasicontinuous with respect to the induced topology on $dom f$.

A function $f : X \to Y$ is subcontinuous at $x \in X$ [11] if for every net (x_i) convergent to x, there is a convergent subnet of $(f(x_i))$. If f is subcontinuous at every $x \in X$, we say that f is subcontinuous. The notion of subcontinuity can be extended for densely defined functions. Let A be a dense subset of a topological space X and Y be a topological space. We say that $f : A \to Y$ is subcontinuous at $x \in X$ ([27]) if for every net $(x_i) \subset A$ convergent to $x \in X$, $(f(x_i))$ has a convergent subnet. A function $f : A \to Y$ is subcontinuous if it is subcontinuous at every $x \in X$.

Let Y be a linear topological space and $B \subset Y$ be a set. By co$B$ we denote the convex hull of the set B and by $\overline{\text{co}}B$ we denote the closure of co$B$.

Theorem 2.1. ([16]) *Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space in which the closed convex hull of a compact set is compact. Let F be a map from X to Y. The following are equivalent:*

1. *F is minimal cusco;*
2. *There is a quasicontinuous subcontinuous selection f of F such that $\overline{\text{co}}\overline{f}(x) = F(x)$ for every $x \in X$;*

3. *There is a densely defined quasicontinuous subcontinuous selection f of F such that $\overline{\mathrm{co}}\overline{f}(x) = F(x)$ for every $x \in X$.*

Remark 2.2. The space $\mathbb{R}$ with the usual topology is a Hausdorff locally convex linear topological space, in which the convex hull of a compact set is compact, so in the previous theorem we can omit the closure of the convex hull of $\overline{f}(x)$.

Let $F \subset X \times \mathbb{R}$ be such that $F(x)$ is a nonempty bounded set for every $x \in X$. Then there are two real-valued functions $\sup F$ and $\inf F$ defined on X by $\sup F(x) = \sup\{t \in \mathbb{R} : t \in F(x)\}$ and $\inf F(x) = \inf\{t \in \mathbb{R} : t \in F(x)\}$.

Theorem 2.3. ([16]) *Let X be a topological space. Let F be a map from X to $\mathbb{R}$. The following are equivalent:*

1. *F is minimal cusco;*
2. *F is nonempty compact, convex valued, F has a closed graph, $\sup F$ and $\inf F$ are quasicontinuous, subcontinuous functions and $\overline{\sup F} = \overline{\inf F}$;*
3. *F is nonempty compact valued, $\sup F$ and $\inf F$ are quasicontinuous, subcontinuous functions and $F(x) = \mathrm{co}\,\overline{\sup F}(x) = \mathrm{co}\,\overline{\inf F}(x)$ for every $x \in X$.*

3. Minimal cusco maps with the topology of uniform convergence on compacta

Let X be a topological space and (Y, d) be a metric space.

The open d-ball with center $z_0 \in Y$ and radius $\varepsilon > 0$ will be denoted by $S_\varepsilon(z_0)$ and the ε-parallel body $\bigcup_{a \in A} S_\varepsilon(a)$ for a subset A of Y will be denoted by $S_\varepsilon(A)$.

Denote by $CL(Y)$ the space of all nonempty closed subsets of Y and by $K(Y)$ the space of all nonempty compact subsets of Y.

If $A \in CL(Y)$, the distance functional $d(\cdot, A) : Y \to [0, \infty)$ is described by the familiar formula

$$d(z, A) = \inf\{d(z, a) : \ a \in A\}.$$

Let A and B be nonempty subsets of (Y, d). The excess of A over B with respect to d is defined by the formula

$$e_d(A, B) = \sup\{d(a, B) : \ a \in A\}.$$

The Hausdorff (extended-valued) metric H_d on $CL(Y)$ [2] is defined by

$$H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}.$$

We will often use the following equality on $CL(Y)$:

$$H_d(A, B) = \inf\{\varepsilon > 0 : \ A \subset S_\varepsilon(B) \text{ and } B \subset S_\varepsilon(A)\}.$$

The topology generated by H_d is called the Hausdorff metric topology.

It is known that if (Y, d) is a complete metric space then $(CL(Y), H_d)$ and $(K(Y), H_d)$ are also complete metric spaces ([2]).

Following [13] we will define the topology τ_p of pointwise convergence on $CL(Y)^X$. The topology τ_p of pointwise convergence on $CL(Y)^X$ is induced by the uniformity $\mathfrak{U}_p$ of pointwise convergence which has a base consisting of sets of the form

$$W(A, \varepsilon) = \{(\Phi, \Psi) : \ \forall\, x \in A \ \ H_d(\Phi(x), \Psi(x)) < \varepsilon\},$$

where A is a finite set in X and $\varepsilon > 0$. The general τ_p-basic neighborhood of $\Phi \in CL(Y)^X$ will be denoted by $W(\Phi, A, \varepsilon)$, where

$$W(\Phi, A, \varepsilon) = \{\Psi : \ \forall\, x \in A \ \ H_d(\Phi(x), \Psi(x)) < \varepsilon\}.$$

We will define the topology τ_{UC} of uniform convergence on compact sets on $CL(Y)^X$ [13]. This topology is induced by the uniformity $\mathfrak{U}_{UC}$ which has a base consisting of sets of the form

$$W(K,\varepsilon) = \{(\Phi,\Psi) : \ \forall\, x \in K \ \ H_d(\Phi(x),\Psi(x)) < \varepsilon\},$$

where $K \in K(X)$ and $\varepsilon > 0$. The general τ_{UC}-basic neighborhood of $\Phi \in CL(Y)^X$ will be denoted by $W(\Phi, K, \varepsilon)$, where

$$W(\Phi,K,\varepsilon) = \{\Psi : \ \forall\, x \in K \ \ H_d(\Phi(x),\Psi(x)) < \varepsilon\}.$$

Denote by $MC(X)$ the space of all minimal cusco maps from X to $\mathbb{R}$, by $MU(X)$ the space of all minimal usco maps from X to $\mathbb{R}$ and by $C(X)$ the space of all continuous functions from X to $\mathbb{R}$.

Let d be the usual Euclidean metric on $\mathbb{R}$ and H_d be the Hausdorff metric induced by d on $CL(\mathbb{R})$.

Remark 3.1. In [17] we proved that for a locally compact space X the spaces $(MC(X), \tau_{UC})$ and $(MU(X), \tau_{UC})$ are homeomorphic. Example 3.2 in [17] shows that $(MC(X), \tau_{UC})$ and $(MU(X), \tau_{UC})$ need not, in general, be homeomorphic.

A topological space X is hemicompact if there is a countable cofinal subfamily in $K(X)$ with respect to the inclusion. If X is hemicompact, then $(MC(X), \tau_{UC})$ is metrizable [21]. We will define a metric on the space $MC(X)$ compatible with the topology τ_{UC}.

Let $\{K_n : \ n \in \mathbb{Z}^+\}$ be a countable cofinal subfamily in $K(X)$ with respect to the inclusion. It is easy to verify that the countable family $\{W(K_m, 1/n) : m, n \in \mathbb{N}\}$ is a base of the uniformity $\mathfrak{U}_{UC}$. Thus the uniformity $\mathfrak{U}_{UC}$ is metrizable [26]. We will define a compatible metric ρ on $MC(X)$.

For every $K \in K(X)$ let p_K be the pseudometric on $MC(X)$ defined by

$$p_K(F,G) = \sup\{H_d(F(x),G(x)) : \ x \in K\}.$$

Notice that for every $F \in MC(X)$ and every $K \in K(X)$ the set $F(K)$ is compact [2].

Then for every $K \in K(X)$ we have the pseudometric h_K defined as

$$h_K(F,G) = \min\{1, p_k(F,G)\}.$$

We define a function $\rho : \ MC(X) \times MC(X) \to \mathbb{R}$ as follows

$$\rho(F,G) = \sum_{n=1}^{\infty} \frac{1}{2^n} h_{K_n}(F,G).$$

It is easy to see that ρ is a metric on $MC(X)$ and uniformity $\mathfrak{U}_{UC}$ is generated by ρ.

Denote by $CK(\mathbb{R})$ the space of all compact intervals in $\mathbb{R}$. Then $(CK(\mathbb{R}), H_d)$ is a complete metric space [2].

Lemma 3.2. ([26], Theorem 7.10 (c)) *Let X be a topological space.* $(CK(\mathbb{R})^X, \mathfrak{U}_{UC})$ *is complete.*

Proposition 3.3. *Let X be a locally compact space. Then $MC(X)$ is a closed subspace of* $(CK(\mathbb{R})^X, \mathfrak{U}_{UC})$.

Proof. $F \in \overline{MC(X)}$ in $(CK(\mathbb{R})^X, \mathfrak{U}_{UC})$. First we prove that F is upper semicontinuous. Suppose that F is not upper semicontinuous at $x \in X$. Let $\varepsilon > 0$ be such that for every open neighbourhood O of x there is

$$y_O \in F(O) \setminus S_{2\varepsilon}(F(x)).$$

Let U be an open neighbourhood of x such that $\overline{U}$ is compact. Let $G \in W(F, \overline{U}, \varepsilon) \cap MC(X)$. The upper semicontinuity of G at x implies that there is an open neighbourhood U_1 of x such that $U_1 \subset U$ and $G(z) \subset S_\varepsilon(F(x))$ for every $z \in U_1$. Let $s \in U_1$ be such that $y_{U_1} \in F(s) \setminus S_{2\varepsilon}(F(x))$, a contradiction, since there must exist $l \in G(s)$ such that $d(y_{U_1}, l) < \varepsilon$.

Now we prove that F is minimal cusco. Suppose that F is cusco but not minimal. Since F is cusco there must exist a minimal cusco L contained in F. Let $x \in X$ be such that there is $y \in F(x) \setminus L(x)$. Let $\varepsilon > 0$ be such that

$$S_\varepsilon(y) \cap S_\varepsilon(L(x)) = \emptyset.$$

Let U be an open neighbourhood of x such that $\overline{U}$ is compact and $L(z) \subset S_\varepsilon(L(x))$ for every $z \in U$. Let $G \in W(F, \overline{U}, \varepsilon/2) \cap MC(X)$. Without loss of generality we can suppose that $\sup L(x) + \varepsilon < y - \varepsilon$. There must exist $z \in G(x)$ such that $y - \varepsilon/2 < z \leq \sup G(x)$. By Theorem 2.3 $\sup G$ is quasicontinuous. Thus there is a nonempty open set U_1 such that $U_1 \subset U$ and

$$y - \varepsilon/2 < \sup G(t) \text{ for every } t \in U_1.$$

By Theorem 2.3 $G(t) = \mathrm{co}\,\overline{\sup G}(t) \subset [y - \varepsilon/2, \infty)$ for every $t \in U_1$, a contradiction. □

Corollary 3.4. *Let X be a locally compact space. Then $(MC(X), \mathfrak{U}_{UC})$ is complete.*

Proof. By Lemma 3.2 $(CK(\mathbb{R})^X, \mathfrak{U}_{UC})$ is complete. By Proposition 3.3 $MC(X)$ is a closed subspace of $(CK(\mathbb{R})^X, \mathfrak{U}_{UC})$. Thus $(MC(X), \mathfrak{U}_{UC})$ is complete too. □

Proposition 3.5. *Let X be a locally compact hemicompact space. Then $(MC(X), \tau_{UC})$ is a completely metrizable locally convex topological vector space.*

Proof. By Theorem 7.5 in [24] if X is locally compact, then $(MC(X), \tau_{UC})$ is a locally convex topological vector space. By Corollary 3.4 $(MC(X), \mathfrak{U}_{UC})$ is complete. Let $\{K_n : n \in \mathbb{Z}^+\}$ be a countable cofinal subfamily in $K(X)$ with respect to the inclusion. The metric ρ defined above with respect to the family $\{K_n : n \in \mathbb{Z}^+\}$ generates the uniformity $\mathfrak{U}_{UC}$. Thus $(MC(X), \tau_{UC})$ is a completely metrizable locally convex topological vector space. □

It is known [33] that if two infinite-dimensional completely metrizable locally convex linear topological spaces have the same density, then they are homeomorphic. In the next section we will study density and other cardinal invariants of $(MC(X), \tau_{UC})$.

4. Cardinal invariants of $(MC(X), \tau_{UC})$

In what follows let X be a Hausdorff topological space. We will consider the cardinal invariants of the space $(MC(X), \tau_{UC})$. Because of simplicity we will omit the specification of the topology τ_{UC}.

We start with the following bound of the cardinality of $MC(X)$.

Proposition 4.1. *Let X be a Baire space. Then $|MC(X)| \leq 2^{w(X)}$.*

Proof. Let $F \in MC(X)$. Denote $S(F) = \{x \in X : |F(x)| = 1\}$. By Theorem 2.7 in [23] $S(F)$ is a dense G_δ subset of X. For every $F \in MC(X)$ the function $F|S(F)$ is a continuous function defined on $S(F)$. Thus $F|S(F) \in C(S(F))$. The cardinality of the family of G_δ subsets of X is less than or equal to $2^{w(X)}$. For each G_δ subset G in X, we have $|C(G)| \leq 2^{d(G)}$ [12]. Thus $|C(G)| \leq 2^{w(G)} \leq 2^{w(X)}$. Denote $\mathcal{C} = \bigcup\{C(G) : G \text{ is a } G_\delta \text{ set}\}$.

Then $|\mathcal{C}| \leq 2^{w(X)}$. Define the mapping $\Psi : MC(X) \to \mathcal{C}$ as follows: $\Psi(F) = F|S(F)$. We show that Ψ is injection. Let $F, G \in MC(X)$ and suppose that $F|S(F) = G|S(G)$. Put $H = F|S(F) = G|S(G)$. Then H is a densely defined continuous and subcontinuous selection of F and G. (H is subcontinuous, since by Proposition 3.3 in [19] every selection of an usco map is subcontinuous.) By Theorem 2.1 $\mathrm{co}\overline{H}(x) = F(x)$ for every $x \in X$ and $\mathrm{co}\overline{H}(x) = G(x)$ for every $x \in X$. Thus $F = G$. We have $|MC(X)| \leq |\mathcal{C}| \leq 2^{w(X)}$. □

Throughout this section we assume that the space Z is not finite, i.e. $|Z| \geq \aleph_0$.

For many basic results relating to the following cardinal invariants, see [12].

For a topological space Z we define:

the weight of Z

$$w(Z) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base in } Z\},$$

the density of Z

$$d(Z) = \aleph_0 + \min\{|\mathcal{D}| : \mathcal{D} \text{ is dense in } Z\},$$

the cellularity of Z

$$c(Z) = \aleph_0 + \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a pairwise disjoint family of nonempty open subset of } Z\},$$

the network weight of Z

$$nw(Z) = \aleph_0 + \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network in } Z\}.$$

They are in general related by the inequalities

$$c(Z) \leq d(Z) \leq nw(Z) \leq w(Z).$$

When Z is metrizable

$$c(Z) = d(Z) = nw(Z) = w(Z).$$

The character of a point z in Z is defined as

$$\chi(Z, z) = \aleph_0 + \min\{|\mathcal{O}| : \mathcal{O} \text{ is a base at } z\},$$

and the character of Z is defined as

$$\chi(Z) = \sup\{\chi(Z, z) : z \in Z\}.$$

The pseudocharacter of a point z in a topological space Z is defined as

$$\psi(Z, z) = \min\{|\mathcal{G}| : \mathcal{G} \text{ is a family of open sets in } Z \text{ such that } \bigcap \mathcal{G} = \{z\}\},$$

and the pseudocharacter of Z is defined as:

$$\psi(Z) = \aleph_0 + \sup\{\psi(Z, z) : z \in Z\},$$

the diagonal degree of Z

$$\Delta(Z) = \aleph_0 + \min\{|\mathcal{G}| : \mathcal{G} \text{ is a family of open sets in } Z \times Z, \bigcap \mathcal{G} = \Delta_Z\}.$$

The pseudocharacter ψ and the diagonal degree Δ of $MC(X)$ can be expressed using the so-called weak k-covering number $wkc(X)$ of X, which is defined as follows

$$wkc(X) = \aleph_0 + \min\{|\beta| : \beta \subset K(X), \overline{\bigcup \beta} = X\}.$$

Theorem 4.2. *Let X be a regular topological space. Then $\psi(MC(X)) = \Delta(MC(X)) = wkc(X)$.*

Proof. To prove that $wkc(X) \leq \psi(MC(X))$, let f be the zero function on X. Let $\{W(f, A_t, \varepsilon_t) : A_t \in K(X), \varepsilon_t > 0, t \in T\}$ be such that

$$f = \bigcap_{t \in T} W(f, A_t, \varepsilon_t) \text{ and } |T| \leq \psi(MC(X)).$$

We claim that $X = \overline{\bigcup A_t : t \in T}$. Suppose there is $x \in X \setminus \overline{\bigcup A_t : t \in T}$. Let V be an open set in X such that $x \in V \subset \overline{V} \subset X \setminus \overline{\bigcup A_t : t \in T}$. Let $h : X \to \mathbb{R}$ be defined as follows:

$$h(z) = \begin{cases} 1, & z \in \overline{V}; \\ 0, & \text{otherwise.} \end{cases}$$

The function h is quasicontinuous and subcontinuous. By Theorem 2.1 $\Phi = \mathrm{co}\overline{h}(x)$ is a minimal cusco map and

$$\Phi \in \bigcap_{t \in T} W(f, A_t, \varepsilon_t),$$

a contradiction since $\Phi(x) = 1$ and $f(x) = 0$. Thus

$$wkc(X) \leq |T| \leq \psi(MC(X)) \leq \Delta(MC(X)).$$

To prove that $\Delta(MC(X)) \leq wkc(X)$, let $\beta \subset K(X)$ be such that $wkc(X) = |\beta|$ and $\overline{\bigcup \beta} = X$. For every $A \in \beta$ and $n \in \mathbb{N}$ put

$$\mathcal{G}_{A,n} = \bigcup \{W(\Psi, A, 1/n) \times W(\Psi, A, 1/n) : \Psi \in MC(X)\}$$

and we claim that

$$\bigcap \{\mathcal{G}_{A,n} : A \in \beta, n \in \mathbb{N}\} = \Delta_{MC(X)}.$$

Let $\Sigma, \Psi \in MC(X)$ be such that $\Sigma \neq \Psi$. Thus there is $z \in X$ such that $\Sigma(z) \neq \Psi(z)$. Suppose that $\sup \Sigma(z) > \sup \Psi(z)$. In the following possible cases: $\sup \Sigma(z) < \sup \Psi(z)$, $\inf \Sigma(z) > \inf \Psi(z)$, $\inf \Sigma(z) < \inf \Psi(z)$ the proof will be analogous. Put $y = \sup \Sigma(z)$. Let $\varepsilon > 0$ be such that $S_{4\varepsilon}(y) \cap \Psi(z) = \emptyset$. Since Ψ is upper semicontinuous at z, there is an open set $O \subset X$ such that $z \in O$ and $\Psi(v) \cap S_{3\varepsilon}(y) = \emptyset$ for every $v \in O$. By Theorem 2.3 $\sup \Sigma$ is quasicontinuous at z, so there is an open set $V \subset O$ such that $\sup \Sigma(l) \in S_\varepsilon(y)$ for every $l \in V$. Since by Theorem 2.3 $\mathrm{co}\,\overline{\sup \Sigma}(l) = \Sigma(l)$, $\Sigma(l) \subseteq \overline{S_\varepsilon(y)}$ for every $l \in V$. There is $A \in \beta$ with $V \cap A \neq \emptyset$. Let $n \in \mathbb{N}$ such that $1/n < \varepsilon/2$. Suppose that there is $\Gamma \in MC(X)$ such that $(\Sigma, \Psi) \in W(\Gamma, A, 1/n) \times W(\Gamma, A, 1/n)$. Let $t \in V \cap A$. Then $\Sigma(t) \subset S_{2\varepsilon}(y)$ and $\Psi(t) \cap S_{3\varepsilon}(y) = \emptyset$. Thus $\Gamma(t) \subset S_{\varepsilon/2}(\Sigma(t)) \subset S_{5\varepsilon/2}(y)$ and simultaneously $\Gamma(t) \cap S_{5\varepsilon/2}(y) = \emptyset$, a contradiction. □

To define the π-character of a topological space Z, we first need a notion of a local π-base. If $z \in Z$, a collection $\mathcal{V}$ of nonempty open subsets of Z is called a local π-base at z provided that for each open neighborhood U of z, there exists a $V \in \mathcal{V}$ which is contained in U.

The π-character of a point z in Z is defined as

$$\pi_\chi(Z, z) = \aleph_0 + \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local } \pi\text{-base at } z\},$$

and the π-character of Z is defined as

$$\pi_\chi(Z) = \sup\{\pi_\chi(Z, z) : z \in Z\}.$$

The k-cofinality of a topological space Z is defined to be

$$kcof(Z) = \aleph_0 + \min\{|\beta| : \beta \text{ is a cofinal family in } K(Z)\}.$$

$kcof(Z) = \aleph_0$ if and only if Z is hemicompact.

Theorem 4.3. *Let X be a topological space. Then $\chi(MC(X)) = \pi_\chi(MC(X)) = kcof(X)$.*

Proof. At first we prove that $kcof(X) \leq \pi_\chi(MC(X))$. Let f be the zero function on X. Then f is a minimal cusco map. Let $\{W(\Phi_t, A_t, \varepsilon_t) : A_t \in K(X), \varepsilon_t > 0, t \in T\}$ be a local π-base of f in $MC(X)$ with $|T| \leq \pi_\chi(MC(X))$.

We claim that $\{A_t : t \in T\}$ is a cofinal family in $K(X)$. Let $A \in K(X)$. There must exist $t \in T$ with

$$W(\Phi_t, A_t, \varepsilon_t) \subset W(f, A, 1).$$

We show that $A \subset A_t$. Suppose there is $a \in A \setminus A_t$. Let U be an open set such that $a \in U$ and $\overline{U} \cap A_t = \emptyset$.

Let $g : X \to \mathbb{R}$ be defined as follows:

$$g(z) = \begin{cases} 1, & z \in \overline{U}; \\ \sup \Phi_t, & \text{otherwise.} \end{cases}$$

The function g is quasicontinuous and subcontinuous (see Theorem 2.3). Put $\Gamma(x) = \overline{co g}(x)$ for every $x \in X$. By Theorem 2.1, $\Gamma \in MC(X)$. It is easy to verify that $\Gamma(s) = \Phi_t(s)$ for every $s \notin \overline{U}$; thus also for every $s \in A_t$. $\Gamma \in W(\Phi_t, A_t, \varepsilon_t)$, but $\Gamma \notin W(f, A, 1)$, a contradiction. Thus

$$kcof(X) \leq \pi_\chi(MC(X)) \leq \chi(MC(X)).$$

To prove that $\chi(MC(X)) \leq kcof(X)$, let $\Phi \in MC(X)$ and let β be a cofinal subfamily of $K(X)$ with $|\beta| = kcof(X)$. It is easy to verify that the family $\{W(\Phi, K, 1/n) : K \in \beta, n \in \mathbb{N}\}$ is a local base at Φ. □

For a Tychonoff space Z we define the uniform weight of Z [10]

$$u(Z) = \aleph_0 + \min\{m : \text{there is a uniformity on } Z \text{ of weight } \leq m\}.$$

Theorem 4.4. *Let X be a topological space. Then $u(MC(X)) = kcof(X)$.*

Proof. Let β be a cofinal family in $K(X)$ such that $kcof(X) = |\beta|$. It is easy to verify that the family $\{W(K, 1/n) : K \in \beta, n \in \mathbb{N}\}$ is a base of the uniformity $\mathfrak{U}_{UC}$. Thus $u(MC(X)) \leq kcof(X)$. For every uniform space X we have $\chi(X) \leq u(X)$. Since by Theorem 4.3 $kcof(X) = \chi(MC(X))$, we have $u(MC(X)) = kcof(X)$. □

Corollary 4.5. *Let X be a topological space. The following are equivalent.*

1. *X is hemicompact,*
2. *$MC(X)$ is metrizable,*
3. *$MC(X)$ is first countable.*

To define the π-weight of a topological space Z, we first need a notion of a π-base. A collection $\mathcal{V}$ of nonempty open subsets of Z is called a π-base provided that for each open set U in Z, there exists a $V \in \mathcal{V}$ which is contained in U.

Define the π-weight of Z by

$$\pi w(Z) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \pi\text{-base in } Z\}.$$

For a topological space Z we define the extent of Z

$$e(Z) = \aleph_0 + \sup\{|E| : E \text{ is a closed discrete set in } Z\}.$$

Theorem 4.6. *For every space X $\pi w(MC(X)) = w(MC(X))$. In fact*
$\pi w(MC(X)) = kcof(X) \cdot d(MC(X))$ and $w(MC(X)) = kcof(X) \cdot c(MC(X)) = kcof(X) \cdot e(MC(X))$.

Proof. It is known (see [10]) that for a Tychonoff space Z, $w(Z) = c(Z) \cdot u(Z)$ and $w(Z) = e(Z) \cdot u(Z)$. Thus by Theorem 4.4 $w(MC(X)) = kcof(X) \cdot c(MC(X))$ and $w(MC(X)) = kcof(X) \cdot e(MC(X))$. Since $\pi w(MC(X)) \geq \pi\chi(MC(X)) = \chi(MC(X)) = kcof(X)$ and $\pi w(MC(X)) \geq d(MC(X))$ we have

$$\pi w(MC(X)) \geq kcof(X) \cdot d(MC(X)) \geq w(MC(X)).$$

□

Lemma 4.7. *Let X be a topological space which contains an infinite compact set. Then $c(MC(X)) \geq \mathfrak{c}$ and also $e(MC(X)) \geq \mathfrak{c}$.*

Proof. Let K be an infinite compact set in X. There is a pairwise disjoint sequence $\{U_n : n \in \mathbb{N}\}$ of open sets such that $U_n \cap K \neq \emptyset$ for every $n \in \mathbb{N}$. Let $2^{\mathbb{N}}$ denote the set of all functions from $\mathbb{N}$ to $\{0, 1\}$. Put $A = (\bigcup_{n\in\mathbb{N}} U_n) \cup (X \setminus \overline{\bigcup_{n\in\mathbb{N}} U_n})$. For every $\varphi \in 2^{\mathbb{N}}$ let $f_\varphi : A \to \{0, 1\}$ be a function defined as follows:

$$f_\varphi(x) = \begin{cases} \varphi(n), & \text{if } x \in U_n \text{ for some } n \in \mathbb{N}; \\ 0, & x \in (X \setminus \overline{\bigcup_{n\in\mathbb{N}} U_n}). \end{cases}$$

Then f_φ is a densely defined quasicontinuous subcontinuous function. Thus by Theorem 2.1 F_φ defined by $F_\varphi(x) = \overline{\text{co}f_\varphi}(x)$ for every $x \in X$, is a minimal cusco map.

For every $\varphi \in 2^{\mathbb{N}}$ define $B_\varphi = W(F_\varphi, K, 1/4)$. Then $\{B_\varphi : \varphi \in 2^{\mathbb{N}}\}$ is a pairwise disjoint family of open sets in $(MC(X), \tau_{UC})$. Thus $c(MC(X)) \geq \mathfrak{c}$. The set $\{F_\varphi : \varphi \in 2^{\mathbb{N}}\}$ is a discrete set. We show that $\{F_\varphi : \varphi \in 2^{\mathbb{N}}\}$ is a closed set in $(MC(X), \tau_{UC})$. Let $G \in MC(X) \setminus \{F_\varphi : \varphi \in 2^{\mathbb{N}}\}$. Suppose that $F_\psi \in W(G, K, 1/4)$ for some $\psi \in 2^{\mathbb{N}}$. Since there may be at most one $\psi \in 2^{\mathbb{N}}$ such that $F_\psi \in W(G, K, 1/4)$, the set $W(G, K, 1/4) \setminus \{F_\psi\}$ is an open neighborhood of G such that $\{F_\varphi : \varphi \in 2^{\mathbb{N}}\} \cap (W(G, K, 1/4) \setminus \{F_\psi\}) = \emptyset$. Thus $e(MC(X)) \geq \mathfrak{c}$. □

For a topological space Z the Lindelöf degree of Z is

$$L(Z) = \aleph_0 + \min\{\kappa : \text{every open cover of } Z \text{ has a subcover of cardinality at most } \kappa\}.$$

For a topological space Z the spread of Z is

$$s(Z) = \aleph_0 + \sup\{|E| : E \text{ is a discrete set in } Z\}.$$

It is known that if a topological space Z is metrizable, then all cardinal invariants $c, d, nw, s, e, L, \pi w, w$ coincide on Z.

If X is hemicompact, i.e. $kcof(X) = \aleph_0$, then by Corollary 4.5 $MC(X)$ is metrizable, thus all cardinal invariants $c, d, nw, s, e, L, \pi w, w$ coincide on $MC(X)$. The following theorem gives other conditions on X under which the cardinal invariants coincide on $MC(X)$.

Theorem 4.8. *Let X be a topological space which contains an infinite compact set and let $kcof(X) \leq \mathfrak{c}$. Then we have*

$$c(MC(X)) = d(MC(X)) = nw(MC(X)) = L(MC(X)) =$$
$$s(MC(X)) = e(MC(X)) = \pi w(MC(X)) = w(MC(X)).$$

Proof. Recall that for a Tychonoff space Z, $w(Z) = c(Z) \cdot u(Z)$, $w(Z) = e(Z) \cdot u(Z)$. By Theorem 4.4 $u(MC(X)) = kcof(X)$, thus

$$kcof(X) \cdot e(MC(X)) = w(MC(X)) = kcof(X) \cdot c(MC(X)).$$

By Lemma 4.7 we have $e(MC(X)) = w(MC(X)) = c(MC(X))$. Since other cardinal invariants are between c, w and e we are done. □

Corollary 4.9. *Let X be a discrete topological space. Then $c(MC(X)) = \aleph_0$ and $w(MC(X)) = kcof(X) = |X|$.*

Proof. If X is a discrete topological space, then the topology τ_{UC} coincides with the topology τ_p on $MC(X)$ and $MC(X) = \mathbb{R}^X$, so $c(MC(X)) = \aleph_0$. Since by Theorem 4.6 $w(MC(X)) = kcof(X) \cdot c(MC(X))$ we have that $w(MC(X)) = kcof(X) \cdot c(MC(X)) = |X|$. □

Theorem 4.10. *Let X be a Tychonoff topological space. The following are equivalent:*

1. $w(MC(X)) = \aleph_0$,
2. *X is countable and every compact set in X is finite.*

Proof. (1) ⇒ (2) If $w(MC(X)) = \aleph_0$, then $c(MC(X)) = \aleph_0$, thus by Lemma 4.7 every compact set in X must be finite. Then the topology τ_{UC} coincides with the topology τ_p on $MC(X)$. Thus $w(C(X)) = \aleph_0$ in the topology τ_p. By Corollary 4.5.4 in [29] X must be countable.

(2) ⇒ (1) If every compact set in X is finite, the topology $\tau_{UC} = \tau_p$ on $MC(X)$. Since $MC(X) \subset K(\mathbb{R})^X$, $w(MC(X)) \leq w(K(\mathbb{R})^X)$, where the space $K(\mathbb{R})$ is equipped with the Hausdorff distance induced by the Euclidean metric on $\mathbb{R}$. X is countable, thus $w(K(\mathbb{R})^X) = \aleph_0$. □

Theorem 4.11. *Let X be a first countable topological space. The following are equivalent:*

1. $c(MC(X)) = \aleph_0$,
2. *X is discrete.*

Proof. (2) ⇒ (1) by Corollary 4.9.

(1) ⇒ (2) By Lemma 4.7 every compact set in X must be finite. Suppose there is a non-isolated point $x \in X$. Then we can find a sequence $\{x_n : n \in \mathbb{N}\}$ of different points which converges to x. The set $K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is an infinite compact set in X, a contradiction. □

Theorem 4.12. *Let X be a first countable topological space. The following are equivalent:*

1. $nw(MC(X)) = \aleph_0$,
2. *X is countable and discrete.*

Proof. (2) ⇒ (1) By Theorem 4.10 $w(MC(X)) = \aleph_0$, so we are done.

(1) ⇒ (2) If $nw(MC(X)) = \aleph_0$, then also $c(MC(X)) = \aleph_0$, thus by Theorem 4.11 X must be discrete. $MC(X) = C(X)$ and the topology τ_{UC} coincides with the topology τ_p on $MC(X)$. By Corollary 4.1.3 in [29] X has a countable network, i.e. X must be countable. □

Corollary 4.13. *If X is a nondiscrete locally compact second countable space, then*

$$c(MC(X)) = d(MC(X)) = e(MC(X)) = L(MC(X)) =$$
$$s(MC(X)) = nw(MC(X)) = \pi w(MC(X)) = w(MC(X)) = \mathfrak{c},$$

and $|MC(X)| = \mathfrak{c}$.

Proof. Use Proposition 4.1, Lemma 4.7 and the fact that $MC(X)$ is metrizable. □

If X is a nondiscrete locally compact second countable space, then by Theorem 4.4.2 in [29] $C(X)$ is metrizable and by Corollary 4.2.2 in [29] $d(C(X)) = \aleph_0$. Thus by Theorem 8.1 (c) in [12] we have

$$c(C(X)) = d(C(X)) = e(C(X)) = L(C(X)) =$$
$$s(C(X)) = nw(C(X)) = \pi w(C(X)) = w(C(X)) = \aleph_0.$$

By Theorem 10.1 in [12] $|C(X)| = \mathfrak{c}$.

Theorem 4.14. *For every two nondiscrete locally compact second countable spaces X and Y, MC(X) and MC(Y) are homeomorphic and they are homeomorphic to* $C(I^{\mathfrak{c}})$.

Proof. If X and Y are two nondiscrete locally compact second countable spaces, then by Corollary 4.13 $d(MC(X)) = d(MC(Y)) = \mathfrak{c}$. By Proposition 3.5 both $MC(X)$ and $MC(Y)$ are completely metrizable locally convex topological vector spaces. By Corollary 5.2.2 in [29] $C(I^{\mathfrak{c}})$ is a completely metrizable space and by a sentence before Theorem 1.1.7 in [29] it is also a locally convex topological vector space. By Theorem 4.2.4 in [29] $d(C(I^{\mathfrak{c}})) = \mathfrak{c}$. By the result of Torunczyk [33] we are done. □

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