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Actions of R-module groupoids

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Abstract. Let R be a ring, 1_R be the identity of ring R and N be an R-module on R. In this work, we are going to give a new definition that it is called an action of R-module groupoids. First, we are going to give the definition of the action of the R-module groupoids on the R-module N. Then we obtained a new category $RMGpdOp(\Omega)$ of actions of Ω on R-modules. We also find that there is a groupoid $\Omega \ltimes N$ on N. $\Omega \ltimes N$ is called action R-module groupoid. Finally, we prove that the category $RMGpdOp(\Omega)$ of actions of R-module groupoids is equivalent to $RMGpdCov(\Omega)$ of coverings of R-module groupoids.

1. Introduction

First of all, the groupoid was given by Brandt [4]. Then Ehresmann offered the other versions of the groupoid in the 1950s [11]. So the groupoids theory has been improved and it has been found in many applications such as noncommutative geometry, differential topology, algebraic topology and theoretical physics.

The action is the major appliance in the algebraic and differential topology. This structure plays a major role in the category theory when it is studied groupoids. Ehresmann has defined a groupoid action over a set [11]. Then the action of groupoids has been pointed to most work in the differential and algebraic topology. Mathematicians have studied with different viewpoint of groupoid [8, 9, 14, 15, 17, 20].

Privately, to provide that two categories are equivalent is an important problem in the algebraic topology. Algebraic topology has searched some categories that equivalent to the categories of action groupoids. Such as, Gabriel et al. have proved that the category $GpdCov(\Omega)$ of covering groupoids of Ω and the category $GpdOp(\Omega)$ of actions of Ω are equivalent [12]. Brown et al. have studied this equivalence as topological [7]. Moreover, there are other the categories that are equivalence to groupoids [10, 13, 16].

On the other hand, the notion of groups with operations which originally comes from Higgins and Orzech is adapted in the following paper [21] unifying groups, rings, associative algebras, associative commutative algebras, Lie algebras, Leibniz algebras, alternative algebras and others. Then in the paper [1] the result was generalised to the internal groupoids and groups with operations (Theorem 4.2).

In this presentation, we investigate the actions of the R-module groupoid and R-module action groupoid. First of all, we defined groupoids and their actions of groupoids. Secondly, we have given the definition of R-module groupoids described in [2]. And then we described the action of the R-module groupoid and we obtained the action R-module groupoid. We proved that $RMGpdCov(\Omega)$ and $RMGpdOp(\Omega)$ categories are eqivalent.

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2. Groupoids

Definition 2.1. ([5, 6]) Let Ω be a category. If all morphisms of Ω are isomorphisms, then we say that Ω is a groupoid. Ω_0 is the object set, Ω is the all of morphisms set, for $x, y \in \Omega_0$, from x to y morphisms set is $\Omega(x, y)$, on the brink of structure maps are given as follow;

1. For $\forall\alpha \in \Omega(x, y)$ and $s, t : \Omega \to \Omega_0$ such that $s(\alpha) = x$ and $t(\alpha) = y$ are called source and target maps, respectively.
2. For $\forall x \in \Omega_0$, a map $\varepsilon : \Omega_0 \to \Omega$, $x \mapsto 1_x$ is called the object (unit) map.

In a groupoid a composition operation $\circledast : \Omega_s \times_t \Omega \to \Omega$, $(\beta, \alpha) \mapsto \beta \circledast \alpha$ is defined on the pull back

$$\Omega_s \times_t \Omega = \{(\beta, \alpha) | s(\beta) = t(\alpha)\}$$

These have to satisfy the following:

1. $\forall(\beta, \alpha) \in \Omega_s \times_t \Omega$, $s(\beta \circledast \alpha) = s(\alpha)$ and $t(\beta \circledast \alpha) = t(\beta)$.
2. $\forall\alpha, \beta, \gamma \in \Omega$, $\gamma \circledast (\beta \circledast \alpha) = (\gamma \circledast \beta) \circledast \alpha$ so that $s(\beta) = t(\alpha)$ and $s(\gamma) = t(\beta)$.
3. $\forall x \in \Omega_0$, $s(1_x) = t(1_x) = x$.
4. $\forall\alpha \in \Omega$, $\alpha \circledast 1_{s(\alpha)} = \alpha$ and $1_{t(\alpha)} \circledast \alpha = \alpha$.
5. $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$, $\alpha^{-1} \circledast \alpha = 1_{s(\alpha)}$ and $\alpha \circledast \alpha^{-1} = 1_{t(\alpha)}$.

We denote a groupoid over Ω_0 by Ω.

Let Ω be a groupoid and $x, y \in \Omega_0$. $\Omega(x, y)$ is a set of all morphisms defined from x to y such that $s(\alpha) = x$, $t(\alpha) = y$. For $x \in \Omega_0$, Ω_x is a set of morphisms that is started at x and Ω^y is a set of morphisms that is ended at y. Finally, the object group or the vertex group at x is the set $\Omega(x, x) = \{\alpha \in \Omega | s(\alpha) = t(\alpha) = x\}$ and is denoted by $\Omega\{x\}$. If Ω' is the most widely connected subgroupoid of Ω, then it is called the connected component of Ω [6].

Example 2.2. ([19]) Let S be a set and $(N, +)$ be a group. Then $S \times N \times S$ be a groupoid over S. This is clear from the following items.

1. For $x, y \in S$ and $m \in N$, a morphism from x to y is $(y, m, x) \in S \times N \times S$.
2. For $x, y, z \in S$ and $m, n \in N$, $(z, n, y) \circledast (y, m, x) = (z, m + n, x)$ gives the composition of morphisms.
3. Let $e \in N$ be the unit element of N. For $x \in S$, a unit morphism in x is $(x, e, x) \in S \times N \times S$.
4. An inverse morphism is $(x, -m, y) \in S \times N \times S$, for $(y, m, x) \in S \times N \times S$.

So $S \times N \times S$ is a groupoid.

Definition 2.3. ([6]) Let Ω and Ω' be two groupoids. A groupoid morphism is a pair (F, F_0) of maps $F : \Omega' \to \Omega$ and $F_0 : \Omega'_0 \to \Omega_0$ such that $s_\Omega \circledast F = F_0 \circledast s_{\Omega'}$, $t_\Omega \circledast F = F_0 \circledast t_{\Omega'}$ and $F(\beta \circledast \alpha) = F(\beta) \circledast F(\alpha)$ for all $(\beta, \alpha) \in \Omega'_s \times_t \Omega'$. The (F, F_0) is represented by F shortly. If F is injective and surjective then its called a groupoid isomorphism.

Thus, from the above definition, we have the category *Gpd* of the groupoids, such that objects are all of groupoids and morphisms are groupoids morphisms [6].

Let Ω be a groupoid. Ω' is called a subgroupoid of Ω if Ω' is a subcategory that is also a groupoid. That is the condition $\alpha \in \Omega \Rightarrow \alpha^{-1} \in \Omega$ is satisfied. For $\forall x, y \in \Omega'$ if $\Omega'(x, y) = \Omega(x, y)$ is hold then we say that Ω' is full, and if $\Omega_0 = \Omega'_0$ is hold we say that Ω' is wide [6].

Let $\alpha \in \Omega(x, y)$ is given. The left-translation $L_\alpha : \Omega^x \to \Omega^y$, $\beta \mapsto \alpha \circledast \beta$ and the right-translation $R_\alpha : \Omega_y \to \Omega_x$, $\beta \mapsto \beta \circledast \alpha$ are isomorphisms. In addition to the map $I_\alpha : \Omega(x, x) \to \Omega(y, y)$, $\beta \mapsto \alpha \circledast \beta \circledast \alpha^{-1}$ is the inner automorphism [6].

Definition 2.4. ([7, 12, 18]) Let Ω be a groupoid and N be a set. Let $\lambda : N \to \Omega_0$ is a map. A left action of Ω on N via λ is a map $\varphi : \Omega_s \times_\lambda N \to N$; $(\alpha, m) \mapsto^\alpha m$ which satisfies the following:

1. $\lambda({}^{\alpha}m) = t(\alpha)$,
2. ${}^{\beta}({}^{\alpha}m) =^{\beta\circledast\alpha} m$,
3. ${}^{1_{\lambda(m)}}m = m$

for $\alpha, \beta \in \Omega$ and $m \in N$. N is called a left Ω-set. We also can define a right action. Let $\xi : N_{\lambda} \times_t \Omega \to N$; $(m, \alpha) \mapsto m^{\alpha}$ satisfying the following conditions:

1. $\lambda(m^{\alpha}) = s(\alpha)$,
2. $(m^{\alpha})^{\beta} = m^{\alpha\circledast\beta}$,
3. $m^{1_{\lambda(m)}} = m$

for $\alpha, \beta \in \Omega$ and $m \in N$. In this case, N is called a right Ω-set.

Example 2.5. ([5]) We can easily show that every groupoid acts on itself from left and right sides by the composition of Ω. Here s and t are supposed that left and right actions, respectively.

Example 2.6. ([19]) Let Ω be a groupoid and S be a set. Let Ω act over S via $\lambda : S \to \Omega_0$. So, $\Omega \bowtie S$ action groupoid is obtained with the object set S via this action. This action groupoid's morphisms set is $G_{\alpha} \times_{\lambda} X$. Namely, a morphism is (a, x) such that ${}^{\alpha}x = y$ for $x, y \in S$ from x to y. The source and target maps are given by $s(\alpha, x) = x$ and $t(\alpha, x) =^{\alpha} x = y$. Unit and Inverse maps are defined by $x \mapsto (1_{\lambda(x)}, x)$ and $(\alpha, x)^{-1} = (\alpha^{-1}, ^{\alpha} x)$, respectively. Finally, $(\beta, y) \circledast (\alpha, x) = (\beta \circledast \alpha, x)$ is the composition over $\Omega \bowtie S$. So, $\Omega \bowtie S$ is a groupoid over S and this groupoid is called action groupoid.

3. R-module groupoids

Definition 3.1. ([3]) Let $(R, \oplus, \cdot)$ be a ring with unity 1_R and $(N, +)$ be an abelian group. Let $\delta : R \times N \to N$, $(r, m) \mapsto r \odot m$ be a map of R on N. If the following condition provide, N is (left) R-module for $r, s \in R$, $m, n \in N$.

1. $r \odot (m + n) = (r \odot m) + (r \odot b)$,
2. $(r \oplus s) \odot m = (r \odot m) \oplus (s \odot m)$,
3. $(r \cdot s) \odot m = r \cdot (s \odot m)$,
4. $1_R \odot m = m$.

Definition 3.2. ([2]) Let R be a ring with identity 1_R. An R-module groupoid is a Ω groupoid in which morphisms set Ω and objects set Ω_0 are both R-modules such that source and target maps $s, t : \Omega \to \Omega_0$, unit map $\varepsilon : \Omega_0 \to \Omega$, an inverse map $u : \Omega \to \Omega$, $\alpha \mapsto \alpha^{-1}$, a composition $\circledast : \Omega_s \times_t \Omega \to \Omega$, $(\alpha, \beta) \mapsto \alpha \circledast \beta$ are all R-module morphisms. These morphisms are have to keep the following items. For $r \in R$, $x \in \Omega_0$ and $\alpha, \beta \in \Omega$ and composition $\alpha \circledast \beta$ is defined.

1. $s(r \odot \alpha) = r \cdot s(\alpha), t(r \odot \alpha) = r \cdot t(\alpha)$.
2. $(r \odot \alpha)^{-1} = r \odot (\alpha^{-1})$.
3. $\varepsilon(r \cdot x) = r \odot \varepsilon(x) = r \odot 1_x$.
4. $(r \odot \alpha) \circledast (r \odot \beta) = r \odot (\alpha \circledast \beta)$.

So, an R-module groupoid Ω is a Ω-groupoid such that the above conditions $1-4$ are satisfied.

Considering the internal categories, the R-module groupoid is a groupoid object in the R-modules category. So it is an internal category in the category of R-modules.

Example 3.3. ([2]) Let R be a topological ring, 1_R be a unit element of R and N is a topological R-module. So, the fundamental groupoid $\pi_1(N)$ of N is an R-module groupoid: since N is a topological R-module, we have continuous operations as follows: group addition $+ : N \times N \to N$, $(m, n) \mapsto m + n$, inverse map $u : N \to N$; $m \mapsto -m$ and $\delta : R \times N \to N$, $(r, m) \mapsto r \odot m$. Therefore we have the following.

$$\pi_1(+) : \pi_1(N) \times \pi_1(N) \to \pi_1(N), ([m], [n]) \mapsto [m+n]$$
$$\pi_1(u) : \pi_1(N) \to \pi_1(N), [m] \mapsto [-m] = -[m]$$
$$R \times \pi_1(N) \to \pi_1(N), (r, [m]) \mapsto r \odot [m] = [r \odot m]$$

where the $r\odot\alpha$ defined by $(r\odot\alpha)(x) = r\cdot\alpha(x)$, for $x \in [0,1]$. We know that $\pi_1(N)$ is a Ω-groupoid. Furthermore This action makes $\pi_1(N)$ an R-module groupoid, as is required.

Example 3.4. ([19]) If N is an R-module, we know that $\Omega = N \times N$ on N is a Ω-groupoid. Further for $r \in R, m, n, k \in N$ and $\alpha = (m,n), \beta = (n,k)$ we have that $s(r \odot \alpha) = r \odot s(\alpha), t(r \odot \alpha) = r \odot t(\alpha), (r \odot \alpha)^{-1} = r \odot (\alpha^{-1}), 1_{r\cdot m} = r \odot 1_m$ and $(r \odot \alpha) \circledast (r \odot \beta) = r \odot (\alpha \circledast \beta)$. Therefore Ω is an R-module groupoid.

Proposition 3.5. *Let Ω be an R-module groupoid. Then the following assertions hold:*

1. $1_m \circledast \alpha = \alpha, \forall \alpha \in s^{-1}(1_m)$,
2. $\alpha \circledast 1_m = \alpha, \forall \alpha \in t^{-1}(1_m)$.

Proof. If $\alpha \in s^{-1}(1_m)$, then $s(\alpha) = m = t(1_m)$. So $(1_m, \alpha) \in \Omega \times \Omega$ and, using the condition associativity from groupoid definition, one obtains that $1_m \circledast \alpha = \alpha$. In the same way, we obtain $\alpha \circledast 1_m = \alpha$ □

Definition 3.6. Let Ω be an R-module groupoid on Ω_0. An R-module subgroupoid of Ω is a pair of sub R-modules $\Omega' \subset \Omega, \Omega_0 \subset \Omega_0$ such that $s(\Omega') \subset \Omega'_0, t(\Omega') \subset \Omega'_0$, $1_h \in \Omega'$ for all $h \in \Omega_0$, and Ω' is closed under the composition and the inversion in Ω. An R-module subgroupoid Ω' of Ω is called wide if $\Omega'_0 = \Omega_0$, and is called full if $\Omega'(h,k) = \Omega(h,k)$ for all $h, k \in \Omega'_0$.

The identity R-module subgroupoid of Ω is the R-module subgroupoid $\triangle_\Omega = \{1_m | m \in \Omega_0\}$. The inner R-module subgroupoid of Ω is the R-module subgroupoid $I\Omega = \bigcup_{m\in\Omega_0} \Omega(m,m)$

Definition 3.7. Let Ω be an R-module groupoid and $\mathcal{N}$ be a wide R-module subgroupoid. If $\alpha \odot n \odot \alpha^{-1} \in \mathcal{N}$ is hold then we say that $\mathcal{N}$ is a normal R-module subgroupoid, for any $n \in \mathcal{N}$ and any $\alpha \in \Omega$ with $s(\alpha) = s(n) = t(n)$.

Definition 3.8. ([2]) Let Ω_1 and Ω_2 be two R-module groupoids. An R-module groupoid morphism is a groupoid morphism $F : \Omega_1 \to \Omega_2$ such that F is a homo- morphism. Namely, F preserves the all algebraic structures. So, we have the category *RMGpd* of the R-module groupoids, such that objects are all of R-module groupoids and morphisms are R-module groupoids morphisms.

Definition 3.9. Let Ω be an R-module groupoid on Ω_0. Ω is transitive if $\Omega(x,y) \neq \emptyset$ for all $x, y \in \Omega_0$. Ω is totally intransitive if $\Omega(x,y) = \emptyset$ for all $x, y \in \Omega_0$.

For example, it can easily be shown that the inner and the identity R-module subgroupoid of an R-module groupoid Ω are totally intransitive.

4. Actions of R-module groupoids

Definition 4.1. Let Ω be an R-module groupoid over Ω_0 and N be an R-module. Let $\lambda : N \to \Omega_0$ be an R-module morphism. If there exists an R-module morphism $\varphi : \Omega_s \times_\lambda N \to N, (\alpha, m) \mapsto^\alpha m$ such that this morphism satisfies the following conditions, we say that Ω acts on n via λ and φ is the left action. This action is shown with (N, λ).

i) $\lambda(^\alpha m) = t(\alpha)$,
ii) $^\beta(^\alpha m) =^{(\beta\circledast\alpha)} m$,
iii) $^{1_{\lambda(m)}} m = m$.

Similarly, we can do it for right action of R-module. Namely; let $\lambda : N \to \Omega_0$ be an R-module morphism and $\varphi : N_s \times_\lambda \Omega \to N$ be an R-module groupoid morphism. In addition, φ is has to keep the following items.

i) $\lambda(m^\alpha) = s(\alpha)$,
ii) $(m^\alpha)^\beta = m^{(\alpha\circledast\beta)}$,
iii) $m^{1_{\lambda(m)}} = m$.

Lemma 4.2. *Every R-module groupoid object set Ω_0 acts on its R-module groupoid Ω on both sides.*

Proof. Let Ω be an R-module groupoid over Ω_0. We know that Ω_0 is an R-module. We can choose $N = \Omega_0$. So, assume that $\lambda = Id : N = \Omega_0 \to \Omega_0$ be an R-module morphism. There exists $\varphi : \Omega_s \times_\lambda \Omega_0 \to \Omega_0$, $(\alpha, x) \mapsto \varphi(\alpha, x) =^\alpha x = t(\alpha)$. Now we show that this structure provides the above conditions.

i) $Id(^\alpha x) =^\alpha x = t(\alpha)$ is apparent by our acception.

ii) $\varphi(\alpha, \varphi(\beta, x)) = \varphi(\alpha,^\beta x) = \varphi(\alpha, t(\beta)) =^\alpha t(\beta) = t(\alpha)$

and

$\varphi((\alpha \circledast \beta), x) =^{(\alpha\circledast\beta)} x = t(\alpha \circledast \beta) = t(\alpha)$,

so, $\varphi(\alpha, \varphi(\beta, x)) = \varphi((\alpha \circledast \beta), x)$

iii)$\varphi(1_{Id(x)}, x) =^{1_{Id(x)}} x = t(1_{Id(x)}) = t(1_x) = x$.

Now, we show that φ is an R-module groupoid morphism. $\varphi : \Omega_s \times_\lambda \Omega_0 \to \Omega_0$ and $r \in R$ and all $(\alpha, x), (\beta, y) \in \Omega_s \times_\lambda \Omega_0$

$$\begin{aligned}
\varphi((\alpha, x) + (\beta, y) &= \varphi((\alpha + \beta), (x + y)) \\
&=^{(\alpha+\beta)} (x + y) \\
&= t(\alpha + \beta) \\
&= t(\alpha) + t(\beta) \\
&=^\alpha x +^\beta y \\
&= \varphi((\alpha, x)) + \varphi((\beta, y))
\end{aligned}$$

and

$$\begin{aligned}
\varphi(r \cdot (\alpha, x)) &= \varphi((r \cdot \alpha, r \cdot x)) \\
&=^{(r\cdot\alpha)} (r \cdot x) \\
&= t(r \cdot \alpha) \\
&= r \cdot t(\alpha) \\
&= r \cdot (^\alpha x) \\
&= r \cdot \varphi((\alpha, x))
\end{aligned}$$

Therefore, every R-module groupoid object set acts on its R-module groupoid from the left side. Similarly, we can show that also from the right side. □

Example 4.3. Let Ω be an R-module groupoid and N be an R-module. Assume that Ω acts over N via $\lambda : N \to \Omega_0$. So, $\Omega \bowtie N$ action groupoid is obtained over object set N via this action. This action groupoid's morphisms set is $\Omega_s \times_\lambda N$. Namely, a morphism is (α, m) such that $^\alpha m = n$ for $m, n \in N$ from m to n. Source map is $s(\alpha, m) = m$, target map is $t(\alpha, m) =^\alpha m = n$. Unit map is $m \mapsto (1_{\lambda(m)}, m)$. Inverse map is $(\alpha, m)^{-1} = (\alpha^{-1},^\alpha m)$. Composition is defined by $(\beta, n) \circledast (\alpha, m) = (\beta \circledast \alpha, m)$. We have shown that $\Omega \bowtie N$ is a Ω-groupoid in [5]. Now we show that $\Omega \bowtie N$ is an R-module groupoid. $(\Omega \bowtie N)_0$ has an R-module structure, since it is defined by N. R-module operations on $\Omega \bowtie N$ are defined as follows:

$$+ : (\Omega \bowtie N) \times (\Omega \bowtie N) \to \Omega \bowtie N, ((\alpha, m), (\beta, n)) \mapsto (\alpha, m) + (\beta, n) = (\alpha + \beta, m + n)$$
$$u : \Omega \bowtie N \to \Omega \bowtie N, (\alpha, m) \mapsto (-\alpha, -m) \text{ and}$$
$$\delta : R \times (\Omega \bowtie N) \to \Omega \bowtie N, (r, (\alpha, m)) \mapsto r \odot (\alpha, m) = (r \odot \alpha, r \odot m)$$

where $+, u$ and δ are the R-module operations on Ω and N.

For $r \in R, m \in N = (\Omega \bowtie N)_0$ and $(\alpha, m), (\beta, n) \in \Omega \bowtie N$,
$\alpha(r \odot (\alpha, m)) = s(r \odot \alpha, r \odot m) = r \odot m = r \cdot s(\alpha, m)$,
$t(r \odot (\alpha, m)) = t(r \odot \alpha, r \odot m) =^{r\odot\alpha} (r \odot m) = r \cdot ({}^{\alpha}m) = r \cdot t(\alpha, m)$,
$(r \odot (\alpha, m))^{-1} = (r \odot \alpha, r \odot m)^{-1} = (r \odot \alpha^{-1}, r \odot m) = r \odot (\alpha^{-1}, m) = r \odot (\alpha, m)^{-1}$
$\varepsilon(r \cdot m) = (1_{\lambda(r \cdot m}, r \cdot m) = r \odot (1_{\lambda(m)}, m) = r \odot 1_m$
$(r \odot (\beta, n)) \circledast (r \odot (\alpha, m)) = (r \odot \beta, r \odot n) \circledast (r \odot \alpha, r \odot m) = ((r \odot \beta) \circledast (r \odot \alpha), r \odot m) = (r \odot (\beta \circledast \alpha), r \odot m) = r \odot (\beta \circ \alpha, m)$.
Hence the R-module groupoid conditions are satisfied. As a result, $\Omega \bowtie N$ is an R-module groupoid over N.

Definition 4.4. Let Ω be an R-module groupoid on Ω_0 acting on R-modules N and N'. An R-module morphism $\phi : N \to N'$ is called equivariant if and only if $\lambda_N(m) = \lambda_{N'}(\phi(m))$ and $\phi({}^{\alpha}m) =^{\alpha} \phi(m)$, for all $m \in N$ and $\alpha \in \Omega_{\lambda(m)}$.

Thus we obtain a category whose objects are all actions of the R-module groupoid Ω denoted by $RMGpdOp(\Omega)$ The objects of this category are actions (N, λ) and morphisms are as the above definition. Namely, A morphism from (N, λ_N) to $(N', \lambda_{N'})$ is the following commutative diagram.

$$\begin{tikzcd} N \arrow[rr, "\phi"] \arrow[dr, "\lambda_N"'] & & N' \arrow[dl, "\lambda_{N'}"] \\ & \Omega_0 & \end{tikzcd} \qquad \begin{tikzcd} \Omega \times N \arrow[r, "Id\times\phi"] \arrow[d, "\phi_N"'] & \Omega \times N' \arrow[d, "\phi_{N'}"] \\ N \arrow[r, "\phi"] & N' \end{tikzcd}$$

Source and target maps are defined by $s(\phi) = (M, \lambda_N)$ and $T(\phi) = (N', \lambda_{N'})$, respectively. A unit map is defined by $1_{(N,\lambda_N)} : (N, \lambda_N) \to (N, \lambda_N)$. Finally, composition operation is defined by the following commutative diagram.

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[dr, "\lambda_N"'] & N' \arrow[r, "\phi'"] \arrow[d, "\lambda_{N'}"] & N'' \arrow[dl, "\lambda_{N''}"] \\ & \Omega_0 & \end{tikzcd} \qquad \begin{tikzcd} \Omega_s \times_{\lambda_N} N \arrow[r, "\varphi"] \arrow[d, "1\times\phi"'] & N \arrow[d, "\phi"] \\ \Omega_s \times_{\lambda_{N'}} N' \arrow[r, "\varphi'"] \arrow[d, "1\times\phi'"'] & N' \arrow[d, "\phi'"] \\ \Omega_s \times_{\lambda_{N''}} N'' \arrow[r, "\varphi''"] & N'' \end{tikzcd}$$

Let Ω_1 and Ω_2 be groupoids. A morphism $p : \Omega_1 \to \Omega_2$ of groupoids is called a covering morphism if for each $x \in (\Omega_1)_0$ the restriction $(\Omega_1)_x \to (\Omega_2)_{p(x)}$ of p is bijective. Let $p : \Omega_1 \to \Omega_2$ be covering morphism of groupoids and $(\Omega_2)_s \times_{p_0} (\Omega_1)_0 = \{(\alpha, x) \in \Omega_2 \times (\Omega_2)_0 | s(\alpha) = p_0(x)\}$ be pullback. Function $s_p : (\Omega_2)_s \times_{p_0} (\Omega_1)_0 \to \Omega_1$ is inverse $(p, s) : \Omega_1 \to (\Omega_2)_s \times_{p_0} (\Omega_1)_0$. So p is a covering morphism if and only if (p, s) is bijective.

Definition 4.5. ([2]) Let Ω_1 and Ω_2 be R-module groupoids. A morphism $p : \Omega_1 \to \Omega_2$ of R-module groupoids is called a covering morphism if p is a covering morphism on the underlying groupoids.

Example 4.6. Let $p : \tilde{\Omega} \to \Omega$ be a morphism of R-module groupoids. If it is defined $S = \tilde{\Omega}_0$ and $\lambda = p_0 : \tilde{\Omega}_0 \to \Omega_0$ then Ω acts $S = \tilde{\Omega}_0$ via $\lambda = p_0$. The action is defined by $\varphi : \Omega_s \times_{p_0} \tilde{\Omega}_0 \to \tilde{\Omega}_0, (\alpha, \tilde{x}) \mapsto^{\alpha} \tilde{x} = \tilde{t}(\tilde{\alpha})$. since p is the covering morphism, there is unique $\tilde{\alpha}$ lifting of α ($s(\tilde{\alpha}) = \tilde{x}$) such that for $\tilde{x} \in S = \tilde{\Omega}_0$ and $\alpha \in \Omega_{p_0(\tilde{x})}$, $p(\tilde{\alpha}) = \alpha$ and $p_0(\tilde{x}) = x$. Now we show that action conditions are satisfied. $\lambda({}^{\alpha}\tilde{x}) = p_0({}^{\alpha}\tilde{x}) = p_0(\tilde{t}(\tilde{\alpha})) = t(\alpha)$. ${}^{\beta}({}^{\alpha}\tilde{x}) =^{\beta} \tilde{t}(\tilde{\alpha}) = \tilde{t}(\tilde{\beta})$ and ${}^{\beta\circledast\alpha}\tilde{x} = \tilde{t}(\tilde{\beta} \circledast \tilde{\alpha}) = \tilde{t}(\tilde{\beta})$. So we have that ${}^{\beta}({}^{\alpha}\tilde{x}) =^{\beta\circledast\alpha} \tilde{x}$. Finally, ${}^{1_{p_0(\tilde{x})}}\tilde{x} = \tilde{t}(\tilde{\gamma}) = \tilde{x}$. So, the action conditions are satisfied. Since p is the covering morphism of the R-module groupoid and λ is defined by p_0, it is an R-module morphism. Similarly, it is an R-module morphism since φ is defined by $\tilde{t}$ target morphism of $\tilde{\Omega}$.

So, we have a category of coverings of R-module groupoid Ω denoted by $RMGpdCov(\Omega)$. The objects of this category are covering morphisms of Ω. A morphism from $p : \Omega' \to \Omega$ to $q : \Omega'' \to \Omega$ is defined by the following commutative diagram.

$$\begin{tikzcd} \Omega' \arrow[rr, "r"] \arrow[dr, "p"'] & & \Omega'' \arrow[dl, "q"] \\ & \Omega & \end{tikzcd}$$

Structure maps of the category are defined by $s(r) = p$, $t(r) = q$ and $1_{(p)} : \Omega' \to \Omega'$. For $r : \Omega' \to \Omega''$ and $r' : \Omega'' \to \Omega'''$, the composition map is defined by the following commutative diagram

$$\begin{tikzcd} \Omega' \arrow[r, "r"] \arrow[dr, "q"'] & \Omega'' \arrow[r, "r'"] \arrow[d, "p"] & \Omega''' \arrow[dl, "q'"] \\ & \Omega & \end{tikzcd}$$

Theorem 4.7. *Let Ω be an R-module groupoid. $RMGpdCov(\Omega)$ category of coverings of R-module groupoid and $RMGpdOp(\Omega)$ category of actions of R-module groupoid are equivalent.*

Proof. A functor $\Gamma : RMGpdOp(\Omega) \to RMGpdCov(\Omega)$ is defined as follows: Suppose that Ω acts on R-module N via $\lambda : N \to \Omega_0$. This action is given with $\varphi : \Omega_s \times_\lambda N \to N$, $(\alpha, m) \mapsto \varphi(\alpha, m) =^\alpha m$. In this case, we have the action R-module groupoid $\Omega \bowtie N$ from Example 4.3. If a morphism $p : \Omega \bowtie N \to \Omega$ is defined by $(\alpha, m) \mapsto \alpha$ and λ on the morphisms and the objects, respectively, then p is a covering morphism of R-module groupoids. We have that $p((\beta, n) \circledast (\alpha, m)) = p(\beta \circledast \alpha, m) = \beta \circledast \alpha = p(\beta, n) \circledast p(\alpha, m)$, from the definition of p. Since p is defined by λ on objects, we have that $p(1_{\lambda(m)}, m) = 1_{\lambda(m)} = 1_{p(1_{\lambda(m)}, m)}$. So, p is a groupoid morphism. At the same time it is a covering morphism, since (α, m') is only one element of Ω with $s(\alpha, m') = m'$ and $p(\alpha, m) = \alpha$, for $\alpha \in \Omega(x, y)$ and $m' \in \lambda^{-1}(x)$. Thus (p, s) is bijective. Furthermore we have that $p((\alpha, m)+(\beta, n)) = p(\alpha+\beta, m+n) = \alpha+\beta = p(\alpha, m)+p(\beta, n)$ and $p(r \odot (\alpha, m)) = p(r \odot \alpha, r \odot m) = r \odot \alpha = r \odot p(\alpha, m)$. Therefore p also $\Gamma(N, \lambda)$ is an R-module groupoid morphism. If (N, λ) and (N', λ') are actions of R-module grupoid Ω, then $\Gamma(N, \lambda)$ and $\Gamma(N', \lambda')$ are coverings of R-module groupoid Ω. Let these coverings be $p : \Omega \bowtie N \to \Omega$ and $q : \Omega \bowtie N' \to \Omega$. If $\phi : N \to N'$ is a morphism of actions, then $\Gamma(\phi) = r$ is a morphism of covering morphisms with $r_0 = \phi$ and $r = 1 \times \phi$. This situation is given in the following commutative diagram

$$\begin{tikzcd} \Omega \bowtie N \arrow[rr, "r=1\times\phi"] \arrow[dr, "p"'] & & \Omega \bowtie N' \arrow[dl, "q"] \\ & \Omega & \end{tikzcd} \qquad \begin{tikzcd} (\alpha, m) \arrow[rr, "r=1\times\phi"] \arrow[dr, "p"'] & & (\alpha, \phi(m)) \arrow[dl, "q"] \\ & \alpha & \end{tikzcd}$$

In addition, if $\phi : N \to N'$ and $\phi' : N' \to N''$ are morphisms of R-module groupoid actions then we have that $\Gamma(\phi' \circledast \phi) = \Gamma(\phi') \circledast \Gamma(\phi)$. For $\Gamma(N, \lambda) = \Omega \bowtie N$, $\Gamma(N', \lambda') = \Omega \bowtie N'$, $\Gamma(N'', \lambda'') = \Omega \bowtie N''$, $\Gamma(\phi) = r$ and $\Gamma(\phi') = r'$, we have that $\phi' \circledast \phi : N \to N''$ and $\Gamma(\phi' \circledast \phi) = r' \circledast r = \Gamma(\phi') \circledast \Gamma(\phi)$. This situation is given in the following commutative diagram.

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[dr, "\lambda"'] & N' \arrow[r, "\phi'"] \arrow[d, "\lambda'"] & N'' \arrow[dl, "\lambda''"] \\ & \Omega_0 & \end{tikzcd} \qquad \begin{tikzcd} \Omega \bowtie N \arrow[r, "r"] \arrow[dr, "p"'] & \Omega \bowtie N' \arrow[r, "r'"] \arrow[d, "q"] & \Omega \bowtie N'' \arrow[dl, "q'"] \\ & \Omega & \end{tikzcd}$$

Therefore Γ is a functor.

Define a functor $\Phi : RMGpdCov(\Omega) \to RMGpdOp(\Omega)$ as follows:

Let $p : \tilde{\Omega} \to \Omega$ be a covering morphism of R-module groupoids. If we suppose that $N = \tilde{\Omega}_0$ and $\lambda = p_0 : \tilde{\Omega}_0 \to \Omega_0$ then (N, λ) is an action of R-module groupoid over N from Example 4.6. Namely $\Phi(p)$ is an action of R-module groupoid Ω over N. Thus if $p : \tilde{\Omega} \to \Omega$ and $q : \Omega' \to \Omega$ are covering morphisms of R-module groupoids then $\Phi(p)$ and $\Phi(q)$ are action of R-module groupoid Ω over $\tilde{\Omega}_0$ and Ω'_0 R-modules via p_0 and q_0, respectively. Let these actions be denoted by $(\tilde{\Omega}_0, p_0)$ and (Ω'_0, q_0). We known that if p and q are covering morphisms of R-module groupoids then $r : \tilde{\Omega} \to \Omega'$ is the covering morphisms of R-module groupoids with $p = q \circledast r$. Therefore $r_0 = \phi$ and $\Phi(r) = \phi$ are morphisms of the R-module actions. This situation is given in the following commutative diagram.

$$\begin{tikzcd} \tilde{\Omega}_0 \arrow[rr, "r_0=\phi"] \arrow[dr, "p_0"'] & & \Omega'_0 \arrow[dl, "q_0"] \\ & \Omega_0 & \end{tikzcd}$$

we see that the action is protected from the following commutative diagram since we have that $p = q \circledast r$ and $p_0 = q_0 \circledast r_0$.

$$\begin{tikzcd} \Omega_s \times_{p_0} \tilde{\Omega}_0 \arrow[r, "\varphi"] \arrow[d, "1\times r_0"'] & \tilde{\Omega}_0 \arrow[d, "\phi=r_0"] \\ \Omega_s \times_{q_0} H_0 \arrow[r, "\varphi'"'] & \Omega'_0 \end{tikzcd} \qquad \begin{tikzcd} (\alpha, \tilde{x}) \arrow[r, "\varphi"] \arrow[d, "1\times r_0"'] & {}^{\alpha}\tilde{x} \arrow[d, "\phi=r_0"] \\ (\alpha, r_0(\tilde{x}) = \phi(\tilde{x})) \arrow[r, "\varphi'"'] & \phi({}^{\alpha}\tilde{x}) =^{\alpha} \phi(\tilde{x}) \end{tikzcd}$$

Let $r : \tilde{\Omega} \to \Omega'$ be a morphism from $p : \tilde{\Omega} \to \Omega$ to $q : \Omega' \to \Omega$ and let $r' : \Omega' \to \Omega''$ be a morphism from $q : \Omega' \to \Omega$ to $p' : \Omega'' \to \Omega$. Then we have $\Phi(r' \circledast r) = \Phi(r') \circledast \Phi(r)$. For $\Phi(p) = (\tilde{\Omega}_0, p_0)$, $\Phi(q) = (\Omega'_0, q_0)$, $\Phi(p') = (\Omega''_0, p'_0)$, $\Phi(r) = \phi$ and $\Phi(r') = \phi'$,

$$r' \circledast r : \tilde{\Omega} \to \Omega''$$

a covering morphism of R-module groupoids. So we find $\Phi(r' \circledast r) = \phi' \circledast \phi = \Phi(r') \circledast \Phi(r)$. This is seen from the following commutative diagram.

$$\begin{tikzcd} \tilde{\Omega} \arrow[r, "r"] \arrow[dr, "p"'] & \Omega' \arrow[r, "r'"] \arrow[d, "q"] & \Omega'' \arrow[dl, "p'"] \\ & \Omega & \end{tikzcd} \qquad \begin{tikzcd} \tilde{\Omega}_0 \arrow[r, "\phi"] \arrow[dr, "p_0"'] & \Omega'_0 \arrow[r, "\phi'"] \arrow[d, "q_0"] & \Omega''_0 \arrow[dl, "p'_0"] \\ & \Omega_0 & \end{tikzcd}$$

Therefore Φ is a functor.

It is obvious that $\Gamma\Phi \cong 1_{RMGpdCov(\Omega)}$ and $\Phi\Gamma \cong 1_{RMGpdOp(\Omega)}$. $\square$

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