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Nonlinear bi-skew Jordan-type derivations on $*$-algebras

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Abstract. Let $\mathcal{A}$ be a unital $*$-algebra. In this paper, under some mild conditions on $\mathcal{A}$, it is shown that Φ is a nonlinear bi-skew Jordan-type derivations on $\mathcal{A}$ if and only if Φ is an additive $*$−derivation. As applications, the nonlinear bi-skew Jordan-type derivations on prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras and standard operator algebras are characterized.

1. Introduction

Let $\mathcal{A}$ be an algebra. Recall that a linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$ and a Lie derivation if $\delta([A, B]) = [\delta(A), B]+[A, \delta(B)]$ for all $A, B \in \mathcal{A}$, where $[A, B] = AB-BA$ is the usual Lie product of A and B. The question of characterizing Lie derivations and revealing the relationship between Lie derivations and derivations has attracted the attentions of many researchers, see for example [2, 3, 21–23, 25, 27, 28]. Furthermore, we say that a map (without the additivity or linearity assumption) $\delta : \mathcal{A} \to \mathcal{A}$ is a nonlinear Lie derivation or a Lie derivable map if $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for all $A, B \in \mathcal{A}$. Undoubtedly, it is interesting to study nonlinear Lie derivations. Recently, many mathematicians devoted themselves to study the characterizations of nonlinear Lie derivations, see for example [1, 8–14, 16, 19, 20, 24, 33].

Recently, many authors have studied derivations related to some new products, such as the nonlinear skew Lie derivations (see [7, 15, 17, 34]), the nonlinear Jordan $*$-derivations (see [15, 30, 35–39]), the noninear bi-skew Lie derivations (see [26, 32]) and so on. Let $\mathcal{A}$ be a $*$-algebra. For $A, B \in \mathcal{A}$, define the bi-skew Jordan product of A and B by $A \circ B = A^*B + B^*A$. It is clear that the bi-skew Jordan product is different from the Jordan product $AB + BA$, the Lie product $AB - BA$, the skew Lie product $AB - BA^*$, the Jordan $*$-product $AB + BA^*$ and the bi-skew Lie product $A^*B - B^*A$. Quite recently, the bi-skew Jordan products have attracted many scholars to study. C. Li et al. [18] proved that every bijective map preserving bi-skew Jordan product between von Neumann algebras with no central abelian projections is just a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. A. Taghavi and S. Gholampoor [31] studied surjective maps preserving bi-skew Jordan product between C^*-algebras. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear bi-skew Jordan derivation if

$$\Phi(A \circ B) = \Phi(A) \circ B + A \circ \Phi(B)$$

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for all $A, B \in \mathcal{A}$. V. Darvish et al. [5] proved any nonlinear bi-skew Jordan derivation on prime $*$-algebra is an additive $*$-derivation. Similarly, a map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a bi-skew Jordan triple derivation if

$$\Phi(A \circ B \circ C) = \Phi(A) \circ B \circ C + A \circ \Phi(B) \circ C + A \circ B \circ \Phi(C)$$

for all $A, B, C \in \mathcal{A}$, where $A \circ B \circ C := (A \circ B) \circ C$. In [6], V. Darvish et al. proved any nonlinear bi-skew Jordan triple derivation on prime $*$-algebra is an additive $*$-derivation.

Given the consideration of nonlinear bi-skew Jordan derivations and nonlinear bi-skew Jordan triple derivations, we can further develop them in one natural way. Suppose that $n \geq 2$ is a fixed positive integer. Accordingly, a nonlinear bi-skew Jordan-type derivation is a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfying the condition

$$\Phi(A_1 \circ A_2 \circ \cdots \circ A_n) = \sum_{k=1}^{n} A_1 \circ \cdots \circ A_{k-1} \circ \Phi(A_k) \circ A_{k+1} \circ \cdots \circ A_n$$

for all $A_1, A_2, \cdots, A_n \in \mathcal{A}$, where $A_1 \circ A_2 \circ \cdots \circ A_n = (\cdots((A_1 \circ A_2) \circ A_3) \cdots \circ A_n)$. By the definition, it is clear that every bi-skew Jordan derivation is a bi-skew Jordan-2 derivation and every bi-skew Jordan triple derivation is a bi-skew Jordan-3 derivation. It is obvious that every nonlinear bi-skew Jordan derivation on any $*$-algebra is a bi-skew Jordan-n derivation. But we do not know whether the converse is true.

Motivated by the above mentioned works, we will concentrate on giving a description of nonlinear bi-skew Jordan-type derivations on $*$-algebras. In this paper, our main results not only improve the results of the previous articles [5, 6], but also, most importantly, the methods used in our article are different from theirs.

2. The main result and its proof

Our main theorem in this paper is as follows.

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with the unit I. Assume that $\mathcal{A}$ contains a nontrivial projection P which satisfies*

$$(\spadesuit) \quad X\mathcal{A}P = 0 \quad \text{implies} \quad X = 0$$

and

$$(\clubsuit) \quad X\mathcal{A}(I-P) = 0 \quad \text{implies} \quad X = 0.$$

Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\Phi(A_1 \circ A_2 \circ \cdots \circ A_n) = \sum_{k=1}^{n} A_1 \circ \cdots \circ A_{k-1} \circ \Phi(A_k) \circ A_{k+1} \circ \cdots \circ A_n$$

for all $A_1, A_2, \cdots A_n \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.

In the following, let $P_1 = P$ and $P_2 = I - P$. Denote $\mathcal{A}^a = \{A \in \mathcal{A} : A = A^*\}$, $\mathcal{A}_{11} = P_1\mathcal{A}^aP_1$, $\mathcal{A}_{12} = \{P_1AP_2 + P_2AP_1 : A \in \mathcal{A}^a\}$ and $\mathcal{A}_{22} = P_2\mathcal{A}^aP_2$. For every $A \in \mathcal{A}^a$, we may write $A = A_{11} + A_{12} + A_{22}$, where $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}$ and $A_{22} \in \mathcal{A}_{22}$. Clearly, we only need to prove the necessity. We will complete the proof by several lemmas.

Lemma 2.2. $\Phi(0) = 0$.

Proof. Indeed, we have

$$\begin{aligned}
\Phi(0) &= \Phi(0 \circ 0 \circ I \circ \cdots \circ I) \\
&= \Phi(0) \circ 0 \circ I \circ \cdots \circ I + \cdots + 0 \circ 0 \circ I \circ \cdots \circ I \circ \Phi(I) \\
&= 0.
\end{aligned}$$

□

Lemma 2.3. *For any $A \in \mathcal{A}^a$, we have $\Phi(A) \in \mathcal{A}^a$.*

Proof. For any $A \in \mathcal{A}^a$, $A = A \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}$. Since $B \circ C \in \mathcal{A}^a$ for any $B, C \in \mathcal{A}$, we obtain

$$\begin{aligned}\Phi(A) &= \Phi(A \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2})\\ &= \Phi(A) \circ \frac{I}{2} \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} + \cdots + A \circ \frac{I}{2} \circ \cdots \circ \Phi(\frac{I}{2}) \in \mathcal{A}^a.\end{aligned}$$

□

Lemma 2.4. *For any $A_{11} \in \mathcal{A}_{11}, A_{22} \in \mathcal{A}_{22}$ and $B_{12} \in \mathcal{A}_{12}$, we have*

$$\Phi(A_{11} + B_{12}) = \Phi(A_{11}) + \Phi(B_{12})$$

and

$$\Phi(A_{22} + B_{12}) = \Phi(A_{22}) + \Phi(B_{12}).$$

Proof. Let $T = \Phi(A_{11} + B_{12}) - \Phi(A_{11}) - \Phi(B_{12})$. By Lemma 2.3, we have $T^* = T$. So we only need to prove $T = T_{11} + T_{12} + T_{22} = 0$. Since $P_2 \circ A_{11} = 0$, we obtain

$$\begin{aligned}&\Phi(P_2) \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ I + P_2 \circ \Phi(A_{11} + B_{12}) \circ I \circ \cdots \circ I + \cdots\\ &+ P_2 \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ \Phi(I)\\ &= \Phi(P_2 \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ I)\\ &= \Phi(P_2 \circ A_{11} \circ I \circ \cdots \circ I) + \Phi(P_2 \circ B_{12} \circ I \circ \cdots \circ I)\\ &= \Phi(P_2) \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ I + P_2 \circ (\Phi(A_{11}) + \Phi(B_{12})) \circ I \circ \cdots \circ I + \cdots\\ &+ P_2 \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ \Phi(I).\end{aligned}$$

Hence $P_2 \circ T \circ I \circ \cdots \circ I = 0$, and then it yields that $T_{12} = T_{22} = 0$.

It follows from $(P_1 - P_2) \circ B_{12} = 0$ that

$$\begin{aligned}&\Phi(P_1 - P_2) \circ (A_{11} + B_{12}) + (P_1 - P_2) \circ \Phi(A_{11} + B_{12}) + \cdots\\ &+ (P_1 - P_2) \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ \Phi(I)\\ &= \Phi((P_1 - P_2) \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ I))\\ &= \Phi((P_1 - P_2) \circ A_{11} \circ I \circ \cdots \circ I) + \Phi((P_1 - P_2) \circ B_{12} \circ I \circ \cdots \circ I)\\ &= \Phi(P_1 - P_2) \circ (A_{11} + B_{12}) + (P_1 - P_2) \circ (\Phi(A_{11}) + \Phi(B_{12})) + \cdots\\ &+ (P_1 - P_2) \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ \Phi(I),\end{aligned}$$

which implies that $(P_1 - P_2) \circ T \circ I \circ \cdots \circ I = 0$. So $T_{11} = 0$, and then $T = 0$. □

Lemma 2.5. *For any $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}$ and $C_{22} \in \mathcal{A}_{22}$, we have*

$$\Phi(A_{11} + B_{12} + C_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{22}).$$

Proof. Let

$$T = \Phi(A_{11} + B_{12} + C_{22}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{22}).$$

By Lemma 2.3, we have $T^* = T$. Since $P_1 \circ C_{22} = 0$, it follows from Lemma 2.4 that

$$\begin{aligned}&\Phi(P_1) \circ (A_{11} + B_{12} + C_{22}) \circ I \circ \cdots \circ I + C_{22}) + P_1 \circ \Phi(A_{11} + B_{12} + C_{22}) \circ I \circ \cdots \circ I\\ &+ \cdots + P_1 \circ (A_{11} + B_{12} + C_{22}) \circ I \circ \cdots \circ \Phi(I)\\ &= \Phi(P_1 \circ (A_{11} + B_{12} + C_{22}) \circ I \circ \cdots \circ I)\\ &= \Phi(P_1 \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ I) + \Phi(P_1 \circ C_{22} \circ I \circ \cdots \circ I)\\ &= \Phi(P_1 \circ A_{11} \circ I \circ \cdots \circ I) + \Phi(P_1 \circ B_{12} \circ I \circ \cdots \circ I) + \Phi(P_1 \circ C_{22} \circ I \circ \cdots \circ I)\\ &= \Phi(P_1) \circ (A_{11} + B_{12} + C_{22}) \circ I \circ \cdots \circ I + P_1 \circ (\Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{22})) \circ I \circ \cdots \circ I\\ &+ \cdots + P_1 \circ (A_{11} + B_{12} + C_{22}) \circ I \circ \cdots \circ \Phi(I).\end{aligned}$$

Hence $P_1 \circ T \circ I \circ \cdots \circ I = 0$, and then it yields that $T_{11} = T_{12} = 0$. Similarly, we can get that $T_{22} = 0$. Thus $T = 0$. □

Lemma 2.6. *For any $A_{12}, B_{12} \in \mathcal{A}_{12}$, we have*

$$\Phi(A_{12} + B_{12}) = \Phi(A_{12}) + \Phi(B_{12}).$$

Proof. Let $A_{12}, B_{12} \in \mathcal{A}_{12}$. Then $A_{12} = P_1AP_2 + P_2AP_1, B_{12} = P_1BP_2 + P_2BP_1$, where $A, B \in \mathcal{A}^a$. Since

$$(P_1 + A_{12}) \circ (P_2 + B_{12}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} = A_{12} + B_{12} + A_{12}B_{12} + B_{12}A_{12},$$

where

$$A_{12} + B_{12} \in \mathcal{A}_{12}$$

and

$$A_{12}B_{12} + B_{12}A_{12} = P_1(AP_2B + BP_2A)P_1 + P_2(AP_1B + BP_1A)P_2 \in \mathcal{A}_{11} + \mathcal{A}_{22},$$

by Lemma 2.4, we have

$$\begin{aligned}
&\Phi(A_{12} + B_{12}) + \Phi(A_{12}B_{12} + B_{12}A_{12}) \\
&= \Phi(A_{12} + B_{12} + A_{12}B_{12} + B_{12}A_{12}) \\
&= \Phi((P_1 + A_{12}) \circ (P_2 + B_{12}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}) \\
&= \Phi(P_1 + A_{12}) \circ (P_2 + B_{12}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} + (P_1 + A_{12}) \circ \Phi(P_2 + B_{12}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} \\
&+ \cdots + (P_1 + A_{12}) \circ (P_2 + B_{12}) \circ \frac{I}{2} \circ \cdots \circ \Phi(\frac{I}{2}) \\
&= (\Phi(P_1) + \Phi(A_{12})) \circ (P_2 + B_{12}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} + (P_1 + A_{12}) \circ (\Phi(P_2) + \Phi(B_{12})) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} \\
&+ \cdots + (P_1 + A_{12}) \circ (P_2 + B_{12}) \circ \frac{I}{2} \circ \cdots \circ \Phi(\frac{I}{2}) \\
&= \Phi(P_1 \circ P_2 \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}) + \Phi(P_1 \circ B_{12} \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}) + \Phi(A_{12} \circ P_2 \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}) \\
&+ \Phi(A_{12} \circ B_{12} \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}) \\
&= \Phi(A_{12}) + \Phi(B_{12}) + \Phi(A_{12}B_{12} + B_{12}A_{12}),
\end{aligned}$$

which implies that

$$\Phi(A_{12} + B_{12}) = \Phi(A_{12}) + \Phi(B_{12}).$$

□

Lemma 2.7. *For any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}, i = 1, 2$, we have*

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Proof. Let $T = \Phi(A_{11} + B_{11}) - \Phi(A_{11}) - \Phi(B_{11})$. Since $P_2 \circ A_{11} = P_2 \circ B_{11} = 0$, we have

$$\begin{aligned}
&\Phi(P_2) \circ (A_{11} + B_{11}) \circ I \circ \cdots \circ I + P_2 \circ \Phi(A_{11} + B_{11}) \circ I \circ \cdots \circ I + \\
&+ \cdots + P_2 \circ (A_{11} + B_{11}) \circ I \circ \cdots \circ \Phi(I) \\
&= \Phi(P_2 \circ (A_{11} + B_{11}) \circ I \circ \cdots \circ I) \\
&= \Phi(P_2 \circ A_{11} \circ I \circ \cdots \circ I) + \Phi(P_2 \circ B_{11} \circ I \circ \cdots \circ I) \\
&= \Phi(P_2) \circ (A_{11} + B_{12}) \circ I \circ \cdots \circ I + P_2 \circ (\Phi(A_{11}) + \Phi(B_{11})) \circ I \circ \cdots \circ I \\
&+ \cdots + P_2 \circ (A_{11} + B_{11}) \circ I \circ \cdots \circ \Phi(I),
\end{aligned}$$

which implies that $P_2 \circ T \circ I \circ \cdots \circ I = 0$, and then $T_{22} = T_{12} = 0$.

For any $D \in \mathcal{A}$, let $C_{12} = P_1DP_2 + (P_1DP_2)^*$. Then

$$C_{12}, A_{11} \circ C_{12}, B_{11} \circ C_{12} \in \mathcal{A}_{12}.$$

It follows from Lemma 2.6 that

$$\begin{aligned}
&\Phi(A_{11} + B_{11}) \circ C_{12} \circ I \circ \cdots \circ I + (A_{11} + B_{11}) \circ \Phi(C_{12}) \circ I \circ \cdots \circ I \\
&+ \cdots + (A_{11} + B_{11}) \circ C_{12} \circ I \circ \cdots \circ \Phi(I) \\
&= \Phi((A_{11} + B_{11}) \circ C_{12} \circ I \circ \cdots \circ I) \\
&= \Phi(A_{11} \circ C_{12} \circ I \circ \cdots \circ I) + \Phi(B_{11} \circ C_{12} \circ I \circ \cdots \circ I) \\
&= (\Phi(A_{11}) + \Phi(B_{11})) \circ C_{12} \circ I \circ \cdots \circ I + (A_{11} + B_{11}) \circ \Phi(C_{12}) \circ I \circ \cdots \circ I \\
&+ \cdots + (A_{11} + B_{11}) \circ C_{12} \circ I \circ \cdots \circ \Phi(I),
\end{aligned}$$

which implies that

$$T \circ C_{12} \circ I \circ \cdots \circ I = T_{11} \circ C_{12} \circ I \circ \cdots \circ I = 0,$$

that is $T_{11}P_1DP_2 + (P_1DP_2)^*T_{11} = 0$. Multiplying the above equation by P_2 from the right, we have $T_{11}P_1DP_2 = 0$ for any $D \in \mathcal{A}$. It follows from (♣) that $T_{11} = 0$, and so $T = 0$. Similarly, we can prove that $\Phi(A_{22} + B_{22}) = \Phi(A_{22}) + \Phi(B_{22})$. □

Remark 2.8. *It follows from Lemmas 2.5-2.7 that Φ is additive on $\mathcal{A}^a$.*

Lemma 2.9. *(1) $\Phi(I) = \Phi(iI) = 0$;*

(2) For every $M^* = -M$, we have $\Phi(M)^* = -\Phi(M)$ and $\Phi(iM) = i\Phi(M)$.

Proof. It follows from Lemma 2.3 and Remark 2.8 that

$$\begin{aligned}
2^{n-1}\Phi(I) = \Phi(2^{n-1}I) &= \Phi(I \circ I \circ \cdots \circ I) \\
&= \Phi(I) \circ I \circ \cdots \circ I + \cdots + I \circ \cdots \circ I \circ \Phi(I) \\
&= 2^{n-1}n\Phi(I),
\end{aligned}$$

which implies $\Phi(I) = 0$.

For any $M^* = -M$, we have

$$\begin{aligned}
0 &= \Phi(M \circ I \circ \cdots \circ I) \\
&= \Phi(M) \circ I \circ \cdots \circ I \\
&= 2^{n-2}(\Phi(M) + \Phi(M)^*).
\end{aligned}$$

So $\Phi(M)^* = -\Phi(M)$.

Now, we can obtain that

$$\begin{aligned}
0 &= 2^{n-1}\Phi(I) = \Phi(2^{n-1}I) \\
&= \Phi((iI) \circ (iI) \circ \cdots \circ I) \\
&= \Phi(iI) \circ (iI) \circ \cdots \circ I + (iI) \circ \Phi(iI) \circ \cdots I \\
&= -2^n i\Phi(iI),
\end{aligned}$$

that is $\Phi(iI) = 0$.

For any $M^* = -M$, we have

$$\begin{aligned}
-2^{n-1}\Phi(iM) = \Phi(-2^{n-1}iM) &= \Phi((iI) \circ M \circ I \circ \cdots \circ I) \\
&= (iI) \circ \Phi(M) \circ I \circ \cdots \circ I \\
&= -2^{n-1}i\Phi(M).
\end{aligned}$$

Hence $\Phi(iM) = i\Phi(M)$. □

Lemma 2.10. *For any $A_1^* = -A_1, A_2^* = -A_2$, we have*

$$\Phi(A_1 + A_2) = \Phi(A_1) + \Phi(A_2)$$

and

$$\Phi(A_1 + iA_2) = \Phi(A_1) + i\Phi(A_2).$$

Proof. Let $A_1^* = -A_1, A_2^* = -A_2$. It follows from Remark 2.8 and Lemma 2.9 that

$$\begin{aligned} i\Phi(A_1 + A_2) &= \Phi(i(A_1 + A_2)) \\ &= \Phi(iA_1) + \Phi(iA_2) = i(\Phi(A_1) + \Phi(A_2)), \end{aligned}$$

which implies that $\Phi(A_1 + A_2) = \Phi(A_1) + \Phi(A_2)$.

Now we can obtain that

$$\begin{aligned} 2^{n-1} i\Phi(A_2) &= \Phi((A_1 + iA_2) \circ I \circ \cdots \circ I) \\ &= \Phi(A_1 + iA_2) \circ I \circ \cdots \circ I \\ &= 2^{n-2}(\Phi(A_1 + iA_2)^* + \Phi(A_1 + iA_2)) \end{aligned}$$

and

$$\begin{aligned} -2^{n-1} i\Phi(A_1) &= \Phi((A_1 + iA_2) \circ (iI) \circ I \circ \cdots \circ I) \\ &= \Phi(A_1 + iA_2) \circ (iI) \circ I \circ \cdots \circ I \\ &= 2^{n-2} i(\Phi(A_1 + iA_2)^* - \Phi(A_1 + iA_2)). \end{aligned}$$

Comparing the above two equations, we get that $\Phi(A_1 + iA_2) = \Phi(A_1) + i\Phi(A_2)$. □

Lemma 2.11. *(1) For any $A \in \mathcal{A}$, we have*

$$\Phi(iA) = i\Phi(A)$$

and

$$\Phi(A^*) = \Phi(A)^*.$$

(2) Φ is additive on $\mathcal{A}$.

Proof. (1) For any $A \in \mathcal{A}$, we have $A = A_1 + iA_2$, where $A_1^* = -A_1, A_2^* = -A_2$. It follows from Lemmas 2.9 and 2.10 that

$$\begin{aligned} \Phi(iA) &= \Phi(iA_1 - A_2) = i\Phi(A_1) - \Phi(A_2) \\ &= i(\Phi(A_1) + i\Phi(A_2)) = i\Phi(A_1 + iA_2) \\ &= i\Phi(A) \end{aligned}$$

and

$$\begin{aligned} \Phi(A^*) &= \Phi(-A_1 + iA_2) = -\Phi(A_1) + i\Phi(A_2) \\ &= (\Phi(A_1) + i\Phi(A_2))^* = (\Phi(A_1 + iA_2))^* \\ &= \Phi(A)^*. \end{aligned}$$

(2) For any $A, B \in \mathcal{A}$, we have $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where $A_1^* = -A_1, A_2^* = -A_2, B_1^* = -B_1, B_2^* = -B_2$. It follows from Lemma 2.10 that

$$\begin{aligned} \Phi(A + B) &= \Phi((A_1 + B_1) + i(A_2 + B_2)) \\ &= \Phi(A_1 + B_1) + i\Phi(A_2 + B_2) \\ &= \Phi(A_1) + i\Phi(A_2) + \Phi(B_1) + i\Phi(B_2) \\ &= \Phi(A) + \Phi(B). \end{aligned}$$

□

Lemma 2.12. *Φ is an additive $*$-derivation.*

Proof. For any $A, B \in \mathcal{A}$, by Lemma 2.11, on the one hand, we have

$$\begin{aligned} 2^{n-2} i\Phi(A^*B - B^*A) &= \Phi(2^{n-2} i(A^*B - B^*A)) \\ &= \Phi(A \circ (iB) \circ I \circ \cdots \circ I) \\ &= \Phi(A) \circ (iB) \circ I \circ \cdots \circ I + A \circ \Phi(iB) \circ I \circ \cdots \circ I \\ &= 2^{n-2} i(\Phi(A)^*B - B^*\Phi(A) + A^*\Phi(B) - \Phi(B)^*A), \end{aligned}$$

which impies that

$$\Phi(A^*B - B^*A) = \Phi(A)^*B - B^*\Phi(A) + A^*\Phi(B) - \Phi(B)^*A.$$

On the other hand, we also have

$$\begin{aligned} 2^{n-2}(\Phi(A^*B + B^*A)) &= \Phi(A \circ B \circ I \circ \cdots \circ I) \\ &= \Phi(A) \circ B \circ I \circ \cdots \circ I + A \circ \Phi(B) \circ I \circ \cdots \circ I \\ &= 2^{n-2}(\Phi(A)^*B + B^*\Phi(A) + A^*\Phi(B) + \Phi(B)^*A), \end{aligned}$$

which impies that

$$\Phi(A^*B + B^*A) = \Phi(A)^*B + B^*\Phi(A) + A^*\Phi(B) + \Phi(B)^*A.$$

By summing the above equation, we have

$$\Phi(A^*B) = \Phi(A)^*B + A^*\Phi(B).$$

It follows from Lemma 2.11 (1) that

$$\Phi(AB) = \Phi(A)B + A\Phi(B).$$

□

3. Corollaries

An algebra $\mathcal{A}$ is called prime if $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. Observing that prime $*$-algebras satisfy (♠) and (♣), we have the following corollary.

Corollary 3.1. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and P be a nontrivial projection in $\mathcal{A}$. Then Φ is a nonlinear bi-skew Jordan type derivation on $\mathcal{A}$ if and only if Φ is an additive $*-$derivation.*

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H, and $\mathcal{A} \subseteq B(H)$ be a von Neumann algebra. $\mathcal{A}$ is a factor if its center is $\mathbb{C}I$. It is well known that a factor von Neumann algebra is prime and then we have the following corollary.

Corollary 3.2. *Let $\mathcal{A}$ be a factor von Neumann algebra with $dim\mathcal{A} \geq 2$. Then $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear bi-skew Jordan type derivation if and only if Φ is an additive $*-$derivation.*

We denote the subalgebra of all bounded finite rank operators by $\mathcal{F}(H) \subseteq B(H)$. We call a subalgebra $\mathcal{A}$ of $B(H)$ a standard operator algebra if it contains $\mathcal{F}(H)$. Now we have the following corollary.

Corollary 3.3. *Let H be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on H containing the identity operator I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. Then $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear bi-skew Jordan type derivation if and only if Φ is a linear $*$-derivation. Moreover, there exists an operator $T \in B(H)$ satisfying $T + T^* = 0$ such that $\Phi(A) = AT - TA$ for all $A \in A$, i.e., Φ is inner.*

Proof. Since $\mathcal{A}$ is prime, we know that Φ is an additive $*$-derivation. It follows from [29] that Φ is a linear inner derivation, i.e., there exists an operator $S \in B(\mathcal{H})$ such that $\Phi(A) = AS - SA$. Using the fact $\Phi(A^*) = \Phi(A)^*$, we have

$$A^*S - SA^* = \Phi(A^*) = \Phi(A)^* = -A^*S^* + S^*A^*$$

for all $A \in A$. This leads to $A^*(S + S^*) = (S + S^*)A^*$. Hence, $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Let us set $T = S - \frac{1}{2}\lambda I$. One can check that $T + T^* = 0$ such that $\Phi(A) = AT - TA$. □

It is shown in [4] and [15] that if a von Neumann algebra $\mathcal{A}$ has no central summands of type I_1, then $\mathcal{A}$ satifies (♠) and (♣). Now we have the following corollary.

Corollary 3.4. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1. Then $\Phi : \mathcal{A} \to \mathcal{A}$ is a nnonlinear bi-skew Jordan type derivation if and only if Φ is an additive $*-$derivation.*

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