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Well-posedness and energy decay for full von Kármán system of dynamic theromelasticity damped with distributed delay and past history terms

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Abstract. In this paper, we study the well-posedness and the asymptotic behavior of a one-dimensional von Kármán beam is coupled to a thermal effect and frictional damping with distributed delay and past history. We first give the well-posedness of the system by using the semigroup theory. Then, we establish a decay result by introducing a suitable Lyapunov functionals.

1. Introduction

In this article, we study the full von Kármán beam is coupled to a thermal effect and frictional damping with distributed delay and past history, the system is written as

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x + d_2 w_{xxxx} + \delta w_t = 0, \\ u_{tt} - d_1\left[u_x + \frac{1}{2}(w_x)^2\right]_x + \gamma\theta_{tx} + \mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(\zeta)\, u_t(x, t-\zeta)\, d\zeta = 0, \\ \theta_{tt} - \delta\theta_{xx} + \ell\theta_t + \gamma u_{tx} + \int_0^{\infty} g(s)\, \theta_{xx}(x, t-s)\, ds = 0, \end{cases} \tag{1}$$

in $(0,1) \times (0, +\infty)$, with initial data and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), & x \in (0,1), \\ \theta(x,-t) = \theta_0(x),\ \ \theta_t(x,0) = \theta_1(x), & (x,t) \in (0,1) \times (0,+\infty), \\ u_t(x,-t) = f_0(x,t), & (x,t) \in (0,1) \times (0,\tau_2), \\ u(0,t) = w(0,t) = \theta_x(0,t) = w_x(0,t) = 0, & t \in (0,+\infty), \\ u(1,t) = w(1,t) = \theta_x(1,t) = w_x(1,t) = 0, & t \in (0,+\infty), \end{cases} \tag{2}$$

where w the transversal displacement, u is the longitudinal displacement, θ is the temperature difference, and the coefficients d_1, d_2, δ, γ, ℓ and μ_1 are positive constant coefficients. Moreover, τ_1 and τ_2 are

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two real numbers with $0 \le \tau_1 < \tau_2$, and $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function. The initial data $(u_0, u_1, w_0, w_1, \theta_0, \theta_1, f_0)$ are assumed to belong to a suitable functional space.

Here, we prove the well-posedness and stability results for problem on the following parameter, under the assumption

$$\mu_1 > \int_{\tau_1}^{\tau_2} \left|\mu_2(\zeta)\right| d\zeta, \tag{3}$$

and the relaxation function g satisfies the following assumptions:

(G1): $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a C^1 function satisfying

$$g(0) > 0, \;\; \delta - \int_0^\infty g(s)\, ds = \delta - g_0 > 0.$$

(G2): There exists a positive nonincreasing differentiable function $\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$g'(t) \le -\xi(t)\, g(t), \; t \ge 0.$$

System (1) arises in the theory of the transverse of the beam. As we know, many structures in several fields of engineering are formed by a single or a large number of beams. There are different models for these beams depending on their nature and kinds of vibrations they are subject to. A widely accepted dynamical model describing large deflections of thin plates is the von Kármán system of equations. In [19], Lagnese and Lions studied the controllability and stabilization of the following von Kármán system:

$$\begin{cases} \rho A \eta_{tt}(x,t) + \dfrac{\partial^2}{\partial x^2}\left(EI\eta_{xx}(x,t)\right) - \dfrac{\partial}{\partial x}\left(P(x,t)\,\eta_x(x,t)\right) = 0, \\ \rho A \mu_{tt}(x,t) - \dfrac{\partial}{\partial x} P(x,t) = 0, \end{cases}$$

where $0 < x < L, \; t \ge 0$, with appropriate boundary conditions and initial data and

$$P(x,t) := EA\left(\mu_x(x,t) + \frac{1}{2}\eta_x^2(x,t)\right).$$

Here $\eta(x,t)$, $\mu(x,t)$ and $P(x,t)$ are the transverse displacement of a generic point, the longitudinal displacement of a generic point and the axial tension, respectively. ρA the weight per unit length, EI the beam stiffness or flexural rigidity, E Young's modulus, A the cross-sectional area of the beam, and L is the beam length. There is a large literature on this model, when several authors considered problems of existence, uniqueness and asymptotic behaviour in time (when some damping effect is considered) as well as some other important properties (see [2–4, 9, 10, 20–23, 25] and the references therein). Djebabla and Tatar [9] considered the following one-dimensional full von Kármán beam by coupling the system with thermal effect according to the theory of Green and Naghdi's [12–14] and frictional damping for the other component:

$$\begin{cases} u_{tt} - D_1\left[u_x + \frac{1}{2}(w_x)^2\right]_x + \gamma\theta_{tx} = 0, \\ w_{tt} + K_1 w_t - D_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x + D_2 w_{xxxx} = 0, \\ \theta_{tt} - \ell\theta_{xx} + K_2\theta_t + \gamma u_{tx} = 0, \end{cases} \tag{4}$$

in $\Omega \times (0, +\infty)$, where $\Omega = [0, L]$ and K_1, K_2, D_1, D_2, ℓ and γ are positive constants, with the boundary conditions and the initial data

$$\begin{cases} u = 0, \quad w = 0, \quad \theta_x = 0, \quad w_x = 0, & x = 0, L, \;\; t > 0, \\ u(0,.) = u_0, \; u_t(0,.) = u_1, w(0,.) = w_0, \; w_t(0,.) = w_1, & \\ \theta(0,.) = \theta_0, \;\; \theta_t(0,.) = \theta_1. & \end{cases} \tag{5}$$

For the above full von Kármán system, they used the energy method to prove an exponential decay result. Moreover, Liu, W. et al. [23] considered the asymptotic behavior for a one-dimensional non-autonomous full von Kármán beam with a thermo-viscoelastic damping in the internal feedback. By introducing a suitable energy and some Lyapunov functionals, under some restrictions on the non-autonomous functions and the relaxation function, they obtained the asymptotic behavior of the solution and established a general decay result for the energy.

Delay effects arise in many applications and practical problems, and it has attracted lots of attentions from researchers in diverse elds. In recent years, the stability of evolution systems with time delay effects has become an active area of research (e.g. [1–3, 5, 18, 22, 27, 28]). It may not only destabilize a system which is asymptotically stable in the absence of delay but may also lead to ill-posedness (see [8, 31] and the references therein). Therefore, the stability issue of systems with delay is of theoretical and practical great importance. Nicaise and Pignotti [28] considered wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \triangle u + \mu_1 u_t + a(x) \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s) ds = 0,$$

in $\Omega \times (0, \infty)$, with initial and mixed Dirichlet-Neumann boundary conditions and a is a function chosen in an appropriate space. They established exponential stability of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s) ds < \mu_1.$$

The authors also obtained the same result when the distributed delay acted on the part of the boundary. Recently, Bouzettouta and Djebabla [6] considered the system is coupled to a heat equation modeling an expectedly dissipative effect with distributed delay, which has the form

$$\begin{cases} w_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x + d_2 w_{xxxx} + \mu_1 w_t + \int_{\tau_1}^{\tau_2} \mu_2(s) w_t(x, t-s) ds = 0, \\ u_{tt} - d_1 \left[u_x + \frac{1}{2} (w_x)^2 \right]_x + \delta \theta_{tx} = 0, \\ \theta_t + q_x + \delta u_{tx} = 0, \\ q_t + \gamma q + \theta_x = 0, \end{cases} \tag{6}$$

in $\Omega \times (0, \infty)$, where $\Omega = [0, L]$, the authors studied the well-posedness by using the semigroup theory, and they showed that this system is exponentially stable by using the appropriated multiplies and energy method to build an equivalent Lyapunov functional.

The issue of existence and stability of systems with past history has attracted a great deal of attention in the last decades. Rivera and Fernández [26] considered a Timoshenko-type system with a past history of the form

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \int_0^\infty g(s) \psi_{xx}(t-s, .) ds + K(\varphi_x + \psi)_x = 0, \end{cases}$$

and showed that the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal ($\frac{K}{\rho_1} = \frac{b}{\rho_2}$). Jianghao and Fei [17] considered a Timoshenko system of thermoelasticity of type III with past history and distributive delay of the form

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \beta \theta_{tx} = 0, \ (x,t) \in (0,1) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \int_0^\infty g(s) \psi_{xx}(x, t-s) ds - \beta \theta_t + f(\psi) = 0, (x,t) \in (0,1) \times (0, \infty), \\ \rho_3 \theta_{tt} - \delta \theta_{xx} - \ell \theta_{txx} + \gamma \varphi_{tx} + \gamma \psi_t - \int_{\tau_1}^{\tau_2} \mu(\zeta) \theta_{txx}(x, t-\zeta) d\zeta = 0, \ (x,t) \in (0,1) \times (0, \infty), \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \theta(x,0) = \theta_0(x), \ x \in (0,1), \\ \theta_t(x,0) = \theta_1(x), \psi_t(x,0) = \psi_1(x), \ x \in (0,1), \\ \psi(x,-t) = \psi_0(x,t), \ (x,t) \in (0,1) \times (0, \infty), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0, \ t \in (0, \infty), \\ \theta_{tx}(x,-t) = f_0(x,t), \ (x,t) \in (0,1) \times (0, \tau_2), \end{cases}$$

they established the well-posedness and the stability of the system for the cases of equal and nonequal speeds of wave propagation respectively. And, they obtained that the damping effect is strong enough to uniformly stabilize the system even in the presence of time delay under suitable conditions and improve the related results. Gang et al. [11] considered a transmission problem in the presence of history and delay terms. They proved well posedness by using the semigroup theory, under appropriate hypothesis on the relaxation function and the relationship between the weight of the damping and the weight of the delay. Also they established a decay result by introducing a suitable Lyaponov functional. (For other past history problems, see [15, 16, 24, 30, 32] and the references therein).

Motivated by the above results, in the present work, we study the well-posedness and asymptotic behaviour of solutions for (1)-(2). By using semigroup theory, we prove the existence and uniqueness of the solution. By using the perturbed energy method and construct some Lyapunov functionals, we then obtain the decay result.

The paper is organized as follows, In Section 2, we prove the well-posedness of the problem (1)-(2). In Section 3, we prove that the system is decay result.

2. Well-posedness of the problem

In this section, we give a brief idea about the existence and uniqueness of solutions for (1)-(2) using the semigroup theory [29]. We introduce as in [28] the new variable

$$z(x,\rho,\zeta,t) = u_t\left(x,t-\zeta\rho\right),\ \ x\in(0,1)\,,\ \rho\in(0,1)\,,\ \zeta\in(\tau_1,\tau_2)\,,\ t>0.$$

Then, we have

$$\zeta z_t(x,\rho,\zeta,t) + z_\rho(x,\rho,\zeta,t) = 0,\ \ x\in(0,1)\,,\ \rho\in(0,1)\,,\ \zeta\in(\tau_1,\tau_2)\,,\ t>0. \tag{7}$$

Following the ideal in [7], we set

$$\eta^t(x,s) = \theta(x,t) - \theta(x,t-s), \qquad (x,t,s)\in(0,1)\times\mathbb{R}^+\times\mathbb{R}^+. \tag{8}$$

Hence, we obtain the following equation

$$\eta^t_t(x,s) + \eta^t_s(x,s) = \theta_t(x,t),\ \ (x,t,s)\in(0,1)\times\mathbb{R}^+\times\mathbb{R}^+.$$

Therefore, problem (1) takes the form

$$\begin{cases} w_{tt} - d_1\left[\left(u_x + \frac{1}{2}\left(w_x\right)^2\right)w_x\right]_x + d_2 w_{xxxx} + \delta w_t = 0, \\ u_{tt} - d_1\left[u_x + \frac{1}{2}\left(w_x\right)^2\right]_x + \gamma\theta_{tx} + \mu_1 u_t + \int_{\tau_1}^{\tau_2}\mu_2\left(\zeta\right)z(x,1,\zeta,t)d\zeta = 0, \\ \theta_{tt} - \delta\theta_{xx} + \ell\theta_t + \gamma u_{tx} + g_0\theta_{xx}\left(x,t\right) - \int_0^\infty g\left(s\right)\eta^t_{xx}(x,s)ds = 0, \end{cases} \tag{9}$$

with the initial data and boundary conditions

$$\begin{cases} u\left(x,0\right) = u_0\left(x\right),\ u_t\left(x,0\right) = u_1\left(x\right), & x\in(0,1), \\ w\left(x,0\right) = w_0\left(x\right),\ w_t\left(x,0\right) = w_1\left(x\right), & x\in(0,1), \\ \theta\left(x,-t\right) = \theta_0\left(x\right),\ \ \theta_t\left(x,0\right) = \theta_1\left(x\right), & (x,t)\in(0,1)\times(0,\infty)\,, \\ u_t\left(x,-t\right) = f_0\left(x,t\right), & (x,t)\in(0,1)\times(0,\tau_2), \\ u\left(0,t\right) = w\left(0,t\right) = \theta_x\left(0,t\right) = w_x\left(0,t\right) = 0, & t\in(0,\infty), \\ u\left(1,t\right) = w\left(1,t\right) = \theta_x\left(1,t\right) = w_x\left(1,t\right) = 0, & t\in(0,\infty), \\ z\left(x,0,t,\zeta\right) = u_t\left(x,t\right), & (x,t,\zeta)\in(0,1)\times(0,\infty)\times(\tau_1,\tau_2)\,, \\ z\left(x,\rho,0,\zeta\right) = f_0\left(x,\rho\zeta\right), & (x,\rho,\zeta)\in(0,1)\times(0,1)\times(\tau_1,\tau_2)\,, \\ \eta^t(0,s) = \eta^t(1,s) = 0, & (t,s)\in\mathbb{R}^+\times\mathbb{R}^+, \\ \eta^t(x,0) = 0, & (x,t)\in(0,1)\times\mathbb{R}^+, \\ \eta^0(x,s) = \eta_0(s), & (x,s)\in(0,1)\times\mathbb{R}^+. \end{cases} \tag{10}$$

If we set $U = (w, w_t, u, u_t, \theta, \theta_t, z, \eta^t)^T$, then $\partial_t U = \left(w_t, w_{tt}, u_t, u_{tt}, \theta_t, \theta_{tt}, z_t, \eta_t^t\right)^T$.
Therefore, problem (9)-(10) can be written as

$$\left\{\begin{array}{l} \partial_t U = \mathcal{A}U + \mathcal{F}(U), \\ U(0) = U_0 = (w_0, w_1, u_0, u_1, \theta_0, \theta_1, f_0, \eta_0)^T, \end{array}\right. \tag{11}$$

with the linear problem

$$\left\{\begin{array}{l} \partial_t U = \mathcal{A}U, \\ U(0) = U_0 = (w_0, w_1, u_0, u_1, \theta_0, \theta_1, f_0, \eta_0)^T, \end{array}\right. \tag{12}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} w \\ w_t \\ u \\ u_t \\ \theta \\ \theta_t \\ z \\ \eta^t \end{pmatrix} = \begin{pmatrix} w_t \\ -d_2 w_{xxxx} - \delta w_t \\ u_t \\ d_1 u_{xx} - \gamma\theta_{tx} - \mu_1 u_t - \int_{\tau_1}^{\tau_2} \mu_2(\zeta) z(x,1,\zeta,t) d\zeta \\ \theta_t \\ \delta\theta_{xx} - \ell\theta_t - \gamma u_{tx} - g_0\theta_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s) ds \\ -\zeta^{-1} z_\rho \\ \theta_t - \eta_s^t \end{pmatrix}, \tag{13}$$

and

$$\mathcal{F}\begin{pmatrix} w \\ w_t \\ u \\ u_t \\ \theta \\ \theta_t \\ z \\ \eta^t \end{pmatrix} = \begin{pmatrix} 0 \\ d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x \\ \frac{d_1}{2}\left[(w_x)^2\right]_x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{14}$$

Next, we define the energy space as

$$\begin{aligned} \mathcal{H} \quad := \quad & H_*^2(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\ & \times L_g^2\left(\mathbb{R}^+, H_*^1(0,1)\right), \end{aligned}$$

where

$$\begin{aligned} H_*^1(0,1) \quad &:= \quad \left\{\phi \in H^1(0,1) : \phi_x(0) = \phi_x(1) = 0\right\}, \\ H_*^2(0,1) \quad &:= \quad \left\{\phi \in H^2(0,1) : \phi(0) = \phi(1) = \phi_x(0) = \phi_x(1) = 0\right\}, \end{aligned}$$

and $L_g^2\left(\mathbb{R}^+, H_*^1(0,1)\right)$ denotes the Hilbert space of H_*^1−valued functions on $\mathbb{R}^+$, endowed with the inner product

$$\langle \varphi_1, \varphi_2 \rangle_{L_g^2(\mathbb{R}^+, H_*^1(0,1))} = \int_0^1 \int_0^\infty g(s)\varphi_{1x}(s)\varphi_{2x}(s) ds dx.$$

For any $U = (w, w_t, u, u_t, \theta, \theta_t, z, \eta^t)^T \in \mathcal{H}, \widetilde{U} = \left(\widetilde{w}, \widetilde{w}_t, \widetilde{u}, \widetilde{u}_t, \widetilde{\theta}, \widetilde{\theta}_t, \tilde{z}, \tilde{\eta}^t\right)^T \in \mathcal{H}$ and for $\mu_1 > \int_{\tau_1}^{\tau_2} \left|\mu_2(\zeta)\right| d\zeta$, we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} \left\langle U, \widetilde{U} \right\rangle_{\mathcal{H}} \quad = \quad & \int_0^1 [u_t \widetilde{u}_t + w_t \widetilde{w}_t + \theta_t \widetilde{\theta}_t + \delta\theta_x \widetilde{\theta}_x + d_1 u_x \widetilde{u}_x - g_0 \theta_x \widetilde{\theta}_x + d_2 w_{xx} \widetilde{w}_{xx}] dx \\ & + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \zeta \left|\mu_2(\zeta)\right| z(x,\rho,\zeta,t) \widetilde{z}(x,\rho,\zeta,t) d\zeta d\rho dx + \int_0^1 \int_0^\infty g(s) \eta_x^t(s) \tilde{\eta}_x^t(s) ds dx. \end{aligned}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A})=\left\{\begin{array}{c} U\in\left(H^4(0,1)\cap H_*^2(0,1)\right)\times H_*^2(0,1)\times\left(H^2(0,1)\cap H_0^1(0,1)\right)\times H_0^1(0,1)\times\left(H^2(0,1)\cap H_*^1(0,1)\right)\\ \times H_*^1(0,1)\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\times L_g^2\left(\mathbb{R}^+,H^2(0,1)\cap H_*^1(0,1)\right),\\ z_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),u_t(x)=z(x,0,\zeta,t),\ \eta^t(x,0)=0 \text{ in } (0,1)\end{array}\right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

Theorem 2.1. *Suppose that $\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|d\zeta<\mu_1$, **(G1)** and **(G2)** hold. For all $U_0\in\mathcal{H}$, problem (11) possesses then a unique solution $U\in C(\mathbb{R}^+;\mathcal{H})$. Moreover, if $U_0\in D(\mathcal{A})$, the solution satisfies*

$$U\in C(\mathbb{R}^+;D(\mathcal{A}))\cap C^1(\mathbb{R}^+;\mathcal{H}).$$

Proof. We use the semigroup approach. So, we prove that $\mathcal{A}$ is a maximal monotone operator and that $\mathcal{F}$ is a Lipschitz continuous operator. First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U=(w,w_t,u,u_t,\theta,\theta_t,z,\eta^t)^T\in D(\mathcal{A})$, by using the inner product and integrating by parts

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=\left\langle\begin{pmatrix} w_t\\ -d_2w_{xxxx}-\delta w_t\\ u_t\\ d_1u_{xx}-\gamma\theta_{tx}-\mu_1u_t-\int_{\tau_1}^{\tau_2}\mu_2(\zeta)z(x,1,\zeta,t)d\zeta\\ \theta_t\\ \delta\theta_{xx}-\ell\theta_t-\gamma u_{tx}-g_0\theta_{xx}+\int_0^\infty g(s)\eta_{xx}^t(x,s)ds\\ -\zeta^{-1}z_\rho\\ \theta_t-\eta_s^t\end{pmatrix},\begin{pmatrix} w\\ w_t\\ u\\ u_t\\ \theta\\ \theta_t\\ z\\ \eta^t\end{pmatrix}\right\rangle$$

$$\begin{aligned}\langle AU,U\rangle_{\mathcal{H}} &= -\mu_1\int_0^1 u_t^2dx-\delta\int_0^1 w_t^2dx-\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(\zeta)z(x,1,\zeta,t)u_td\zeta dx\\ &\quad-\ell\int_0^1\theta_t^2dx-\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|\int_0^1 z_\rho(x,\rho,\zeta,t)z(x,\rho,\zeta,t)d\rho d\zeta dx\\ &\quad+\int_0^1\theta_t\int_0^\infty g(s)\eta_{xx}^t(x,s)dsdx+\int_0^1\int_0^\infty g(s)\left(\theta_t-\eta_s^t\right)_x\eta_x^t(s)dsdx.\end{aligned}$$

Integrating by parts in ρ, we have

$$\int_0^1 z_\rho(x,\rho,\zeta,t)z(x,\rho,\zeta,t)d\rho=\frac{1}{2}\left[z^2(x,1,\zeta,t)-z^2(x,0,\zeta,t)\right],$$

that is

$$\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|\int_0^1 z_\rho(x,\rho,\zeta,t)z(x,\rho,\zeta,t)d\rho d\zeta dx=\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|\left[z^2(x,1,\zeta,t)-z^2(x,0,\zeta,t)\right]d\zeta dx.$$

Consequently, using the fact that $z(x,0,\zeta,t)=u_t(x)$ and $\eta^t(x,0)=0$ (definition of $D(\mathcal{A})$), we get

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}} &= -\mu_1\int_0^1 u_t^2dx-\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(\zeta)z(x,1,\zeta,t)u_td\zeta dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|z^2(x,1,\zeta,t)d\zeta dx\\ &\quad-\delta\int_0^1 w_t^2dx-\ell\int_0^1\theta_t^2dx+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|u_t^2(x)d\zeta dx+\frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned}$$

Now, using Young's inequality, we can estimate

$$-\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(\zeta)\,z(x,1,\zeta,t)\,u_t d\zeta dx \leq \frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|d\zeta\int_0^1 u_t^2dx+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|z^2(x,1,\zeta,t)\,d\zeta dx.$$

Therefore, from the assumption (3) and (G2) into account, we conclude that

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}} &\leq -\delta\int_0^1 w_t^2dx-\ell\int_0^1\theta_t^2dx-\left(\mu_1-\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|d\zeta\right)\int_0^1 u_t^2dx+\frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx\\ &\leq 0,\end{aligned}$$

that is, $\mathcal{A}$ is a dissipative operator.

Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $\lambda I-\mathcal{A}$ is surjective for a fixed $\lambda>0$. Indeed, given $F=(f_1,...,f_8)^T\in\mathcal{H}$, we prove that there exists a unique $U=(w,w_t,u,u_t,\theta,\theta_t,z,\eta^t)^T\in D(\mathcal{A})$ such that

$$(\lambda I-\mathcal{A})\,U=F, \tag{15}$$

that is,

$$\begin{cases}\lambda w-w_t=f_1,\\ \lambda w_t+d_2w_{xxxx}+\delta w_t=f_2,\\ \lambda u-u_t=f_3,\\ \lambda u_t-d_1u_{xx}+\gamma\theta_{tx}+\mu_1u_t+\int_{\tau_1}^{\tau_2}\mu_2(\zeta)\,z(x,1,\zeta,t)d\zeta=f_4,\\ \lambda\theta-\theta_t=f_5,\\ \lambda\theta_t-\delta\theta_{xx}+\ell\theta_t+\gamma u_{tx}+g_0\theta_{xx}-\int_0^\infty g(s)\,\eta_{xx}^t(x,s)ds=f_6,\\ \lambda z+\zeta^{-1}z_\rho=f_7,\\ \lambda\eta^t+\eta_s^t-\theta_t=f_8.\end{cases} \tag{16}$$

From $(16)_1$, $(16)_3$ and $(16)_5$ we have

$$\begin{cases}\lambda w-w_t=f_1,\\ \lambda u-u_t=f_3,\\ \lambda\theta-\theta_t=f_5.\end{cases} \tag{17}$$

Inserting (17) into $(16)_2$, $(16)_4$ and $(16)_6$, we get

$$\begin{cases}\left(\lambda^2+\delta\lambda\right)w+d_2w_{xxxx}=(\lambda+\delta)\,f_1+f_2,\\ \left(\lambda^2+\lambda\mu_1\right)u-d_1u_{xx}+\gamma\lambda\theta_x+\int_{\tau_1}^{\tau_2}\mu_2(\zeta)\,z(x,1,\zeta,t)d\zeta=(\lambda+\mu_1)\,f_3+f_4+\gamma\,(f_5)_x\,,\\ \left(\lambda^2+\lambda\ell\right)\theta-(\delta-g_0)\,\theta_{xx}+\lambda\gamma u_x-\int_0^\infty g(s)\,\eta_{xx}^t(x,s)ds=\gamma\,(f_3)_x+(\lambda+\ell)\,f_5+f_6,\\ \lambda z+\zeta^{-1}z_\rho=f_7,\\ \lambda\eta^t+\eta_s^t-\theta_t=f_8.\end{cases} \tag{18}$$

Furthermore, by (16) we can find as

$$z(x,0,\zeta,t)=u_t(x)\ \text{ for } x\in(0,1)\,,\zeta\in(\tau_1,\tau_2)\,, \tag{19}$$

and from (16), we have

$$\lambda z(x,\rho,\zeta,t)+\zeta^{-1}z_\rho(x,\rho,\zeta,t)=f_7(x,\rho,\zeta)\ \text{ for } x\in(0,1)\,,\ \rho\in(0,1)\,,\ \zeta\in(\tau_1,\tau_2)\,. \tag{20}$$

Then, by (19) and (20), we obtain

$$z(x,\rho,\zeta,t)=u_t(x)\,e^{-\lambda\rho\zeta}+\zeta e^{-\lambda\rho\zeta}\int_0^\rho f_7(x,\delta,\zeta)\,e^{\lambda\delta\zeta}d\delta.$$

So, from (16) on $(0,1)\times(0,1)\times(\tau_1,\tau_2)$,

$$z(x,\rho,\zeta,t)=\lambda u(x)e^{-\lambda\rho\zeta}-f_3e^{-\lambda\rho\zeta}+\zeta e^{-\lambda\rho\zeta}\int_0^{\rho}f_7(x,\delta,\zeta)e^{\lambda\delta\zeta}d\delta, \tag{21}$$

and in particular, $z(x,1,\zeta,t)=\lambda u(x)e^{-\lambda\zeta}+z_0(x,\zeta)$, with $z_0\in L^2((0,1)\times(\tau_1,\tau_2))$, defined by

$$z_0(x,\zeta)=-f_3e^{-\lambda\zeta}+\zeta e^{-\lambda\zeta}\int_0^{\rho}f_7(x,\delta,\zeta)e^{\lambda\delta\zeta}d\delta.$$

We note that the last equation in (18) with $\eta^t(x,0)=0$ has a unique solution

$$\begin{aligned}\eta^t(x,s) &= \left(\int_0^s e^{\lambda y}\left(f_8(x,y)+\theta_t(x)\right)dy\right)e^{-\lambda s}\\ &= \left(\int_0^s e^{\lambda y}\left(f_8(x,y)+\lambda\theta(x)-f_5(x)\right)dy\right)e^{-\lambda s}.\end{aligned}$$

Multiplying the third equations of system $(18)_1$, $(18)_2$ and $(18)_3$ by $\widetilde{w},\widetilde{u}$ and $\widetilde{\theta}$ respectively, and integrating over $(0,1)$, we arrive at

$$\begin{cases}\int_0^1\left(\lambda^2+\delta\lambda\right)w\widetilde{w}dx+\int_0^1 d_2w_{xxxx}\widetilde{w}dx=\int_0^1(\lambda+\delta)f_1\widetilde{w}dx+\int_0^1 f_2\widetilde{w}dx,\\ \int_0^1\left(\lambda^2+\lambda\mu_1\right)u\widetilde{u}dx-\int_0^1 d_1u_{xx}\widetilde{u}dx+\int_0^1\gamma\lambda\theta_x\widetilde{u}dx+\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(\zeta)z(x,1,\zeta,t)\widetilde{u}d\zeta dx\\ =\int_0^1(\lambda+\mu_1)f_3\widetilde{u}dx+\int_0^1 f_4\widetilde{u}dx+\int_0^1\gamma(f_5)_x\widetilde{u}dx,\\ \int_0^1\left(\lambda^2+\lambda\ell\right)\theta\widetilde{\theta}dx-\int_0^1(\delta-g_0)\theta_{xx}\widetilde{\theta}dx+\int_0^1\lambda\gamma u_x\widetilde{\theta}dx-\int_0^1\int_0^{\infty}g(s)\eta^t_{xx}(x,s)\widetilde{\theta}dsdx\\ =\int_0^1\gamma(f_3)_x\widetilde{\theta}dx+\int_0^1(\lambda+\ell)f_5\widetilde{\theta}dx+\int_0^1 f_6\widetilde{\theta}dx.\end{cases} \tag{22}$$

Consequently, problem (22) is equivalent to the problem

$$B\left((w,u,\theta)^T,\left(\widetilde{w},\widetilde{u},\widetilde{\theta}\right)^T\right)=L\left(\widetilde{w},\widetilde{u},\widetilde{\theta}\right)^T, \tag{23}$$

where $B:\left[H_*^2(0,1)\times H_0^1(0,1)\times H_*^1(0,1)\right]^2\longrightarrow\mathbb{R}$, is the bilinear form given by

$$\begin{aligned}&B\left((w,u,\theta)^T,\left(\widetilde{w},\widetilde{u},\widetilde{\theta}\right)^T\right)\\ =\;&\int_0^1\left(\lambda^2+\delta\lambda\right)w\widetilde{w}dx+\int_0^1 d_2w_{xx}\widetilde{w}_{xx}dx+\int_0^1 d_1u_x\widetilde{u}_xdx+\int_0^1\gamma\lambda\theta_x\widetilde{u}dx\\ &+\int_0^1\left(\lambda^2+\lambda\mu_1+\int_{\tau_1}^{\tau_2}\mu_2(\zeta)\lambda e^{-\lambda\zeta}d\zeta\right)u\widetilde{u}dx+\int_0^1\left(\lambda^2+\lambda\ell\right)\theta\widetilde{\theta}dx\\ &+\int_0^1\left(\delta-g_0+\lambda\int_0^{\infty}g(s)e^{-\lambda s}\left(\int_0^s e^{\lambda y}dy\right)ds\right)\theta_x\widetilde{\theta}_xdx+\int_0^1\lambda\gamma u_x\widetilde{\theta}dx,\end{aligned}$$

and $L : \left[H_*^2(0,1) \times H_0^1(0,1) \times H_*^1(0,1)\right] \longrightarrow \mathbb{R}$, is the linear form defined by

$$\begin{aligned} & L\left(\widetilde{w},\widetilde{u},\widetilde{\theta}\right)^T \\ = & \int_0^1 (\lambda+\delta) f_1 \widetilde{w} dx + \int_0^1 f_2 \widetilde{w} dx + \int_0^1 (\lambda+\mu_1) f_3 \widetilde{u} dx + \int_0^1 f_4 \widetilde{u} dx + \int_0^1 \gamma (f_5)_x \widetilde{u} dx \\ & + \int_0^1 \gamma (f_3)_x \widetilde{\theta} dx + \int_0^1 (\lambda+\ell) f_5 \widetilde{\theta} dx + \int_0^1 f_6 \widetilde{\theta} dx - \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(\zeta) z_0(x,\zeta) \widetilde{u} d\zeta dx \\ & + \int_0^1 \int_0^\infty g(s) e^{-\lambda s} \left(\int_0^s e^{\lambda y} \left(f_8(x,y) - f_5(x,y)\right)_{xx} dy \right) ds \widetilde{\theta} dx. \end{aligned}$$

It is easy to see that $B(.,.)$ is continuous and coercive, and $L(.)$ is continuous. Applying the Lax-Milgram theorem, we deduce that problem (23) admits a unique solution $(w,u,\theta) \in H_*^2(0,1) \times H_0^1(0,1) \times H_*^1(0,1)$ for all $\left(\widetilde{w},\widetilde{u},\widetilde{\theta}\right) \in H_*^2(0,1) \times H_0^1(0,1) \times H_*^1(0,1)$. Applying the classical elliptic regularity, it follows from (22) that $(w,u,\theta) \in \left(H^4(0,1) \cap H_*^2(0,1)\right) \times \left(H^2(0,1) \cap H_0^1(0,1)\right) \times \left(H^2(0,1) \cap H_*^1(0,1)\right)$. Thus, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Therefore, $\mathcal{A}$ is a maximal monotone operator.

Now, we prove that the operator $\mathcal{F}$ defined in (14) is locally Lipschitz continuous in $\mathcal{H}$. Let $U = (w, w_t, u, u_t, \theta, \theta_t, z, \eta^t)^T$ and $\tilde{U} = \left(\tilde{w}, \tilde{w}_t, \tilde{u}, \tilde{u}_t, \tilde{\theta}, \tilde{\theta}_t, \tilde{z}, \tilde{\eta}^t\right)^T$ belong to $\mathcal{H}$, then we have

$$\left\|\mathcal{F}(U) - \mathcal{F}\left(\tilde{U}\right)\right\|_{\mathcal{H}} = d_1 \left|\left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x - \left[\left(\tilde{u}_x + \frac{1}{2}(\tilde{w}_x)^2\right) \tilde{w}_x\right]_x\right| + \frac{d_1}{2}\left|\left[(w_x)^2 - (\tilde{w}_x)^2\right]_x\right|. \tag{24}$$

Let's estimate the first term in the right-hand side of (24), adding and subtracting the term $\left(u_x + \frac{1}{2}(w_x)^2\right)\widetilde{w}_x$, we obtain that

$$\begin{aligned} & \left|\left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x - \left[\left(\tilde{u}_x + \frac{1}{2}(\tilde{w}_x)^2\right) \tilde{w}_x\right]_x\right| \\ \leq & \left\|w_x - \widetilde{w}_x\right\|_{L^\infty(0,1)} \left|u_x + \tfrac{1}{2}(w_x)^2\right| + \left\|\widetilde{w}_x\right\|_{L^\infty(0,1)} \left|u_x - \widetilde{u}_x\right| + \tfrac{1}{2}\left\|\widetilde{w}_x\right\|_{L^\infty(0,1)} \left|w_x + \widetilde{w}_x\right| \left\|w_x - \widetilde{w}_x\right\|_{L^\infty(0,1)}. \end{aligned} \tag{25}$$

Using the embedding of $H^1(0,1)$ into $L^\infty(0,1)$ and deduce from (25) that

$$\left|\left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x - \left[\left(\tilde{u}_x + \frac{1}{2}(\tilde{w}_x)^2\right) \tilde{w}_x\right]_x\right| \leq C\left(\|U\|_{\mathcal{H}}, \left\|\widetilde{U}\right\|_{\mathcal{H}}\right) \left\|U - \widetilde{U}\right\|_{\mathcal{H}}. \tag{26}$$

The second term on the right side of (24) is estimated as follows

$$\begin{aligned} \left|\left[(w_x)^2 - (\tilde{w}_x)^2\right]_x\right| &= \left|\left[(w_x + \tilde{w}_x)(w_x - \tilde{w}_x)\right]_x\right| \\ &\leq |w_{xx} - \tilde{w}_{xx}| \left(\|w_x\|_{L^\infty(0,1)} + \|\tilde{w}_x\|_{L^\infty(0,1)}\right) + \|w_x - \tilde{w}_x\|_{L^\infty(0,1)} \left(|w_{xx}| + |\tilde{w}_{xx}|\right), \end{aligned} \tag{27}$$

again, we use the embedding $H^1(0,1)$ into $L^\infty(0,1)$, one also sees that

$$\left|\left[(w_x)^2 - (\tilde{w}_x)^2\right]_x\right| \leq C\left(\|U\|_{\mathcal{H}}, \left\|\widetilde{U}\right\|_{\mathcal{H}}\right) \left\|U - \widetilde{U}\right\|_{\mathcal{H}}. \tag{28}$$

Combining (24), (26) and (28), shows that $\mathcal{F}(U)$ is locally Lipschitz continuous in $\mathcal{H}$. The proof complete. □

3. Decay of the solution

In this section, we state and prove the stability result for the energy of the system (9)-(10). For the regular solution of the system (9)-(10), we define the energy functional $E(t)$ as

$$\begin{aligned} E(t) := & \frac{1}{2}\int_0^1 \left[u_t^2 + w_t^2 + \theta_t^2 + (\delta - g_0)\,\theta_x^2 + d_1\left(u_x + \tfrac{1}{2}(w_x)^2\right)^2 + d_2 w_{xx}^2\right] dx \\ & + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t)\, d\zeta d\rho dx + \frac{1}{2}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned} \tag{29}$$

The stability result reads as follows.

Theorem 3.1. *Let (w,u,θ) be the solution of (9)-(10). Assume that **(G1)** and **(G2)** hold, that $\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right| d\zeta < \mu_1$, that for some $c_0 \geq 0$,*

$$\int_0^1 \theta_{0x}^2(x,s)\, dx \leq c_0, \quad \forall s > 0, \tag{30}$$

there exist constants $\frac{\alpha_1}{\gamma_2}$, $c_2 > 0$ such that, for all $t \in \mathbb{R}^+$ and for all $c_1 \in \left(0, \frac{\alpha_1}{\gamma_2}\right)$,

$$E(t) \leq c_2\left(1 + \int_0^t (g(t))^{1-c_1}\, dt\right) e^{-c_1\int_0^t \xi(s)ds} + c_2\int_t^\infty g(s)\, ds. \tag{31}$$

In order to prove this result, we need the following lemmas.

Lemma 3.2. *Let (w,u,θ) be the solution of (9)-(10) and assume (3) holds. Then the energy functional, defined by (29) satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) & \leq -\delta\int_0^1 w_t^2 dx - \ell\int_0^1 \theta_t^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right| d\zeta\right)\int_0^1 u_t^2 dx + \frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\ & \leq 0, \quad \forall t \geq 0. \end{aligned} \tag{32}$$

Proof. Multiplying the first equation in (9) by w_t, the second by u_t and the third by θ_t, integrating over $(0,1)$ with respect to x, we obtain

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{2}\int_0^1 \left(u_t^2 + w_t^2 + \theta_t^2 + (\delta - g_0)\,\theta_x^2 + d_2 w_{xx}^2 + d_1\left(u_x + \frac{1}{2}(w_x)^2\right)\right) dx\right] + \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\ = & -\delta\int_0^1 w_t^2 dx - \mu_1\int_0^1 u_t^2 dx - \ell\int_0^1 \theta_t^2 dx - \int_0^1\int_{\tau_1}^{\tau_2} \mu_2(\zeta)\, z(x,1,\zeta,t)\, u_t d\zeta dx \\ & + \frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned} \tag{33}$$

On the other hand, multiplying (7) by $\left|\mu_2(\zeta)\right| z(x,\rho,\zeta,t)$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$ with respect to ρ, x and ζ, we obtain

$$\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta\left|\mu_2(\zeta)\right| z(x,\rho,\zeta,t)\, z_t(x,\rho,\zeta,t)\, d\zeta d\rho dx + \int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right| z(x,\rho,\zeta,t)\, z_\rho(x,\rho,\zeta,t)\, d\zeta d\rho dx = 0,$$

which gives

$$\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t)\, d\zeta d\rho dx = -\frac{1}{2}\frac{d}{d\rho}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t)\, d\zeta d\rho dx.$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\zeta\left|\mu_2(\zeta)\right|z^2(x,\rho,\zeta,t)\,d\zeta d\rho dx = -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|z^2(x,1,\zeta,t)\,d\zeta dx + \frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|d\zeta\int_0^1 u_t^2dx. \tag{34}$$

Summing up (33)-(34), we arrive at

$$\begin{aligned}\frac{d}{dt}E(t) &= -\delta\int_0^1 w_t^2dx - \mu_1\int_0^1 u_t^2dx - \ell\int_0^1\theta_t^2dx - \int_0^1\int_{\tau_1}^{\tau_2}\mu_2(\zeta)\,z(x,1,\zeta,t)\,u_td\zeta dx\\ &\quad -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|z^2(x,1,\zeta,t)\,d\zeta dx + \frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|d\zeta\int_0^1 u_t^2dx\\ &\quad +\frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx.\end{aligned} \tag{35}$$

Using integration by parts and Young's inequality, we have

$$-\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(\zeta)\,z(x,1,\zeta,t)\,u_td\zeta dx \le \frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|d\zeta\int_0^1 u_t^2dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|z^2(x,1,\zeta,t)\,d\zeta dx. \tag{36}$$

Simple substitution of (36) into (35) and using (3) give (32), which concludes the proof. □

Now, we are going to construct a Lyapunov functional equivalent to the energy. For this, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.3. *Let (w,u,θ) be the solution of (9)-(10). Then the functional*

$$F_1(t) := \int_0^1\left(u_tu + \frac{1}{2}w_tw + \frac{\mu_1}{2}u^2 + \frac{\delta}{4}w^2\right)dx \tag{37}$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$\begin{aligned}F_1'(t) &\le -d_1\int_0^1\left(u_x + \frac{1}{2}(w_x)^2\right)^2dx - \frac{d_2}{2}\int_0^1 w_{xx}^2dx + \int_0^1 u_t^2dx + \frac{1}{2}\int_0^1 w_t^2dx\\ &\quad +\varepsilon_1\int_0^1 u_x^2dx + \frac{\gamma^2}{2\varepsilon_1}\int_0^1\theta_t^2dx + \frac{\mu_1}{2\varepsilon_1}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|z^2(x,1,\zeta,t)\,d\zeta dx.\end{aligned} \tag{38}$$

Proof. By differentiating $F_1(t)$ with respect to t, using the first and the second equation of (9), and integrating by parts, we obtain

$$\begin{aligned}F_1'(t) &= \int_0^1\left[d_1\left[u_x + \frac{1}{2}(w_x)^2\right]_x - \gamma\theta_{tx} - \mu_1u_t - \int_{\tau_1}^{\tau_2}\mu_2(\zeta)\,z(x,1,\zeta,t)d\zeta\right]udx + \int_0^1 u_t^2dx + \frac{1}{2}\int_0^1 w_t^2dx\\ &\quad +\frac{1}{2}\int_0^1\left[d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)w_x\right]_x - d_2w_{xxxx} - \delta w_t\right]wdx + \mu_1\int_0^1 u_tudx + \frac{\delta}{2}\int_0^1 w_twdx\\ &= -d_1\int_0^1\left(u_x + \frac{1}{2}(w_x)^2\right)^2dx - \frac{d_2}{2}\int_0^1 w_{xx}^2dx + \int_0^1 u_t^2dx + \frac{1}{2}\int_0^1 w_t^2dx + \gamma\int_0^1\theta_tu_xdx\\ &\quad -\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(\zeta)\,z(x,1,\zeta,t)ud\zeta dx.\end{aligned}$$

By using Young's and Poincaré inequalities, and (3), we obtain for any $\varepsilon_1 > 0$

$$\begin{aligned}\gamma \int_0^1 \theta_t u_x dx &\leq \frac{\varepsilon_1}{2}\int_0^1 u_x^2 dx + \frac{\gamma^2}{2\varepsilon_1}\int_0^1 \theta_t^2 dx,\\ -\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(\zeta) z(x,1,\zeta,t) u d\zeta dx &\leq \frac{\varepsilon_1}{2}\int_0^1 u_x^2 dx + \frac{\mu_1}{2\varepsilon_1}\int_0^1\int_{\tau_1}^{\tau_2} \left|\mu_2(\zeta)\right| z^2(x,1,\zeta,t) d\zeta dx.\end{aligned}$$

Then, (38) is established. □

Lemma 3.4. *Let (w,u,θ) be the solution of (9)-(10). Then the functional*

$$F_2(t) := \int_0^1 \left(\theta_t\theta + \gamma u_x\theta + \frac{\ell}{2}\theta^2\right) dx \tag{39}$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$\begin{aligned}F_2'(t) &\leq -\frac{(\delta - g_0)}{2}\int_0^1 \theta_x^2 dx + \left(1 + \frac{\gamma^2}{4\varepsilon_2}\right)\int_0^1 \theta_t^2 dx + \varepsilon_2 \int_0^1 u_x^2 dx\\ &\quad + \frac{g_0}{2(\delta - g_0)}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx.\end{aligned} \tag{40}$$

Proof. By differentiating $F_2(t)$ with respect to t, then exploiting the third equation in (9), and integrating by parts, we obtain

$$\begin{aligned}F_2'(t) &= \int_0^1 \left[\delta\theta_{xx} - \ell\theta_t - \gamma u_{tx} - g_0\theta_{xx}(x,t) + \int_0^\infty g(s)\eta_{xx}^t(x,s)ds\right]\theta dx + \int_0^1 \theta_t^2 dx + \gamma\int_0^1 u_{xt}\theta dx\\ &\quad + \gamma\int_0^1 u_x\theta_t dx + \ell\int_0^1 \theta_t\theta dx\\ &= -(\delta - g_0)\int_0^1 \theta_x^2 dx + \int_0^1 \theta_t^2 dx + \gamma\int_0^1 u_x\theta_t dx - \int_0^1\int_0^\infty g(s)\eta_x^t(x,s)\theta_x ds dx.\end{aligned}$$

Young's inequality, Holder's inequality and (G2) imply that

$$-\int_0^1\int_0^\infty g(s)\eta_x^t(x,s)\theta_x ds dx \leq \frac{(\delta - g_0)}{2}\int_0^1 \theta_x^2 dx + \frac{g_0}{2(\delta - g_0)}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx.$$

By using Young's inequality, we obtain for any $\varepsilon_2 > 0$,

$$\gamma\int_0^1 u_x\theta_t dx \leq \varepsilon_2\int_0^1 u_x^2 dx + \frac{\gamma^2}{4\varepsilon_2}\int_0^1 \theta_t^2 dx.$$

Then, (40) is established. □

Lemma 3.5. *Let (w,u,θ) be the solution of (9)-(10) and (7). Then the functional*

$$F_3(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta e^{-\zeta\rho}\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t) d\zeta d\rho dx \tag{41}$$

satisfies, for some positive constant n_1, the following estimate

$$\begin{aligned}F_3'(t) &\leq -n_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t) d\zeta d\rho dx - n_1\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right| z^2(x,1,\zeta,t) d\zeta dx\\ &\quad + \ell\int_0^1 u_t^2 dx.\end{aligned} \tag{42}$$

Proof. By differentiating $F_3(t)$ with respect to t, and using the equation (7), we obtain

$$\begin{aligned} F_3'(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\zeta\rho}\left|\mu_2(\zeta)\right| z(x,\rho,\zeta,t)\, z_\rho(x,\rho,\zeta,t)\, d\zeta d\rho dx \\ &= -\frac{d}{d\rho}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\zeta\rho}\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t)\, d\zeta d\rho dx - \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta e^{-\zeta\rho}\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t)\, d\zeta d\rho dx \\ &= -\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right|\left[e^{-\zeta} z^2(x,1,\zeta,t) - z^2(x,0,\zeta,t)\right] d\zeta dx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta e^{-\zeta\rho}\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t)\, d\zeta d\rho dx. \end{aligned}$$

Using the fact that $z(x,0,\zeta,t) = u_t$ and $e^{-\zeta} \le e^{-\zeta\rho} \le 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} F_3'(t) &\le -\int_0^1\int_{\tau_1}^{\tau_2} e^{-\zeta}\left|\mu_2(\zeta)\right| z^2(x,1,\zeta,t)\, d\zeta dx + \int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right| d\zeta \int_0^1 \theta_t^2 dx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta e^{-\zeta}\left|\mu_2(\zeta)\right| z^2(x,\rho,\zeta,t)\, d\zeta d\rho dx. \end{aligned}$$

Because $-e^{-\zeta}$ is an increasing function, we have $-e^{-\zeta} \le -e^{-\tau_2}$, for all $\zeta \in [\tau_1, \tau_2]$.
Finally, setting $n_1 = e^{-\tau_2}$ and recalling (3), we obtain (46). □

Now, we are ready to state and prove the main result of this section.

Proof. (Of **Theorem 3.1**) To finalize the proof, we define the Lyapunov functional $L(t)$ as follows

$$L(t) := NE(t) + F_1(t) + F_2(t) + N_3F_3(t), \tag{43}$$

where N and N_3 are positive constants to be chosen properly later.
By differentiating (43) and recalling (32), (38), (40), (42) and the relations

$$\begin{aligned} \int_0^1 u_x^2 dx &= \int_0^1 \left[\left(u_x + \frac{1}{2}w_x^2\right) - \frac{1}{2}w_x^2\right]^2 dx \\ &\le 2\int_0^1 \left(u_x + \frac{1}{2}w_x^2\right)^2 dx + \frac{1}{2}\int_0^1 w_x^4 dx \\ &\le 2\int_0^1 \left(u_x + \frac{1}{2}w_x^2\right)^2 dx + \frac{1}{4}\int_0^1 w_{xx}^2 dx, \end{aligned}$$

we arrive at

$$\begin{aligned} L'(t) &\le -\left[\delta N - \frac{1}{2}\right]\int_0^1 w_t^2 dx - \left[\left(\mu_1 - \int_{\tau_1}^{\tau_2}\left|\mu_2(\zeta)\right| d\zeta\right)N - 1 - \ell N_3\right]\int_0^1 u_t^2 dx \\ &\quad -\left[\frac{d_2}{2} - \frac{\varepsilon_1}{4} - \frac{\varepsilon_2}{4}\right]\int_0^1 w_{xx}^2 dx - \left[d_1 - 2\varepsilon_1 - 2\varepsilon_2\right]\int_0^1 \left(u_x + \frac{1}{2}(w_x)^2\right)^2 dx \\ &\quad -\left[\ell N - \frac{\gamma^2}{2\varepsilon_1} - \left(1 + \frac{\gamma^2}{4\varepsilon_2}\right)\right]\int_0^1 \theta_t^2 dx + \frac{N}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 ds dx \\ &\quad -\frac{(\delta - g_0)}{2}\int_0^1 \theta_x^2 dx - n_1N_3\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \zeta\left|\mu(\zeta)\right| z^2(x,\rho,\zeta,t)\, d\zeta d\rho dx \\ &\quad -\left[n_1N_3 - \frac{\mu_1}{2\varepsilon_1}\right]\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu(\zeta)\right| z^2(x,1,\zeta,t)\, d\zeta dx + \frac{g_0}{2(\delta - g_0)}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx. \end{aligned} \tag{44}$$

At this point, we need to choose our constants very carefully. First, we take $\varepsilon_1 = \varepsilon_2$ and choose ε_2 so small that $\varepsilon_2 < \min\left\{\frac{d_1}{4}, d_2\right\}$.
Then, we choose N_3 large enough, so that $n_1 N_3 - \frac{\mu_1}{2\varepsilon_1} > 0$.
Finally, we choose N so large, such that $\delta N - \frac{1}{2} > 0, \left(\mu_1 - \int_{\tau_1}^{\tau_2} \left|\mu_2(\zeta)\right| d\zeta\right) N - 1 - \ell N_3 > 0,\ \ell N - \frac{\gamma^2}{2\varepsilon_1} - \left(1 + \frac{\gamma^2}{4\varepsilon_2}\right) > 0.$
By (29) and (G2), we deduce that there exists two positive constants α_1 and α_2 such that (44) becomes

$$L'(t) \leq -\alpha_1 E(t) + \alpha_2 \int_0^1 \int_0^\infty g(s) \left|\eta_x^t(x,s)\right|^2 dsdx, \tag{45}$$

and further, for some $\beta_1, \beta_2 > 0$, we have

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t),\ \forall t \geq 0. \tag{46}$$

To finish the proof of the stability estimates, we need to estimate the last term in (45).
Using (G2) and (32), we obtain that for all $t \in \mathbb{R}^+$,

$$\begin{aligned} \xi(t) \int_0^1 \int_0^t g(s) \left|\eta_x^t(x,s)\right|^2 dsdx &\leq \int_0^1 \int_0^t \xi(t) g(s) \left|\eta_x^t(x,s)\right|^2 dsdx \\ &\leq -\int_0^1 \int_0^t g'(s) \left|\eta_x^t(x,s)\right|^2 dsdx \\ &\leq -\int_0^1 \int_0^\infty g'(s) \left|\eta_x^t(x,s)\right|^2 dsdx \\ &\leq -2E'(t). \end{aligned} \tag{47}$$

Moreover, using the definition of $E(t)$ and the fact that $E(t)$ is nonincreasing imply that

$$\int_0^1 \theta_x^2(x,t)dx \leq \frac{2}{\delta - g_0} E(t) \leq \frac{2}{\delta - g_0} E(0),\ \ \forall t \in \mathbb{R}^+.$$

Using (8), (29) and (30), we arrive at

$$\begin{aligned} \xi(t) \int_0^1 \left|\eta_x^t(x,s)\right|^2 dx &= \xi(t) \int_0^1 (\theta_x(x,t) - \theta_x(x,t-s))^2 dx \\ &\leq 2\xi(t) \int_0^1 \theta_x^2(x,t)dx + 2\xi(t) \int_0^1 \theta_x^2(x,t-s)dx \\ &\leq \frac{8}{\delta - g_0} E(0)\xi(t) + 2c_0 \xi(t),\ \ \forall t, s \in \mathbb{R}^+. \end{aligned}$$

Then, we obtain

$$\xi(t) \int_0^1 \int_t^\infty g(s) \left|\eta_x^t(x,s)\right|^2 dsdx \leq \left(\frac{8}{\delta - g_0} E(0) + 2c_0\right) \xi(t) \int_t^\infty g(s) ds.$$

Then, we deduce that, for all $t \in \mathbb{R}^+$,

$$\xi(t) \int_0^1 \int_0^\infty g(s) \left|\eta_x^t(x,s)\right|^2 dsdx \leq -2E'(t) + \left(\frac{8}{\delta - g_0} E(0) + 2c_0\right) \xi(t) \int_t^\infty g(s) ds. \tag{48}$$

Multiplying (45) by $\xi(t)$ and using (45), we obtain

$$\xi(t) L'(t) + 2\alpha_2 E'(t) \leq -\alpha_1 \xi(t) E(t) + \left(\frac{8}{\delta - g_0} E(0) + 2c_0\right) \alpha_2 \xi(t) \int_t^\infty g(s) ds. \tag{49}$$

Now, we define

$$\mathcal{L}(t) = \xi(t) L(t) + 2\alpha_2 E(t), \quad h(t) = \xi(t) \int_t^\infty g(s)\,ds.$$

Clearly, $\mathcal{L}(t)$ and $E(t)$ are equivalent, that is, exist positive constants γ_1 and γ_2, such that

$$\gamma_1 E(t) \le \mathcal{L}(t) \le \gamma_2 E(t), \ \forall t \ge 0. \tag{50}$$

Then using (49) and (50), we have

$$\mathcal{L}'(t) \le -\frac{\alpha_1}{\gamma_2}\xi(t)\mathcal{L}(t) + \left(\frac{8}{\delta - g_0}E(0) + 2c_0\right)\alpha_2 h(t).$$

This inequality still holds, for any $c_1 \in \left(0, \frac{\alpha_1}{\gamma_2}\right)$, that is

$$\mathcal{L}'(t) \le -c_1\xi(t)\mathcal{L}(t) + \left(\frac{8}{\delta - g_0}E(0) + 2c_0\right)\alpha_2 h(t).$$

Therefore, by integrating over $[0, T]$ with $T \ge 0$, we obtain

$$\mathcal{L}(T) \le e^{-c_1\int_0^T \xi(s)ds}\left(\mathcal{L}(0) + \left(\frac{8}{\delta - g_0}E(0) + 2c_0\right)\alpha_2\int_0^T e^{c_1\int_0^t \xi(s)ds}h(t)\,dt\right).$$

Using (50), we have

$$E(T) \le \frac{1}{\gamma_1}e^{-c_1\int_0^T \xi(s)ds}\left(\mathcal{L}(0) + \left(\frac{8}{\delta - g_0}E(0) + 2c_0\right)\alpha_2\int_0^T e^{c_1\int_0^t \xi(s)ds}h(t)\,dt\right). \tag{51}$$

Then, by integration by parts, we obtain

$$\begin{aligned}\int_0^T e^{c_1\int_0^t \xi(s)ds}h(t)\,dt &= \frac{1}{c_1}\int_0^T \left(e^{c_1\int_0^t \xi(s)ds}\right)'\int_t^\infty g(s)\,dsdt \\ &= \frac{1}{c_1}\left(e^{c_1\int_0^T \xi(s)ds}\int_T^\infty g(s)\,ds - \int_0^\infty g(s)\,ds + \int_0^T e^{c_1\int_0^t \xi(s)ds}g(t)\,dt\right).\end{aligned}$$

Consequently, combining with (51), we have

$$\begin{aligned}E(T) \le\ & \frac{\mathcal{L}(0)}{\gamma_1}e^{-c_1\int_0^T \xi(s)ds} + \frac{1}{\gamma_1}\left(\frac{8}{\delta - g_0}E(0) + 2c_0\right)\frac{\alpha_2}{c_1}\int_T^\infty g(s)\,ds \\ & + \frac{1}{\gamma_1}e^{-c_1\int_0^T \xi(s)ds}\left(\frac{8}{\delta - g_0}E(0) + 2c_0\right)\frac{\alpha_2}{c_1}\int_0^T e^{c_1\int_0^t \xi(s)ds}g(t)\,dt.\end{aligned} \tag{52}$$

On the other hand, for all $t \in \mathbb{R}^+$, $\left(e^{c_1\int_0^t \xi(s)ds}(g(t))^{c_1}\right)' \le 0$, and then $e^{c_1\int_0^t \xi(s)ds}(g(t))^{c_1} \le (g(0))^{c_1}$. Therefore

$$\int_0^T e^{c_1\int_0^t \xi(s)ds}g(t)\,dt \le (g(0))^{c_1}\int_0^T (g(t))^{1-c_1}\,dt. \tag{53}$$

Finally, by combining (52) and (53) we obtain (31) with

$$c_2 = \frac{1}{\gamma_1}\max\left\{\mathcal{L}(0), \left(\frac{8}{\delta - g_0}E(0) + 2c_0\right)\frac{\alpha_2}{c_1}, \left(\frac{8}{\delta - g_0}E(0) + 2c_0\right)\frac{\alpha_2}{c_1}(g(0))^{c_1}\right\},$$

which completes the proof. □

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