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Some existence results for the generalized tensor absolute value equation

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Abstract. This paper introduces a new type of tensor absolute value equation, namely the generalized tensor absolute value equation (GTAVE), and studies its equivalence to the generalized tensor complementarity problem (GTCP). With the help of this equivalence and the degree theory approach, some existence results for the GTAVE are achieved.

1. Introduction

For given real matrices $\mathbf{A}$ and $\mathbf{B}$ of order n and vector $\mathbf{b} \in \mathbb{R}^n$, the absolute value equation (AVE) was introduced by Rohn [17], and defined as

$$\mathbf{A}\mathbf{w} + \mathbf{B}|\mathbf{w}| = \mathbf{b}, \tag{1}$$

where $\mathbf{w} = (w_1, w_2, ..., w_n)^T \in \mathbb{R}^n$. After that, this problem was studied by Mangasarian [12], and it was proved that solving AVE (1) is NP-hard. The AVE has many applications in numerical partial differential equations, applied sciences and optimization. When $\mathbf{A}+\mathbf{B}$ is nonsingular, then it was proved by Rohn [17] that AVE (1) can be converted into a linear complementarity problem (LCP) [2], which is to find a solution of the system

$$\mathbf{w} \geq \mathbf{0},\ \mathbf{M}\mathbf{w} + \mathbf{b} \geq \mathbf{0} \text{ and } \mathbf{w}^T(\mathbf{M}\mathbf{w} + \mathbf{b}) = 0, \tag{2}$$

where the matrix $\mathbf{M}$ is a given real matrix of order n and $\mathbf{b} \in \mathbb{R}^n$ is known. Since the LCP incorporates many mathematical programming problems, such as bimatrix games, quadratic programs, linear programming, it is relevant to ask when does the solution of the LCP (2) exists. Due to the nonlinearity and non-differentiability of the term $\mathbf{B}|\mathbf{w}|$ in AVE (1), it is not easy to determine under what conditions the solution of the AVE (1) exists. Over the years, many researchers have obtained significant results about the existence and uniqueness of the solution by introducing some constraints on $\mathbf{A}$ and $\mathbf{B}$ and utilizing the equivalence between the AVE (1) and the LCP (2), see for example [9, 18, 19, 24]. Note that when $\mathbf{B} = -\mathbf{I}$ ($\mathbf{I}$ is the identity matrix), AVE (1) becomes

$$\mathbf{A}\mathbf{w} - |\mathbf{w}| = \mathbf{b}. \tag{3}$$

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The existence and non-existence of the solution of the AVE (3) were discussed by Mangasarian et al. [13]. Further investigations about the existence and uniqueness of the solution of the AVE (3) is given in [23, 26]. Many generalizations of the AVE has been studied due to its importance in the optimization field. We suggest the readers to see [25, 27]. In recent years, a higher-order generalization of matrices, so-called tensors (hyper-matrices), has been well studied in the literature [16] and the corresponding tensor complementarity problem (TCP); find a vector $\mathbf{w}$ in $\mathbb{R}^n$ satisfying

$$\mathbf{w} \geq \mathbf{0}, F(\mathbf{w}) = \mathcal{A}\mathbf{w}^{m-1} + \mathbf{b} \geq \mathbf{0} \text{ and } \mathbf{w}^T F(\mathbf{w}) = 0, \tag{4}$$

(where $\mathbf{b}$ is known) was introduced and studied by Song and Qi [20] as a subclass of the well-known nonlinear complementarity problems [6]. Here

$$(\mathcal{A}\mathbf{w}^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} w_{i_2} \ldots w_{i_m},$$

is a homogeneous polynomial of degree $m-1$ for a given tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ of order m and dimension n, where $a_{i_1 i_2 \ldots i_m} \in \mathbb{R}$. The TCP has many applications in optimization, non-cooperative games, and equilibrium problems. It is easy to observe that the TCP (4) is a natural generalization of the LCP (2). Since LCP (2) can be equivalently formulated as an AVE (3), one may wonder whether there exists such an equivalent formulation for the TCP (4). This equivalent formulation, named as tensor absolute value equation (TAVE), was introduced by Du et al. [5], and the existence of the solution of this type of equation was studied. A more general type of TAVE was given in [11], and the solution analysis of this type of TAVE was presented using degree theoretical ideas. It should be noted that these TAVE's are a natural generalization of the AVE (1) and AVE (3). Recently Shi-Liang Wu [27] introduced a new type of generalized absolute value equation (NGAVE), where the existing results about the generalized order linear complementarity problem [7, 21] were used to determine the unique solution of the NGAVE.
The importance of the TAVE's in various fields such as scientific computing, optimization, linear programming, quadratic programming has motivated us to define a new type of generalized tensor absolute value equation (GTAVE) and study its equivalence with the generalized tensor complementarity problem (GTCP)[1]. Our results comprise of degree theoretical ideas, and we have established some existence results for the solution of the GTCP. The GTAVE considered by us is a generalized form of the TAVE considered in [5] and it is different from the TAVE considered in [11]. Also it subsumes a multilinear system of equations as a particular case, which has numerous applications in tensor complementarity problems [15] and numerical partial differential equations [3].

2. Preliminaries

2.1. Notation

Throughout this paper, we use $\mathbb{R}^n$ to denote the n-dimensional Euclidean space. $[n]$ is used to denote the set $\{1, 2, ..., n\}$. We use bold small letters to denote vectors in $\mathbb{R}^n$, e.g. $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$..., and bold capital letters to denote matrices, e.g. $\mathbf{A}$, $\mathbf{B}$. $\mathbf{0}$ is used to denote the zero vector. For a vector $\mathbf{w} \in \mathbb{R}^n$ and a natural number m, $\mathbf{w}^{[m-1]}$ denotes a vector having its ith component as w_i^{m-1}. We write $\mathbf{w} \geq \mathbf{0}$ if each component of $\mathbf{w}$ is nonnegative. We use $\min\{\mathbf{u}, \mathbf{v}\}$ to denote a vector having its ith component equal to $\min\{u_i, v_i\}$. Note that

$$\min\{\mathbf{u}, \mathbf{v}\} = \mathbf{0} \text{ if and only if } \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0} \text{ and } \mathbf{u}^T\mathbf{v} = 0.$$

For any two natural numbers m and n, an order m and dimension n real tensor $\mathcal{A}$ is a multidimensional array with entries $(a_{i_1 i_2 \ldots i_m})$, where $1 \leq i_j \leq n$, for all $1 \leq j \leq m$ and $a_{i_1 i_2 \ldots i_m} \in \mathbb{R}$. The entries $a_{ii\ldots i}$ are termed as diagonal entries of $\mathcal{A}$, and the remaining entries are referred as off-diagonal entries. We use calligraphic letters to denote tensors, e.g. $\mathcal{A}$, $\mathcal{B}$ and $\mathbb{T}^{m,n}$ is used to denote the set of all mth order n dimensional real tensors. $\mathcal{I}$ denotes the identity tensor (all the diagonal entries are equal to 1 and the off-diagonal entries are zero). A tensor $\mathcal{A}$ in $\mathbb{T}^{m,n}$ is known as Z-tensor, if all of its off-diagonal entries are nonpositive. If all the entries of a tensor are nonnegative, then it is said to be a nonnegative tensor. A tensor $\mathcal{A} \in \mathbb{T}^{m,n}$ is said to be a P-tensor [20], if for each $\mathbf{w}(\neq \mathbf{0}) \in \mathbb{R}^n$, there exists $i \in [n]$ such that $w_i \neq 0$ and $w_i(\mathcal{A}\mathbf{w}^{m-1})_i > 0$. A scalar λ is said to be an H-eigenvalue of a tensor $\mathcal{A}$, if there exists a nonzero vector $\mathbf{w}$ in $\mathbb{R}^n$ such that

$$\mathcal{A}\mathbf{w}^{m-1} = \lambda \mathbf{w}^{[m-1]}.$$

Let the set of all H-eigenvalues of $\mathcal{A}$ be denoted by $\sigma(\mathcal{A})$. The spectral radius of $\mathcal{A}$ is denoted as $\rho(\mathcal{A})$ and defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

We call a Z-tensor $\mathcal{A}$ as a strong M-tensor [4], if there exits a real number θ and a nonnegative tensor $\mathcal{B}$ such that $\mathcal{A} = \theta\mathcal{I} - \mathcal{B}$ and $\theta > \rho(\mathcal{B})$.

2.2. *Degree Theory*

We describe some basic results of degree theory that will be useful to prove our main theorems. For details, we refer the readers to [10]. Let B be a nonempty, bounded, open subset of $\mathbb{R}^n$. $\bar{B}$, ∂B denotes the closure and boundary of B, respectively. Let $\Phi : \bar{B} \to \mathbb{R}^n$ be a continuous function and $\mathbf{u} \notin \Phi(\partial B)$, then the degree of Φ over B with respect to $\mathbf{u}$ is defined. It is denoted by $\deg(\Phi, B, \mathbf{u})$ and it is always an integer. If $\deg(\Phi, B, \mathbf{u}) \neq 0$, then $\Phi(\mathbf{w}) = \mathbf{u}$ has a solution in B. The following properties hold for the $\deg(\Phi, B, \mathbf{u})$.

(i) $\deg(\mathrm{I}, B, \mathbf{u}) = 1$ if $\mathbf{u} \in B$, where I denotes the identity function.

(ii) **(Nearness Property)**. Let $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that $\sup\{\|\Phi(\mathbf{w}) - \Psi(\mathbf{w})\|_\infty : \mathbf{w} \in \bar{B}\}$ is sufficiently small, then $\deg(\Phi, B, \mathbf{u}) = \deg(\Psi, B, \mathbf{u})$, where $\|\cdot\|_\infty$ denotes the max-norm of vectors in $\mathbb{R}^n$.

(iii) **(Homotopy Invariance Property)**. Let $\mathcal{Z}(\mathbf{w}, \theta) : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ be a homotopy and the set $\Delta = \{\mathbf{w} \in \mathbb{R}^n : \mathcal{Z}(\mathbf{w}, \theta) = \mathbf{0}$ for some $0 \leq \theta \leq 1\}$ be bounded. Let S be a bounded set such that $\Delta \subseteq S$, then we have

$$\deg(\mathcal{Z}(\cdot, 0), S, \mathbf{0}) = \deg(\mathcal{Z}(\cdot, 1), S, \mathbf{0}).$$

(iv) Let $\zeta : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and $\zeta(\mathbf{w}) = \mathbf{0}$ if and only if $\mathbf{w} = \mathbf{0}$. Then $\deg(\zeta, S, \mathbf{0})$ is invariant for any bounded open set S containing $\mathbf{0}$, and we denote $\deg(\zeta, S, \mathbf{0})$ as $\deg(\zeta, \mathbf{0})$. Also $\deg(\zeta, \mathbf{0})$ is known as the local degree of ζ at $\mathbf{0}$.

(v) **(Poincar*é*-Bohl Theorem)** Let $B \subseteq \mathbb{R}^n$ be open and bounded and $\mathbf{u} \in \mathbb{R}^n$. Let $\Phi, \Psi : \mathbb{R}^n \to \mathbb{R}^n$ be two continuous functions. If the line segment $[\Phi(\mathbf{w}), \Psi(\mathbf{w})]$ does not contain $\mathbf{u}$, for all $\mathbf{w} \in \partial B$, then $\deg(\Phi, B, \mathbf{u}) = \deg(\Psi, B, \mathbf{u})$.

(vi) If $\Phi'(\mathbf{w})$ is the derivative of Φ at $\mathbf{w}$, then we denote $J_\Phi(\mathbf{w})$ as the Jacobian determinant of the function Φ at the point $\mathbf{w}$. The set $Z_\Phi = \{\tilde{\mathbf{w}} \in B : J_\Phi(\tilde{\mathbf{w}}) = 0\}$ is said to be the set of all critical points of Φ. Let $\mathbf{v} \notin \Phi(Z_\Phi)$, then $\Phi^{-1}(\mathbf{v})$ is a finite set and

$$\deg(\Phi, B, \mathbf{v}) = \sum_{\mathbf{w} \in \Phi^{-1}(\mathbf{v})} \mathrm{sgn} J_\Phi(\mathbf{w}).$$

Now we recall some results that will be useful throughout our paper.

Theorem 2.1. *[8, Theorem 5.1] Let $\mathcal{A}$ in $\mathbb{T}^{m,n}$ be an even order* Z*-tensor. If $\mathcal{A}$ is a strong* M*-tensor, then $\mathcal{A}\mathbf{w}^{m-1} = \mathbf{0}$ implies $\mathbf{w} = \mathbf{0}$ and* $\deg(\mathcal{A}, \mathbf{0}) = 1$.

Proposition 2.2. *[6, Proposition 2.1.4] Let B be a nonempty, bounded, open subset of $\mathbb{R}^n$. Let $\zeta : \bar{B} \to \mathbb{R}^n$ be an injective continuous function. Then for any $\mathbf{u} \in \zeta(B)$, we have* $\deg(\zeta, B, \mathbf{u}) \neq 0$.

3. Results

In this section, we define the generalized tensor absolute value equation and the generalized tensor complementarity problem and establish the equivalence between these two problems.

Definition 3.1. Let $\mathcal{A}, \mathcal{B} \in \mathbb{T}^{m,n}$ and $\mathbf{b} \in \mathbb{R}^n$. The *generalized tensor absolute value equation* is expressed as

$$\mathcal{A}\mathbf{w}^{m-1} - |\mathcal{B}\mathbf{w}^{m-1}| + \mathbf{b} = \mathbf{0}, \tag{5}$$

where $(|\mathcal{B}\mathbf{w}^{m-1}|)_i = |\sum_{i_2,i_3,\ldots,i_m=1}^{n} b_{ii_2\ldots i_m} w_{i_2} w_{i_3} \ldots w_{i_m}|$. We denote this problem as GTAVE($\mathcal{A}$, $\mathcal{B}$, $\mathbf{b}$). The set of all such $\mathbf{w}$ in $\mathbb{R}^n$ that satisfies Eq.(5) is denoted by SOLGTAVE($\mathcal{A}, \mathcal{B}, \mathbf{b}$).

Definition 3.2. Let $\mathcal{A}, \mathcal{B} \in \mathbb{T}^{m,n}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then the *generalized tensor complementarity problem* (GTCP) is to find a vector $\mathbf{w}$ in $\mathbb{R}^n$ satisfying

$$\mathcal{A}\mathbf{w}^{m-1} + \mathbf{a} \geq \mathbf{0},\ \mathcal{B}\mathbf{w}^{m-1} + \mathbf{b} \geq \mathbf{0} \ \text{ and } \ (\mathcal{A}\mathbf{w}^{m-1} + \mathbf{a})^T(\mathcal{B}\mathbf{w}^{m-1} + \mathbf{b}) = 0. \tag{6}$$

This problem is denoted as GTCP($\mathcal{A}$, $\mathcal{B}$, $\mathbf{a}$, $\mathbf{b}$). The set of all such $\mathbf{w}$ in $\mathbb{R}^n$ satisfying Eq.(6) is denoted as SOLGTCP($\mathcal{A}, \mathcal{B}, \mathbf{a}, \mathbf{b}$), and known as the solution set to the GTCP($\mathcal{A}$, $\mathcal{B}$, $\mathbf{a}$, $\mathbf{b}$).

It can be easily seen that $\mathbf{w}$ solves $\min\{\mathcal{A}\mathbf{w}^{m-1} + \mathbf{a}, \mathcal{B}\mathbf{w}^{m-1} + \mathbf{b}\} = \mathbf{0}$, if and only if $\mathbf{w}$ is a solution of the GTCP($\mathcal{A}$, $\mathcal{B}$, $\mathbf{a}$, $\mathbf{b}$). In the following, we prove that the generalized tensor absolute value equation is equivalent to the generalized tensor complementarity problem.

Theorem 3.3. *The* GTAVE($\mathcal{A}$, $\mathcal{B}$, $\mathbf{b}$) *given by (5) is equivalent to the* GTCP($\mathcal{A} - \mathcal{B}, \mathcal{A} + \mathcal{B}, \mathbf{b}, \mathbf{b}$) *for any* $\mathbf{b} \in \mathbb{R}^n$.

Proof. The GTAVE($\mathcal{A}$, $\mathcal{B}$, $\mathbf{b}$) in Eq.(5) is given by

$$\mathcal{A}\mathbf{w}^{m-1} - |\mathcal{B}\mathbf{w}^{m-1}| + \mathbf{b} = \mathbf{0}.$$

For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have

$$\min\{\mathbf{u}, \mathbf{v}\} = \frac{1}{2}(\mathbf{u} + \mathbf{v} - |\mathbf{u} - \mathbf{v}|).$$

This implies

$$\min\{(\mathcal{A} + \mathcal{B})\mathbf{w}^{m-1} + \mathbf{b}, (\mathcal{A} - \mathcal{B})\mathbf{w}^{m-1} + \mathbf{b}\} = \mathcal{A}\mathbf{w}^{m-1} - |\mathcal{B}\mathbf{w}^{m-1}| + \mathbf{b}. \tag{7}$$

From Eq.(7), it is clear that $\mathbf{w}$ solves the GTAVE($\mathcal{A}, \mathcal{B}, \mathbf{b}$) if and only if $\mathbf{w}$ is a solution of the GTCP($\mathcal{A} + \mathcal{B}, \mathcal{A} - \mathcal{B}, \mathbf{b}, \mathbf{b}$). This completes the proof. □

It is clear from the definition of GTCP that the generalized tensor complementarity problem is a particular case of the polynomial complementarity problems (PCP) discussed in [14]. For the GTCP($\mathcal{A}$, $\mathcal{B}$, $\mathbf{a}$, $\mathbf{b}$), we recall the following result from [14].

Proposition 3.4. *[14, Proposition 4.4] Let* $\zeta(\mathbf{w}) = \min\{\Phi(\mathbf{w}), \Psi(\mathbf{w})\}$, *where* $\Phi(\mathbf{w}) = \mathcal{A}\mathbf{w}^{m-1}$ *and* $\Psi(\mathbf{w}) = \mathcal{B}\mathbf{w}^{m-1}$, *for given tensors* $\mathcal{A}$, $\mathcal{B}$ *in* $\mathbb{T}^{m,n}$. *Suppose that the following conditions hold.*

(i) $\zeta(\mathbf{w}) = \mathbf{0} \implies \mathbf{w} = \mathbf{0}$.

(ii) $\deg(\zeta, \mathbf{0}) \neq 0$.

Then the GTCP($\mathcal{A}$, $\mathcal{B}$, $\mathbf{a}$, $\mathbf{b}$) *has a nonempty and compact solution set for each* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

From Theorem 3.3 and Proposition 3.4, we give the following existence theorem for the solution of the GTAVE($\mathcal{A}, \mathcal{B}, \mathbf{b}$).

Theorem 3.5. *Let $\eta : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined as $\eta(\mathbf{w}) = \mathcal{A}\mathbf{w}^{m-1} - |\mathcal{B}\mathbf{w}^{m-1}|$, where $\mathcal{A}, \mathcal{B} \in \mathbb{T}^{m,n}$. Suppose that the following conditions hold.*

(i) $\eta(\mathbf{w}) = \mathbf{0} \implies \mathbf{w} = \mathbf{0}$.

(ii) $\deg(\eta, \mathbf{0}) \neq 0$.

Then the GTAVE$(\mathcal{A}, \mathcal{B}, \mathbf{b})$ *has a nonempty compact solution set for each* $\mathbf{b} \in \mathbb{R}^n$.

We now give an example that illustrates the above theorem.

Example 3.6. Let $\mathcal{A} \in \mathbb{T}^{4,2}$ with $a_{1111} = \frac{3}{4}, a_{2222} = 1, a_{1222} = -1$ and all other zero. Let $\mathcal{B} \in \mathbb{T}^{4,2}$ such that $b_{1111} = \frac{1}{4}, b_{1222} = 1$ and all other zero. Then for any $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$, the function $\eta(\mathbf{w}) = \mathcal{A}\mathbf{w}^3 - |\mathcal{B}\mathbf{w}^3|$ is given as

$$\eta(\mathbf{w}) = (\frac{3}{4}w_1^3 - w_2^3 - |\frac{1}{4}w_1^3 + w_2^3|, w_2^3).$$

It is easy to see that $\eta(\mathbf{w}) = \mathbf{0}$ if and only if $\mathbf{w} = \mathbf{0}$. We now show that $\deg(\eta, \mathbf{0}) \neq 0$. Let us denote $C = \mathcal{A} - \mathcal{B}$ and $\Phi(\mathbf{w}) = C\mathbf{w}^3$. Then $c_{1111} = \frac{1}{2}, c_{2222} = 1, c_{1222} = -2$ and all other entries are zero. So C is an even order Z-tensor. Also, it can be easily verified that C is a P-tensor. From [8, Theorem 4.3], it follows that C is a strong M-tensor. By Theorem 2.1, we get $\Phi(\mathbf{w}) = \mathbf{0} \implies \mathbf{w} = \mathbf{0}$ and $\deg(\Phi, \mathbf{0}) = 1$. As $\eta(\mathbf{w}) - \Phi(\mathbf{w})$ is a continuous function, so there exists an open neighbourhood B of $\mathbf{0}$ such that $\sup\{\|\eta(\mathbf{w}) - \Phi(\mathbf{w})\|_\infty : \mathbf{w} \in \bar{B}\}$ is sufficiently small. So by property 2 (nearness property), we get $\deg(\eta, B, \mathbf{0}) = \deg(\Phi, B, \mathbf{0}) = \deg(\Phi, \mathbf{0})$. As $\eta(\mathbf{w}) = \mathbf{0} \iff \mathbf{w} = \mathbf{0}$, and $\deg(\Phi, \mathbf{0}) \neq 0$, it follows that $\deg(\eta, B, \mathbf{0}) = \deg(\eta, \mathbf{0}) \neq 0$. Therefore by Theorem 3.5, GTAVE$(\mathcal{A}, \mathcal{B}, \mathbf{b})$ has a nonempty compact solution set for each $\mathbf{b} \in \mathbb{R}^2$.

Definition 3.7. A function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a Z^+-*function*, if the system

$$\Phi(\mathbf{w}) + t\mathbf{w} = \mathbf{0}, \ \ t \geq 0, \ \ \mathbf{w} \neq \mathbf{0},$$

does not have a solution in $\mathbb{R}^n$.

We now give some examples of Z^+-functions.

Example 3.8. Let $\mathcal{A} \in \mathbb{T}^{m,n}$ be a Z^+-tensor [22], then it is easy to see that $\Phi(\mathbf{w}) = \mathcal{A}\mathbf{w}^{m-1}$ is a Z^+-function.

Example 3.9. A pair of tensors $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}, \mathcal{B} \in \mathbb{T}^{m,n}$ is said to be a P^*-pair, if for each non-zero $\mathbf{w} \in \mathbb{R}^n$, there exists an index $k \in [n]$ such that

$$w_k(\mathcal{A}\mathbf{w}^{m-1})_k > 0 \ \text{ and } \ w_k(\mathcal{B}\mathbf{w}^{m-1})_k > 0.$$

If $(\mathcal{A}, \mathcal{B})$ is a P^*-pair, then $\Phi(\mathbf{w}) = \min\{\mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\}$ is a Z^+-function. Suppose not, then there exists some $(\theta, \mathbf{w}) \in (\mathbb{R}_+ \times \mathbb{R}^n \setminus \{\mathbf{0}\})$ such that $\Phi(\mathbf{w}) = -\theta\mathbf{w}$. This implies $\min\{\mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\} = -\theta\mathbf{w}$. Therefore $(\mathcal{A}\mathbf{w}^{m-1})_i = -\theta w_i$, for each $i \in S$ and $(\mathcal{B}\mathbf{w}^{m-1})_j = -\theta w_j$, for each $j \in [n] \setminus S$, where S is a subset (may be empty) of $[n]$. Then

$$w_i(\mathcal{A}\mathbf{w}^{m-1})_i = -\theta w_i^2 \leq 0, \ \forall \ i \in S \text{ and } w_j(\mathcal{B}\mathbf{w}^{m-1})_j = -\theta w_j^2 \leq 0, \ \forall \ j \in [n] \setminus S. \tag{8}$$

From Eq.(8), it is easy to see that there is no such $k \in [n]$ satisfying $w_k(\mathcal{A}\mathbf{w}^{m-1})_k > 0$ and $w_k(\mathcal{B}\mathbf{w}^{m-1})_k > 0$, which contradicts the fact that the tensor pair $(\mathcal{A}, \mathcal{B})$ is a P^*-pair. Hence $\Phi(\mathbf{w}) = \min\{\mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\}$ is a Z^+-function.

We now give the solvability of an equation for the positively homogeneous, Z^+-function. Recall that a function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a positively homogeneous function of order d, if $\Phi(\alpha\mathbf{w}) = \alpha^d\Phi(\mathbf{w})$, for all $\alpha \geq 0$ and $\mathbf{w} \in \mathbb{R}^n$. We now prove our result.

Theorem 3.10. *Let $\zeta : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and positively homogeneous function with order d ($d \geq 1$) and $\mathbf{b} \in \mathbb{R}^n$. Then either $\zeta(\mathbf{w}) = \mathbf{b}$ has a solution or there exists $(\bar{t}, \bar{\mathbf{w}}) \in (\mathbb{R}_+ \times \mathbb{R}^n \setminus \{\mathbf{0}\})$ such that $\zeta(\bar{\mathbf{w}}) + \bar{t}\bar{\mathbf{w}} = \mathbf{0}$.*

Proof. Let $\mathbf{b} \in \mathbb{R}^n$ be fixed. For any $r > 0$, let B_r be the open ball with radius r centred at $\mathbf{0}$. So

$$B_r = \{\mathbf{w} \in \mathbb{R}^n : \|\mathbf{w}\| < r\} \text{ and } \partial B_r = \{\mathbf{w} \in \mathbb{R}^n : \|\mathbf{w}\| = r\}.$$

Consider the homotopy

$$\mathcal{Z}(\mathbf{w}, \theta) = \theta\mathbf{w} + (1-\theta)[\zeta(\mathbf{w}) - \mathbf{b}], \text{ for all } \mathbf{w} \in \partial B_r \text{ and } 0 \leq \theta \leq 1.$$

Then $\mathcal{Z}(\mathbf{w}, 0) = \zeta(\mathbf{w}) - \mathbf{b}$ and $\mathcal{Z}(\mathbf{w}, 1) = \mathbf{w}$. We consider the following two cases:
(a) There exists some $r > 0$ such that $\mathcal{Z}(\mathbf{w}, \theta)$ does not vanish on $\partial B_r \times [0, 1]$. By Poincaré-Bohl Theorem, we get $\deg(\mathrm{I}, B_r, \mathbf{0}) = \deg(\mathcal{Z}(\cdot, 0), B_r, \mathbf{0})$. Since $\deg(\mathrm{I}, B_r, \mathbf{0}) = 1$, therefore we get $\deg(\mathcal{Z}(\cdot, 0), B_r, \mathbf{0}) = 1$. Hence there exists at least one solution of the equation $\zeta(\mathbf{w}) - \mathbf{b} = \mathbf{0}$ in B_r.
(b) For each $r > 0$, there exists $\mathbf{w}_r$ in ∂B_r and θ_r in $[0, 1]$ such that

$$\mathcal{Z}(\mathbf{w}_r, \theta_r) = \theta_r\mathbf{w}_r + (1-\theta_r)(\zeta(\mathbf{w}_r) - \mathbf{b}) = \mathbf{0}. \tag{9}$$

Now we have the following sub cases:
(i) When $\theta_r = 0$ for some $r > 0$, then we get $\zeta(\mathbf{w}_r) - \mathbf{b} = \mathbf{0}$. Hence $\mathbf{w}_r$ is a solution of $\zeta(\mathbf{w}) = \mathbf{b}$.
(ii) If $\theta_r = 1$ for some $r > 0$, then we get $\mathbf{w}_r = \mathbf{0}$, but $\|\mathbf{w}_r\| = r > 0$. So we get a contradiction. Thus $\theta_r < 1$.
(iii) If $0 < \theta_r < 1$ for each r, let $\alpha_r = \frac{\theta_r}{(1-\theta_r)} > 0$. Then from Eq.(9), we get

$$\alpha_r\mathbf{w}_r + (\zeta(\mathbf{w}_r) - \mathbf{b}) = \mathbf{0} \implies \alpha_r\frac{\mathbf{w}_r}{\|\mathbf{w}_r\|^d} + \frac{(\zeta(\mathbf{w}_r) - \mathbf{b})}{\|\mathbf{w}_r\|^d} = \mathbf{0}. \tag{10}$$

In Eq.(10), as $r \to \infty$, we can see that $\{\frac{\alpha_r}{\|\mathbf{w}_r\|^{d-1}}\}$ is bounded. Without loss of generality, we assume that $\frac{\alpha_r}{\|\mathbf{w}_r\|^{d-1}} \to \alpha$ and $\frac{\mathbf{w}_r}{\|\mathbf{w}_r\|} \to \bar{\mathbf{w}}$. Clearly $\alpha \geq 0$ and $\bar{\mathbf{w}} \neq \mathbf{0}$. In Eq.(10), letting $r \to \infty$, we get $\zeta(\bar{\mathbf{w}}) + \alpha\bar{\mathbf{w}} = \mathbf{0}$. Hence our conclusion follows. □

Corollary 3.11. *If $\zeta(\mathbf{w}) = \mathcal{A}\mathbf{w}^{m-1} - |\mathcal{B}\mathbf{w}^{m-1}|$, where $m \geq 2$ is a Z^+-function, then* SOLGTAVE$(\mathcal{A}, \mathcal{B}, \mathbf{b})$ *is nonempty for all $\mathbf{b} \in \mathbb{R}^n$.*

Proof. Given that $\zeta(\mathbf{w}) = \mathcal{A}\mathbf{w}^{m-1} - |\mathcal{B}\mathbf{w}^{m-1}|$ is a Z^+-function. Note that ζ is continuous and positively homogeneous of order $m-1$, therefore from Theorem 3.10, for each $\mathbf{b} \in \mathbb{R}^n$ there exists $\mathbf{w} \in \mathbb{R}^n$ such that $\zeta(\mathbf{w}) + \mathbf{b} = \mathbf{0}$. This implies there exists $\mathbf{w} \in \mathbb{R}^n$ such that $\mathcal{A}\mathbf{w}^{m-1} - |\mathcal{B}\mathbf{w}^{m-1}| + \mathbf{b} = \mathbf{0}$. Hence our conclusion follows. □

Definition 3.12. Let $\mathcal{A}, \mathcal{B} \in \mathbb{T}^{m,n}$, then the tensor pair $(\mathcal{A}, \mathcal{B})$ is said to be a Karamardian pair, if the following conditions are satisfied.

(i) SOLGTCP$(\mathcal{A}, \mathcal{B}, \mathbf{0}, \mathbf{0}) = \{\mathbf{0}\}$.

(ii) SOLGTCP$(\mathcal{A}, \mathcal{B}, \mathbf{0}, \mathbf{d}) = \{\mathbf{0}\}$, for some $\mathbf{d} > \mathbf{0}$.

We present a simple example of a Karamardian pair.

Example 3.13. Let m be even. Let $\mathcal{A} \in \mathbb{T}^{m,n}$ be a P-tensor and $\mathcal{B} \in \mathbb{T}^{m,n}$ be a diagonal tensor with positive diagonal entries, say $b_{ii\ldots i}$. Then $(\mathcal{A}, \mathcal{B})$ is a Karamardian pair. It can be seen as follows.

(i) Let $\mathbf{w} \in$ SOLGTCP$(\mathcal{A}, \mathcal{B}, \mathbf{0}, \mathbf{0})$, then we have $\mathcal{A}\mathbf{w}^{m-1} \geq \mathbf{0}$, $\mathcal{B}\mathbf{w}^{m-1} \geq \mathbf{0}$ and $(\mathcal{A}\mathbf{w}^{m-1})^T(\mathcal{B}\mathbf{w}^{m-1}) = 0$. This implies $b_{ii\ldots i}{w_i}^{m-1}(\mathcal{A}\mathbf{w}^{m-1})_i = 0$, for each $i \in [n]$. Since $b_{ii\ldots i} > 0$, for each $i \in [n]$ and m is even, we get $w_i(\mathcal{A}\mathbf{w}^{m-1})_i = 0$, for each $i \in [n]$. Due to $\mathcal{A}$ being a P-tensor, we get $\mathbf{w} = \mathbf{0}$.

(ii) Let $\mathbf{d} \in \mathbb{R}^n$ such that $d_i = 1$, for each $i \in [n]$. Consider a vector $\mathbf{w}$ in SOLGTCP$(\mathcal{A}, \mathcal{B}, \mathbf{0}, \mathbf{d})$. This implies $\mathcal{A}\mathbf{w}^{m-1} \geq \mathbf{0}$, $\mathcal{B}\mathbf{w}^{m-1} + \mathbf{d} \geq \mathbf{0}$ and $(\mathcal{A}\mathbf{w}^{m-1})^T(\mathcal{B}\mathbf{w}^{m-1} + \mathbf{d}) = 0$. This implies $(b_{ii\ldots i}{w_i}^{m-1} + 1)(\mathcal{A}\mathbf{w}^{m-1})_i = 0$, for each $i \in [n]$. Therefore, we get $(b_{ii\ldots i}{w_i}^{m-1})(\mathcal{A}\mathbf{w}^{m-1})_i = -(\mathcal{A}\mathbf{w}^{m-1})_i \leq 0$, for each $i \in [n]$. Since $b_{ii\ldots i} > 0$, for each $i \in [n]$ and m is even, we get $w_i(\mathcal{A}\mathbf{w}^{m-1})_i \leq 0$, for each $i \in [n]$. Due to $\mathcal{A}$ being a P-tensor, we get $\mathbf{w} = \mathbf{0}$.

Theorem 3.14. *Suppose $\mathcal{A}, \mathcal{B} \in \mathbb{T}^{m,n}$, where $m \geq 2$. If there exists C in $\mathbb{T}^{m,n}$ such that*

(i) $C\mathbf{w}^{m-1} = \mathbf{0}$ *implies* $\mathbf{w} = \mathbf{0}$ *and* $\deg(C, \mathbf{0}) \neq 0$.

(ii) *For any* $\alpha \geq 0$, $\min\{(\mathcal{A} + \alpha C)\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\} = \mathbf{0} \implies \mathbf{w} = \mathbf{0}$.

(iii) $(C, \mathcal{B})$ *is a Karamardian pair.*

then SOLGTCP$(\mathcal{A}, \mathcal{B}, \mathbf{a}, \mathbf{b})$ *is nonempty and compact for each* $\mathbf{a}, \mathbf{b}$ *in* $\mathbb{R}^n$.

Proof. Let $H(\mathbf{w}) = \min\{\mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\}$. From condition (ii), we get $H(\mathbf{w}) = \mathbf{0}$ implies $\mathbf{w} = \mathbf{0}$. By Proposition 3.4, it is enough to show that $\deg(H, \mathbf{0}) \neq 0$. Since $(C, \mathcal{B})$ is a Karamardian pair, so there exists $\mathbf{d} > \mathbf{0}$ such that SOLGTCP$(C, \mathcal{B}, \mathbf{0}, \mathbf{d}) = \{\mathbf{0}\}$. Let this $\mathbf{d} > \mathbf{0}$ be fixed and $\mathcal{Z}(\mathbf{w}, \theta) : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ be such that

$$\mathcal{Z}(\mathbf{w}, \theta) = \min\{\theta(C\mathbf{w}^{m-1}) + (1 - \theta)\mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1} + \theta\mathbf{d}\}$$

Then $\mathcal{Z}(\mathbf{w}, 0) = H(\mathbf{w})$ and $\mathcal{Z}(\mathbf{w}, 1) = \min\{C\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1} + \mathbf{d}\}$. Let $\Delta = \{\mathbf{w} \in \mathbb{R}^n : \mathcal{Z}(\mathbf{w}, \theta) = \mathbf{0}$ for some $\theta \in [0, 1]\}$. We claim that Δ is bounded. Assume contrary. Suppose that there exists an unbounded sequence $\{\mathbf{w}_n\}$ in Δ and a sequence $\{\theta_n\} \subseteq [0, 1]$ such that $\mathcal{Z}(\mathbf{w}_n, \theta_n) = \mathbf{0}, \forall\, n$. Then

$$\begin{aligned} &\min\{\theta_n(C\mathbf{w}_n^{m-1}) + (1 - \theta_n)\mathcal{A}(\mathbf{w}_n)^{m-1}, \mathcal{B}(\mathbf{w}_n)^{m-1} + \theta_n\mathbf{d}\} = \mathbf{0} \\ \implies &\min\{\frac{(\theta_n(C\mathbf{w}_n^{m-1}) + (1 - \theta_n)\mathcal{A}(\mathbf{w}_n)^{m-1})}{\|\mathbf{w}_n\|^{m-1}}, \frac{(\mathcal{B}(\mathbf{w}_n)^{m-1} + \theta_n\mathbf{d})}{\|\mathbf{w}_n\|^{m-1}}\} = \mathbf{0}. \end{aligned} \tag{11}$$

Assume (without loss of generality) that $\frac{\mathbf{w}_n}{\|\mathbf{w}_n\|} \to \mathbf{w}$, and $\theta_n \to \theta$ as $n \to \infty$. Then from Eq.(11), as $n \to \infty$, we get

$$\min\{\theta(C\mathbf{w}^{m-1}) + (1 - \theta)\mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\} = \mathbf{0}.$$

We have the following cases:
(i): If $\theta = 0$, then using the condition (ii), we get

$$\min\{\mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\} = \mathbf{0} \text{ implies } \mathbf{w} = \mathbf{0}.$$

(ii): When $\theta = 1$, then from the condition (iii),

$$\min\{C\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\} = \mathbf{0} \text{ implies } \mathbf{w} = \mathbf{0}.$$

(iii): When $\theta \in (0, 1)$, we have

$$\begin{aligned} &\min\{\Big(\frac{\theta}{1 - \theta}\Big)C\mathbf{w}^{m-1} + \mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\} = \mathbf{0} \\ \implies &\min\{\alpha(C\mathbf{w}^{m-1}) + \mathcal{A}\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\} = \mathbf{0} \\ \implies &\min\{(\mathcal{A} + \alpha C)\mathbf{w}^{m-1}, \mathcal{B}\mathbf{w}^{m-1}\} = \mathbf{0}, \end{aligned}$$

where $\alpha = \left(\frac{\theta}{1-\theta}\right) > 0$. From the condition (ii), we get $\mathbf{w} = \mathbf{0}$. Since $\|\mathbf{w}\| = 1$, we arrive at a contradiction in each of the above cases. Hence Δ is bounded. Suppose that B is a bounded open set containing Δ, then by property 3 (homotopy invariance), we get

$$\deg(\mathcal{Z}(\cdot, 1), B, \mathbf{0}) = \deg(\mathcal{Z}(\cdot, 0), B, \mathbf{0}). \tag{12}$$

Note that condition (iii) yields $\mathcal{Z}(\mathbf{w},1)=\mathbf{0}$ if and only if $\mathbf{w}=\mathbf{0}$. Therefore $\deg(\mathcal{Z}(\cdot,1),B,\mathbf{0})=\deg(\mathcal{Z}(\cdot,1),\mathbf{0})$. Now we claim that $\deg(\mathcal{Z}(\cdot,1),\mathbf{0})\neq 0$. It is easy to verify that that $\mathcal{B}\mathbf{w}^{m-1}+\mathbf{d}$ is close to $\mathbf{d}>\mathbf{0}$, whenever $\mathbf{w}$ is near to zero. Hence there exists an open neighbourhood Ω of $\mathbf{0}$ such that $\sup\{\|\mathcal{Z}(\mathbf{w},1)-C\mathbf{w}^{m-1}\|:\mathbf{w}\in\bar{\Omega}\}$ is sufficiently small. So by property 2 (nearness property) and condition (i), we get

$$\deg(\mathcal{Z}(\cdot,1),\mathbf{0})=\deg(\mathcal{Z}(\cdot,1),\Omega,\mathbf{0})=\deg(C,\Omega,\mathbf{0})=\deg(C,\mathbf{0}). \tag{13}$$

Also from condition (i), we have $\deg(C,\mathbf{0})\neq 0$. Therefore, from Eq.(13), we get $\deg(\mathcal{Z}(\cdot,1),\mathbf{0})\neq 0$. Now from the Eqs.(12) and (13), we get $\deg(H,\mathbf{0})\neq 0$. Hence our claim is proved. □

Corollary 3.15. *Let $\mathcal{A},\mathcal{B}\in\mathbb{T}^{m,n}$ and $m\ (\geq 2)$ be even. Assume that the following statements are true.*

(i) *For any $\alpha\geq 0$, $\min\{(\mathcal{A}+\alpha\mathcal{I})\mathbf{w}^{m-1},\mathcal{B}\mathbf{w}^{m-1}\}=\mathbf{0}\implies\mathbf{w}=\mathbf{0}$.*

(ii) *$(\mathcal{I},\mathcal{B})$ is a Karamardian pair.*

Then SOLGTCP$(\mathcal{A},\mathcal{B},\mathbf{a},\mathbf{b})$ *is nonempty and compact for each $\mathbf{a},\mathbf{b}$ in $\mathbb{R}^n$. Hence* SOLGTAVE$(\mathcal{A}+\mathcal{B},\mathcal{A}-\mathcal{B},\mathbf{b})$ *is nonempty and compact for any $\mathbf{b}\in\mathbb{R}^n$.*

Proof. Let us denote $\Phi(\mathbf{w})=\mathcal{I}\mathbf{w}^{m-1}$. Then $\Phi(\mathbf{w})=\mathbf{0}$ implies $\mathbf{w}=\mathbf{0}$. Due to m being even, it follows that $\Phi(\mathbf{w})=\mathbf{w}^{[m-1]}$ is an injective continuous function on $\mathbb{R}^n$. From Proposition 2.2, we get $\deg(\Phi,\mathbf{0})\neq 0$. Our Proof follows by taking $C=\mathcal{I}$ in Theorem 3.14. □

Corollary 3.16. *Let $\mathcal{A},\mathcal{B}\in\mathbb{T}^{m,n}$ $(m\geq 2)$ with $\mathcal{A}$ be an even order* Z*-tensor. Suppose that the following conditions hold.*

(i) *$\mathcal{A}$ is a strong* M*-tensor.*

(ii) *$(\mathcal{A},\mathcal{B})$ is a Karamardian pair.*

Then SOLGTCP$(\mathcal{A},\mathcal{B},\mathbf{a},\mathbf{b})$ *is nonempty and compact for each $\mathbf{a},\mathbf{b}$ in $\mathbb{R}^n$. Hence* SOLGTAVE$(\mathcal{A}+\mathcal{B},\mathcal{A}-\mathcal{B},\mathbf{b})$ *is nonempty and compact for any $\mathbf{b}\in\mathbb{R}^n$.*

Proof. Given that $\mathcal{A}$ is an even order Z-tensor. From condition (i), we have $\mathcal{A}$ is a strong M-tensor. By Theorem 2.1, it follows that $\mathcal{A}\mathbf{w}^{m-1}=\mathbf{0}$ implies $\mathbf{w}=\mathbf{0}$ and $\deg(\mathcal{A},\mathbf{0})=1$. By replacing $C=\mathcal{A}$ in Theorem 3.14, our conclusion follows. □

Proposition 3.17. *Let $m\geq 2$ and $\mathcal{A}\in\mathbb{T}^{m,n}$ be a* Z$^+$*-tensor, then the local degree of the mapping $\Phi(\mathbf{w})=\mathcal{A}\mathbf{w}^{m-1}$ at $\mathbf{0}$ is well-defined and nonzero.*

Proof. As $\mathcal{A}$ is a Z$^+$-tensor, so $\Phi(\mathbf{w})=\mathbf{0}$ implies $\mathbf{w}=\mathbf{0}$. Therefore local degree of Φ at $\mathbf{0}$ is well-defined. We claim that $\deg(\Phi,\mathbf{0})\neq 0$. To prove this, let us consider $\mathcal{Z}(\mathbf{w},\theta):\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$ defined as

$$\mathcal{Z}(\mathbf{w},\theta)=\mathcal{A}\mathbf{w}^{m-1}+\theta\mathbf{w}.$$

Then $\mathcal{Z}(\mathbf{w},0)=\mathcal{A}\mathbf{w}^{m-1}$ and $\mathcal{Z}(\mathbf{w},1)=\mathcal{A}\mathbf{w}^{m-1}+\mathbf{w}$. Let $\Delta=\{\mathbf{w}\in\mathbb{R}^n:\mathcal{Z}(\mathbf{w},\theta)=\mathbf{0}$ for some $0\leq\theta\leq 1\}$. We claim that Δ is bounded. Assume Contrary. Suppose there exists an unbounded sequence $\{\mathbf{w}_n\}$ in Δ and a sequence $\{\theta_n\}\subseteq[0,1]$ such that $\mathcal{Z}(\mathbf{w}_n,\theta_n)=\mathbf{0}$. This gives that $\mathcal{A}(\mathbf{w}_n)^{m-1}+\theta_n\mathbf{w}_n=\mathbf{0}$, for large n. This implies

$$\frac{\mathcal{A}\mathbf{w}_n^{m-1}+\theta_n\mathbf{w}_n}{\|\mathbf{w}_n\|^{m-1}}=\mathbf{0}\implies\mathcal{A}\mathbf{y}_n^{m-1}+\theta_n\frac{\mathbf{y}_n}{\|\mathbf{w}_n\|^{m-2}}=\mathbf{0}, \tag{14}$$

where $\{\mathbf{y}_n\}=\{\frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}\}$. As $n\to\infty$, we assume (without loss of generality) that $\theta_n\to\theta$, $\mathbf{y}_n\to\mathbf{y}$ and $\frac{\theta_n}{\|\mathbf{w}_n\|^{m-2}}\to\beta$. Clearly $\beta\geq 0$ and $\|\mathbf{y}\|=1$. As $m\geq 2$, in Eq.(14), letting $n\to\infty$, we get $\mathcal{A}\mathbf{y}^{m-1}+\beta\mathbf{y}=\mathbf{0}$. Since $\mathcal{A}$ is a

Z^+-tensor, we get $\mathbf{y} = \mathbf{0}$, which is a contradiction. Hence the set Δ must be bounded. Suppose that B is a bounded open set containing Δ, then by property 3 (homotopy invariance), we get

$$\deg(\mathcal{Z}(\cdot,1), B, \mathbf{0}) = \deg(\mathcal{Z}(\cdot,0), B, \mathbf{0}) \implies \deg(\mathcal{Z}(\cdot,1), B, \mathbf{0}) = \deg(\Phi, \mathbf{0}). \tag{15}$$

Since $\mathcal{A}$ is a Z^+-tensor, so $\mathcal{Z}(\mathbf{w},1) = \mathbf{0}$ implies $\mathbf{w} = \mathbf{0}$. Therefore $\deg(\mathcal{Z}(\cdot,1), B, \mathbf{0}) = \text{sgn}\,(\det(J(\mathcal{A}\mathbf{w}^{m-1} + \mathbf{w}))|_{\mathbf{w}=\mathbf{0}}) = \text{sgn}\,((\det(J(\mathcal{A}\mathbf{w}^{m-1}) + J(\mathbf{w}))|_{\mathbf{w}=\mathbf{0}}) = 1$, where J denotes the Jacobian matrix. Therefore we get $\deg(\mathcal{Z}(\cdot,1), B, \mathbf{0}) \neq 0$ and hence from Eq.(15), we get $\deg(\Phi, \mathbf{0}) \neq 0$, which proves our claim. $\square$

Theorem 3.18. *Let $m \geq 2$ and $\mathcal{A}, \mathcal{B} \in \mathbb{T}^{m,n}$ such that $(\mathcal{A}, \mathcal{B})$ be a Karamardian pair and let $\mathcal{A}$ be a Z^+-tensor. Then* SOLGTCP$(\mathcal{A}, \mathcal{B}, \mathbf{a}, \mathbf{b})$ *is nonempty for each $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^n$. Hence the* GTAVE$(\mathcal{A}-\mathcal{B}, \mathcal{A}+\mathcal{B}, \mathbf{b})$ *has a solution for every $\mathbf{b} \in \mathbb{R}^n$.*

Proof. Since $\mathcal{A}$ is a Z^+-tensor, from Proposition 3.17, it follows that $\deg(\mathcal{A}, \mathbf{0}) \neq 0$. Replacing C with $\mathcal{A}$ in Theorem 3.14, our conclusion is straightforward. $\square$

We now give a class of tensor pair $(\mathcal{A}, \mathcal{B})$ such that GTAVE$(\mathcal{A}-\mathcal{B}, \mathcal{A}+\mathcal{B}, \mathbf{b})$ has a solution for every $\mathbf{b} \in \mathbb{R}^n$.

Example 3.19. Let $m \geq 2$ be even. Let $\mathcal{A} \in \mathbb{T}^{m,n}$ be a P-tensor and $\mathcal{B} \in \mathbb{T}^{m,n}$ be a diagonal tensor with positive diagonal entries. It can be seen from Example 3.13 that $(\mathcal{A}, \mathcal{B})$ is a Karamardian pair. Also it is easy to verify that a P-tensor must be a Z^+-tensor, so $\mathcal{A}$ is a Z^+-tensor. Hence from Theorem 3.18, we can conclude that SOLGTAVE$(\mathcal{A}-\mathcal{B}, \mathcal{A}+\mathcal{B}, \mathbf{b})$ is nonempty for every $\mathbf{b} \in \mathbb{R}^n$.

4. Conclusion

We have defined a new type of generalized tensor absolute value equation (GTAVE) and obtained its equivalence with the generalized tensor complementarity problem (GTCP). With the help of some existing results about the solution set of the GTCP and using degree theoretic ideas, we establish sufficient conditions to guarantee the existence of the solution of the GTAVE.

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