A note on Kantorovich and Ando inequalities

Mohammad Sababheh[a], Hamid Reza Moradi[b], Ibrahim Halil Gümüş[c], Shigeru Furuichi[d]

[a]*Department of basic sciences, Princess Sumaya University For Technology, Al Jubaiha, Amman 11941, Jordan*
[b]*Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
[c]*Department of Mathematics, Faculty of Arts and Sciences, Adıaman University, Adıyaman, Turkey*
[d]*Department of Information Science, College of Humanities and Sciences, Nihon University, 3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan*

Abstract. The main goal of this exposition is to present further analysis of the Kantorovich and Ando operator inequalities. In particular, a new proof of Ando's inequality is given, a new non-trivial refinement of Kantorovich inequality is shown, and some equivalent forms of the Kantorovich inequality are presented with a Minkowski-type application.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$, with identity $I_{\mathcal{H}}$ (or I if no confusion arises). For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a linear mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is said to be positive if it preserves positive operators. That is, if $\Phi(A) \geq 0$ whenever $A \geq 0$, where an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive, denoted $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. In addition, if the positive linear mapping Φ satisfies $\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$, it is said to be a unital (or normalized) positive linear mapping.

Operator convex and operator concave functions have played a major role in understanding the geometry of $\mathcal{B}(\mathcal{H})$. In this context, a function $f : J \to \mathbb{R}$ is said to be operator convex if $f((1 - t)A + tB) \leq (1 - t)f(A) + tf(B)$ for all $0 \leq t \leq 1$ and self adjoint operators A, B with spectra in the interval J. Operator concave functions are defined similarly. On the other hand, operator monotone functions have a strong relation with operator concave functions. Recall that $f : J \to \mathbb{R}$ is said to be operator monotone if $f(A) \leq f(B)$ for all self-adjoint operators A, B with spectra in the interval J, such that $A \leq B$ (i.e., $B - A \geq 0$). Operator monotone decreasing functions are defined similarly.

Unlike scalar monotony and convexity, operator monotony and convexity are strongly related, as stated in the next proposition, which can be found in [23, Theorem 2.4], and [1, Theorem 2.1, Theorem 3.1, Theorem 2.3 and Theorem 3.7].

Proposition 1.1. *Let $f : (0, \infty) \to [0, \infty)$ be continuous. Then*

  1. *f is operator monotone decreasing if and only if f is operator convex and $f(\infty) < \infty$.*

2. *f is operator monotone increasing if and only if f is operator concave.*

The Choi-Davis inequality states that [3, 5]

$$f(\Phi(A)) \le \Phi(f(A)), \tag{1}$$

for all self adjoint operators $A \in \mathcal{B}(\mathcal{H})$ with spectra in the interval J, all operator convex functions $f : J \to \mathbb{R}$ and all unital positive linear mappings $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$.

In particular, if $A > 0$ (i.e., $A \ge 0$ is invertible), then

$$\Phi(A)^{-1} \le \Phi(A^{-1}), \tag{2}$$

since $f(t) = t^{-1}$ is operator convex on $(0, \infty)$. The inequality (2) can be reversed under the additional condition that $0 < mI \le A \le MI$, for some scalars m, M as follows [8, 16, 20]

$$\Phi\left(A^{-1}\right) \le \frac{(M+m)^2}{4Mm} \Phi(A)^{-1}. \tag{3}$$

Among many other equivalences, we shall prove that (3) is equivalent to

$$\Phi\left(A^2\right) \le \frac{(M+m)^2}{4Mm} \Phi(A)^2. \tag{4}$$

Recalling that the geometric mean of two positive invertible operators A, B is defined by

$$A \sharp B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2} A^{1/2},$$

it is shown in [16] that (3) implies

$$\Phi\left(A^{-1}\right) \sharp \Phi(A) \le \frac{M+m}{2\sqrt{Mm}} I. \tag{5}$$

The inequality (3) is usually referred to as the Kantorovich inequality.

In [6, Lemma 2.1] it is proved that if $f : J \to \mathbb{R}$ is a convex function and $A \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator with spectrum in the interval J, then for any unital positive linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$,

$$f\left(\langle \Phi(A)x, x\rangle\right) \le \langle \Phi(f(A))x, x\rangle \quad (x \in \mathcal{K}, \|x\| = 1).$$

As a corollary (see [7, Theorem 1.4]), we see that if A is a positive operator, then

$$\langle Ax, x\rangle^r \le \langle A^r x, x\rangle \quad (r \ge 1). \tag{6}$$

If the operator is positive and invertible, (6) is also true for $r < 0$.

A strongly related inequality that we will discuss is the celebrated Ando's inequality stating [2]

$$\Phi(A \sharp B) \le \Phi(A) \sharp \Phi(B), \tag{7}$$

where A and B are two positive operators and Φ is a unital positive linear map. In addition to the aforementioned references that have been cited, we refer the reader to [11, 13–15, 18, 19, 22] for further and related discussions.

In this article, we first present a new proof of (7). This will help better understand this celebrated inequality. Then, we use Kantorovich-type inequalities to provide the reverse of Ando's inequality. Once this is done, we present a non-trivial refinement of (5). Further discussion of the Kantorovich inequality is presented via several equivalent forms. Some applications are given, including a submultiplicative inequality for unital positive linear mappings and an operator Minkowski-type inequality.

2. Ando's inequality

In this section, we first present a new proof of Ando's inequality; then, we prove a reversed version of Ando's inequality.

Recall that for positive invertible operators A and B, the Riccati equation $XA^{-1}X = B$ has the geometric mean $A\sharp B$ as a unique positive solution [21, Theorem 2.2].

Let $X = A\sharp B$ and let Φ be a unital positive linear map. It follows from Choi's inequality [4, Proposition 4.3],

$$\Phi(X)\Phi(A)^{-1}\Phi(X) \le \Phi\left(XA^{-1}X\right) = \Phi(B).$$

Therefore,

$$\left(\Phi(A)^{-\frac{1}{2}}\Phi(X)\Phi(A)^{-\frac{1}{2}}\right)^2 \le \Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}.$$

Since $f(t) = t^{\frac{1}{2}}$ is operator monotone [7, Corollary 1.16], we get

$$\Phi(A)^{-\frac{1}{2}}\Phi(X)\Phi(A)^{-\frac{1}{2}} \le \left(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Consequently,

$$\Phi(X) \le \Phi(A)^{\frac{1}{2}}\left(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}}\Phi(A)^{\frac{1}{2}},$$

which is equivalent to

$$\Phi(A\sharp B) \le \Phi(A)\sharp\Phi(B).$$

This proves Ando's inequality.

Next, we utilize (4) to prove a reversed version of Ando's inequality under the sandwich condition. We remark that this reversed version has been shown in [10, Theorem 4] using a completely different method. In this article, we utilize the Kantorovich-type inequalities to offer this version. This helps understand the relation between Ando-type and Kantorovich-type inequalities.

Proposition 2.1. *Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a unital positive linear mapping and let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $m^2A \le B \le M^2A$, for some positive scalars $0 < m < M$. Then*

$$\Phi(A)\sharp\Phi(B) \le \frac{M + m}{2\sqrt{mM}}\Phi(A\sharp B).$$

Proof. For the given Φ and A, define the positive unital linear mapping Ψ by $\Psi(X) \equiv \Phi(A)^{-\frac{1}{2}}\Phi\left(A^{\frac{1}{2}}XA^{\frac{1}{2}}\right)\Phi(A)^{-\frac{1}{2}}$ and let $C = \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}$. Since $m^2A \le B \le M^2A$, it follows that $mI \le C \le MI$. Therefore, we may apply the inequality $\Psi(C^2) \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2\Psi(C)^2$ to obtain

$$\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}} \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2\left(\Phi(A)^{-\frac{1}{2}}\Phi(A\sharp B)\Phi(A)^{-\frac{1}{2}}\right)^2.$$

Since the function $f(t) = t^{\frac{1}{2}}$ is operator monotone, it follows that

$$\left(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}} \le \frac{M+m}{2\sqrt{Mm}}\Phi(A)^{-\frac{1}{2}}\Phi(A\sharp B)\Phi(A)^{-\frac{1}{2}},$$

which implies the desired inequality. $\square$

In fact, Ando's inequality follows from a more general result that

$$\Phi(A\sigma_f B) \le \Phi(A)\sigma_f \Phi(B), \tag{8}$$

where A, B are positive and σ_f is an operator mean with representing function f. In the next result, we show that if f is operator convex, then Ando's inequality is reversed, then we show that this reversed Ando inequality implies (2). We point out here that Theorem 2.2 does not follow from (8), as f is a positive function. So, multiplying (8) with -1 does not imply Theorem 2.2.

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible, $f : (0, \infty) \to (0, \infty)$ be a given operator convex function and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a positive unital linear mapping. Then*

$$\Phi(A\sigma_f B) \ge \Phi(A)\sigma_f \Phi(B),$$

where the connection σ_f is defined by

$$A\sigma_f B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

Proof. For the given parameters, define

$$\Psi(X) \equiv \Phi(A)^{-\frac{1}{2}} \Phi\left(A^{\frac{1}{2}} X A^{\frac{1}{2}}\right) \Phi(A)^{-\frac{1}{2}} ; \ X \in \mathcal{B}(\mathcal{H}).$$

Then Ψ is positive unital. Since f is operator convex, (1) implies $f(\Psi(X)) \le \Psi(f(X))$, for any self adjoint $X \in \mathcal{B}(\mathcal{H})$. Let $X = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ and apply this latter inequality to get

$$f\left(\Phi(A)^{-\frac{1}{2}} \Phi(B) \Phi(A)^{-\frac{1}{2}}\right) \le \Phi(A)^{-\frac{1}{2}} \Phi\left(A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}\right) \Phi(A)^{-\frac{1}{2}},$$

which is equivalent to

$$\Phi(A)\sigma_f \Phi(B) \le \Phi(A\sigma_f B),$$

as desired. $\square$

Interestingly, Theorem 2.2 implies (2), as follows: In Theorem 2.2, let $f(t) = t^2$ and $B = I$. Since f is operator convex, we may apply the theorem. Direct computations show that $A\sigma_f B = A^{-1}$. Consequently,

$$\Phi(A^{-1}) = \Phi(A\sigma_f B) \ge \Phi(A)\sigma_f \Phi(B) = \Phi(A)^{-1};$$

as desired.

Remark 2.3. *We remark that in Theorem 2.2, if we let $A = I$, we get*

$$f(\Phi(B)) \le \Phi(f(B));$$

an inequality that is equivalent to the fact that f is operator convex. This shows that the inequality in Theorem 2.2 is equivalent to the fact that f is operator convex.

3. Further analysis of the Kantorovich inequality

This section is devoted to the study of the Kantorovich inequality (3), where we begin by giving multiple equivalent statements. It should be remarked that these individual statements are well known, but their equivalence is the aim of Theorem 3.1.

We will use the following observation to prove the next result. Let Φ be a given unital positive linear map and let Φ' be another unital positive linear map. Then (7) implies

$$\Phi'\left(\Phi(A \sharp B)\right) \le \Phi'\left(\Phi(A) \sharp \Phi(B)\right) \le \Phi'\left(\Phi(A)\right) \sharp \Phi'\left(\Phi(B)\right).$$

Defining $\Phi'(T) \equiv \langle Tx, x \rangle$, with $x \in \mathcal{H}$, $\|x\| = 1$, we get

$$
\begin{aligned}
\langle \Phi(A \sharp B) x, x \rangle &\leq \langle \Phi(A) \sharp \Phi(B) x, x \rangle \\
&\leq \langle \Phi(A) x, x \rangle \sharp \langle \Phi(B) x, x \rangle \\
&= \sqrt{\langle \Phi(A) x, x \rangle \langle \Phi(B) x, x \rangle}.
\end{aligned}
\tag{9}
$$

As we mentioned earlier, these statements are all known. We have already seen (i), (iii) and (iv) (from Theorem 3.1) in (3), (5) and (4) respectively. For (ii) in Theorem 3.1, it can be easily deduced from Proposition 2.1 on letting $B = A^{-1}$ and defining the new mapping $\Phi'(X) = \langle \Phi(X)x, x \rangle$.

Theorem 3.1. *Let $A \in \mathcal{B}(\mathcal{H})$ satisfying $mI \leq A \leq MI$ for some scalars $0 < m < M$. Then the following assertions are equivalent.*

(i) $\Phi\left(A^{-1}\right) \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A)^{-1}$ *for any unital positive linear mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$.*

(ii) $\left\langle \Phi\left(A^{-1}\right) x, x \right\rangle \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \langle \Phi(A) x, x \rangle^{-1}$ *for any unit vector $x \in \mathcal{K}$ and any positive unital linear mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$.*

(iii) $\Phi\left(A^{-1}\right) \sharp \Phi(A) \leq \frac{M+m}{2\sqrt{Mm}} I$ *for any unital positive linear mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$.*

(iv) $\Phi\left(A^2\right) \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A)^2$ *for any unital positive linear mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$.*

Proof. $(ii) \Rightarrow (i)$
Assuming that (ii) is true. Using (6) we see that for any unit vector $x \in \mathcal{K}$,

$$
\begin{aligned}
\left\langle \Phi\left(A^{-1}\right) x, x \right\rangle &\leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \langle \Phi(A) x, x \rangle^{-1} \\
&\leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \left\langle \Phi(A)^{-1} x, x \right\rangle.
\end{aligned}
$$

This implies the desired result.
$(ii) \Rightarrow (iii)$
Since

$$
\left\langle \Phi\left(A^{-1}\right) x, x \right\rangle \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \langle \Phi(A) x, x \rangle^{-1},
$$

we get

$$
\sqrt{\langle \Phi(A^{-1}) x, x \rangle \langle \Phi(A) x, x \rangle} \leq \frac{M+m}{2\sqrt{Mm}}.
$$

On the other hand, by (9),

$$
\left\langle \Phi\left(A^{-1}\right) \sharp \Phi(A) x, x \right\rangle \leq \sqrt{\langle \Phi(A^{-1}) x, x \rangle \langle \Phi(A) x, x \rangle}
$$

we get for any unit vector $x \in \mathcal{K}$,

$$
\left\langle \Phi\left(A^{-1}\right) \sharp \Phi(A) x, x \right\rangle \leq \frac{M+m}{2\sqrt{Mm}}.
$$

(*i*) $\Rightarrow$ (*iii*)

As shown in [20], (*i*) implies (*iii*), but here we give another proof. For positive linear functional $\Psi : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}^+$ defined by $\Psi(A) := \langle \Phi(A)x, x \rangle$ (with the understanding $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ is a unital positive linear map and $x \in \mathcal{K}$ is a unit vector), (*i*) implies

$$\left\langle \Phi\left(A^{-1}\right)x, x \right\rangle \langle \Phi(A)x, x \rangle \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2.$$

This yields

$$\sqrt{\left\langle \Phi\left(A^{-1}\right)x, x \right\rangle \langle \Phi(A)x, x \rangle} \le \frac{M+m}{2\sqrt{Mm}}.$$

By (9), we have

$$\left\langle \Phi\left(A^{-1}\right) \sharp \Phi(A)x, x \right\rangle \le \frac{M+m}{2\sqrt{Mm}},$$

as desired.

(*iii*) $\Rightarrow$ (*ii*)

We may take the unital positive linear map $\Psi : \mathcal{B}(\mathcal{K}) \rightarrow \mathbb{R}^+$ defined by $\Psi(A) := \langle \Phi(A)x, x \rangle$ for any unital positive map Φ, $A > 0$ and a unit vector $x \in \mathcal{K}$. From the assumption (*iii*) with Ψ, we have for any unit vector $x \in \mathcal{K}$,

$$\sqrt{\langle \Phi(A)x, x \rangle \langle \Phi(A^{-1})x, x \rangle} = \Psi(A) \sharp \Psi(A^{-1}) \le \frac{M+m}{2\sqrt{Mm}},$$

which implies (*ii*):

$$\langle \Phi(A^{-1})x, x \rangle \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \langle \Phi(A)x, x \rangle^{-1}.$$

(*i*) $\Rightarrow$ (*iv*)

By taking $\Psi(X) \equiv \Phi(A)^{-\frac{1}{2}} \Phi\left(A^{\frac{1}{2}} X A^{\frac{1}{2}}\right) \Phi(A)^{-\frac{1}{2}}$, where Φ is an arbitrary unital positive linear map in (*i*), we obtain

$$\Phi(A)^{-1} \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \left(\Phi(A)^{-\frac{1}{2}} \Phi\left(A^2\right) \Phi(A)^{-\frac{1}{2}}\right)^{-1}$$

$$= \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A)^{\frac{1}{2}} \Phi\left(A^2\right)^{-1} \Phi(A)^{\frac{1}{2}}.$$

This implies

$$\Phi(A)^{-2} \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi\left(A^2\right)^{-1}.$$

By taking the inverse, we infer

$$\Phi\left(A^2\right) \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A)^2.$$

(*iv*) $\Rightarrow$ (*i*)

Assuming (*iv*) and replacing A with A^{-1}, we obtain

$$\Phi((A^{-1})^2) \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A^{-1})^2,$$

for any unital positive linear mapping Φ. Again, defining $\Psi(X) \equiv \Phi(A)^{-\frac{1}{2}} \Phi\left(A^{\frac{1}{2}} X A^{\frac{1}{2}}\right) \Phi(A)^{-\frac{1}{2}}$ and applying this latter inequality to Ψ, we obtain

$$\Psi((A^{-1})^2) \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Psi(A^{-1})^2,$$

which is equivalent to

$$\Phi(A)^{-\frac{1}{2}} \Phi(A^{-1}) \Phi(A)^{-\frac{1}{2}} \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A)^{-2}.$$

This implies

$$\Phi(A^{-1}) \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A)^{-1},$$

as required. This completes the proof.
$\square$

We have seen that the Kantorovich inequality (3) is equivalent to the inequality

$$\Phi\left(A^{-1}\right) \sharp \Phi(A) \le \frac{M+m}{2\sqrt{Mm}} I, \tag{10}$$

which in turn implies

$$\Phi\left(A^{-1}\right) \sharp \Phi(A) \le \left\| \Phi\left(A^{-1}\right) \sharp \Phi(A) \right\| I \le \frac{M+m}{2\sqrt{Mm}} I.$$

Next, we present a more precise estimate than (10), as follows.

Theorem 3.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator such that $mI \le A \le MI$, for some scalars $0 < m < M$. If $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a positive unital linear mapping, then*

$$\Phi\left(A^{-1}\right) \sharp \Phi(A) \le \left\| \left(\Phi(A)^{\frac{1}{2}} \Phi\left(A^{-1}\right) \Phi(A)^{\frac{1}{2}}\right)^{\frac{1}{2}} \right\| I \le \frac{M+m}{2\sqrt{Mm}} I.$$

Proof. The Kantorovich inequality states that if $mI \le A \le MI$ and Φ is a unital positive linear map, then

$$\Phi\left(A^{-1}\right) \le \frac{(M+m)^2}{4Mm} \Phi(A)^{-1}.$$

This implies

$$\Phi(A)^{\frac{1}{2}} \Phi\left(A^{-1}\right) \Phi(A)^{\frac{1}{2}} \le \frac{(M+m)^2}{4Mm} I.$$

Noting operator monotony of the function $f(t) = t^{\frac{1}{2}}$, we have

$$\left(\Phi(A)^{\frac{1}{2}} \Phi\left(A^{-1}\right) \Phi(A)^{\frac{1}{2}}\right)^{\frac{1}{2}} \le \frac{M+m}{2\sqrt{Mm}} I.$$

Whence

$$\left\| \left(\Phi(A)^{\frac{1}{2}} \Phi\left(A^{-1}\right) \Phi(A)^{\frac{1}{2}}\right)^{\frac{1}{2}} \right\| \le \frac{M+m}{2\sqrt{Mm}}.$$

On the other hand, from [12, Corollary 2.13], we infer that

$$\left\| \Phi\left(A^{-1}\right) \sharp \Phi\left(A\right) \right\| \le \left\| \left(\Phi(A)^{\frac{1}{2}} \Phi\left(A^{-1}\right) \Phi(A)^{\frac{1}{2}}\right)^{\frac{1}{2}} \right\|.$$

Consequently,

$$\Phi\left(A^{-1}\right) \sharp \Phi\left(A\right) \le \left\| \left(\Phi(A)^{\frac{1}{2}} \Phi\left(A^{-1}\right) \Phi(A)^{\frac{1}{2}}\right)^{\frac{1}{2}} \right\| I \le \frac{M+m}{2\sqrt{Mm}} I.$$

This completes the proof. □

In what follows, we present a reversed version of [4, Proposition 4.3] using (*iv*) in Theorem 3.1. We remark that this proposition has already been shown in [15, Corollary 3.11], using a different technique.

Proposition 3.3. *Let* $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *be a unital positive linear mapping and let* $A, B \in \mathcal{B}(\mathcal{H})$ *be such that* $mA \le B \le MA$, *for some scalars* $0 < m < M$. *Then*

$$\Phi(BA^{-1}B) \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(B)\Phi(A)^{-1}\Phi(B).$$

Proof. From Theorem 3.1, we have $\Phi\left(A^2\right) \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi\left(A\right)^2$ for any Φ and $mI \le A \le MI$. When $mA \le B \le MA$, we get $mI \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le MI$. Therefore, any positive unital linear mapping Φ satisfies

$$\Phi\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^2\right) \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^2.$$

Letting $\Psi\left(X\right) \equiv \Phi\left(A\right)^{-\frac{1}{2}} \Phi\left(A^{\frac{1}{2}}XA^{\frac{1}{2}}\right) \Phi\left(A\right)^{-\frac{1}{2}}$ and applying the latter inequality for Ψ, we obtain

$$\Phi(A)^{-\frac{1}{2}}\Phi(BA^{-1}B)\Phi(A)^{-\frac{1}{2}} \le \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-1}\Phi(B)\Phi(A)^{-\frac{1}{2}},$$

which implies the desired inequality. □

We notice that (*ii*) in Theorem 3.1 is a particular case of the following more general result, whose proof is an implementation of the well-known Mond-Pečarić method. We remark that this theorem follows from [15, Theorem 2.2] upon letting $\Psi(X) = \langle \Phi(X)x, x \rangle$, but we present the proof here for the reader's convenience.

Theorem 3.4. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be a self-adjoint operator with the spectra in the interval* $[m, M]$ *and let* Φ *be a unital positive linear mapping on* $\mathcal{B}(\mathcal{H})$. *If* $f : [m, M] \to \mathbb{R}$ *is a convex function, then for any unit vector* $x \in \mathcal{H}$ *and* $\alpha \ge 0$

$$\langle \Phi\left(f\left(A\right)\right)x, x \rangle \le \beta + \alpha f\left(\langle \Phi\left(A\right)x, x \rangle\right)$$

holds, where $\beta = \max_{m \le t \le M}\left\{a_f t + b_f - \alpha f\left(t\right)\right\}$ *with* $a_f = (f\left(M\right) - f\left(m\right))/(M - m)$ *and* $b_f = (Mf\left(m\right) - Mf\left(m\right))/(M - m)$.

Proof. Since f is convex on $[m, M]$, we have for any $m \le t \le M$,

$$f\left(t\right) \le a_f t + b_f.$$

It follows from the continuous functional calculus that

$$f\left(A\right) \le a_f A + b_f I.$$

The assumptions on Φ implies

$$\Phi(f(A)) \leq a_f \Phi(A) + b_f I.$$

Thus, for any unit vector $x \in \mathcal{H}$

$$\langle \Phi(f(A)) x, x \rangle \leq a_f \langle \Phi(A) x, x \rangle + b_f.$$

Therefore,

$$\langle \Phi(f(A)) x, x \rangle - \alpha f(\langle \Phi(A) x, x \rangle) \leq a_f \langle \Phi(A) x, x \rangle + b_f - \alpha f(\langle \Phi(A) x, x \rangle)$$
$$\leq \max_{m \leq t \leq M} \left\{ a_f t + b_f - \alpha f(t) \right\}$$
$$= \beta.$$

$\square$

Corollary 3.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator satisfying $mI \leq A \leq MI$ for some scalars $0 < m < M$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a unital positive linear map. Then, for any unit vector $x \in \mathcal{K}$,*

$$\langle \Phi(A^p) x, x \rangle \leq K(p, m, M) \langle \Phi(A) x, x \rangle^p \tag{11}$$

where $K(p, m, M)$ is the generalized Kantorovich constant defined by

$$K(p, m, M) := \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{(p-1)}{p} \frac{(M^p - m^p)}{(mM^p - Mm^p)} \right)^p. \tag{12}$$

Proof. If we take $f(t) = t^p$, $(t > 0)$, for $p \geq 1$ or $p \leq 0$, we obtain (11). $\square$

Remark 3.6. *We know that if A is a positive operator, then for any $p \geq 1$*

$$\Phi(A^p) \leq K(p, m, M) \Phi(A)^p. \tag{13}$$

If the operator A is positive and invertible, (13) is also true for $p < 0$. Evidently, (13) implies

$$\langle \Phi(A^p) x, x \rangle \leq K(p, m, M) \langle \Phi(A)^p x, x \rangle \tag{14}$$

for any unit vector $x \in \mathcal{K}$. Thus, (11) can be considered as an improvement of (14), thanks to (6).

Notice that the case $p = 2$ in (13) reduces to

$$\Phi(A^2) \leq \left(\frac{M+m}{2\sqrt{Mm}} \right)^2 \Phi(A)^2. \tag{15}$$

4. Related results via operator convex and operator monotone functions

An additive form (see [17, Theorem 2]) of (15) is incorporated in

$$\Phi(A^2)^{\frac{1}{2}} \leq \frac{(M-m)^2}{4(M+m)} I + \Phi(A). \tag{16}$$

In this section we present a two-term version of this inequality in a more general setting; where this inequality is looked at as $f^{-1}(\Phi(f(A)))$ where $f(t) = t^2$. Then, we present a Minkowski-type inequality for

tuples of operators.

For the used notation in the next theorem, we shall adopt the following notations

$$\alpha[f;m,M] = \max\left\{\frac{1}{f(t)}\left(\frac{f(M)-f(m)}{M-m}t + \frac{Mf(m)-mf(M)}{M-m}\right): m \le t \le M\right\},$$

and

$$\beta_0[f;m,M] = \max\left\{f(t) - \frac{f(M)-f(m)}{M-m}t - \frac{Mf(m)-mf(M)}{M-m} : m \le t \le M\right\}, \tag{17}$$

where $f : [m, M] \to (0, \infty)$ is a given function.

Theorem 4.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators satisfying $mI \le A, B \le MI$ for some scalars $0 < m < M$ and let Φ be a unital positive linear mapping on $\mathcal{B}(\mathcal{H})$. If $f : (0, \infty) \to (0, \infty)$ is a 1-1 operator convex function such that f^{-1} is operator monotone, then*

$$f^{-1}(\Phi(f(A))) + f^{-1}(\Phi(f(B))) \le \alpha f^{-1}(\Phi(f(A+B))),$$

and

$$f^{-1}(\Phi(f(A))) + f^{-1}(\Phi(f(B))) \le \beta I + f^{-1}(\Phi(f(A+B))),$$

where $\alpha = \alpha[f;m,M], m' = \min_{t \in [m,M]} f(t), M' = \max_{t \in [m,M]} f(t)$ and $\beta = 2\beta_0[f^{-1};m',M']$.

Proof. From the Choi-Davis inequality, we have

$$f(\Phi(A+B)) \le \Phi(f(A+B)),$$

which then implies

$$\Phi(A+B) \le f^{-1}(\Phi(f(A+B))), \tag{18}$$

since, by assumption, f^{-1} is operator monotone. Furthermore, by [15, Corollary 2.5], we have

$$\Phi(f(A)) \le \alpha f(\Phi(A)) \text{ and } \Phi(f(B)) \le \alpha f(\Phi(B)) \tag{19}$$

since f is convex. Now since f^{-1} is operator monotone, $\alpha > 1$, the latter inequalities imply

$$f^{-1}(\Phi(f(A))) \le \alpha \Phi(A) \text{ and } f^{-1}(\Phi(f(B))) \le \alpha \Phi(B). \tag{20}$$

Combining (20) and (18) imply

$$\begin{aligned} f^{-1}(\Phi(f(A))) + f^{-1}(\Phi(f(B))) &\le \alpha(\Phi(A) + \Phi(B)) \\ &= \alpha\Phi(A+B) \\ &\le \alpha f^{-1}(\Phi(f(A+B))). \end{aligned}$$

This proves the first inequality. To prove the second inequality, recall that if g is operator concave then $g(\Phi(A)) \ge \Phi(g(A))$. Further, we know, from [15, Remark 2.3], that if $mI \le A \le MI$, then

$$g(\Phi(A)) \le \beta_0[g;m,M]I + \Phi(g(A)). \tag{21}$$

By assumption, $f^{-1} : (0, \infty) \to (0, \infty)$ is operator monotone, hence it is operator concave (Proposition 1.1). So, applying (21) with $g = f^{-1}$, we obtain

$$f^{-1}(\Phi(f(A)) \le \beta_0[f^{-1};m',M']I + \Phi(A) \quad \text{and} \quad f^{-1}(\Phi(f(B)) \le \beta_0[f^{-1};m',M']I + \Phi(B), \tag{22}$$

where $m' = \min\{f(t) : m \le t \le M\}$ and $M' = \max\{f(t) : m \le t \le M\}$. Adding the two inequalities we get

$$f^{-1}(\Phi(f(A))) + f^{-1}(\Phi(f(B))) \le 2\beta_0[f^{-1}; m', M']I + \Phi(A + B).$$

But we know that

$$\Phi(A + B) = \Phi(f^{-1}(f(A + B))) \le f^{-1}(\Phi(f(A + B)))$$

since f^{-1} is operator concave. Thus, we have shown that

$$f^{-1}(\Phi(f(A))) + f^{-1}(\Phi(f(B))) \le \beta I + f^{-1}(\Phi(f(A + B))),$$

where $\beta = 2\beta_0[f^{-1}; m', M']$. This completes the proof. $\square$

Notice that if $f(t) = t^2$, then $m' = m^2, M' = M^2$ and $f^{-1}(t) = \sqrt{t}$. Calculating the maximum in (17), we obtain

$$\beta_0[f^{-1}; m', M'] = \frac{(M - m)^2}{4(m + M)}.$$

Therefore, the inequality (16) follows from (22).

In general, if $f(t) = t^p, p \ge 1$, we can show that

$$\beta_0[f^{-1}; m', M'] = g_p\left(\left\{p\frac{M - m}{M^p - m^p}\right\}^{\frac{p}{1-p}}\right),$$

where

$$g_p(t) = t^{1/p} - \frac{M - m}{M^p - m^p}t - \frac{mM^p - Mm^p}{M^p - m^p}.$$

We will use the notation

$$\beta_p = 2\beta_0[t^{1/p}; m^p, M^p]. \tag{23}$$

Remark 4.2. *Tracing the proof of Theorem 4.1, one can see that if f^{-1} is a power function, then (19) implies*

$$f^{-1}\left(\Phi\left(f\left(A\right)\right)\right) \le f^{-1}(\alpha)\Phi\left(A\right) \text{ and } f^{-1}\left(\Phi\left(f\left(B\right)\right)\right) \le f^{-1}(\alpha)\Phi\left(B\right).$$

This implies

$$f^{-1}(\Phi(f(A))) + f^{-1}(\Phi(f(B))) \le f^{-1}(\alpha)f^{-1}(\Phi(f(A + B))).$$

In particular, letting $f(t) = t^p, 1 \le p \le 2$ in Remark 4.2, we obtain the following special cases. We refer the reader to [9] for a detailed discussion of the next corollary.

Corollary 4.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two self-adjoint operators satisfying $mI \le A, B \le MI$ for some scalars $0 < m < M$ and let Φ be a unital positive linear mapping on $\mathcal{B}(\mathcal{H})$. If $1 \le p \le 2$, then*

$$\Phi\left(A^p\right)^{\frac{1}{p}} + \Phi\left(B^p\right)^{\frac{1}{p}} \le K_p^{\frac{1}{p}}\Phi\left((A + B)^p\right)^{\frac{1}{p}},$$

and

$$\Phi\left(A^p\right)^{\frac{1}{p}} + \Phi\left(B^p\right)^{\frac{1}{p}} \le \beta_p I + \Phi\left((A + B)^p\right)^{\frac{1}{p}},$$

where $K_p = K(p, m, M)$ is defined as in (12) and β_p is as in (23). In particular, when $p = 2$,

$$\Phi\left(A^2\right)^{\frac{1}{2}} + \Phi\left(B^2\right)^{\frac{1}{2}} \le \frac{M + m}{2\sqrt{Mm}}\Phi\left((A + B)^2\right)^{\frac{1}{2}},$$

and

$$\Phi\left(A^2\right)^{\frac{1}{2}} + \Phi\left(B^2\right)^{\frac{1}{2}} \le \frac{(M - m)^2}{2(M + m)}I + \Phi\left((A + B)^2\right)^{\frac{1}{2}}.$$

We conclude this section by presenting the following Minkowski-type inequalities as an application of Corollary 4.3.

Corollary 4.4. *Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be Hermitian matrices satisfying $mI \leq A_i, B_i \leq MI$ for $i = 1, \ldots, k$ and some scalars $0 < m < M$, and let $\Phi_1, \ldots, \Phi_k : \mathcal{M}_n \to \mathcal{M}_\ell$ be positive linear mappings with $\sum_{i=1}^{k} \Phi_i(I) = I$. Then*

$$\left(\sum_{i=1}^{k} \Phi_i \left(A_i^2 \right) \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{k} \Phi_i \left(B_i^2 \right) \right)^{\frac{1}{2}} \leq \frac{M + m}{2\sqrt{Mm}} \left(\sum_{i=1}^{k} \Phi_i \left((A_i + B_i)^2 \right) \right)^{\frac{1}{2}}, \tag{24}$$

and

$$\left(\sum_{i=1}^{k} \Phi_i \left(A_i^2 \right) \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{k} \Phi_i \left(B_i^2 \right) \right)^{\frac{1}{2}} \leq \frac{(M - m)^2}{2(M + m)} + \left(\sum_{i=1}^{k} \Phi_i \left((A_i + B_i)^2 \right) \right)^{\frac{1}{2}}. \tag{25}$$

Proof. If $A_1, \ldots, A_k \in \mathcal{M}_n$ are positive matrices, then $A = A_1 \oplus \cdots \oplus A_k$ is a positive matrix in $\mathcal{M}_k(\mathcal{M}_n)$. Let the unital positive linear mapping $\Phi : \mathcal{M}_k(\mathcal{M}_n) \to \mathcal{M}_\ell$ be defined by $\Phi(A) = \sum_{i=1}^{k} \Phi_i(A_i)$. Utilizing Corollary 4.3, we obtain the desired inequalities (24) and (25). □

In particular, we have the following.

Corollary 4.5. *Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be Hermitian matrices satisfying $mI \leq A_i, B_i \leq MI$ for $i = 1, \ldots, k$ and some scalars $0 < m < M$, and let $w_1, \ldots, w_k$ be positive scalars satisfying $\sum_{i=1}^{k} w_i = 1$. Then*

$$\left(\sum_{i=1}^{k} w_i A_i^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{k} w_i B_i^2 \right)^{\frac{1}{2}} \leq \frac{M + m}{2\sqrt{Mm}} \left(\sum_{i=1}^{k} w_i (A_i + B_i)^2 \right)^{\frac{1}{2}} \tag{26}$$

and

$$\left(\sum_{i=1}^{k} w_i A_i^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{k} w_i B_i^2 \right)^{\frac{1}{2}} \leq \frac{(M - m)^2}{2(M + m)} + \left(\sum_{i=1}^{k} w_i (A_i + B_i)^2 \right)^{\frac{1}{2}}. \tag{27}$$

Proof. By applying inequalities (24) and (25) for positive linear mappings $\Phi_i : \mathcal{M}_n \to \mathcal{M}_n$ determined by $\Phi_i : T \mapsto w_i T$, $i = 1, \ldots, k$, we get (26) and (27). □

5. A counterexample

In studying the equivalence of inequalities (3) and (5), we first tried to prove the following inequality:

$$\Phi \left(A^{-1} \right)^2 \leq \frac{(M + m)^2}{4Mm} \Phi(A)^{-\frac{1}{2}} \Phi \left(A^{-1} \right) \Phi(A)^{-\frac{1}{2}}.$$

This inequality is not true, as noticed by Yamazaki [24]. To show this, let $A = \begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix}$, and define

$$\Phi(A) \equiv \frac{1}{2} U^* A U + \frac{1}{2} V^* A V,$$

where U and V are 2×2 unitary matrices. Then Φ is a unital positive linear map. We set unitary matrices

$$U = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad V = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{R}$. Then for $x > 0$, we have only to check

$$T(x, \alpha, \beta) \equiv \frac{(1+x)^2}{4x} \Phi(A)^{-\frac{1}{2}} \Phi\left(A^{-1}\right) \Phi(A)^{-\frac{1}{2}} - \Phi\left(A^{-1}\right)^2 \geq 0.$$

With the help of Mathematica, we get

$$T\left(2, \frac{\pi}{3}, \frac{\pi}{6}\right) = \begin{pmatrix} -0.0842034 & -0.185577 \\ -0.185577 & -0.826511 \end{pmatrix}$$

and its eigenvalues are -0.87032, -0.0403946.

Acknowledgment

References

[1] T. Ando and F. Hiai, *Operator log–convex functions and operator means*, Math. Ann., **350**(3) (2011), 611–630.
[2] T. Ando, C.-K. Li, and R. Mathias, *Geometric means*, Linear Algebra Appl., **385** (2004), 305–334.
[3] M. D. Choi, *A Schwarz inequality for positive linear maps on C*-algebras*, Illinois J. Math., **18** (1974), 565–574.
[4] M. D. Choi, *Some assorted inequalities for positive linear maps on C*-algebras*, J. Operator Theory., **4** (1980), 271–285.
[5] C. Davis, *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc., **8** (1957), 42–44.
[6] S. Furuichi, H. R. Moradi and A. Zardadi, *Some new Karamata type inequalities and their applications to some entropies*, Rep. Math. Phys., **84**(2) (2019), 201–214.
[7] T. Furuta, J. Mićić, J. Pečarić, and Y. Seo, *Mond–Pečarić method in operator inequalities*, Element, Zagreb, 2005.
[8] W. Greub and W. Rheinboldt, *On a generalization of an inequality of L.V.Kantorovich*, Proc. Amer. Math. Soc., **10** (1959), 407–415.
[9] M. Kian and Y. Seo, *Jointly convex mappings related to the Lieb's functional and Minkowski type operator inequalities*, Anal. Math. Phys., (2021) 11: 72.
[10] E. Y. Lee, *A matrix reverse Cauchy–Schwarz inequality*, Linear Algebra Appl., **430** (2009), 805–810.
[11] C. K. Li and R. Mathias, *Matrix inequalities involving a positive linear map*, Linear Multilinear Algebra., **41** (1996), 221–231.
[12] J. S. Matharu and J. S. Aujla, *Some inequalities for unitarily invariant norms*, Linear Algebra Appl., **436**(6) (2012), 1623–1631.
[13] J. Mićić, J. Pečarić, Y. Seo and M. Tominaga, *Inequalities for positive linear maps on Hermitian matrices*, Math. Inequal. Appl., **3** (2000), 559–591.
[14] J. Mićić and J. Pečarić, *Order among power means of positive operators, II*, Sci. Math. Jpn., **71**(1) (2010), 93–109.
[15] J. Mićić, J. Pečarić and Y. Seo, *Complementary inequalities to inequalities of Jensen and Ando based on the Mond–Pečarić method*, Linear Algebra Appl., **318** (2000), 87–107.
[16] B. Mond and J. E. Pečarić, *Bounds for Jensen's inequality for several operators*, Houston J. Math., **20** (1994), 645–651.
[17] B. Mond and J. Pečarić, *Converses of Jensen's inequality for linear maps of operators*, An. Univ. Timisoara Ser. Mat. Inform., **31**(2) (1993), 223–228.
[18] B. Mond and J. E. Pečarić, *Operator convex functions of several variables*, Soochow J. Math., **24** (1998), 239–254.
[19] H. R. Moradi, I. H. Gümüş and Z. Heydarbeygi, *A glimpse at the operator Kantorovich inequality*, Linear Multilinear Algebra., **67**(5) (2019), 1031–1036.
[20] R. Nakamoto and M. Nakamura, *Operator mean and Kantorovich inequality*, Math. Japon., **44**(3) (1996), 495–498.
[21] N. Nakamura, *Geometric operator mean induced from the Riccati equation*, Sci. Math. Japon., **66** (2007), 83–87.
[22] M. Sababheh, H. R. Moradi and S. Furuichi, *Exponential inequalities for positive linear mappings*, J. Funct. Spaces., (2018), Article ID 5467413.
[23] M. Uchiyama, *Operator monotone functions, positive definite kernels and majorization*, Proc. Amer. Math. Soc., **138**(11) (2010), 3985–3996.
[24] T. Yamazaki, *Private communications*.