Filomat 37:13 (2023), 4155–4170
https://doi.org/10.2298/FIL2313155P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An n-dimensional pseudo-differential operator involving linear canonical transform and some applications in quantum mechanics

Tusharakanta Pradhan[a], Manish Kumar[a,*]

[a]*Department of Mathematics, Birla Institute of Technology and Science-Pilani, Hyderabad Campus, Hyderabad-500078, Telangana, India*

Abstract. In this work, an n-dimensional pseudo-differential operator involving the n-dimensional linear canonical transform associated with the symbol $\sigma(x_1,\ldots,x_n;y_1,\ldots,y_n)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ is defined. We have introduced various properties of the n-dimensional pseudo-differential operator on the Schwartz space using linear canonical transform. It has been shown that the product of two n-dimensional pseudo-differential operators is an n-dimensional pseudo-differential operator. Further, we have investigated formal adjoint operators with a symbol $\sigma\in\mathcal{S}^m$ using the n-dimensional linear canonical transform, and the $L^p(\mathbb{R}^n)$ boundedness property of the n-dimensional pseudo-differential operator is provided. Furthermore, some applications of the n-dimensional linear canonical transform are given to solve generalized partial differential equations and their particular cases that reduce to well-known n-dimensional time-dependent Schrödinger-type-I/Schrödinger-type-II/Schrödinger equations in quantum mechanics for one particle with a constant potential.

1. Introduction, definitions, and preliminaries

The one-dimensional linear canonical transform (LCT) was introduced in 1970 as an integral transform. The linear canonical transform (LCT) [2–4, 6] plays a vital role in many fields of quantum mechanics, optics, signal processing, image processing, and engineering sciences, which is a generalization of many integral transform, including the Fourier transform (FT) [7, 9, 10], the fractional Fourier transform (FRFT) [1, 2, 8], the Fresnel transform. Most recent developments in such integral transforms include (for example) the short-time special affine Fourier transform, the discrete quadratic-phase Fourier transform, the Mehler-Foke type index transform, the non-separable linear canonical wavelet transform, the ridgelet and linear canonical transform, the quantum representation of the linear canonical wavelet transform, wavelet multipliers involving the Watson transform, the quadratic-phase wave-packet transform, continuous fractional Bessel wavelet transform, the solution of a non-linear Hunter-Saxton equation using Fibonacci wavelet method, the Kontorovich-Lebedev transform, general families of integral transformations and so on can be found in [11–23]. It has found many applications, such as optics systems, filter design, signal synthesis, time-frequency analysis, phase retrieval, pattern recognition, and many other areas. The classical multi-dimensional separable LCT is reported in [24]. Let us begin by recalling [24, Definition 2.1, p.4] and a matrix $M=$

2020 *Mathematics Subject Classification*. Primary 47G30; Secondary 46A11; 53D22; 35Bxx; 35Q41
Keywords. Pseudo-differential operator, Schwartz space, Linear canonical transform, Generalized solutions to paritial differential equations, Time-dependent Schrödinger equations
Received: 02 March 2022; Revised: 23 August 2022; Accepted: 28 November 2022
Communicated by Hari M. Srivastava
* Corresponding author: Manish Kumar
Email address: manishkumar@hyderabad.bits-pilani.ac.in; manish.math.bhu@gmail.com (Manish Kumar)

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^{-1} \in SL(2,\mathbb{F})$ (a special linear group of order 2), where $\mathbb{F}$ is a field with $ad - bc = 1$ defined in [29]. Now we define an n-dimensional linear canonical transform for a function $\varphi(x_1,\ldots,x_n) \in L^1(\mathbb{R}^n)$ associated with a matrix M is given by

$$\mathcal{L}_M(\varphi(x_1,\ldots,x_n))(y_1,\ldots,y_n) = \varphi_M(y_1,\ldots,y_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n} K_M(x_k,y_k)\right\}\varphi(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n, \tag{1.1}$$

where $\mathrm{i} = \sqrt{-1}$ and the kernel $K_M(x_k,y_k)$ for each $k = 1,\ldots,n$, is defined by

$$K_M(x_k,y_k) = C_M\, \mathrm{e}^{\mathrm{i}\left(\frac{ax_k^2}{2b} - \frac{x_k y_k}{b} + \frac{dy_k^2}{2b}\right)}, \quad (b \neq 0).$$

In the case when $b = 0$, then the LCT of a function $\varphi_M(y_1,\ldots,y_n)$ is given by

$$\varphi_M(y_1,\ldots,y_n) = \sqrt{d}\, \mathrm{e}^{\frac{\mathrm{i}cd}{2}\sum_{k=1}^{n} y_k^2}\varphi(dy_1,\ldots,dy_n),$$

and $C_M = \frac{1}{(2\pi \mathrm{i} b)^{\frac{n}{2}}}$. The corresponding *inverse* of Eq. (1.1) is defined by

$$\mathcal{L}_{M^{-1}}(\varphi_M(y_1,\ldots,y_n))(x_1,\ldots,x_n) = \varphi(x_1,\ldots,x_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n} K_{M^{-1}}(y_k,x_k)\right\}\varphi_M(y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n, \tag{1.2}$$

where the kernel $\overline{K_M(x_k,y_k)}$ for each $k = 1,\ldots,n$, that is, $K_{M^{-1}}(y_k,x_k)$, is given by

$$K_{M^{-1}}(y_k,x_k) = \overline{C_M}\, \mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b} - \frac{x_k y_k}{b} + \frac{dy_k^2}{2b}\right)}, \quad (b \neq 0),$$

when $b = 0$, then the n-dimensional *inverse* LCT of a function $\varphi(x_1,\ldots,x_n)$ is defined by

$$\varphi(x_1,\ldots,x_n) = \sqrt{a}\, \mathrm{e}^{\frac{-\mathrm{i}ac}{2}\sum_{k=1}^{n} x_k^2}\varphi(ax_1,\ldots,ax_n),$$

with $\overline{C_M} = \frac{1}{(-2\pi \mathrm{i} b)^{\frac{n}{2}}}$, and the *overbar* denotes the complex conjugate. Throughout this paper, we shall assume $b \neq 0$.

Definition 1.1. Schwartz space $\mathcal{S}(\mathbb{R}^n)$: *The Schwartz space consists of all C^∞-functions φ on $\mathbb{R}^n$ such that*

$$\gamma_{\alpha,\beta}(\varphi) = \sup_{(x_1,\ldots,x_n)\in\mathbb{R}^n}\left|\prod_{k=1}^{n} x_k^{\alpha_k}\left(\frac{\partial}{\partial x_k}\right)^{\beta_k}\varphi(x_1,\ldots,x_n)\right| < \infty, \tag{1.3}$$

where the letters α and β are denoted by non-negative integers in $\mathbb{N}_0^n$, that is α_k and β_k $(k = 1,\ldots,n)$ are non-negative integers such that $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $|\beta| = \beta_1 + \cdots + \beta_n$.

Definition 1.2. Schwartz-type space $\mathcal{S}^M(\mathbb{R}^n)$: *The space $\mathcal{S}^M(\mathbb{R}^n)$ is defined as follows; φ is member of $\mathcal{S}^M(\mathbb{R}^n)$ iff it is a complex valued $\mathbb{C}^\infty$-function on $\mathbb{R}^n$, which satisfy the following condition:*

$$\mathcal{S}^M_{\alpha,\beta}(\varphi) = \sup_{(x_1,\ldots,x_n)\in\mathbb{R}^n}\left|\prod_{k=1}^{n}\left\{x_k^{\alpha_k}\left(\Delta_{x_k,a,b}\right)^{\beta_k}\right\}\varphi(x_1,\ldots,x_n)\right| < \infty, \tag{1.4}$$

for every choice of non-negative integers α and β in $\mathbb{N}_0^n$, where $\Delta_{x_k,a,b}$ is defined by:

$$\Delta_{x_k,a,b} = \left(\frac{\partial}{\partial x_k} - \mathrm{i}\frac{a}{b}x_k\right), \tag{1.5}$$

if we put $a = 0$ in (1.4)*, then it reduces to Definition 1.1.*

Definition 1.3. *Let $m \in \mathbb{R}$. Then we define $\mathcal{S}^m$ to be the set of all functions $\sigma(x_1, \ldots, x_n; y_1, \ldots, y_n)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any two multi-indices α and β, there is a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which*

$$\left|\prod_{k=1}^{n}\left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}\left(\frac{\partial}{\partial y_k}\right)^{\beta_k} \sigma(x_1,\ldots,x_n;y_1,\ldots,y_n)\right| \leq C_{\alpha,\beta}\left(1+\prod_{k=1}^{n}|y_k|\right)^{m-|\beta|}. \tag{1.6}$$

Definition 1.4. *Let $\sigma(x_1, \ldots, x_n; y_1, \ldots, y_n) \in \mathcal{S}^m$. Then the n-dimensional pseudo-differential operator $T_{\sigma,M}$ associated with symbol $\sigma(x_1, \ldots, x_n; y_1, \ldots, y_n)$, is defined by*

$$\begin{aligned}(T_{\sigma,M}\varphi)(x_1,\ldots,x_n) &= \overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n} e^{-i\left(\frac{ax_k^2}{2b}-\frac{x_k y_k}{b}+\frac{dy_k^2}{2b}\right)}\right\}\sigma(x_1,\ldots,x_n;y_1,\ldots,y_n)\\ &\times \varphi_M(y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n, \quad \forall\varphi\in\mathcal{S}(\mathbb{R}^n),\end{aligned} \tag{1.7}$$

where $\varphi_M(y_1, \ldots, y_n)$ is the n-dimensional linear canonical transform of a function φ. Let $\sigma(x_1, \ldots, x_n; y_1, \ldots, y_n) \in \mathcal{S}^m$ for $r = 0, 1, 2, \ldots$, which is given by:

$$\sigma_r(x_1,\ldots,x_n;y_1,\ldots,y_n) = \sigma(x_1,\ldots,x_n;y_1,\ldots,y_n)\varphi_r(y_1,\ldots,y_n), \quad \forall\,(x_1,\ldots,x_n),(y_1,\ldots,y_n)\in\mathbb{R}^n, \tag{1.8}$$

then we define

$$K_{r,M}(x_1,\ldots,x_n;z_1,\ldots,z_n) = \overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{e^{-i\left(\frac{az_k^2}{2b}-\frac{z_k y_k}{b}+\frac{dy_k^2}{2b}\right)}\right\}\sigma_r(x_1,\ldots,x_n;y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n, \tag{1.9}$$

where φ_r is the partition of the unity constructed in the book of Wong [28, p. 40].

Theorem 1.5 (Taylor's Formula [28]). *Suppose $f \in C^\infty(\mathbb{R}^n)$ and for all positive integer l, we have*

$$f(y_1+\eta_1,\ldots,y_n+\eta_n) = \sum_{\alpha_1=l}^{\infty}\cdots\sum_{\alpha_n=l}^{\infty}\frac{\eta_1^{\alpha_1}\cdots\eta_n^{\alpha_n}}{\alpha_1!\ldots\alpha_n!}\frac{(\partial^{\alpha_1+\cdots+\alpha_n}f)(y_1,\ldots,y_n)}{\partial y_1^{\alpha_1}\ldots\partial y_n^{\alpha_n}} + R_l(y_1,\ldots,y_n;\eta_1,\ldots,\eta_n), \tag{1.10}$$

where $R_l(y_1, \ldots, y_n; \eta_1, \ldots, \eta_n) = l\sum_{\nu_1=l}^{\infty}\cdots\sum_{\nu_n=l}^{\infty}\frac{\eta_1^{\nu_1}\ldots\eta_n^{\nu_n}}{\nu_1!\ldots\nu_n!}\int_0^1(1-\theta)^{l-1}\frac{(\partial^{\nu_1+\cdots+\nu_n}f)(y_1,\ldots,y_n)}{\partial y_1^{\nu_1}\ldots\partial y_n^{\nu_n}}(y_1+\theta\eta_1,\ldots,y_n+\theta\eta_n)\,\mathrm{d}\theta$, for all $(y_1, \ldots, y_n), (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$.

Cordero and Gröchenig [26], Du and Wong [27] found the product formula for localization operators (sub-class of pseudo-differential operators) on modulation space and Gelfand-Shilov spaces. Motivated by the work of Du and Wong [27, 28], our main goal in this paper is to find the characterization of n-dimensional pseudo-differential operator associated with the symbol $\sigma(x_1, \ldots, x_n; y_1, \ldots, y_n) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ involving n-dimensional linear canonical transform. Further, we have derived some new results of the n-dimensional linear canonical transform in a distributional sense. These new findings have become an elegant tool for solving generalized partial differential equations. Furthermore, we have investigated some applications of the n-dimensional linear canonical transform to solve generalized partial differential equations and their particular cases that reduce to well-known n-dimensional time-dependent Schrödinger-type-I/Schrödinger-type-II/Schrödinger equations in quantum mechanics for one particle with a constant potential.

2. Properties of the n-dimensional pseudo-differential operator

This section proves that the n-dimensional pseudo-differential operator is a continuous linear map from Schwartz space into itself. Then we have shown that the product of two n-dimensional pseudo-differential operators is again an n-dimensional pseudo-differential operator. We have defined the formal adjoint of an n-dimensional pseudo-differential operator. Using this fact, we have derived an asymptotic expansion for the symbol of the formal adjoint. The boundedness on $L^p(\mathbb{R}^n)$ has been investigated by using the n-dimensional linear canonical transform technique.

Proposition 2.1. *Let $\sigma(x_1,\dots,x_n;y_1,\dots,y_n)\in S^m$, $m\in\mathbb{R}$. Then the n-dimensional pseudo-differential operators $T_{\sigma,M}:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is a continuous linear map.*

Proof. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Then for any two multi-indices α and β, we have to show that

$$\sup_{(x_1,\dots,x_n)\in\mathbb{R}^n}\left|\left(\prod_{k=1}^n x_k^{\alpha_k}\left(\frac{\partial}{\partial x_k}\right)^{\beta_k} T_{\sigma,M}\varphi\right)(x_1,\dots,x_n)\right|<\infty.$$

Multiplying both sides of Eq. (1.7) by $\prod_{k=1}^n x_k^{\alpha_k}\left(\frac{\partial}{\partial x_k}\right)^{\beta_k}$ and using the Leibniz rule followed by integration by parts with the setting $\alpha_k+\eta_k=\delta_k$, $(k=1,\dots,n)$, we have

$$=(-1)^{|\delta|}\overline{C_M}\prod_{k=1}^n\left\{\sum_{\beta_k'=0}^{\beta_k}\binom{\beta_k}{\beta_k'}\sum_{\beta_k''=0}^{\beta_k'}\binom{\beta_k'}{\beta_k''}\right\}\left(\mathrm{i}\frac{1}{b}\right)^{\beta_k''-\delta_k}\left\{\prod_{k=1}^n\sum_{\eta_k=0}^{\beta_k'-\beta_k''}A_{\eta_k}\left(\frac{a}{b}\right)\right\}\left\{\prod_{k=1}^n\sum_{\eta_k'=0}^{\beta_k'-\beta_k''}A_{\eta_k'}\left(\frac{d}{b}\right)\right\}^{-1}$$
$$\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^n \mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b}+\frac{dy_k^2}{2b}-\frac{x_ky_k}{b}\right)}\right\}\prod_{k=1}^n\sum_{\delta_k'=0}^{\delta_k}\binom{\delta_k}{\delta_k'}\left(\left(\frac{\partial}{\partial y_k}\right)^{\delta_k-\delta_k'}\left(\frac{\partial}{\partial x_k}\right)^{\beta-\beta_k'}\sigma\right)(x_1,\dots,x_n;y_1,\dots,y_n)$$
$$\times\prod_{k=1}^n\left(\frac{\partial}{\partial y_k}\right)^{\delta_k'}\left(y_k^{\beta_k''-\eta_k'}\varphi_M(y_1,\dots,y_n)\right)\mathrm{d}y_1\dots\mathrm{d}y_n.$$

with the help of Definition 1.3, we can find a positive constant depending on the two multi-indices $\delta-\delta'$ and $\beta-\beta'$ such that

$$\sup_{(x_1,\dots,x_n)\in\mathbb{R}^n}\left|\prod_{k=1}^n x_k^{\alpha_k}\left(\frac{\partial}{\partial x_k}\right)^{\beta_k}(T_{\sigma,M}\varphi)(x_1,\dots,x_n)\right|$$
$$\le|\overline{C_M}|\prod_{k=1}^n\left\{\sum_{\beta_k'=0}^{\beta_k}\binom{\beta_k}{\beta_k'}\sum_{\beta_k''=0}^{\beta_k'}\binom{\beta_k'}{\beta_k''}\right\}\left(\frac{1}{b}\right)^{\beta_k''-\delta_k}\left\{\prod_{k=1}^n\sum_{\eta_k=0}^{\beta_k'-\beta_k''}A_{\eta_k}\left(\frac{a}{b}\right)\right\}\left\{\prod_{k=1}^n\sum_{\eta_k'=0}^{\beta_k'-\beta_k''}A_{\eta_k'}\left(\frac{d}{b}\right)\right\}^{-1}\prod_{k=1}^n\left(\sum_{\delta_k'=0}^{\delta_k}\binom{\delta_k}{\delta_k'}\right)$$
$$\times C_{\beta-\beta',\delta-\delta'}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(1+\prod_{k=1}^n|y_k|\right)^{m-|\delta|+|\delta'|}\left|\prod_{k=1}^n\left(\frac{\partial}{\partial y_k}\right)^{\delta_k'}\left(y_k^{\beta_k''-\eta_k'}\varphi_M(y_1,\dots,y_n)\right)\right|\mathrm{d}y_1\dots\mathrm{d}y_n. \tag{2.1}$$

Since $\varphi\in\mathcal{S}(\mathbb{R}^n)$, it follows from Eq. (2.1), and there exist is a positive constant $C_{M,\beta-\beta',\delta-\delta'}$ depending on $C_{\beta-\beta',\delta-\delta'}$ and C_M, we have

$$\sup_{(x_1,\dots,x_n)\in\mathbb{R}^n}\left|\prod_{k=1}^n x_k^{\alpha_k}\left(\frac{\partial}{\partial x_k}\right)^{\beta_k}(T_{\sigma,M}\varphi)(x_1,\dots,x_n)\right|\le C_{M,\beta-\beta',\delta-\delta'}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(1+\prod_{k=1}^n|y_k|\right)^{m-|\delta|+|\delta'|}\mathrm{d}y_1\dots\mathrm{d}y_n.$$

Since the integrals on the right-hand side are convergent for sufficiently large multi-index δ, we obtained the desired result. □

Theorem 2.2. *Let $\sigma\in\mathcal{S}^{m_1}$ and $\tau\in\mathcal{S}^{m_2}$, $(m_1,m_2\in\mathbb{R})$. Then the product of two n-dimensional pseudo-differential operators $T_{\sigma,M}T_{\tau,M}$ is again an n-dimensional pseudo-differential operator $T_{\lambda,M}$, where λ is a symbol in $\mathcal{S}^{m_1+m_2}$ and has the following asymptotic expansion*

$$\lambda(x_1,\dots,x_n;y_1,\dots,y_n)\sim\prod_{k=1}^n\sum_{\mu_k=0}^{\infty}\frac{(-\mathrm{i})^{\mu_k}}{\mu_k!}\left(\frac{\partial\sigma}{\partial y_k}\right)^{\mu_k}\left(\frac{\partial\tau}{\partial x_k}\right)^{\mu_k}(x_1,\dots,x_n;y_1,\dots,y_n), \tag{2.2}$$

here, Eq. (2.2) *means that*

$$\lambda(x_1,\ldots,x_n;y_1,\ldots,y_n)-\prod_{k=1}^{n}\sum_{\mu_k=0}^{l-1}\frac{(-\mathrm{i})^{\mu_k}}{\mu_k!}\left(\frac{\partial\sigma}{\partial y_k}\right)^{\mu_k}\left(\frac{\partial\tau}{\partial x_k}\right)^{\mu_k}(x_1,\ldots,x_n;y_1,\ldots,y_n), \tag{2.3}$$

is a symbol in $\mathcal{S}^{m_1+m_2-l}$ for every positive integer l.

Proof. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$. From Eq. (1.7), we have

$$\begin{aligned}T_{\sigma,M}\varphi(x_1,\ldots,x_n)&=\sum_{r=0}^{\infty}(T_{\sigma_r,M}\varphi)(x_1,\ldots,x_n)\\
&=\overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b}+\frac{dy_k^2}{2b}-\frac{x_ky_k}{b}\right)}\right\}\left(\sum_{r=0}^{\infty}\sigma_r(x_1,\ldots,x_n;y_1,\ldots,y_n)\right)\varphi_M(y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n\\
&=\overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b}+\frac{dy_k^2}{2b}-\frac{x_ky_k}{b}\right)}\right\}\sigma(x_1,\ldots,x_n;y_1,\ldots,y_n)\varphi_M(y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n.\end{aligned}$$

For $\tau\in\mathcal{S}^{m_2}$, we obtain

$$\begin{aligned}(T_{\sigma_r,M}T_{\tau,M}\varphi)(x_1,\ldots,x_n)&=\overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b}+\frac{dy_k^2}{2b}-\frac{x_ky_k}{b}\right)}\right\}\\
&\times\ \sigma_r(x_1,\ldots,x_n;y_1,\ldots,y_n)\mathcal{L}_M(T_{\tau,M}\varphi)(y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n.\end{aligned}$$

Using Eq. (1.1), (1.9) and Fubini's theorem on the right-hand side of the above expression, we get

$$\begin{aligned}&=\overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b}+\frac{dy_k^2}{2b}\right)+\mathrm{i}\frac{x_ky_k}{b}}\right\}\left(C_M\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{\mathrm{i}\left(\frac{a(x_k-t_k)^2}{2b}+\frac{dy_k^2}{2b}\right)-\mathrm{i}\frac{y_k(x_k-t_k)}{b}}\right\}\right.\\
&\times\ \left.K_{r,M}(x_1,\ldots,x_n;x_1-t_1,\ldots,x_n-t_n)\tau(y_1,\ldots,y_n;t_1,\ldots,t_n)\mathrm{d}t_1\ldots\mathrm{d}t_n\right)\varphi_M(y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n\\
&=\overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b}+\frac{dy_k^2}{2b}\right)+\mathrm{i}\frac{x_ky_k}{b}}\right\}\lambda_r(x_1,\ldots,x_n;y_1,\ldots,y_n)\varphi_M(y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n,\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}\lambda_r(x_1,\ldots,x_n;y_1,\ldots,y_n)=C_M\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}&\left\{\prod_{k=1}^{n}\mathrm{e}^{\mathrm{i}\left(\frac{a(x_k-t_k)^2}{2b}+\frac{dy_k^2}{2b}\right)-\mathrm{i}\frac{y_k(x_k-t_k)}{b}}\right\}\\
&\times K_{r,M}(x_1,\ldots,x_n;x_1-t_1,\ldots,x_n-t_n)\tau(y_1,\ldots,y_n;t_1,\ldots,t_n)\mathrm{d}t_1\ldots\mathrm{d}t_n.\end{aligned}\tag{2.5}$$

Therefore, Eq. (2.4) becomes

$$(T_{\sigma_r,M}T_{\tau,M}\varphi)(x_1,\ldots,x_n)=(T_{\lambda,M}\varphi)(x_1,\ldots,x_n), \tag{2.6}$$

where

$$\lambda(x_1,\ldots,x_n;y_1,\ldots,y_n)=\sum_{r=0}^{\infty}\lambda_r(x_1,\ldots x_n;y_1,\ldots,y_n). \tag{2.7}$$

Now, Eq. (2.5) can be re-written as

$$\begin{aligned}\lambda_r(x_1,\ldots,x_n;y_1,\ldots,y_n)&=C_M\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{\mathrm{i}\left(\frac{az_k^2}{2b}+\frac{dy_k^2}{2b}\right)-\mathrm{i}\frac{y_kz_k}{b}}\right\}K_{r,M}(x_1,\ldots,x_n;z_1,\ldots,z_n)\\
&\times\ \tau(x_1-z_1,\ldots,x_n-z_n;y_1,\ldots,y_n)\mathrm{d}z_1\ldots\mathrm{d}z_n.\end{aligned}\tag{2.8}$$

Using Taylor's formula provided in Theorem 1.5, we can write the right-hand side of the above expression as

$$\begin{aligned}\tau(x_1-z_1,\dots,x_n-z_n;y_1,\dots,y_n) &= \prod_{k=1}^{n}\sum_{\mu_k=0}^{l_1-1}\frac{(-z_k)^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\tau(x_1,\dots,x_n;y_1,\dots,y_n)\\ &+ R_{l_1}(x_1,\dots,x_n;y_1,\dots,y_n;z_1,\dots,z_n). \end{aligned} \tag{2.9}$$

Using Eq. (2.8) and Eq. (2.9), we have

$$\begin{aligned}\lambda_r(x_1,\dots,x_n;y_1,\dots,y_n) &= \prod_{k=1}^{n}\sum_{\mu_k=0}^{l_1-1}\frac{1}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\tau(x_1,\dots,x_n;y_1,\dots,y_n)C_M\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}e^{i\left(\frac{az_k^2}{2b}+\frac{dy_k^2}{2b}\right)-i\frac{y_kz_k}{b}}\right\}\\ &\times K_{r,M}(x_1,\dots,x_n;z_1,\dots,z_n)[(-z_1)^{\mu_1}\dots(-z_k)^{\mu_k}]dz_1\dots dz_n+T^r_{l_1,M}(x_1,\dots,x_n;y_1,\dots,y_n),\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}T^r_{l_1,M}(x_1,\dots,x_n;y_1,\dots,y_n) &= C_M\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}e^{i\left(\frac{az_k^2}{2b}+\frac{dy_k^2}{2b}\right)-i\frac{y_kz_k}{b}}\right\}K_{r,M}(x_1,\dots,x_n;z_1,\dots,z_n)\\ &\times R_{l_1}(x_1,\dots,x_n;y_1,\dots,y_n;z_1,\dots,z_n)dz_1\dots dz_n.\end{aligned} \tag{2.11}$$

Using Definition 1.1 and Eq. (2.10), we get

$$\begin{aligned}\lambda_r(x_1,\dots,x_n;y_1,\dots,y_n) &= \prod_{k=1}^{n}\sum_{\mu_k=0}^{l_1-1}\frac{(-z_k)^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\sigma_r(x_1,\dots,x_n;y_1,\dots,y_n)\\ &\times \prod_{k=1}^{n}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\tau(x_1,\dots,x_n;y_1,\dots,y_n)+T^r_{l_1,M}(x_1,\dots,x_n;y_1,\dots,y_n).\end{aligned} \tag{2.12}$$

For any integer $l_1 > l$, the function λ given by Eq. (2.7) satisfies

$$\begin{aligned}&\lambda(x_1,\dots,x_n;y_1,\dots,y_n)-\prod_{k=1}^{n}\sum_{\mu_k=0}^{l-1}\frac{(-i_k)^{\mu_k}}{\mu_k!}\left(\frac{\partial\sigma_r}{\partial y_k}\right)^{\mu_k}\left(\frac{\partial\tau}{\partial x_k}\right)^{\mu_k}(x_1,\dots,x_n;y_1,\dots,y_n)\\ =\ &\lambda(x_1,\dots,x_n;y_1,\dots,y_n)-\prod_{k=1}^{n}\sum_{\mu_k=0}^{l_1-1}\frac{(-i_k)^{\mu_k}}{\mu_k!}\left(\frac{\partial\sigma_r}{\partial y_k}\right)^{\mu_k}\left(\frac{\partial\tau}{\partial x_k}\right)^{\mu_k}(x_1,\dots,x_n;y_1,\dots,y_n)\\ +\ &\prod_{k=1}^{n}\sum_{\mu_k=l}^{l_1-1}\frac{(-i_k)^{\mu_k}}{\mu_k!}\left(\frac{\partial\sigma_r}{\partial y_k}\right)^{\mu_k}\left(\frac{\partial\tau}{\partial x_k}\right)^{\mu_k}(x_1,\dots,x_n;y_1,\dots,y_n).\end{aligned} \tag{2.13}$$

From Wong [28, p. 57], we have

$$\prod_{k=1}^{n}\sum_{\mu_k=l}^{l_1-1}\frac{(-i_k)^{\mu_k}}{\mu_k!}\left(\frac{\partial\sigma_r}{\partial y_k}\right)^{\mu_k}\left(\frac{\partial\tau}{\partial x_k}\right)^{\mu_k}(x_1,\dots,x_n;y_1,\dots,y_n)\in\mathcal{S}^{m_1+m_2-l}. \tag{2.14}$$

From Wong [28, p. 58–59] and for all multi-indices α and β there exist a positive constant $C_{\alpha,\beta} > 0$, such that

$$\left|\partial_x^\alpha\partial_y^\beta\left(\lambda-\prod_{k=1}^{n}\sum_{\mu_k=0}^{l-1}\frac{(-i_k)^{\mu_k}}{\mu_k!}\left(\frac{\partial\sigma_r}{\partial y_k}\right)^{\mu_k}\left(\frac{\partial\tau}{\partial x_k}\right)^{\mu_k}\right)(x_1,\dots,x_n;y_1,\dots,y_n)\right| \le C_{\alpha,\beta}\left(1+\prod_{k=1}^{n}|y_k|\right)^{m_1+m_2-l-|\beta|}. \tag{2.15}$$

From Eq. (2.13), Eq. (2.14) and Eq. (2.15), we conclude that $\lambda\in\mathcal{S}^{m_1+m_2}$. □

Definition 2.3. *Let σ be a symbol in $\mathcal{S}^m$ and $T_{\sigma,M}$ is the n-dimensional pseudo-differential operator. Suppose there exist a linear operator $T^{\star}_{\sigma,M} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, such that*

$$\left\langle T_{\sigma,M}\varphi, \psi \right\rangle = \left\langle \varphi, T^{\star}_{\sigma,M}\psi \right\rangle, \quad \forall\ \varphi, \psi \in \mathcal{S}(\mathbb{R}^n), \tag{2.16}$$

then, $T^{\star}_{\sigma,M}$ is said to be formal adjoint of the operator $T_{\sigma,M}$.

Theorem 2.4. *Let $\sigma \in \mathcal{S}^m$. Then the formal adjoint of the n-dimensional pseudo-differential operator $T_{\sigma,M}$ is again an n-dimensional pseudo-differential operator $T_{\tau,M}$ and has the following asymptotic expansion*

$$\tau(x_1, \dots, x_n; y_1, \dots, y_n) \sim \prod_{k=1}^{n} \sum_{\mu_k=0}^{\infty} \frac{(-\mathrm{i}_k)^{\mu_k}}{\mu_k!} \left(\frac{\partial}{\partial x_k}\right)^{\mu_k} \left(\frac{\partial}{\partial y_k}\right)^{\mu_k} \overline{\sigma}(x_1, \dots, x_n; y_1, \dots, y_n), \tag{2.17}$$

here, Eq. (2.17) *means that*

$$\tau(x_1, \dots, x_n; y_1, \dots, y_n) - \prod_{k=1}^{n} \sum_{\mu_k=0}^{l-1} \frac{(-\mathrm{i}_k)^{\mu_k}}{\mu_k!} \left(\frac{\partial}{\partial x_k}\right)^{\mu_k} \left(\frac{\partial}{\partial y_k}\right)^{\mu_k} \overline{\sigma}(x_1, \dots, x_n; y_1, \dots, y_n) \in \mathcal{S}^{m-l}, \tag{2.18}$$

where $\tau \in \mathcal{S}^m$, and for all positive integer l.

Proof. For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, then we have

$$\begin{aligned} \left\langle T_{\sigma_r,M}\varphi, \psi \right\rangle &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\overline{C_M} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{n} \mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b} + \frac{dy_k^2}{2b}\right) + \mathrm{i}\frac{x_k y_k}{b}} \right\} \right. \\ &\times \left. \sigma_r(x_1, \dots, x_n; y_1, \dots, y_n)\varphi_M(y_1, \dots y_n) \right) \mathrm{d}y_1 \dots \mathrm{d}y_n \overline{\psi(x_1, \dots, x_n)} \mathrm{d}x_1 \dots \mathrm{d}x_n. \end{aligned}$$

Using Eq. (1.9), Eq. (2.16), and Fubini's theorem on the right-hand side of the above expression, we have

$$\begin{aligned} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(C_M \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{n} \mathrm{e}^{-\mathrm{i}\left(\frac{a(x_k^2 - z_k^2)}{2b}\right) + \mathrm{i}\left(\frac{a(x_k - z_k)^2}{2b} + \frac{dy_k^2}{2b}\right)} \right\} \right. \\ &\times \left. K_{r,M}(x_1, \dots, x_n; x_1 - z_1, \dots, x_n - z_n) \overline{\psi(x_1, \dots, x_n)} \mathrm{d}x_1 \dots \mathrm{d}x_n \right) \varphi(z_1, \dots, z_n) \mathrm{d}z_1 \dots \mathrm{d}z_n \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(z_1, \dots, z_n) \overline{(T^{\star}_{\sigma_r,M}\psi)(z_1, \dots, z_n)} \mathrm{d}y_1 \dots \mathrm{d}y_n, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \left(T^{\star}_{\sigma_r,M}\psi\right)(x_1, \dots, x_n) = \overline{C_M} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{n} \mathrm{e}^{\mathrm{i}\left(\frac{a(z_k^2 - x_k^2)}{2b}\right) - \mathrm{i}\left(\frac{a(z_k - x_k)^2}{2b} + \frac{dy_k^2}{2b}\right)} \right\} \\ \times \overline{K_{r,M}}(z_1, \dots, z_n; z_1 - x_1, \dots, z_n - x_n)\psi(z_1, \dots, z_n) \mathrm{d}z_1 \dots \mathrm{d}z_n. \end{aligned} \tag{2.20}$$

Using the n-dimensional *inverse* LCT on the right-hand side of the above expression and applying change of variables (let us take $z_k - x_k = \zeta_k$, for $k = 1, \dots, n$), we get

$$\begin{aligned} &= \overline{C_M} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{n} \mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b} + \frac{dy_k^2}{2b}\right) + \mathrm{i}\frac{x_k y_k}{b}} \right\} \left(\overline{C_M} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{n} \mathrm{e}^{-\mathrm{i}\left(\frac{a\zeta_k^2}{2b} + \frac{dy_k^2}{2b}\right) + \mathrm{i}\frac{\zeta_k y_k}{b}} \right\} \right. \\ &\times \left. \overline{K_{r,M}}(x_1 + \zeta_1, \dots, x_n + \zeta_n; \zeta_1, \dots, \zeta_n) \mathrm{d}\zeta_1 \dots \mathrm{d}\zeta_n \right) \psi_M(y_1, \dots, y_n) \mathrm{d}y_1 \dots \mathrm{d}y_n \\ &= \overline{C_M} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{n} \mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b} + \frac{dy_k^2}{2b}\right) + \mathrm{i}\frac{x_k y_k}{b}} \right\} \tau_r(x_1, \dots, x_n; y_1, \dots, y_n)\psi_M(y_1, \dots, y_n) \mathrm{d}y_1 \dots \mathrm{d}y_n \\ &= \left(T_{\tau_r,M}\psi\right)(x_1, \dots, x_n), \end{aligned} \tag{2.21}$$

where

$$\tau_r(x_1,\dots,x_n;y_1,\dots,y_n) = \overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n} e^{-i\left(\frac{a\zeta_k^2}{2b}+\frac{dy_k^2}{2b}\right)+i\frac{y_k\zeta_k}{b}}\right\} \times \overline{K_{r,M}}(x_1+\zeta_1,\dots,x_n+\zeta_n;\zeta_1,\dots,\zeta_n)d\zeta_1\dots d\zeta_n. \tag{2.22}$$

Therefore, from Eq. (2.21), $T^{\star}_{\sigma_r,M}$ is again an n-dimensional pseudo-differential operator. Since

$$\langle T_{\sigma,M}\varphi,\psi\rangle = \sum_{r=0}^{\infty}\langle T_{\sigma_r,M}\varphi,\psi\rangle.$$

Therefore,

$$\tau(x_1,\dots,x_n;y_1,\dots,y_n) = \sum_{r=0}^{\infty}\tau_r(x_1,\dots,x_n;y_1,\dots,y_n), \tag{2.23}$$

and

$$\left\langle T_{\sigma,M}\varphi,\psi\right\rangle = \left\langle\varphi, T_{\tau,M}\psi\right\rangle. \tag{2.24}$$

Then by Taylor's formula with the integral remainder given by Eq. (1.10), we get

$$K_{r,M}(x_1+\zeta_1,\dots,x_n+\zeta_n;\zeta_1,\dots,\zeta_n) = \prod_{k=1}^{n}\sum_{\mu_k=0}^{l_1-1}\frac{\zeta_k^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}K_{r,M}(x_1,\dots,x_n;\zeta_1,\dots,\zeta_n) + R^r_{l_1}(x_1,\dots,x_n;\zeta_1,\dots,\zeta_n), \tag{2.25}$$

Therefore, from Eq. (2.22), we have

$$\tau_r(x_1,\dots,x_n;y_1,\dots,y_n) = \prod_{k=1}^{n}\sum_{\mu_k=l_1}^{\infty}\frac{(-i)^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\left(\frac{\partial}{\partial y_k}\right)^{\mu_k}\overline{\sigma_r}(x_1,\dots,x_n;y_1,\dots,y_n) + T^r_{l_1}(x_1,\dots,x_n;y_1,\dots,y_n), \tag{2.26}$$

where

$$T^r_{l_1}(x_1,\dots,x_n;y_1,\dots,y_n) = \overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n} e^{-i\left(\frac{a\zeta_k^2}{2b}+\frac{dy_k^2}{2b}\right)+i\frac{\zeta_k y_k}{b}}\right\} \times \overline{R^r_{l_1}(x_1,\dots,x_n;\zeta_1,\dots,\zeta_n)}d\zeta_1\dots d\zeta_n. \tag{2.27}$$

For any positive integer $l_1 > l$, the function τ is given by Eq. (2.23) satisfies

$$\begin{aligned}
&\tau(x_1,\dots,x_n;y_1,\dots,y_n) - \prod_{k=1}^{n}\sum_{\mu_k=0}^{l-1}\frac{(-i)^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\left(\frac{\partial}{\partial y_k}\right)^{\mu_k}\overline{\sigma}(x_1,\dots,x_n;y_1,\dots,y_n)\\
= {}& \tau(x_1,\dots,x_n;y_1,\dots,y_n) - \prod_{k=1}^{n}\sum_{\mu_k=0}^{l_1-1}\frac{(-i)^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\left(\frac{\partial}{\partial y_k}\right)^{\mu_k}\overline{\sigma}(x_1,\dots,x_n;y_1,\dots,y_n)\\
+ {}& \prod_{k=1}^{n}\sum_{\mu_k=l}^{l_1-1}\frac{(-i)^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\left(\frac{\partial}{\partial y_k}\right)^{\mu_k}\overline{\sigma}(x_1,\dots,x_n;y_1,\dots,y_n).
\end{aligned} \tag{2.28}$$

Hence,

$$\prod_{k=1}^{n}\sum_{\mu_k=l}^{l_1-1}\frac{(-\mathrm{i})^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\left(\frac{\partial}{\partial y_k}\right)^{\mu_k}\overline{\sigma}(x_1,\ldots,x_n;y_1,\ldots,y_n)\in\mathcal{S}^{m-l}. \tag{2.29}$$

From Wong [28, p. 58–59] and for all multi-indices α and β, there exists a constant $C_{\alpha,\beta}>0$, such that

$$\left|\left(\left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}\left(\frac{\partial}{\partial y_k}\right)^{\beta_k}\left(\tau-\prod_{k=1}^{n}\sum_{\mu_k=0}^{l-1}\frac{(-\mathrm{i})^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\left(\frac{\partial}{\partial y_k}\right)^{\mu_k}\overline{\sigma}\right)\right)(x_1,\ldots,x_n;y_1,\ldots,y_n)\right|\leq C_{\alpha,\beta}\left(1+\prod_{k=1}^{n}|y_k|\right)^{m-l-|\beta|}. \tag{2.30}$$

Therefore, from Eq. (2.30), we get the desired result as follows

$$\tau(x_1,\ldots,x_n;y_1,\ldots,y_n)-\prod_{k=1}^{n}\sum_{\mu_k=0}^{l-1}\frac{(-\mathrm{i})^{\mu_k}}{\mu_k!}\left(\frac{\partial}{\partial x_k}\right)^{\mu_k}\left(\frac{\partial}{\partial y_k}\right)^{\mu_k}\overline{\sigma}(x_1,\ldots,x_n;y_1,\ldots,y_n)\in S^{m-l}.$$

□

Theorem 2.5. *Let $\sigma\in\mathcal{S}^0$, then $T_{\sigma,M}:L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ is a bounded linear operator for $1\leq p<\infty$.*

Proof. From Eq. (1.7), we have

$$\begin{aligned}(T_{\sigma,M}\varphi)(x_1,\ldots,x_n)&=\sum_{r=0}^{\infty}(T_{\sigma_r,M}\varphi)(x_1,\ldots,x_n)\\ &=\overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{-\mathrm{i}\left(\frac{ax_k^2}{2b}+\frac{dy_k^2}{2b}\right)+\mathrm{i}\frac{x_ky_k}{b}}\right\}\sigma(x_1,\ldots,x_n;y_1,\ldots,y_n)\varphi_M(y_1,\ldots,y_n)\mathrm{d}y_1\ldots\mathrm{d}y_n,\end{aligned} \tag{2.31}$$

applying n-dimensional LCT then after n-dimensional *inverse* LCT on the symbol $\sigma(x_1,\ldots,x_n;y_1,\ldots,y_n)$, and using Fubini's theorem, we have

$$(T_{\sigma,M}\varphi)(x_1,\ldots,x_n)=\overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}\mathrm{e}^{-\mathrm{i}\left(\frac{a\lambda_k^2}{2b}+\frac{dx_k^2}{2b}\right)+\mathrm{i}\frac{\lambda_kx_k}{b}}\right\}(T_{\lambda,M}\varphi)(x_1,\ldots,x_n)\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_n. \tag{2.32}$$

Since $\sigma\in\mathcal{S}^0$ and from Upadhyay and Dubey [25] for $1\leq p<\infty$, we have

$$\left\|T_{\lambda,M}\varphi\right\|_p=C_{M,\beta}\left(1+\prod_{k=1}^{n}|\lambda_k|\right)^{-|\beta|}\left\|\varphi\right\|_p. \tag{2.33}$$

Using Minkowski's inequality in Eq. (2.32), we obtain

$$\left\|T_{\lambda,M}\varphi\right\|_p\leq\overline{C_M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\|T_{\lambda,M}\varphi\right\|_p\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_n. \tag{2.34}$$

From Eq. (2.33) and Eq. (2.34), we get

$$\left\|T_{\sigma,M}\varphi\right\|_p\leq\overline{C_M}C_{M,\beta}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(1+\prod_{k=1}^{n}|\lambda_k|\right)^{-|\beta|}\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_n\right)\left\|\varphi\right\|_p,$$

since the last integrals are bounded, therefore we have

$$\left\|T_{\sigma,M}\varphi\right\|_p\leq\overline{C_{M,\beta}}\left\|\varphi\right\|_p,\quad\text{for all }1\leq p<\infty.$$

□

3. Important results of an n-dimensional linear canonical transform

This section has derived some new results of an n-dimensional linear canonical transform in a distributional sense. In the next section, these new findings become elegant tools for solving generalized partial differential equations and particular cases in quantum mechanics.

Proposition 3.1. *Suppose $\varphi, \psi \in \mathcal{S}^M(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$, then following results can be obtained*

(i) $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\left\{\prod_{k=1}^{n}\Delta_{x_k,a,b}\right\}\varphi(x_1,\ldots,x_n)\right)\psi(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n$
$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(x_1,\ldots,x_n)\left(\left\{\prod_{k=1}^{n}\Delta^{\star}_{x_k,a,b}\right\}\psi(x_1,\ldots,x_n)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n,$

(ii) $\left\{\prod_{k=1}^{n}\left(\Delta_{x_k,a,b}\right)K_M(x_k,y_k)\right\}^j = \left(\prod_{k=1}^{n}\frac{-\mathrm{i}y_k}{b}\right)^j\{K_M(x_k,y_k)\},$

(iii) $\mathcal{L}_M\left\{\left(\prod_{k=1}^{n}\Delta^{\star}_{x_k,a,b}\right)^j\varphi(x_1,\ldots,x_n)\right\} = \left(\prod_{k=1}^{n}\frac{-\mathrm{i}y_k}{b}\right)^j(\mathcal{L}_M\varphi(x_1,\ldots,x_n))(y_1,\ldots,y_n),$

where $\Delta^{\star}_{x_k,a,b} = -\left(\frac{\partial}{\partial x_k}+\mathrm{i}\frac{a}{b}x_k\right)$ is the adjoint operator of $\Delta_{x_k,a,b} = \left(\frac{\partial}{\partial x_k}-\mathrm{i}\frac{a}{b}x_k\right)$.

Proof. One can derive this proposition easily by extending (one dimension to n-dimension) results provided in [4, 5]. □

Proposition 3.2. *For any function $f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n,a,b}) \in \mathcal{S}(\mathbb{R}^n)$, then we have the following result*

$$f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n,a,b})\left\{\prod_{k=1}^{n}K_M(x_k,y_k)\right\} = f\left(-\frac{\mathrm{i}y_1}{b},\ldots,-\frac{\mathrm{i}y_n}{b}\right)\left\{\prod_{k=1}^{n}K_M(x_k,y_k)\right\}.$$

Proof. As $f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n,a,b}) \in \mathcal{S}(\mathbb{R}^n)$, then $f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n,a,b})$ can be written in the form of n-dimensional Taylor's series expansion about the origin by

$$f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n,a,b}) = \sum_{r_1=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\Delta^{r_1}_{x_1,a,b}\cdots\Delta^{r_n}_{x_n,a,b}}{r_1!\ldots r_n!}\frac{\partial^{|r|}f(0,\ldots,0)}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}, \tag{3.1}$$

using Proposition 3.1 (ii) part, we can write

$$f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n a,b})\left\{\prod_{k=1}^{n}K_M(x_k,y_k)\right\} = \sum_{r_1=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\Delta^{r_1}_{x_1,a,b}K_M(x_1,y_1)\ldots\Delta^{r_n}_{x_n,a,b}K_M(x_n,y_n)}{r_1!\ldots r_n!}\frac{\partial^{|r|}f(0,\ldots,0)}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}$$
$$= \sum_{r_1=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\left(-\frac{\mathrm{i}y_1}{b}\right)^{r_1}\cdots\left(-\frac{\mathrm{i}y_n}{b}\right)^{r_n}}{r_1!\ldots r_n!}\frac{\partial^{|r|}f(0,\ldots,0)}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}\left\{\prod_{k=1}^{n}K_M(x_k,y_k)\right\} = f\left(-\frac{\mathrm{i}y_1}{b},\ldots,-\frac{\mathrm{i}y_n}{b}\right)\left\{\prod_{k=1}^{n}K_M(x_k,y_k)\right\}.$$

□

Lemma 3.3. *For all $f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n,a,b}) \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n,a,b})\left\{\prod_{k=1}^{n}K_M(x_k,y_k)\right\}\varphi(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n$$
$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n}K_M(x_k,y_k)\right\}f(\Delta^{\star}_{x_1,a,b},\ldots,\Delta^{\star}_{x_n,a,b})\varphi(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{3.2}$$

Proof. To prove Lemma 3.3, we first start with the left-hand side of Eq. (3.2) as below

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\Delta_{x_1,a,b},\ldots,\Delta_{x_n,a,b})\left\{\prod_{k=1}^{n} K_M(x_k,y_k)\right\}\varphi(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n,$$

using Eq. (3.1) and Proposition 3.1 (iii) part, we have

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left\{\prod_{k=1}^{n} K_M(x_k,y_k)\right\}\left(\sum_{r_1=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{\left(\Delta^{\star}_{x_1,a,b}\right)^{r_1}\ldots\left(\Delta^{\star}_{x_n,a,b}\right)^{r_n}}{r_1!\ldots r_n!}\frac{\partial^{|r|}f(0,\ldots,0)}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}\right)$$
$$\varphi(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n.$$

So, we got the desired result. □

4. Applications

This section uses the n-dimensional linear canonical transform and significant results obtained in the previous section to solve the initial value problem of n-dimensional Generalized Partial Differential Equations (n-GPDE) and their particular cases.

4.1. *An n-dimensional generalized partial differential equations*

The n-dimensional Generalized Partial Differential Equations (n-GPDE) are as follows:

$$\left.\begin{array}{ll}\text{n-GPDE}: & \mathcal{A}\frac{\partial\varphi(x_1,\ldots,x_n,t)}{\partial t} = \mathcal{B}\left\{f\left(\Delta^{\star}_{x_1,a,b},\ldots,\Delta^{\star}_{x_n,a,b}\right)\varphi\right\}(x_1,\ldots,x_n,t) + C\varphi(x_1,\ldots,x_n,t),\\ \text{IC}: & \text{Initial } \varphi(x_1,\ldots,x_n,0) = g(x_1,\ldots,x_n),\quad \forall\,(x_1,\ldots,x_n)\in\mathbb{R}^n,\ t\geq 0,\end{array}\right\}\tag{4.1}$$

where $\mathcal{A}\neq 0,\mathcal{B}$ and C are governing parameters of the n-GPDE. Also, $\Delta^{\star}_{x_k,a,b}$ and $f\left(\Delta^{\star}_{x_1,a,b},\ldots,\Delta^{\star}_{x_n,a,b}\right)$ are defined in Proposition 3.1, and Eq. (3.1) (where $\Delta_{x_k,a,b}$ is replaced by $\Delta^{\star}_{x_k,a,b}$ for $k=1,2,\ldots,n$), respectively. Taking n-dimensional LCT on n-GPDE of Eq. (4.1) and using Proposition 3.2, Lemma 3.3, we have

$$\frac{\partial\varphi_M(y_1,\ldots,y_n,t)}{\partial t} = \left(\frac{\mathcal{B}}{\mathcal{A}}f\left(-\frac{\mathrm{i}y_1}{b},\ldots,-\frac{\mathrm{i}y_1}{b}\right)+\frac{C}{\mathcal{A}}\right)\varphi_M(y_1,\ldots,y_n,t).$$

Therefore,

$$\varphi_M(y_1,\ldots,y_n,t) = g_M(y_1,\ldots,y_n)\,\mathrm{e}^{\left(\frac{\mathcal{B}}{\mathcal{A}}f\left(-\frac{\mathrm{i}y_1}{b},\ldots,-\frac{\mathrm{i}y_1}{b}\right)+\frac{C}{\mathcal{A}}\right)t}.\tag{4.2}$$

Taking n-dimensional *inverse* LCT of both sides of Eq. (4.2), we have

$$\varphi(x_1,\ldots,x_n,t) = \frac{\mathrm{e}^{\frac{C}{\mathcal{A}}t}\mathrm{e}^{-\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n}x_k^2\right)}}{(2\pi b)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{1}{b}\left(\sum_{k=1}^{n}x_ky_k\right)}\mathrm{e}^{\left(\frac{\mathcal{B}}{\mathcal{A}}f\left(-\frac{\mathrm{i}y_1}{b},\ldots,-\frac{\mathrm{i}y_1}{b}\right)\right)t}\mathrm{d}y_1\ldots\mathrm{d}y_n$$
$$\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{1}{b}\left(\sum_{k=1}^{n}y_kz_k\right)}\mathrm{e}^{\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n}z_k^2\right)}g(z_1,\ldots,z_n)\mathrm{d}z_1\ldots\mathrm{d}z_n.$$

4.2. *An n-dimensional generalized time-dependent Schrödinger-type-I equations in quantum mechanics and their particular cases*

In this section, we have established the n-dimensional generalized time-dependent Schrödinger-type-I equations in quantum mechanics for one particle with a constant potential and its particular cases while choosing suitable governing parameters and differential operators.

If we take the governing parameters as $\mathcal{A} = \mathrm{i}\hbar, \mathcal{B} = -\frac{\hbar^2}{2m}$ (where $\hbar = \frac{h}{2\pi}$ is the reduced Planck's constant and m is the mass of the particle), and $C = V(x_1, \dots, x_n) = \beta \neq 0$ is the constant potential energy influencing the particle along with the Laplacian-type-I operator $f\left(\Delta^{\star}_{x_1,a,b}, \dots, \Delta^{\star}_{x_n,a,b}\right) = \left(\Delta^{\star}_{x_1,a,b}\right)^2 + \cdots + \left(\Delta^{\star}_{x_n,a,b}\right)^2$ in the n-GPDE as defined in Eq. (4.1), then it reduces to the n-dimensional generalized time-dependent Schrödinger-type-I equations in quantum mechanics for one particle with non-zero constant potential as follows

$$\left.\begin{array}{ll} \text{n-GPDE}: & \mathrm{i}\hbar\frac{\partial\varphi(x_1,\dots,x_n,t)}{\partial t} = -\frac{\hbar^2}{2m}\left\{\left(\Delta^{\star}_{x_1,a,b}\right)^2 + \cdots + \left(\Delta^{\star}_{x_n,a,b}\right)^2\right\}\varphi(x_1,\dots,x_n,t) \\ & +\beta\varphi(x_1,\dots,x_n,t), \quad \forall\,(x_1,\dots,x_n) \in \mathbb{R}^n,\ t \geq 0, \\ & \\ \text{IC}: & \text{Initial } \varphi(x_1,\dots,x_n,0) = g(x_1,\dots,x_n), \end{array}\right\} \tag{4.3}$$

where $\Delta^{\star}_{x_k,a,b} = -\left(\frac{\partial}{\partial x_k} + \mathrm{i}\frac{a}{b}x_k\right)$ is defined in Proposition 3.1, and taking n-dimensional LCT on n-GPDE of Eq. (4.3) and using Proposition 3.1, we have

$$\frac{\partial\varphi_M(y_1,\dots,y_n,t)}{\partial t} = \left\{-\frac{\mathrm{i}\hbar}{2mb^2}\left(\sum_{k=1}^{n} y_k^2\right) - \frac{\mathrm{i}\beta}{\hbar}\right\}\varphi_M(y_1,\dots,y_n,t).$$

Therefore,

$$\varphi_M(y_1,\dots,y_n,t) = g_M(y_1,\dots,y_n)\,\mathrm{e}^{\left\{-\frac{\mathrm{i}\hbar}{2mb^2}\left(\sum_{k=1}^{n} y_k^2\right) - \frac{\mathrm{i}\beta}{\hbar}\right\}t}. \tag{4.4}$$

Taking n-dimensional *inverse* LCT of both sides of Eq. (4.4), we have

$$\begin{aligned}\varphi(x_1,\dots,x_n,t) = \frac{\mathrm{e}^{-\mathrm{i}\frac{\beta}{\hbar}t}\mathrm{e}^{-\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n} x_k^2\right)}}{(2\pi b)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{1}{b}\left(\sum_{k=1}^{n} x_k y_k\right)}\mathrm{e}^{\left\{-\frac{\mathrm{i}\hbar}{2mb^2}\left(\sum_{k=1}^{n} y_k^2\right)-\frac{\mathrm{i}\beta}{\hbar}\right\}t}\mathrm{d}y_1\dots\mathrm{d}y_n \\ \times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{1}{b}\left(\sum_{k=1}^{n} y_k z_k\right)}\mathrm{e}^{\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n} z_k^2\right)}g(z_1,\dots,z_n)\mathrm{d}z_1\dots\mathrm{d}z_n.\end{aligned}$$

Let us assume that $\Lambda_M(z_1,\dots,z_n) = \mathrm{e}^{\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n} z_k^2\right)}g(z_1,\dots,z_n)$, and the setting $\frac{y_k}{b} = v_k$, then the above expression becomes

$$\varphi(x_1,\dots,x_n,t) = \frac{m\mathrm{e}^{-\mathrm{i}\frac{\beta}{\hbar}t}\mathrm{e}^{-\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n} x_k^2\right)}}{\mathrm{i}\hbar(\pi \mathrm{t})^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\left(\sum_{k=1}^{n} x_k v_k\right)}\mathcal{F}\left(\mathrm{e}^{-\frac{\mathrm{i}\hbar x_k^2}{8mt}}\right)(v_1,\dots,v_n)\mathcal{F}(\Lambda_M(z))(v_1,\dots,v_n)\mathrm{d}v_1\dots\mathrm{d}v_n,$$

where $\mathcal{F}(f)$ denotes the Fourier transform of a function f. We have

$$\begin{aligned}&\varphi(x_1,\dots,x_n,t) \\ &= \frac{m\mathrm{e}^{-\mathrm{i}\frac{\beta}{\hbar}t}\mathrm{e}^{-\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n} x_k^2\right)}}{\mathrm{i}\hbar(\pi \mathrm{t})^{\frac{n}{2}}}\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\left(\sum_{k=1}^{n} x_k v_k\right)}\mathcal{F}\left\{\left(\mathrm{e}^{-\frac{\mathrm{i}\hbar x_k^2}{8mt}}\right)*(\Lambda_M(x))\right\}(v_1,\dots,v_n)\mathrm{d}v_1\dots\mathrm{d}v_n \\ &= \frac{m\mathrm{e}^{-\mathrm{i}\frac{\beta}{\hbar}t}\mathrm{e}^{-\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n} x_k^2\right)}}{\mathrm{i}\hbar(\pi \mathrm{t})^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\frac{\hbar}{8tm}\left(\sum_{k=1}^{n}(x_k-w_k)^2\right)}\Lambda_M(w_1,\dots,w_n)\mathrm{d}w_1\dots\mathrm{d}w_n.\end{aligned}$$

Hence,

$$\varphi(x_1,\dots,x_n,t) = \frac{m\mathrm{e}^{-\mathrm{i}\frac{\beta}{\hbar}t}\mathrm{e}^{-\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n} x_k^2\right)}}{\mathrm{i}\hbar(\pi \mathrm{t})^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\frac{\hbar}{8tm}\left(\sum_{k=1}^{n}(x_k-w_k)^2\right)}\mathrm{e}^{\mathrm{i}\frac{a}{2b}\left(\sum_{k=1}^{n} w_k^2\right)}g(w_1,\dots,w_n)\mathrm{d}w_1\dots\mathrm{d}w_n.$$

If we take the governing parameters as $\mathcal{A} = \mathrm{i}\hbar, \mathcal{B} = -\frac{\hbar^2}{2m}$ (where $\hbar = \frac{h}{2\pi}$ is the reduced Planck's constant and m is the mass of the particle), and $C = V(x_1,\dots,x_n) = \beta = 0$ is the constant potential energy influencing the particle along with the Laplacian-type-I operator $f\left(\Delta^{\star}_{x_1,a,b},\ldots,\Delta^{\star}_{x_n,a,b}\right)=\left(\Delta^{\star}_{x_1,a,b}\right)^2+\cdots+\left(\Delta^{\star}_{x_n,a,b}\right)^2$ in the n-GPDE as defined in Eq. (4.1), then it reduces to the n-dimensional generalized time-dependent Schrödinger-type-I equations in quantum mechanics for one particle with zero potential as follows:

$$\left.\begin{array}{ll}\text{n-GPDE}: & i\hbar\frac{\partial\varphi(x_1,\ldots,x_n,t)}{\partial t}=-\frac{\hbar^2}{2m}\left\{\left(\Delta^{\star}_{x_1,a,b}\right)^2+\cdots+\left(\Delta^{\star}_{x_n,a,b}\right)^2\right\}\varphi(x_1,\ldots,x_n,t),\\ & \forall\,(x_1,\ldots,x_n)\in\mathbb{R}^n,\ t\geq 0,\\ \text{IC}: & \text{Initial } \varphi(x_1,\ldots,x_n,0)=g(x_1,\ldots,x_n),\end{array}\right\} \tag{4.5}$$

where $\Delta^{\star}_{x_k,a,b}=-\left(\frac{\partial}{\partial x_k}+i\frac{a}{b}x_k\right)$ is defined in Proposition 3.1. Then the required solutions for Eq. (4.5) are as follows

$$\varphi(x_1,\ldots,x_n,t)=\frac{m e^{-i\frac{a}{2b}\left(\sum_{k=1}^n x_k^2\right)}}{i\hbar(\pi t)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{-i\frac{\hbar}{8tm}\left(\sum_{k=1}^n(x_k-w_k)^2\right)}e^{i\frac{a}{2b}\left(\sum_{k=1}^n w_k^2\right)}g(w_1,\ldots,w_n)dw_1\ldots dw_n.$$

4.3. An n-dimensional generalized time-dependent Schrödinger-type-II equations in quantum mechanics and their particular cases

In this section, we have established the n-dimensional generalized time-dependent Schrödinger-type-II equations in quantum mechanics for one particle with a constant potential and its particular cases while choosing suitable governing parameters and differential operators. If we take the governing parameters as $\mathcal{A}=i\hbar,\mathcal{B}=-\frac{\hbar^2}{2m}$ (where $\hbar=\frac{h}{2\pi}$ is the reduced Planck's constant and m is the mass of the particle), and $C=V(x_1,\ldots,x_n)=\beta\neq 0$ is the constant potential energy influencing the particle along with the Laplacian-type-II operator $f\left(\Delta^{\star}_{x_1,\cos\theta,\sin\theta},\ldots,\Delta^{\star}_{x_n,\cos\theta,\sin\theta}\right)=\left(\Delta^{\star}_{x_1,\cos\theta,\sin\theta}\right)^2+\cdots+\left(\Delta^{\star}_{x_n,\cos\theta,\sin\theta}\right)^2$ in the n-GPDE as defined in Eq. (4.1), then it reduces to the n-dimensional generalized time-dependent Schrödinger-type-II equations in quantum mechanics for one particle with non-zero potential as follows:

$$\left.\begin{array}{ll}\text{n-GPDE}: & i\hbar\frac{\partial\varphi(x_1,\ldots,x_n,t)}{\partial t}=-\frac{\hbar^2}{2m}\left\{\left(\Delta^{\star}_{x_1,\cos\theta,\sin\theta}\right)^2+\cdots+\left(\Delta^{\star}_{x_n,\cos\theta,\sin\theta}\right)^2\right\}\varphi(x_1,\ldots,x_n,t)\\ & +\beta\varphi(x_1,\ldots,x_n,t),\quad \forall\,(x_1,\ldots,x_n)\in\mathbb{R}^n,\ t\geq 0,\\ \text{IC}: & \text{Initial } \varphi(x_1,\ldots,x_n,0)=g(x_1,\ldots,x_n),\end{array}\right\} \tag{4.6}$$

where $\Delta^{\star}_{x_k,\cos\theta,\sin\theta}=-\left(\frac{\partial}{\partial x_k}+ix_k\cot\theta\right)$ is defined in Proposition 3.1, and the Laplacian-type-II operator is obtained by considering the entries of the matrix as $M=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ and $\theta\neq n\pi$, $n\in\mathbb{Z}$. Then the required solutions for Eq. (4.6) are as follows

$$\varphi(x_1,\ldots,x_n,t)=\frac{m e^{-i\frac{\beta}{\hbar}t}e^{-i\frac{\cot\theta}{2}\left(\sum_{k=1}^n x_k^2\right)}}{i\hbar(\pi t)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{-i\frac{\hbar}{8tm}\left(\sum_{k=1}^n(x_k-w_k)^2\right)}e^{i\frac{\cot\theta}{2}\left(\sum_{k=1}^n w_k^2\right)}g(w_1,\ldots,w_n)dw_1\ldots dw_n.$$

If we take the governing parameters as $\mathcal{A}=i\hbar,\mathcal{B}=-\frac{\hbar^2}{2m}$ (where $\hbar=\frac{h}{2\pi}$ is the reduced Planck's constant and m is the mass of the particle), and $C=V(x_1,\ldots,x_n)=\beta=0$ is the constant potential energy influencing the particle along with the Laplacian-type-II operator $f\left(\Delta^{\star}_{x_1,\cos\theta,\sin\theta},\ldots,\Delta^{\star}_{x_n,\cos\theta,\sin\theta}\right)=\left(\Delta^{\star}_{x_1,\cos\theta,\sin\theta}\right)^2+\cdots+\left(\Delta^{\star}_{x_n,\cos\theta,\sin\theta}\right)^2$ in the n-GPDE as defined in Eq. (4.1), then it reduces to the n-dimensional generalized time-dependent Schrödinger-type-II equations in quantum mechanics for one particle with zero potential as follows:

$$\left.\begin{array}{ll}\text{n-GPDE}: & i\hbar\frac{\partial\varphi(x_1,\ldots,x_n,t)}{\partial t}=-\frac{\hbar^2}{2m}\left\{\left(\Delta^{\star}_{x_1,\cos\theta,\sin\theta}\right)^2+\cdots+\left(\Delta^{\star}_{x_n,\cos\theta,\sin\theta}\right)^2\right\}\varphi(x_1,\ldots,x_n,t)\\ & \forall\,(x_1,\ldots,x_n)\in\mathbb{R}^n,\ t\geq 0,\\ \text{IC}: & \text{Initial } \varphi(x_1,\ldots,x_n,0)=g(x_1,\ldots,x_n),\end{array}\right\} \tag{4.7}$$

where $\Delta^{\star}_{x_k,\cos\theta,\sin\theta} = -\left(\frac{\partial}{\partial x_k} + \mathrm{i}x_k\cot\theta\right)$ is defined in Proposition 3.1, and the Laplacian-type-II operator is obtained by considering the entries of the matrix as $M = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $\theta \neq n\pi$, $n \in \mathbb{Z}$. Then the required solutions for Eq. (4.7) are as follows

$$\varphi(x_1,\ldots,x_n,t) = \frac{m\mathrm{e}^{-\mathrm{i}\frac{\cot\theta}{2}\left(\sum_{k=1}^{n} x_k^2\right)}}{\mathrm{i}\hbar(\pi \mathrm{t})^{\frac{n}{2}}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\frac{\hbar}{8tm}\left(\sum_{k=1}^{n}(x_k-w_k)^2\right)} \mathrm{e}^{\mathrm{i}\frac{\cot\theta}{2}\left(\sum_{k=1}^{n} w_k^2\right)} g(w_1,\ldots,w_n)\mathrm{d}w_1\ldots\mathrm{d}w_n.$$

4.4. An n-dimensional time-dependent Schrödinger equations in quantum mechanics and their particular cases

In this section, we have established the n-dimensional generalized time-dependent Schrödinger equations in quantum mechanics for one particle with a constant potential and its particular cases while choosing suitable governing parameters and differential operators. If we take the governing parameters as $\mathcal{A} = \mathrm{i}\hbar, \mathcal{B} = -\frac{\hbar^2}{2m}$ (where $\hbar = \frac{h}{2\pi}$ is the reduced Planck's constant and m is the mass of the particle), and $C = V(x_1,\ldots,x_n) = \beta \neq 0$ is the constant potential energy influencing the particle along with the Laplacian operator $f\left(\Delta^{\star}_{x_1,0,1},\ldots,\Delta^{\star}_{x_n,0,1}\right) = \left(\Delta^{\star}_{x_1,0,1}\right)^2 + \cdots + \left(\Delta^{\star}_{x_n,0,1}\right)^2$ in the n-GPDE as defined in Eq. (4.1), then it reduces to the n-dimensional generalized time-dependent Schrödinger equations in quantum mechanics for one particle with non-zero potential as follows:

$$\left.\begin{array}{ll} \text{n-GPDE}: & \mathrm{i}\hbar\frac{\partial\varphi(x_1,\ldots,x_n,t)}{\partial t} = -\frac{\hbar^2}{2m}\left\{\left(\Delta^{\star}_{x_1,0,1}\right)^2 + \cdots + \left(\Delta^{\star}_{x_n,0,1}\right)^2\right\}\varphi(x_1,\ldots,x_n,t) \\ & +\beta\varphi(x_1,\ldots,x_n,t), \quad \forall\,(x_1,\ldots,x_n) \in \mathbb{R}^n,\ t \geq 0, \\ \text{IC}: & \text{Initial } \varphi(x_1,\ldots,x_n,0) = g(x_1,\ldots,x_n), \end{array}\right\} \tag{4.8}$$

where $\Delta^{\star}_{x_k,0,1} = -\left(\frac{\partial}{\partial x_k}\right)$ is defined in Proposition 3.1, and the Laplacian operator is obtained by considering the entries of the matrix as $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then the required solutions for Eq. (4.8) are as follows

$$\varphi(x_1,\ldots,x_n,t) = \frac{m\mathrm{e}^{-\mathrm{i}\frac{\beta}{\hbar}t}}{\mathrm{i}\hbar(\pi \mathrm{t})^{\frac{n}{2}}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\frac{\hbar}{8tm}\left(\sum_{k=1}^{n}(x_k-w_k)^2\right)} g(w_1,\ldots,w_n)\mathrm{d}w_1\ldots\mathrm{d}w_n.$$

If we take the governing parameters as $\mathcal{A} = \mathrm{i}\hbar, \mathcal{B} = -\frac{\hbar^2}{2m}$ (where $\hbar = \frac{h}{2\pi}$ is the reduced Planck's constant and m is the mass of the particle), and $C = V(x_1,\ldots,x_n) = \beta = 0$ is the constant potential energy influencing the particle along with the Laplacian operator $f\left(\Delta^{\star}_{x_1,0,1},\ldots,\Delta^{\star}_{x_n,0,1}\right) = \left(\Delta^{\star}_{x_1,0,1}\right)^2 + \cdots + \left(\Delta^{\star}_{x_n,0,1}\right)^2$ in the n-GPDE as defined in Eq. (4.1), then it reduces to the n-dimensional generalized time-dependent Schrödinger equations in quantum mechanics for one particle with zero potential as follows:

$$\left.\begin{array}{ll} \text{n-GPDE}: & \mathrm{i}\hbar\frac{\partial\varphi(x_1,\ldots,x_n,t)}{\partial t} = -\frac{\hbar^2}{2m}\left\{\left(\Delta^{\star}_{x_1,0,1}\right)^2 + \cdots + \left(\Delta^{\star}_{x_n,0,1}\right)^2\right\}\varphi(x_1,\ldots,x_n,t) \\ & \forall\,(x_1,\ldots,x_n) \in \mathbb{R}^n,\ t \geq 0, \\ \text{IC}: & \text{Initial } \varphi(x_1,\ldots,x_n,0) = g(x_1,\ldots,x_n), \end{array}\right\} \tag{4.9}$$

where $\Delta^{\star}_{x_k,0,1} = -\left(\frac{\partial}{\partial x_k}\right)$ is defined in Proposition 3.1, and the Laplacian operator is obtained by considering the entries of the matrix as $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then the required solutions for Eq. (4.9) are as follows

$$\varphi(x_1,\ldots,x_n,t) = \frac{m}{\mathrm{i}\hbar(\pi \mathrm{t})^{\frac{n}{2}}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\frac{\hbar}{8tm}\left(\sum_{k=1}^{n}(x_k-w_k)^2\right)} g(w_1,\ldots,w_n)\mathrm{d}w_1\ldots\mathrm{d}w_n.$$

Conflict of interest:

The authors declare that they do not have any conflict of interest.

Acknowledgments:

We are grateful to the anonymous referees and Prof. H. M. Srivastava, Editor, for their careful reading, valuable comments, and helpful suggestions that have helped us to improve this work's presentation. This work has been supported by BITS Pilani, Hyderabad Campus, India, under the Senior Research Fellowship Scheme (2018PHXF0028H).

References

[1] L.B. Almeida, The fractional Fourier transform and time-frequency representations. *IEEE Trans. Signal Process.*, **42**(11):3084–3091,1994.

[2] A. Bultheel, and H. Martinez-Sulbaran, Recent developments in the theory of the fractional Fourier and linear canonical transforms. *Bull. Belg. Math. Soc. Simon Stevin*, **13**(5):971–1005,2007.

[3] M. Moshinsky, and C. Quesne, Linear canonical transformations and their unitary representations. *J. Math. Phy.*, **12**(5):1772–1780,1971.

[4] M. Kumar, A new class of pseudo-differential operators involving linear canonical transform. *Int. J. Appl. Comput. Math.*, **6**:1–23,2020. Article ID 165.

[5] A. Prasad, and Z.A. Ansari, Continuous wavelet transform involving linear canonical transform. *Natl. Acad. Sci. Lett.*, **42**(4):337–344,2019.

[6] J.J. Healy, M.A. Kutay, H.M. Ozaktas, and J.T Sheridan, Linear canonical transforms: Theory and applications. New York (NY): Springer; *(Springer Ser. Optical sci.; 198)*,2016.

[7] V. Namias, The fractional order Fourier transform and its application to quantum mechanics. *J. Inst. Math. Appl.*, **25**(3):241–265,1980.

[8] A.I. Zayed, Fractional Fourier transform of generalized functions. *Integral Transforms Spec. Funct.*, **7**(3-4):299–312,1998.

[9] E.C. Titchmarsh, *Introduction to the theory of Fourier integrals*. Oxford University Press (Clarendon Press), Oxford, London and New York, 1948.

[10] I.N. Sneddon, *Fourier Transforms*. McGraw-Hill Book Company, New York, Toronto and London, Reprinted by Courier Corporation, Chelmsford, 1995.

[11] H.M. Srivastava, F.A. Shah, and A.A. Teali, Short-time special affine Fourier transform for quaternion-valued functions. *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat.* (RACSAM), 116:1–20,2022. Article ID 66.

[12] H.M. Srivastava, W.Z. Lone, F.A. Shah, and A.I. Zayed, Discrete quadratic-phase Fourier transform: Theory and convolution structures, *Entropy*, **24**: 1–14,2022. Article ID 1340.

[13] H.M. Srivastava, B.J. González, and E. R. Negrín, An operational calculus for a Mehler-Fock type index transform on distributions of compact support, *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat.* (RACSAM) **117**: 1–11,2023. Article ID 3.

[14] H.M. Srivastava, F.A. Shah, T.K. Garg, W.Z. Lone, and H.L. Qadri, Non-separable linear canonical wavelet transform. *Symmetry*, **13**(11):2021. Article ID 2182.

[15] H.M. Srivastava, A.Y. Tantary, F.A. Shah, and A.I. Zayed, An interplay of ridgelet and linear canonical transforms. *Mathematics*. **10**(12):2022. Article ID 1986.

[16] H.M. Srivastava, F.A. Shah, and A.A. Teali, On quantum representation of the linear canonical wavelet transform. *Universe* **8**: 1–11,2022. Article ID 477.

[17] H.M. Srivastava, P. Shukla, and S.K. Upadhyay, The localization operator and wavelet multipliers involving the Watson transform. *J. Pseudo-Differ. Oper. Appl.* **13**: 1–21,2022. Article ID 46.

[18] H.M. Srivastava, F.A. Shah, and W.Z. Lone, Quadratic-phase wave-packet transform in $L^2(\mathbb{R})$, *Symmetry* **14**: 1–16,2022. Article ID 2018.

[19] H.M. Srivastava, K.K. Mishra, and S.K. Upadhyay, Characterizations of continuous fractional Bessel wavelet transforms. *Mathematics* **10**: 1–11,2022. Article ID 3084.

[20] H.M. Srivastava, F.A. Shah, and N.A. Nayied, Fibonacci wavelet method for the solution of the non-linear Hunter-Saxton equation. *Appl. Sci.* **12**: 1–17, 2022. Article ID 7738.

[21] H.M. Srivastava, M.S. Chauhan, and S.K. Upadhyay, Asymptotic series of a general symbol and pseudo-differential operators involving the Kontorovich-Lebedev transform. *J. Nonlinear Convex Anal.*, **22**(11):2461–2478,2021.

[22] H.M. Srivastava, M.S. Chauhan, and S.K. Upadhyay, L_p-boundedness of the pseudo-differential operators associated with the Kontorovich-Lebedev transform. *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat.* (RACSAM), 114:1–18,2020. Article ID 80.

[23] H.M. Srivastava, Some general families of integral transformations and related results. *Appl. Math. Comput. Sci.*, 6:27–41,2022. Article ID 7738.

[24] D. Wei, W. Yang, and Y.M. Li, Lattices sampling and sampling rate conversion of multidimensional bandlimited signals in the linear canonical transform domain. *J. Franklin Inst.*, Elsevier, **356**(13):7571–7607,2019.

[25] S.K. Upadhyay, A. Kumar, and J.K. Dubey, Characterization of spaces of type W and pseudo-differential operators of infinite order involving fractional Fourier transform. *J. Pseudo-Differ. Oper. Appl.*, **5**:215–230,2014.

[26] E. Cordero, and K. Gröchenig, On the product of localization operators. *Modern Trends in Pseudo-Differential Operators*, Springer, **172**:279–295,2006.
[27] J.D. Du, and M.W. Wong, A product formula for localization operators. *Bull. Korean Math. Soc.*, **37**:77–84,2000.
[28] M.W. Wong, *Introduction to pseudo-differential operators*. 3rd edition. World Scientific, Singapore, 2014.
[29] H.M. Srivastava, M. Kumar, and T. Pradhan, A framework of linear canonical Hankel transform pairs in distribution spaces and their applications. *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat.* (RACSAM). **115**(4): 1–18,2021. Article ID 171.