



Generalized fractional midpoint type inequalities for co-ordinated convex functions

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Abstract. In this research paper, we investigate generalized fractional integrals to obtain midpoint type inequalities for the co-ordinated convex functions. First of all, we establish an identity for twice partially differentiable mappings. By utilizing this equality, some midpoint type inequalities via generalized fractional integrals are proved. We also show that the main results reduce some midpoint inequalities given in earlier works for Riemann integrals and Riemann-Liouville fractional integrals. Finally, some new inequalities for k -Riemann-Liouville fractional integrals are presented as special cases of our results.

1. Introduction

The inequalities, introduced by C. Hermite and J. Hadamard for convex functions, are significant issue in the literature. These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \quad (1)$$

is valid. If f is concave, then both inequalities in (1) hold to the reverse direction. With the help of the convex functions, Dragomir and Agarwal [12] first obtained an upper bound for

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx,$$

which is the right-hand side of inequality (1). In addition to this, Kırmacı [20] first obtained upper bound for

$$\frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right),$$

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which is the left-hand side of inequality (1). These inequalities are called by trapezoid type inequality and midpoint type inequality, respectively. Many researchers have been studied extensively the trapezoid inequalities and midpoint inequalities for various types of convex functions [3, 21, 28, 33, 35, 45]. In 2013, Sarikaya et al. first proved Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals and the authors also gave some corresponding trapezoid type inequalities [39]. With the help of the results of Sarikaya et al., some fractional midpoint type inequalities for convex functions were established in [17]. Thereupon, many researchers obtained fractional midpoint inequalities and trapezoid inequalities using different kind of fractional integrals [1, 2, 4, 5, 8, 11, 27, 29, 32, 36, 37, 46]. On the other hand, Dragomir first proved Hermite-Hadamard inequalities for co-ordinated convex mappings in [13]. The midpoint and trapezoid type inequalities for co-ordinated convex functions were established in the papers [23] and [38], respectively. Furthermore, the Hermite-Hadamard inequalities for functions with two variables by utilizing Riemann-Liouville fractional integrals were obtained in [40]. Although Sarikaya gave the corresponding fractional trapezoid inequalities for co-ordinated convex functions in [40], Tunç et al. presented fractional midpoint type inequalities for co-ordinated convex functions in [43]. In the literature, there are great number of papers about to Hermite-Hadamard inequalities for several type co-ordinated convex functions. Because of these reasons, the reader is referred to [9, 10, 19, 22, 24, 30, 31, 42] and the references therein for additional information and unexplained subjects about these topics.

The generalized fractional integrals, which will be used frequently throughout this paper, were introduced by Sarikaya and Ertuğral in [41]. In the same paper, the authors also established Hermite-Hadamard inequalities and introduced several trapezoid and midpoint type inequalities for this kind fractional integrals. In addition to this, Turkay et al. defined the generalized fractional integrals for functions with two variables in [44]. These authors presented Hermite-Hadamard and trapezoid type inequalities for this kind of fractional integrals.

The purpose of this paper is to establish some generalized midpoint type inequalities for co-ordinated convex functions involving generalized fractional integrals. The general structure of our article contains four parts, including the introduction. The rest of the paper continues as follows: In Section 2, the definitions of generalized fractional integrals are given. Moreover, relations between generalized fractional integrals and other type fractional integrals are introduced. In Section 3, we first prove an equality involving for twice partially differentiable functions. Then, we establish several generalized midpoint type inequalities whose partially derivatives in absolute value are co-ordinated convex. We show that our main results are reduced to inequalities from earlier studies by looking at their particular case. Furthermore, some new midpoint type inequalities for k -Riemann-Liouville fractional integrals are given. Finally, some results and further aspects of research are discussed in the last section.

2. Generalized fractional Integrals

In this section, we will give the necessary definition of generalized fractional integrals introduced by Sarikaya and Ertuğral in [41]. Moreover, the relation between generalized fractional integrals and other type integrals are considered.

Definition 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ denote a integrable function. The left-sided and right-sided generalized fractional integral operators are given by

$${}_{a+}I_{\varphi}f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \quad x > a \quad (2)$$

and

$${}_{b-}I_{\varphi}f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \quad x < b, \quad (3)$$

respectively. Here, the function $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying the condition

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

Remark 2.2. With the help of the given Definition 2.1, the following cases are provided:

1. If we choose $\varphi(t) = t$, the operators (2) and (3) reduce to the Riemann integral.
2. Let us consider $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\alpha > 0$. Then, the operators (2) and (3) reduce to the Riemann-Liouville fractional integrals $J_{a+}^\alpha f(x)$ and $J_{b-}^\alpha f(x)$, respectively. Here, Γ is Gamma function.
3. Let us define $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$ and $\alpha, k > 0$. Then, the operators (2) and (3) reduce to the k -Riemann-Liouville fractional integrals $J_{a+,k}^\alpha f(x)$ and $J_{b-,k}^\alpha f(x)$, respectively. Here, Γ_k is k -Gamma function.

There are several papers on inequalities for generalized fractional integrals in the literature. In [41], Sarikaya and Ertuğral also proved Hermite-Hadamard inequalities for generalized fractional integrals. In addition, Budak et al. proved midpoint type inequalities and extensions of Hermite-Hadamard inequalities in the papers [6] and [7], respectively. In [14], Ertuğral and Sarikaya presented some Simpson type inequalities for these fractional integral operators. For some of other papers on inequalities for generalized fractional integrals, please refer to [15, 16, 18, 25, 26, 34, 47].

Generalized double fractional integrals are given by Turkay et al. in [44], as follows:

Definition 2.3. Let $f : \Omega := [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a integrable function. The generalized double fractional integrals ${}_{a+,c+}I_{\varphi,\psi}$, ${}_{a+,d-}I_{\varphi,\psi}$, ${}_{b-,c+}I_{\varphi,\psi}$, ${}_{b-,d-}I_{\varphi,\psi}$ are defined by

$${}_{a+,c+}I_{\varphi,\psi} f(x, y) = \int_a^x \int_c^y \frac{\varphi(x-t)}{x-t} \frac{\psi(y-s)}{y-s} f(t, s) ds dt, x > a, y > c, \tag{4}$$

$${}_{a+,d-}I_{\varphi,\psi} f(x, y) = \int_a^x \int_y^d \frac{\varphi(x-t)}{x-t} \frac{\psi(s-y)}{s-y} f(t, s) ds dt, x > a, y < d, \tag{5}$$

$${}_{b-,c+}I_{\varphi,\psi} f(x, y) = \int_x^b \int_c^y \frac{\varphi(t-x)}{t-x} \frac{\psi(y-s)}{y-s} f(t, s) ds dt, x < b, y > c, \tag{6}$$

and

$${}_{b-,d-}I_{\varphi,\psi} f(x, y) = \int_x^b \int_y^d \frac{\varphi(t-x)}{t-x} \frac{\psi(s-y)}{s-y} f(t, s) ds dt, x < b, y < d. \tag{7}$$

Here, the function $\varphi : [0, \infty) \rightarrow [0, \infty)$ and the function $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfy the conditions $\int_0^1 \frac{\varphi(t)}{t} dt < \infty$ and $\int_0^1 \frac{\psi(s)}{s} ds < \infty$, respectively.

Remark 2.4. By using the Definition 2.3, the following conditions are ensured:

1. If we take $\varphi(t) = t$ and $\psi(s) = s$, then the operators (4), (5), (6), and (7) reduce to the double Riemann integral.
2. Let us consider $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ for $\alpha, \beta > 0$. Then, the operators (4), (5), (6), and (7) reduce to the Riemann-Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta} f(x, y)$, $J_{a+,d-}^{\alpha,\beta} f(x, y)$, $J_{b-,c+}^{\alpha,\beta} f(x, y)$, and $J_{b-,d-}^{\alpha,\beta} f(x, y)$, respectively.
3. Let us note that $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s) = \frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ for $\alpha, \beta, k > 0$. Then, the operators (4), (5), (6), and (7) reduce to the k -Riemann-Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta,k} f(x, y)$, $J_{a+,d-}^{\alpha,\beta,k} f(x, y)$, $J_{b-,c+}^{\alpha,\beta,k} f(x, y)$, and $J_{b-,d-}^{\alpha,\beta,k} f(x, y)$, respectively.

3. Generalized midpoint type Inequalities for co-ordinated convex functions

To make the presentation easier and compact to understand, we make some symbolic representation:

$$\begin{aligned}
 I_1 &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt, \\
 I_2 &= - \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt, \\
 I_3 &= - \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt, \\
 I_4 &= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) ds dt, \\
 I_5 &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (tb + (1-t)a, sd + (1-s)c) ds dt, \\
 I_6 &= - \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (tb + (1-t)a, sd + (1-s)c) ds dt, \\
 I_7 &= - \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (tb + (1-t)a, sd + (1-s)c) ds dt, \\
 I_8 &= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (tb + (1-t)a, sd + (1-s)c) ds dt, \\
 I_9 &= - \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sd + (1-s)c) ds dt, \\
 I_{10} &= \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sd + (1-s)c) ds dt, \\
 I_{11} &= \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sd + (1-s)c) ds dt, \\
 I_{12} &= - \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sd + (1-s)c) ds dt, \\
 I_{13} &= - \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (tb + (1-t)a, sc + (1-s)d) ds dt, \\
 I_{14} &= \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (tb + (1-t)a, sc + (1-s)d) ds dt, \\
 I_{15} &= \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (tb + (1-t)a, sc + (1-s)d) ds dt, \\
 I_{16} &= - \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s} (tb + (1-t)a, sc + (1-s)d) ds dt,
 \end{aligned}$$

where

$$\begin{cases} \Lambda_1(t) = \int_0^t \frac{\varphi((b-a)u)}{u} du, \\ \Lambda_2(s) = \int_0^s \frac{\psi((d-c)u)}{u} du, \end{cases}$$

and

$$\begin{cases} \Delta_1(t) = \int_t^1 \frac{\varphi((b-a)u)}{u} du, \\ \Delta_2(s) = \int_s^1 \frac{\psi((d-c)u)}{u} du. \end{cases}$$

In order to prove our main results, we need the following Lemma.

Lemma 3.1. *Let $f : \Omega \rightarrow \mathbb{R}$ be a partial differentiable mapping on Ω and let $\frac{\partial^2 f}{\partial t \partial s} \in L(\Omega)$. Then, the following equality holds:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2\Lambda_1(1)} \left[{}_{a+}I_{\varphi}f\left(b, \frac{c+d}{2}\right) + {}_{b-}I_{\varphi}f\left(a, \frac{c+d}{2}\right) \right] \\ & - \frac{1}{2\Lambda_2(1)} \left[{}_{c+}I_{\psi}f\left(\frac{a+b}{2}, d\right) + {}_{d-}I_{\psi}f\left(\frac{a+b}{2}, c\right) \right] \\ & + \frac{1}{4\Lambda_1(1)\Lambda_2(1)} \left[{}_{a+,c+}I_{\varphi,\psi}f(b, d) + {}_{a+,d-}I_{\varphi,\psi}f(b, c) + {}_{b-,c+}I_{\varphi,\psi}f(a, d) + {}_{b-,d-}I_{\varphi,\psi}f(a, c) \right] \\ & = \frac{(b-a)(d-c)}{4\Lambda_1(1)\Lambda_2(1)} \sum_{k=1}^{16} I_k. \end{aligned} \tag{8}$$

Proof. With the help of the integration by parts, we obtain

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \Lambda_1(t) \Lambda_2(s) \frac{\partial^2 f}{\partial t \partial s}(ta + (1-t)b, sc + (1-s)d) ds dt \\ &= \int_0^{\frac{1}{2}} \Lambda_1(t) \left\{ \Lambda_2(s) \frac{1}{c-d} \frac{\partial f}{\partial t}(ta + (1-t)b, sc + (1-s)d) \Big|_0^{\frac{1}{2}} \right. \\ &\quad \left. - \frac{1}{c-d} \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} \frac{\partial f}{\partial t}(ta + (1-t)b, sc + (1-s)d) ds \right\} dt \\ &= \frac{\Lambda_2\left(\frac{1}{2}\right)}{c-d} \int_0^{\frac{1}{2}} \Lambda_1(t) \frac{\partial f}{\partial t}\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\ &\quad - \frac{1}{c-d} \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} \int_0^{\frac{1}{2}} \Lambda_1(t) \frac{\partial f}{\partial t}(ta + (1-t)b, sc + (1-s)d) dt ds \\ &= \frac{\Lambda_2\left(\frac{1}{2}\right)}{c-d} \left[\frac{\Lambda_1\left(\frac{1}{2}\right)}{a-b} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{a-b} \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \right] \end{aligned} \tag{9}$$

$$\begin{aligned}
 & -\frac{1}{c-d} \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} \left[\frac{\Lambda_1\left(\frac{1}{2}\right)}{a-b} f\left(\frac{a+b}{2}, sc + (1-s)d\right) \right. \\
 & \left. -\frac{1}{a-b} \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} \frac{\partial f}{\partial s}(ta + (1-t)b, sc + (1-s)d) dt \right] ds \\
 & = \frac{1}{(b-a)(d-c)} \left[\Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & \quad -\Lambda_2\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\
 & \quad -\Lambda_1\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sc + (1-s)d\right) ds \\
 & \quad \left. + \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} \frac{\psi((d-c)s)}{s} f(ta + (1-t)b, sc + (1-s)d) ds dt \right].
 \end{aligned}$$

Similarly, by using integration by parts it follows that

$$\begin{aligned}
 I_2 & = \frac{1}{(b-a)(d-c)} \left[\Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & \quad -\Lambda_2\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\
 & \quad -\Lambda_1\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sc + (1-s)d\right) ds \\
 & \quad \left. + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} \frac{\psi((d-c)s)}{s} f(ta + (1-t)b, sc + (1-s)d) ds dt \right], \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 I_3 & = \frac{1}{(b-a)(d-c)} \left[\Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & \quad -\Lambda_2\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\
 & \quad -\Lambda_1\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sc + (1-s)d\right) ds \\
 & \quad \left. + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} \frac{\psi((d-c)s)}{s} f(ta + (1-t)b, sc + (1-s)d) ds dt \right], \tag{11}
 \end{aligned}$$

$$\begin{aligned}
 I_4 & = \frac{1}{(b-a)(d-c)} \left[\Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & \quad \left. -\Lambda_2\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \right] \tag{12}
 \end{aligned}$$

$$-\Delta_1\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1\frac{\psi((d-c)s)}{s}f\left(\frac{a+b}{2},sc+(1-s)d\right)ds$$

$$+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\frac{\varphi((b-a)t)}{t}\frac{\psi((d-c)s)}{s}f\left(ta+(1-t)b,sc+(1-s)d\right)dsdt\Big],$$

$$I_5 = \frac{1}{(b-a)(d-c)}\left[\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \tag{13}$$

$$-\Lambda_2\left(\frac{1}{2}\right)\int_0^{\frac{1}{2}}\frac{\varphi((b-a)t)}{t}f\left(tb+(1-t)a,\frac{c+d}{2}\right)dt$$

$$-\Lambda_1\left(\frac{1}{2}\right)\int_0^{\frac{1}{2}}\frac{\psi((d-c)s)}{s}f\left(\frac{a+b}{2},sd+(1-s)c\right)ds$$

$$\left.+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{\varphi((b-a)t)}{t}\frac{\psi((d-c)s)}{s}f\left(tb+(1-t)a,sd+(1-s)c\right)dsdt\right],$$

$$I_6 = \frac{1}{(b-a)(d-c)}\left[\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \tag{14}$$

$$-\Delta_2\left(\frac{1}{2}\right)\int_0^{\frac{1}{2}}\frac{\varphi((b-a)t)}{t}f\left(tb+(1-t)a,\frac{c+d}{2}\right)dt$$

$$-\Lambda_1\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1\frac{\psi((d-c)s)}{s}f\left(\frac{a+b}{2},sd+(1-s)c\right)ds$$

$$\left.+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\frac{\varphi((b-a)t)}{t}\frac{\psi((d-c)s)}{s}f\left(tb+(1-t)a,sd+(1-s)c\right)dsdt\right],$$

$$I_7 = \frac{1}{(b-a)(d-c)}\left[\Delta_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \tag{15}$$

$$-\Lambda_2\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1\frac{\varphi((b-a)t)}{t}f\left(tb+(1-t)a,\frac{c+d}{2}\right)dt$$

$$-\Delta_1\left(\frac{1}{2}\right)\int_0^{\frac{1}{2}}\frac{\psi((d-c)s)}{s}f\left(\frac{a+b}{2},sd+(1-s)c\right)ds$$

$$\left.+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\frac{\varphi((b-a)t)}{t}\frac{\psi((d-c)s)}{s}f\left(tb+(1-t)a,sd+(1-s)c\right)dsdt\right],$$

$$I_8 = \frac{1}{(b-a)(d-c)}\left[\Delta_1\left(\frac{1}{2}\right)\Delta_2\left(\frac{1}{2}\right)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \tag{16}$$

$$-\Delta_2\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1\frac{\varphi((b-a)t)}{t}f\left(tb+(1-t)a,\frac{c+d}{2}\right)dt \tag{17}$$

$$-\Delta_1\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1\frac{\psi((d-c)s)}{s}f\left(\frac{a+b}{2},sd+(1-s)c\right)ds$$

$$+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\frac{\varphi((b-a)t)}{t}\frac{\psi((d-c)s)}{s}f\left(tb+(1-t)a,sd+(1-s)c\right)dsdt\Big],$$

$$\begin{aligned}
 I_9 = & \frac{1}{(b-a)(d-c)} \left[\Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & - \Lambda_2\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\
 & - \Lambda_1\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sd + (1-s)c\right) ds \\
 & \left. + \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} \frac{\psi((d-c)s)}{s} f\left(ta + (1-t)b, sd + (1-s)c\right) ds dt \right], \tag{18}
 \end{aligned}$$

$$\begin{aligned}
 I_{10} = & \frac{1}{(b-a)(d-c)} \left[\Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & - \Lambda_2\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\
 & - \Lambda_1\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sd + (1-s)c\right) ds \\
 & \left. + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} \frac{\psi((d-c)s)}{s} f\left(ta + (1-t)b, sd + (1-s)c\right) ds dt \right], \tag{19}
 \end{aligned}$$

$$\begin{aligned}
 I_{11} = & \frac{1}{(b-a)(d-c)} \left[\Delta_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & - \Lambda_2\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\
 & - \Delta_1\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sd + (1-s)c\right) ds \\
 & \left. + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} \frac{\psi((d-c)s)}{s} f\left(ta + (1-t)b, sd + (1-s)c\right) ds dt \right], \tag{20}
 \end{aligned}$$

$$\begin{aligned}
 I_{12} = & \frac{1}{(b-a)(d-c)} \left[\Delta_1\left(\frac{1}{2}\right) \Delta_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & - \Delta_2\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\
 & - \Delta_1\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sd + (1-s)c\right) ds \\
 & \left. + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} \frac{\psi((d-c)s)}{s} f\left(ta + (1-t)b, sd + (1-s)c\right) ds dt \right], \tag{21}
 \end{aligned}$$

$$\begin{aligned}
 I_{13} = & \frac{1}{(b-a)(d-c)} \left[\Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & - \Lambda_2\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f\left(tb + (1-t)a, \frac{c+d}{2}\right) dt \\
 & \left. + \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} \frac{\psi((d-c)s)}{s} f\left(tb + (1-t)a, sd + (1-s)c\right) ds dt \right], \tag{22}
 \end{aligned}$$

$$-\Lambda_1\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sc+(1-s)d\right) ds + \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)\psi((d-c)s)}{t} f(tb+(1-t)a, sc+(1-s)d) dsdt \Big],$$

$$I_{14} = \frac{1}{(b-a)(d-c)} \left[\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \Lambda_2\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)}{t} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt - \Lambda_1\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sc+(1-s)d\right) ds + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)\psi((d-c)s)}{t} f(tb+(1-t)a, sc+(1-s)d) dsdt \Big], \tag{23}$$

$$I_{15} = \frac{1}{(b-a)(d-c)} \left[\Delta_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \Lambda_2\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt - \Delta_1\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sc+(1-s)d\right) ds + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \frac{\varphi((b-a)t)\psi((d-c)s)}{t} f(tb+(1-t)a, sc+(1-s)d) dsdt \Big], \tag{24}$$

and

$$I_{16} = \frac{1}{(b-a)(d-c)} \left[\Delta_1\left(\frac{1}{2}\right)\Delta_2\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \Delta_2\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)}{t} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt - \Delta_1\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 \frac{\psi((d-c)s)}{s} f\left(\frac{a+b}{2}, sc+(1-s)d\right) ds + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \frac{\varphi((b-a)t)\psi((d-c)s)}{t} f(tb+(1-t)a, sc+(1-s)d) dsdt \Big]. \tag{25}$$

By using the equations (9)-(25), the change of the variable $x = ta + (1-t)b$, and $y = sc + (1-s)d$ for $t, s \in [0, 1]$, it can be rewritten as follows

$$I_1 + I_2 + \dots + I_{16} = \frac{4\Lambda_1(1)\Lambda_2(1)}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \tag{26}$$

$$\begin{aligned}
 & - \frac{2\Lambda_2(1)}{(b-a)(d-c)} \left[{}_{a+}I_\varphi f \left(b, \frac{c+d}{2} \right) + {}_{b-}I_\varphi f \left(a, \frac{c+d}{2} \right) \right] \\
 & - \frac{2\Lambda_1(1)}{(b-a)(d-c)} \left[{}_{c+}I_\psi f \left(\frac{a+b}{2}, d \right) + {}_{d-}I_\psi f \left(\frac{a+b}{2}, c \right) \right] \\
 & + \frac{1}{(b-a)(d-c)} \left[{}_{a+,c+}I_{\varphi,\psi} f(b, d) + {}_{a+,d-}I_{\varphi,\psi} f(b, c) + {}_{b-,c+}I_{\varphi,\psi} f(a, d) + {}_{b-,d-}I_{\varphi,\psi} f(a, c) \right].
 \end{aligned}$$

Multiplying the both sides of (26) by $\frac{(b-a)(d-c)}{4\Lambda_1(1)\Lambda_2(1)}$, we get equation (8). This ends the proof of Lemma 3.1. \square

Next, we start to state the first theorem containing the Hermite-Hadamard type inequality for fractional integrals.

Theorem 3.2. Suppose $f : \Omega \rightarrow \mathbb{R}$ is a partial differentiable mapping on Ω . Suppose also $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$ is a convex function on the co-ordinates on Ω . Then, the following inequality

$$\begin{aligned}
 & \left| f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{1}{2\Lambda_1(1)} \left[{}_{a+}I_\varphi f \left(b, \frac{c+d}{2} \right) + {}_{b-}I_\varphi f \left(a, \frac{c+d}{2} \right) \right] \right. \\
 & \left. - \frac{1}{2\Lambda_2(1)} \left[{}_{c+}I_\psi f \left(\frac{a+b}{2}, d \right) + {}_{d-}I_\psi f \left(\frac{a+b}{2}, c \right) \right] \right. \\
 & \left. + \frac{1}{4\Lambda_1(1)\Lambda_2(1)} \left[{}_{a+,c+}I_{\varphi,\psi} f(b, d) + {}_{a+,d-}I_{\varphi,\psi} f(b, c) + {}_{b-,c+}I_{\varphi,\psi} f(a, d) + {}_{b-,d-}I_{\varphi,\psi} f(a, c) \right] \right| \\
 & \leq \frac{(b-a)(d-c)}{4\Lambda_1(1)\Lambda_2(1)} [A_1 + A_2] [B_1 + B_2] \\
 & \times \left(\left| \frac{\partial^2 f}{\partial t \partial s} (a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s} (b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s} (a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial s} (b, d) \right| \right)
 \end{aligned}$$

is valid. Here,

$$\begin{cases} A_1 = \int_0^{\frac{1}{2}} |\Lambda_1(t)| dt, & B_1 = \int_0^{\frac{1}{2}} |\Lambda_2(s)| ds, \\ A_2 = \int_{\frac{1}{2}}^1 |\Delta_1(t)| dt, & B_2 = \int_{\frac{1}{2}}^1 |\Delta_2(s)| ds. \end{cases}$$

Proof. Lemma 3.1 yields the following inequality

$$\begin{aligned}
 & \left| f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{1}{2\Lambda_1(1)} \left[{}_{a+}I_\varphi f \left(b, \frac{c+d}{2} \right) + {}_{b-}I_\varphi f \left(a, \frac{c+d}{2} \right) \right] \right. \\
 & \left. - \frac{1}{2\Lambda_2(1)} \left[{}_{c+}I_\psi f \left(\frac{a+b}{2}, d \right) + {}_{d-}I_\psi f \left(\frac{a+b}{2}, c \right) \right] \right|
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 & + \frac{1}{4\Lambda_1(1)\Lambda_2(1)} \left[{}_{a+,c+}I_{\varphi,\psi}f(b,d) + {}_{a+,d-}I_{\varphi,\psi}f(b,c) + {}_{b-,c+}I_{\varphi,\psi}f(a,d) + {}_{b-,d-}I_{\varphi,\psi}f(a,c) \right] \\
 & \leq \frac{(b-a)(d-c)}{4\Lambda_1(1)\Lambda_2(1)} \{|I_1| + |I_2| + \dots + |I_{16}|\}.
 \end{aligned}$$

By using co-ordinated convexity of $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$ function on Ω and calculating the integrals in above inequality, the following inequalities hold:

$$\begin{aligned}
 |I_1| & \leq \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| \left| \frac{\partial^2 f}{\partial t \partial s}(ta + (1-t)b, sc + (1-s)d) \right| ds dt & (28) \\
 & \leq \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| \left\{ ts \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + s(1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| \right. \\
 & \quad \left. + t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| + (1-s)(1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \right\} ds dt \\
 & = \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| ts ds dt \\
 & \quad + \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| s(1-t) ds dt \\
 & \quad + \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & \quad + \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 |I_2| & \leq \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Lambda_2(s)| ts ds dt & (29) \\
 & \quad + \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Lambda_2(s)| s(1-t) ds dt \\
 & \quad + \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & \quad + \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 |I_3| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Delta_2(s)| t s ds dt & (30) \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Delta_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Delta_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Delta_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 |I_4| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| t s ds dt & (31) \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 |I_5| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| t s ds dt & (32) \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$|I_6| \leq \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| t s ds dt \tag{33}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Lambda_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt, \\
 |I_7| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| ts ds dt \tag{34}
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt, \\
 |I_8| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| ts ds dt \tag{35}
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| (1-s)(1-t) ds dt, \\
 |I_9| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| ts ds dt \tag{36}
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| s(1-t) ds dt
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt, \\
 |I_{10}| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| ts ds dt \tag{37} \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 |I_{11}| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| ts ds dt \tag{38} \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 |I_{12}| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| ts ds dt \tag{39} \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| (1-s)(1-t) ds dt, \\
 |I_{13}| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| ts ds dt \tag{40} \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 |I_{14}| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| ts ds dt \tag{41} \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 |\Lambda_1(t)| |\Delta_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 |I_{15}| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| ts ds dt \tag{42} \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} |\Delta_1(t)| |\Lambda_2(s)| (1-s)(1-t) ds dt,
 \end{aligned}$$

and

$$\begin{aligned}
 |I_{16}| \leq & \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| t s ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| s(1-t) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| t(1-s) ds dt \\
 & + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 |\Delta_1(t)| |\Delta_2(s)| (1-s)(1-t) ds dt.
 \end{aligned} \tag{43}$$

With the help of the inequalities (28)-(43), we get

$$\begin{aligned}
 & |I_1| + |I_2| + \dots + |I_{16}| \\
 & \leq [A_1 + A_2][B_1 + B_2] \left(\left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \right).
 \end{aligned} \tag{44}$$

If the inequality (44) is written into (27), then we get desired result. This finish the proof of Theorem 3.2. \square

Remark 3.3. In Theorem 3.2, if we assign $\varphi(t) = t$ and $\psi(s) = s$ for all $(t, s) \in \Omega$, then Theorem 3.2 reduces to [23, Theorem 2].

Remark 3.4. In Theorem 3.2, let us now note that $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ for all $(t, s) \in \Omega$ and $\alpha, \beta > 0$. Then, Theorem 3.2 reduces to [43, Theorem 2.1].

Corollary 3.5. In Theorem 3.2, if we choose $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s) = \frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ for all $(t, s) \in \Omega$ and $\alpha, \beta, k > 0$, then we obtain the following inequality

$$\begin{aligned}
 & \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \left[\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[J_{a+,k}^\alpha f\left(b, \frac{c+d}{2}\right) + J_{b-,k}^\alpha f\left(a, \frac{c+d}{2}\right) \right] \right. \right. \\
 & \quad \left. \left. + \frac{\Gamma_k(\beta+k)}{2(d-c)^{\frac{\beta}{k}}} \left[J_{c+,k}^\beta f\left(\frac{a+b}{2}, d\right) + J_{d-,k}^\beta f\left(\frac{a+b}{2}, c\right) \right] \right] \right| \\
 & \quad + \frac{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{4(b-a)^{\frac{\alpha}{k}}(d-c)^{\frac{\beta}{k}}} \\
 & \quad \times \left[J_{a+,c+}^{\alpha,\beta,k} f(b, d) + J_{a+,d-}^{\alpha,\beta,k} f(b, c) + J_{b-,c+}^{\alpha,\beta,k} f(a, d) + J_{b-,d-}^{\alpha,\beta,k} f(a, c) \right] \\
 & \leq \frac{(b-a)(d-c)}{4} \left(\frac{1}{2} + \frac{k(1-2^{\frac{\alpha}{k}})}{2^{\frac{\alpha}{k}}(\alpha+k)} \right) \left(\frac{1}{2} + \frac{k(1-2^{\frac{\beta}{k}})}{2^{\frac{\beta}{k}}(\beta+k)} \right)
 \end{aligned}$$

$$\times \left(\left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right| \right).$$

Theorem 3.6. Assume $f : \Omega \rightarrow \mathbb{R}$ is a partial differentiable mapping on Ω . Assume also $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q, q > 1$ is a convex function on the co-ordinates on Ω . Then, one obtains the following inequality

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2\Lambda_1(1)} \left[{}_{a+}I_{\varphi} f\left(b, \frac{c+d}{2}\right) + {}_{b-}I_{\varphi} f\left(a, \frac{c+d}{2}\right) \right] \right. \\ & \quad \left. - \frac{1}{2\Lambda_2(1)} \left[{}_{c+}I_{\psi} f\left(\frac{a+b}{2}, d\right) + {}_{d-}I_{\psi} f\left(\frac{a+b}{2}, c\right) \right] \right. \\ & \quad \left. + \frac{1}{4\Lambda_1(1)\Lambda_2(1)} \left[{}_{a+,c+}I_{\varphi,\psi} f(b, d) + {}_{a+,d-}I_{\varphi,\psi} f(b, c) + {}_{b-,c+}I_{\varphi,\psi} f(a, d) + {}_{b-,d-}I_{\varphi,\psi} f(a, c) \right] \right|^{\frac{1}{q}} \\ & \leq \frac{(b-a)(d-c)}{4\Lambda_1(1)\Lambda_2(1)} [C_1 + C_2] [D_1 + D_2] \\ & \quad \times \left\{ \left(\frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\frac{9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

where

$$\begin{cases} C_1 = \left(\int_0^{\frac{1}{2}} |\Lambda_1(t)|^p dt \right)^{\frac{1}{p}}, & D_1 = \left(\int_0^{\frac{1}{2}} |\Lambda_2(s)|^p ds \right)^{\frac{1}{p}}, \\ C_2 = \left(\int_{\frac{1}{2}}^1 |\Lambda_1(t)|^p dt \right)^{\frac{1}{p}}, & D_2 = \left(\int_{\frac{1}{2}}^1 |\Lambda_2(s)|^p ds \right)^{\frac{1}{p}}, \end{cases}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. With the help of the Hölder’s inequality for double integrals and by using the co-ordinated convexity

of $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$ function on Ω , we have

$$|I_1| \leq \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)| |\Lambda_2(s)| \left| \frac{\partial^2 f}{\partial t \partial s}(ta + (1-t)b, sc + (1-s)d) \right| ds dt \tag{45}$$

$$\begin{aligned} &\leq \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |\Lambda_1(t)|^p |\Lambda_2(s)|^p ds dt \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| \frac{\partial^2 f}{\partial t \partial s}(ta + (1-t)b, sc + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \\ &\leq C_1 \cdot D_1 \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left\{ ts \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + s(1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q \right. \right. \\ &\quad \left. \left. + t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q \right\} ds dt \right)^{\frac{1}{q}} \\ &\leq C_1 \cdot D_1 \left(\frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \end{aligned}$$

$$|I_2| \leq C_1 \cdot D_2 \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{46}$$

$$|I_3| \leq C_2 \cdot D_1 \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{47}$$

$$|I_4| \leq C_2 \cdot D_2 \left(\frac{9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{48}$$

$$|I_5| \leq C_1 \cdot D_1 \left(\frac{9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{49}$$

$$|I_6| \leq C_1 \cdot D_2 \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{50}$$

$$|I_7| \leq C_2 \cdot D_1 \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{51}$$

$$|I_8| \leq C_2 \cdot D_2 \left(\frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{52}$$

$$|I_9| \leq C_1 \cdot D_1 \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{53}$$

$$|I_{10}| \leq C_1 \cdot D_2 \left(\frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{54}$$

$$|I_{11}| \leq C_2 \cdot D_1 \left(\frac{9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{55}$$

$$|I_{12}| \leq C_2 \cdot D_2 \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{56}$$

$$|I_{13}| \leq C_1 \cdot D_1 \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{57}$$

$$|I_{14}| \leq C_1 \cdot D_2 \left(\frac{9 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{58}$$

$$|I_{15}| \leq C_2 \cdot D_1 \left(\frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}, \tag{59}$$

and

$$|I_{16}| \leq C_2 \cdot D_2 \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a, c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a, d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b, d) \right|^q}{64} \right)^{\frac{1}{q}}. \tag{60}$$

If the inequalities (45)-(60) are written into (27), then we obtain desired result. This is the end of the proof of Theorem 3.6. \square

Remark 3.7. In Theorem 3.6, let us consider that $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ for all $(t, s) \in \Omega$. Then, Theorem 3.6 reduces to [43, Theorem 2.2].

Corollary 3.8. In Theorem 3.6, if we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s) = \frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ for all $(t, s) \in \Omega$, then we get the following inequality

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \left[\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[J_{a^+,k}^\alpha f\left(b, \frac{c+d}{2}\right) + J_{b^-,k}^\alpha f\left(a, \frac{c+d}{2}\right) \right] \right. \right. \\ & \quad \left. \left. + \frac{\Gamma_k(\beta+k)}{2(d-c)^{\frac{\beta}{k}}} \left[J_{c^+,k}^\beta f\left(\frac{a+b}{2}, d\right) + J_{d^-,k}^\beta f\left(\frac{a+b}{2}, c\right) \right] \right] \right| \\ & \quad + \frac{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{4(b-a)^{\frac{\alpha}{k}}(d-c)^{\frac{\beta}{k}}} \\ & \quad \times \left[J_{a^+,c^+}^{\alpha,\beta,k} f(b,d) + J_{a^+,d^-}^{\alpha,\beta,k} f(b,c) + J_{b^-,c^+}^{\alpha,\beta,k} f(a,d) + J_{b^-,d^-}^{\alpha,\beta,k} f(a,c) \right] \\ & \leq \frac{(b-a)(d-c)}{4} \left[\left(\frac{1}{2} + \frac{1-2^{p\frac{\alpha}{k}+1}}{2^{p\frac{\alpha}{k}+1}(p\frac{\alpha}{k}+1)} \right)^{\frac{1}{p}} + \left(\frac{1}{2^{p\frac{\alpha}{k}+1}(p\frac{\alpha}{k}+1)} \right)^{\frac{1}{p}} \right] \\ & \quad \times \left[\left(\frac{1}{2} + \frac{1-2^{p\frac{\beta}{k}+1}}{2^{p\frac{\beta}{k}+1}(p\frac{\beta}{k}+1)} \right)^{\frac{1}{p}} + \left(\frac{1}{2^{p\frac{\beta}{k}+1}(p\frac{\beta}{k}+1)} \right)^{\frac{1}{p}} \right] \\ & \quad \times \left\{ \frac{9 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{64} \right\}^{\frac{1}{q}} \\ & \quad + \left\{ \frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{64} \right\}^{\frac{1}{q}} \\ & \quad + \left\{ \frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{64} \right\}^{\frac{1}{q}} \\ & \quad + \left\{ \frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{64} \right\}^{\frac{1}{q}} \Bigg\}. \end{aligned}$$

4. Conclusions

In this manuscript, we consider generalized fractional integrals to get midpoint type inequalities for the co-ordinated convex functions. Firstly, it is established an identity for twice partially differentiable mappings. By using the this identity, some midpoint type inequalities via generalized fractional integrals are proved. Furthermore, the main results reduce some midpoint inequalities given in earlier works for Riemann integrals and Riemann-Liouville fractional integrals. In addition to this, it is introduced some new inequalities for k -Riemann-Liouville fractional integrals as special cases of our results. We propose for forthcoming researchers that the methods and techniques used in this study can be established similar inequalities for different kinds of co-ordinated convexity.

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