Filomat 37:13 (2023), 4065–4082
https://doi.org/10.2298/FIL2313065C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some effect of drift of the generalized Brownian motion process: Existence of the operator-valued generalized Feynman integral

Jae Gil Choi[a]

[a] *School of General Education, Dankook University, Cheonan 31116, Republic of Korea*

Abstract. In this paper an analytic operator-valued generalized Feynman integral was studied on a very general Wiener space $C_{a,b}[0,T]$. The general Wiener space $C_{a,b}[0,T]$ is a function space which is induced by the generalized Brownian motion process associated with continuous functions a and b. The structure of the analytic operator-valued generalized Feynman integral is suggested and the existence of the analytic operator-valued generalized Feynman integral is investigated as an operator from $L^1(\mathbb{R},\nu_{\delta,a})$ to $L^\infty(\mathbb{R})$ where $\nu_{\delta,a}$ is a σ-finite measure on $\mathbb{R}$ given by

$$d\nu_{\delta,a} = \exp\{\delta \mathrm{Var}(a)u^2\}du,$$

where $\delta > 0$ and $\mathrm{Var}(a)$ denotes the total variation of the mean function a of the generalized Brownian motion process. It turns out in this paper that the analytic operator-valued generalized Feynman integrals of functionals defined by the stochastic Fourier–Stieltjes transform of complex measures on the infinite dimensional Hilbert space $C'_{a,b}[0,T]$ are elements of the linear space

$$\bigcap_{\delta>0} \mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}), L^\infty(\mathbb{R})).$$

1. Introduction

Before giving a basic survey and a motivation for our topic we fix some notation. Let $\mathbb{C}$, $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively. For all $\lambda \in \widetilde{\mathbb{C}}_+$, $\lambda^{1/2} \equiv \sqrt{\lambda}$ (or $\lambda^{-1/2}$) is always chosen to have positive real part. Furthermore, let $C[0,T]$ denote the space of real-valued continuous functions x on $[0,T]$ and let $C_0[0,T]$ denote those x in $C[0,T]$ such that $x(0) = 0$. The function space $C_0[0,T]$ is referred to as one-parameter Wiener space, and we let m_w denote Wiener measure. Given two Banach spaces X and Y, let $\mathcal{L}(X,Y)$ denote the space of continuous linear operators from X to Y.

2020 *Mathematics Subject Classification*. Primary 28C20 ; Secondary 46G12, 46B09.
Keywords. Generalized Brownian motion process, analytic operator-valued generalized function space integral, analytic operator-valued generalized Feynman integral.
Received: 13 July 2022; Accepted: 01 August 2022
Communicated by Dragan S. Djordjević
This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2021R1F1A1062770).
Email address: jgchoi@dankook.ac.kr (Jae Gil Choi)

Let F be a $\mathbb{C}$-valued measurable functional on $C[0,T]$. For $\lambda > 0$, $\psi \in L^2(\mathbb{R})$, and $\xi \in \mathbb{R}$, consider the Wiener integral

$$(I_\lambda(F)\psi)(\xi) \equiv \int_{C_0[0,T]} F\Big(\lambda^{-1/2}x + \xi\Big)\psi\Big(\lambda^{-1/2}x(T) + \xi\Big)dm_w(x). \tag{1.1}$$

In the application of the Feynman integral to quantum theory, the function ψ in (1.1) corresponds to the initial condition associated with Schrödinger equation.

In [1], Cameron and Storvick considered the following natural and interesting questions. Under what conditions on F will the linear operator $I_\lambda(F)$ given by (1.1) be an element of $\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$? If so, does the operator valued function $\lambda \to I_\lambda(F)$ have an analytic extension, write $I_\lambda^{\text{an}}(F)$ (it is called the analytic operator-valued Wiener integral of F with parameter λ), to $\mathbb{C}_+$? If so, for each nonzero real number q, does the limit

$$J_q^{\text{an}}(F) \equiv \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} I_\lambda^{\text{an}}(F)$$

exist in some topological (normed) structure? The functional $J_q^{\text{an}}(F)$ (if it exists) is called the analytic operator-valued Feynman integral of F with parameter q.

Cameron and Storvick in [1] introduced the analytic operator-valued function space "Feynman integral" $J_q^{\text{an}}(F)$, which mapped an $L^2(\mathbb{R})$ function ψ into an $L^2(\mathbb{R})$ function $J_q^{\text{an}}(F)\psi$. In [1] and several subsequent papers [2, 3, 15–22], the existence of this integral as an element of $\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$ was established for various functionals. Next, the existence of the integral as an element of $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$ was established in [4, 5, 14, 23]. Finally, the $L_p \to L_{p'}$ theory ($1 < p \le 2$ and $1/p + 1/p' = 1$) was developed as an element of $\mathcal{L}(L^p(\mathbb{R}), L^{p'}(\mathbb{R}))$ in [24].

The Wiener space $C_0[0,T]$ can be considered as the space of sample paths of standard Brownian motion process (SBMP). Thus, in various Feynman integration theories, the integrand F of the Feynman integral (1.1) is a functional of the SBMP, see [1–5, 14–24].

Let $D = [0,T]$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. By the definition, a generalized Brownian motion process (GBMP) on $D \times \Omega$ is a Gaussian process $Y \equiv \{Y_t\}_{t\in D}$ such that $Y_0 = 0$ almost surely and for any $0 \le s < t \le T$,

$$Y_t - Y_s \sim N\Big(a(t) - a(s),\, b(t) - b(s)\Big),$$

where $N(m, \sigma^2)$ denotes the normal distribution with mean m and variance σ^2, $a(t)$ is a continuous real-valued function on $[0,T]$ and $b(t)$ is an increasing continuous real-valued function on $[0,T]$. Thus a GBMP is determined by the continuous functions $a(t)$ and $b(t)$. The function space $C_{a,b}[0,T]$, induced by GBMP, was introduced by Yeh [25, 26] and was used extensively in [6–13]. The function space $C_{a,b}[0,T]$ used in [6–13] can be considered as the space of sample paths of the GBMP.

The generalized Feynman integral studied in [6, 7, 9, 10] are scalar-valued. In this paper, *the analytic operator-valued generalized Feynman integral* (AOVG'Feynman'I) of functionals F on the general Wiener space $C_{a,b}[0,T]$ is investigated as an element of $\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R}))$, where $\nu_{\delta,a}$ is a measure on $\mathbb{R}$ given by

$$d\nu_{\delta,a} = \exp\{\delta \mathrm{Var}(a) u^2\} du,$$

and where $\delta > 0$ and $\mathrm{Var}(a)$ denotes the total variation of the mean function a of the GBMP. It turns out in this paper that the AOVG'Feynman'Is of functionals defined by the stochastic Fourier–Stieltjes transform of complex measures on the infinite dimensional Hilbert space $C'_{a,b}[0,T]$, the space of absolutely continuous functions in $C_{a,b}[0,T]$, are elements of the linear space

$$\bigcap_{\delta>0} \mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R})).$$

Note that when $a(t) \equiv 0$ and $b(t) = t$, the GBMP is an SBMP, and so the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. But we are obliged to point out that an SBMP used in [1–5, 14–24] is stationary in time and is free of drift. While, the GBMP used in this paper, as well as in [6–13], is nonstationary in time and is subject to a drift $a(t)$. It turns out, as noted in Remark 4.2 below, that including a drift term $a(t)$ makes establishing the existence of *the analytic operator-valued generalized function space integral* (AOVGFSI) and AOVG'Feynman'I of functionals on $C_{a,b}[0,T]$ very difficult.

In [13], Chang, Skoug and the current author introduced the analytic operator-valued Feynman integrals $J_q^{\mathrm{an}}(F)$ of finite-dimensional functionals $F : C_{a,b}[0,T] \to \mathbb{C}$, having the form

$$F(x) = f\Big(\int_0^T \theta_1(t)dx(t), \dots, \int_0^T \theta_n(t)dx(t)\Big)$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function, $\int_0^T \theta(t)dx(t)$ denotes the Paley–Wiener–Zygmund stochastic integral with x in $C_{a,b}[0,T]$, and $\{\theta_1, \dots, \theta_n\}$ is a set of functions of bounded variation on the time interval $[0,T]$ such that

$$\int_0^T \theta_j(t)\theta_l(t)dm_{b,|a|}(t) = \delta_{jl} \quad \text{(Kronecker delta)},$$

and where $m_{b,|a|}$ denots the Lebesgue–Stieltjes measure induced by the variance function b and the mean function a of the GBMP Y. But, in [13], the topological structures between the domain and the codomain of the operator "J_q^{an}" was not discussed.

The results in this paper are quite a lot more complicated because the GBMP Y referred to above is neither stationary nor centered. We refer to the reference [6, 7, 9] for an unusual behavior of the GBMP.

2. Preliminaries

In this section, we briefly list some of the preliminaries from [6, 7, 9, 10] that we need to establish our results in next sections; for more details, see [6, 7, 9, 10].

Let $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ denote the function space induced by the GBMP Y determined by continuous functions $a(t)$ and $b(t)$, where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-field induced by the sup-norm, see [25, 26]. We assume in this paper that $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is an increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$. We complete this function space to obtain the complete probability measure space $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all μ-Carathéodory measurable subsets of $C_{a,b}[0,T]$.

We can consider the coordinate process $X : [0,T] \times C_{a,b}[0,T] \to \mathbb{R}$ given by $X(t,x) = x(t)$ which is a continuous process. The separable process X induced by Y [26] also has the following properties:

(i) $X(0,x) = x(0) = 0$ for every $x \in C_{a,b}[0,T]$.

(ii) For any $s,t \in [0,T]$ with $s \le t$ and $x \in C_{a,b}[0,T]$, $x(t) - x(s) \sim N(a(t) - a(s), b(t) - b(s))$.

Thus it follows that for $s,t \in [0,T]$, $\mathrm{Cov}(X(s,x), X(t,x)) = \min\{b(s), b(t)\}$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{W}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho\ \cdot)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho > 0$.

Let $L_{a,b}^2[0,T]$ be the separable Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on $[0,T]$ induced by $b(t)$ and $a(t)$: i.e.,

$$L_{a,b}^2[0,T] = \Big\{ v : \int_0^T v^2(s)db(s) < +\infty \ \text{ and } \ \int_0^T v^2(s)d|a|(s) < +\infty \Big\}$$

where $|a|(t)$ denotes the total variation function of $a(t)$ on $[0,T]$. The inner product on $L^2_{a,b}[0,T]$ is defined by $(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t)+|a|(t)]$. Note that $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}} = 0$ if and only if $u(t)=0$ a.e. on $[0,T]$ and that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t)\equiv 0$ and $b(t)=t$, then $L^2_{a,b}[0,T]=L^2[0,T]$. In fact,

$$(L^2_{a,b}[0,T], \|\cdot\|_{a,b}) \subset (L^2_{0,b}[0,T], \|\cdot\|_{0,b}) = (L^2[0,T], \|\cdot\|_2)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

Throughout the rest of this paper, we consider the linear space

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t\in[0,T]$, let $D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1,w_2)_{C'_{a,b}} = \int_0^T Dw_1(t)Dw_2(t)db(t) = \int_0^T z_1(t)z_2(t)db(t)$$

is also a separable Hilbert space.

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are topologically homeomorphic under the linear operator given by equation (2.1). The inverse operator of D is given by

$$(D^{-1}z)(t) = \int_0^t z(s)db(s)$$

for $t\in[0,T]$.

In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty. \tag{2.2}$$

Then, the function $a:[0,T]\to\mathbb{R}$ satisfies the condition (2.2) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$, see [11, 12]. Under the condition (2.2), we observe that for each $w\in C'_{a,b}[0,T]$ with $Dw=z$,

$$(w,a)_{C'_{a,b}} = \int_0^T Dw(t)Da(t)db(t) = \int_0^T z(t)\frac{a'(t)}{b'(t)}db(t) = \int_0^T z(t)da(t).$$

Next we will define a Paley–Wiener–Zygmund (PWZ) stochastic integral. Let $\{g_j\}_{j=1}^\infty$ be a complete orthonormal set in $C'_{a,b}[0,T]$ such that for each $j=1,2,\ldots$, $Dg_j=\alpha_j$ is of bounded variation on $[0,T]$. For each $w=D^{-1}z\in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w,x)^\sim$ is defined by the formula

$$(w,x)^\sim = \lim_{n\to\infty}\int_0^T \sum_{j=1}^n (w,g_j)_{C'_{a,b}} Dg_j(t)dx(t) = \lim_{n\to\infty}\int_0^T \sum_{j=1}^n \int_0^T z(s)\alpha_j(s)db(s)\alpha_j(t)dx(t)$$

for all $x\in C_{a,b}[0,T]$ for which the limit exists.

It is known that for each $w\in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w,x)^\sim$ exists for μ-a.e. $x\in C_{a,b}[0,T]$. If $Dw=z\in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w,x)^\sim$ equals the

Riemann–Stieltjes integral $\int_0^T z(t)dx(t)$. It also follows that for $w, x \in C'_{a,b}[0,T]$, $(w,x)^\sim = (w,x)_{C'_{a,b}}$. For each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w,x)^\sim$ is a Gaussian random variable on $C_{a,b}[0,T]$ with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$. Note that for all $w_1, w_2 \in C'_{a,b}[0,T]$,

$$\int_{C_{a,b}[0,T]} (w_1,x)^\sim (w_2,x)^\sim d\mu(x) = (w_1,w_2)_{C'_{a,b}} + (w_1,x)_{C'_{a,b}}(w_2,x)_{C'_{a,b}}.$$

Hence we see that for $w_1, w_2 \in C'_{a,b}[0,T]$, $(w_1,w_2)_{C'_{a,b}} = 0$ if and only if $(w_1,x)^\sim$ and $(w_2,x)^\sim$ are independent random variables. We thus have the following function space integration formula: let $\{e_1,\dots,e_n\}$ be an orthonormal set in $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$, and given a Lebesgue measurable function $r : \mathbb{R}^n \to \mathbb{C}$, let $R : C_{a,b}[0,T] \to \mathbb{C}$ be given by equation

$$R(x) = r((e_1,x)^\sim, \dots, (e_n,x)^\sim).$$

Then

$$\begin{aligned}\int_{C_{a,b}[0,T]} R(x)d\mu(x) &\equiv \int_{C_{a,b}[0,T]} r((e_1,x)^\sim,\dots,(e_n,x)^\sim)d\mu(x)\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} r(u_1,\dots,u_n)\exp\Big\{-\sum_{j=1}^n \frac{(u_j-(e_j,a)_{C'_{a,b}})^2}{2}\Big\}du_1\cdots du_n\end{aligned} \tag{2.3}$$

in the sense that if either side of equation (2.3) exists, both sides exist and equality holds.

The following integration formula is also used in this paper:

$$\int_{\mathbb{R}} \exp\{-\alpha u^2 + \beta u\}du = \sqrt{\frac{\pi}{\alpha}}\exp\Big\{\frac{\beta^2}{4\alpha}\Big\} \tag{2.4}$$

for complex numbers α and β with $\mathrm{Re}(\alpha) > 0$.

3. Analytic operator-valued generalized function space integral

In this section, we introduce the definition of the AOVGFSI as an element of $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$. The definition below is based on the previous definitions in [3–5, 14, 22–24].

Definition 3.1. *Let $F : C[0,T] \to \mathbb{C}$ be a scale-invariant measurable functional and let h be an element of $C'_{a,b}[0,T]\backslash\{0\}$. Given $\lambda > 0$, $\psi \in L^1(\mathbb{R})$ and $\xi \in \mathbb{R}$, let*

$$(I_\lambda(F;h)\psi)(\xi) \equiv \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}(h,x)^\sim + \xi)d\mu(x). \tag{3.1}$$

If $I_\lambda(F;h)\psi$ is in $L^\infty(\mathbb{R})$ as a function of ξ and if the correspondence $\psi \to I_\lambda(F;h)\psi$ gives an element of $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$, we say that the operator-valued generalized function space integral (OVGFSI) $I_\lambda(F;h)$ exists.

Let Γ be a region in $\mathbb{C}_+$ such that $\mathrm{Int}(\Gamma)$ *is a simply connected domain in $\mathbb{C}_+$ and* $\mathrm{Int}(\Gamma) \cap (0,+\infty)$ *is a nonempty open interval of positive real numbers. Suppose that there exists an $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$-valued function which is analytic in λ on* $\mathrm{Int}(\Gamma)$ *and agrees with $I_\lambda(F;h)$ on* $\mathrm{Int}(\Gamma) \cap (0,+\infty)$*, then this $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$-valued function is denoted by $I_\lambda^{\mathrm{an}}(F;h)$ and is called the AOVGFSI of F associated with λ.*

The notation $\|\cdot\|_o$ will be used for the norm of operators in $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$.

Remark 3.2. (i) *In equation* (3.1) *above, choosing* $h(t) = \int_0^t db(s) = b(t) \in C'_{a,b}[0,T]$*, we obtain*

$$(h, x)^{\sim} = (b, x)^{\sim} = \int_0^T Db(t)dx(t) = \int_0^T dx(t) = x(T).$$

In this case, equation (3.1) *is rewritten by*

$$(I_\lambda(F; b)\psi)(\xi) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(T) + \xi)d\mu(x). \tag{3.2}$$

Moreover, if $a(t) \equiv 0$ *and* $b(t) = t$ *on* $[0, T]$*, then the function space* $C_{a,b}[0, T]$ *reduces to the classical Wiener space* $C_0[0, T]$ *and the definition of the OVGFSI* $I_\lambda(F; b)$ *in equation* (3.2) *agrees with the definitions of the operator-valued function space integrals* $I_\lambda(F)$ *with* $\lambda > 0$ *defined in [1–5, 14–24].*

(ii) *In the case that* $a(t) \equiv 0$ *and* $h(t) = b(t) = t$ *on* $[0, T]$*, choosing* $\Gamma = \mathbb{C}_+ \cap \{\lambda \in \mathbb{C} : |\lambda| < \lambda_0\}$ *for some* $\lambda_0 \in (0, +\infty)$*, then the definition of the AOVGFSI* $I_\lambda^{\mathrm{an}}(F; b)$ *(if it exists) agrees with the definitions of the analytic operator-valued function space integral* $I_\lambda^{\mathrm{an}}(F)$ *associated with* $\lambda > 0$ *defined in [23, 24].*

4. The $\mathcal{F}(C_{a,b}[0, T])$ theory

In [6, 8], Chang, Choi and Lee introduced a Banach algebra $\mathcal{F}(C_{a,b}[0, T])$ of functionals on function space $C_{a,b}[0, T]$, each of which is a stochastic Fourier transform of $\mathbb{C}$-valued Borel measure on $C'_{a,b}[0, T]$, and showed that it contains many functionals of interest in Feynman integration theory. In [6, 7], the authors showed that the analytic (but scalar-valued) generalized Feynman integral exists for functionals in $\mathcal{F}(C_{a,b}[0, T])$. In this section, we show that the AOVGFSI $I_\lambda^{\mathrm{an}}(F; h)$ is in $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$ for functionals F in $\mathcal{F}(C_{a,b}[0, T])$.

Let $\mathcal{M}(C'_{a,b}[0, T])$ denote the space of $\mathbb{C}$-valued, countably additive (and hence finite) Borel measures on $C'_{a,b}[0, T]$. We define the Fresnel type class $\mathcal{F}(C_{a,b}[0, T])$ of functionals on $C_{a,b}[0, T]$ as the space of all stochastic Fourier–Stieltjes transforms of elements of $\mathcal{M}(C'_{a,b}[0, T])$; that is, $F \in \mathcal{F}(C_{a,b}[0, T])$ if and only if there exists a measure f in $\mathcal{M}(C'_{a,b}[0, T])$ such that

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i(w, x)^{\sim}\}df(w) \tag{4.1}$$

for s-a.e. $x \in C_{a,b}[0, T]$.

More precisely, since we shall identify functionals which coincide s-a.e. on $C_{a,b}[0, T]$, $\mathcal{F}(C_{a,b}[0, T])$ can be regarded as the space of all s-equivalence classes of functionals having the form (4.1).

We note that $\mathcal{M}(C'_{a,b}[0, T])$ is a Banach algebra under the total variation norm and with convolution as multiplication. The Fresnel type class $\mathcal{F}(C_{a,b}[0, T])$ also is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{C'_{a,b}[0,T]} d|f|(w).$$

In fact, the correspondence $f \mapsto F$ is injective, carries convolution into pointwise multiplication and is a Banach algebra isomorphism where f and F are related by (4.1). For a more detailed study of functionals in $\mathcal{F}(C_{a,b}[0, T])$, see [6, 8].

Remark 4.1. *If* F *is in* $\mathcal{F}(C_{a,b}[0, T])$*, then* F *is scale-invariant measurable and s-a.e. defined on* $C_{a,b}[0, T]$*. If* x *in* $C_{a,b}[0, T]$ *is such that* $F(x)$ *is defined, then by* (4.1) *and the definition of the PWZ stochastic integral, it follows that* $F(x + \xi) = F(x)$ *for all* $\xi \in \mathbb{R}$*.*

Let h be a (fixed) function in $C'_{a,b}[0,T]\backslash\{0\}$. Then for any function w in $C'_{a,b}[0,T]$, we obtain an orthonormal set $\{e_1, e_2(w)\}$ in $C'_{a,b}[0,T]$, by the Gram–Schmidt process, such that $h = \|h\|_{C'_{a,b}} e_1$ and

$$w = (w, e_1)_{C'_{a,b}} e_1 + \beta_w e_2(w) \tag{4.2}$$

where

$$\beta_w = \left\| w - (w, e_1)_{C'_{a,b}} e_1 \right\|_{C'_{a,b}} = \left[\|w\|^2_{C'_{a,b}} - (w, e_1)^2_{C'_{a,b}} \right]^{1/2}.$$

Throughout this paper, we will use the following notations for convenience:

$$M(\lambda; h) = \left(\frac{\lambda}{2\pi\|h\|^2_{C'_{a,b}}} \right)^{1/2}, \tag{4.3}$$

$$V(\lambda; \xi, v; h, w) = \exp\left\{ \frac{1}{2\lambda\|h\|^2_{C'_{a,b}}} \left[\left(i\lambda(v-\xi) + (h, w)_{C'_{a,b}} \right)^2 - \|h\|^2_{C'_{a,b}} \|w\|^2_{C'_{a,b}} \right] \right\}, \tag{4.4}$$

$$L(\lambda; \xi, v; h) = \exp\left\{ \frac{\lambda}{2} \frac{(v-\xi)^2}{\|h\|^2_{C'_{a,b}}} \right\}, \tag{4.5}$$

$$H(\lambda; \xi, v; h) = \exp\left\{ -\frac{\left(\sqrt{\lambda}(v-\xi) - (h, a)_{C'_{a,b}} \right)^2}{2\|h\|^2_{C'_{a,b}}} \right\}, \tag{4.6}$$

$$A(\lambda; w) = \exp\left\{ \frac{i}{\sqrt{\lambda}} \beta_w (e_2(w), a)_{C'_{a,b}} \right\} = \exp\left\{ \frac{i}{\sqrt{\lambda}} \left[\|w\|^2_{C'_{a,b}} - (w, e_1)^2_{C'_{a,b}} \right]^{1/2} (e_2(w), a)_{C'_{a,b}} \right\} \tag{4.7}$$

and

$$(K_\lambda(F; h)\psi)(\xi) = M(\lambda; h) \int_{C'_{a,b}[0,T]} \int_{\mathbb{R}} \psi(v) V(\lambda; \xi, v; h, w) L(\lambda; \xi, v; h) H(\lambda; \xi, v; h) A(\lambda; w) dv df(w) \tag{4.8}$$

for $(\lambda, \xi, v, h, w, \psi) \in \widetilde{\mathbb{C}}_+ \times \mathbb{R}^2 \times (C'_{a,b}[0,T]\backslash\{0\}) \times C'_{a,b}[0,T] \times L^1(\mathbb{R})$. In equation (4.7) above, w, e_1 and e_2 are related by equation (4.2).

Remark 4.2. *Clearly, for $\lambda > 0$, $|H(\lambda; \xi, v; h)| \le 1$ for all $(\xi, v, h) \in \mathbb{R}^2 \times (C'_{a,b}[0,T]\backslash\{0\})$. But for $\lambda \in \widetilde{\mathbb{C}}_+$, $|H(\lambda; \xi, v; h)|$ is not necessarily bounded by 1. Note that for each $\lambda \in \widetilde{\mathbb{C}}_+$, $\mathrm{Re}(\lambda) \ge 0$ and $\mathrm{Re}(\sqrt{\lambda}) \ge |\mathrm{Im}(\sqrt{\lambda})| \ge 0$. Hence for each $\lambda \in \widetilde{\mathbb{C}}_+$,*

$$H(\lambda; \xi, v; h) = \exp\left\{ -\frac{[\mathrm{Re}(\lambda) + i\mathrm{Im}(\lambda)](v-\xi)^2}{2\|h\|^2_{C'_{a,b}}} + \frac{[\mathrm{Re}(\sqrt{\lambda}) + i\mathrm{Im}(\sqrt{\lambda})](v-\xi)(h, a)_{C'_{a,b}}}{\|h\|^2_{C'_{a,b}}} - \frac{(h, a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}} \right\}, \tag{4.9}$$

and so

$$\left| H(\lambda; \xi, v; h) \right| = \exp\left\{ -\frac{\mathrm{Re}(\lambda)(v-\xi)^2}{2\|h\|^2_{C'_{a,b}}} + \frac{\mathrm{Re}(\sqrt{\lambda})(v-\xi)(h, a)_{C'_{a,b}}}{\|h\|^2_{C'_{a,b}}} - \frac{(h, a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}} \right\}. \tag{4.10}$$

Note that for $\lambda \in \mathbb{C}_+$, the case we consider throughout Section 4, $\mathrm{Re}(\sqrt{\lambda}) > |\mathrm{Im}(\sqrt{\lambda})| \ge 0$, which implies that $\mathrm{Re}(\lambda) = [\mathrm{Re}(\sqrt{\lambda})]^2 - [\mathrm{Im}(\sqrt{\lambda})]^2 > 0$. Hence for each $\lambda \in \mathbb{C}_+$, $0 < |\mathrm{Arg}(\lambda)| < \pi/2$ and so

$$\frac{[\mathrm{Re}(\sqrt{\lambda})]^2}{\mathrm{Re}(\lambda)} = \frac{1}{2}\left(\frac{|\lambda|}{\mathrm{Re}(\lambda)} + 1 \right) = \frac{1}{2}(\sec \mathrm{Arg}(\lambda) + 1). \tag{4.11}$$

For $(\lambda, h) \in \mathbb{C}_+ \times (C'_{a,b}[0,T]\backslash\{0\})$, *let*

$$S(\lambda; h) = \exp\Big\{(\sec \operatorname{Arg}(\lambda) + 1)\frac{(h,a)^2_{C'_{a,b}}}{4\|h\|^2_{C'_{a,b}}}\Big\}. \tag{4.12}$$

Using (4.10), (4.11), *and* (4.12), *we obtain that for all* $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} &\Big|H(\lambda; \xi, v; h)\Big| \\ &= \exp\Big\{-\frac{\operatorname{Re}(\lambda)(v-\xi)^2}{2\|h\|^2_{C'_{a,b}}} + \frac{\operatorname{Re}(\sqrt{\lambda})(v-\xi)(h,a)_{C'_{a,b}}}{\|h\|^2_{C'_{a,b}}} - \frac{(h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}}\Big\} \\ &= \exp\Big\{-\frac{\operatorname{Re}(\lambda)}{2\|h\|^2_{C'_{a,b}}}\Big[(v-\xi) - \frac{\operatorname{Re}(\sqrt{\lambda})}{\operatorname{Re}(\lambda)}(h,a)_{C'_{a,b}}\Big]^2 + \frac{[\operatorname{Re}(\sqrt{\lambda})]^2}{\operatorname{Re}(\lambda)}\frac{(h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}} - \frac{(h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}}\Big\} \\ &\le S(\lambda; h). \end{aligned} \tag{4.13}$$

These observations are critical to the development of the existence of the AOVGFSI $I^{\mathrm{an}}_\lambda(F; h)$.

One can see that for all $(\lambda, \xi, v, h, w) \in \mathbb{C}_+ \times \mathbb{R}^2 \times (C'_{a,b}[0,T]\backslash\{0\}) \times C'_{a,b}[0,T]$,

$$\begin{aligned} &\Big|V(\lambda; \xi, v; h, w)L(\lambda; \xi, v; h)\Big| \\ &= \Big|\exp\Big\{\frac{\Big[\Big(i\lambda(v-\xi) + (h,w)_{C'_{a,b}}\Big)^2 - \|h\|^2_{C'_{a,b}}\|w\|^2_{C'_{a,b}}\Big]}{2\lambda\|h\|^2_{C'_{a,b}}} + \frac{\lambda}{2}\Big(\frac{v-\xi}{\|h\|_{C'_{a,b}}}\Big)^2\Big\}\Big| \\ &= \exp\Big\{-\frac{\operatorname{Re}(\lambda)}{2|\lambda|^2\|h\|^2_{C'_{a,b}}}\Big[\|h\|^2_{C'_{a,b}}\|w\|^2_{C'_{a,b}} - (h,w)^2_{C'_{a,b}}\Big]\Big\} \\ &\le 1, \end{aligned} \tag{4.14}$$

because $(h,w)^2_{C'_{a,b}} \le \|h\|^2_{C'_{a,b}}\|w\|^2_{C'_{a,b}}$. *However, the expression* (4.7) *is an unbounded function of* w *for* $w \in C'_{a,b}[0,T]$, *because* $\beta_w(e_2(w), a)_{C'_{a,b}}$ *with*

$$e_2(w) = \frac{1}{\beta_w}\Big[w - (w, e_1)_{C'_{a,b}}e_1\Big] = \frac{1}{\beta_w}\Big[w - \frac{1}{\|h\|^2_{C'_{a,b}}}(h,w)_{C'_{a,b}}h\Big] \tag{4.15}$$

is an unbounded function of w *for* $w \in C'_{a,b}[0,T]$. *Throughout this section, we thus will need to put additional restrictions on the complex measure* f *corresponding to* F *in order to obtain the existence of our AOVGFSI* $I^{\mathrm{an}}_\lambda(F; h)$ *of* F *in* $\mathcal{F}(C_{a,b}[0,T])$.

In order to obtain the existence of the AOVGFSI, we need to impose additional restrictions on the functionals in $\mathcal{F}(C_{a,b}[0,T])$.

For a positive real number q_0, let

$$k(q_0; w) = \exp\Big\{(2q_0)^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\Big\} \tag{4.16}$$

and let

$$\Gamma_{q_0} = \Big\{\lambda \in \widetilde{\mathbb{C}}_+ : |\operatorname{Im}(\lambda^{-1/2})| = \sqrt{\tfrac{|\lambda| - \operatorname{Re}(\lambda)}{2|\lambda|^2}} < (2q_0)^{-1/2}\Big\}. \tag{4.17}$$

Define a subclass $\mathcal{F}^{q_0}$ of $\mathcal{F}(C_{a,b}[0,T])$ by $F \in \mathcal{F}^{q_0}$ if and only if

$$\int_{C'_{a,b}[0,T]} k(q_0; w)\, d|f|(w) < +\infty. \tag{4.18}$$

Then for all $\lambda \in \Gamma_{q_0}$,

$$|A(\lambda; w)| < k(q_0; w). \tag{4.19}$$

Remark 4.3. *The region Γ_{q_0} given by* (4.17) *satisfies the conditions stated in Definition 3.1; i.e.,* $\mathrm{Int}(\Gamma_{q_0})$ *is a simple connected domain in $\mathbb{C}_+$ and* $\mathrm{Int}(\Gamma_{q_0}) \cap (0, +\infty)$ *is an open interval. We note that for all real q with $|q| > q_0$, $(-iq)^{-1/2} = 1/\sqrt{2|q|} + i\mathrm{sign}(q)/\sqrt{2|q|}$. Also, by a close examination of* (4.17)*, it follows that $-iq$ is an element of the region Γ_{q_0}. In fact, Γ_{q_0} is a simple connected neighborhood of $-iq$ in $\widetilde{\mathbb{C}}_+$.*

Lemma 4.4. *Let q_0 be a positive real number and let F be an element of $\mathcal{F}^{q_0}$. Let h be an element of $C'_{a,b}[0,T]\backslash\{0\}$ and let Γ_{q_0} be given by* (4.17)*. Let $(K_\lambda(F;h)\psi)(\xi)$ be given by equation* (4.8) *for $(\lambda, \xi, \psi) \in \Gamma_{q_0} \times \mathbb{R} \times L^1(\mathbb{R})$. Then $K_\lambda(F;h)$ is an element of $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$ for each $\lambda \in \mathrm{Int}(\Gamma_{q_0})$.*

Proof. Let Γ_{q_0} be given by (4.17). Using (4.8), (4.3), (4.4), (4.5), (4.6), (4.7), (4.14), the Fubini theorem, (4.13), and (4.19), we observe that for all $(\lambda, \xi, \psi) \in \mathrm{Int}(\Gamma_{q_0}) \times \mathbb{R} \times L^1(\mathbb{R})$,

$$\begin{aligned}
&\big|(K_\lambda(F;h)\psi)(\xi)\big| \\
&\le M(|\lambda|;h)\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}}\big|\psi(v)\big|\big|V(\lambda;\xi,v;h,w)L(\lambda;\xi,v;h)\big| \times \big|H(\lambda;\xi,v;h)\big|\big|A(\lambda;w)\big|dvd|f|(w) \\
&\le M(|\lambda|;h)\int_{\mathbb{R}}\big|\psi(v)\big|\big|H(\lambda;\xi,v;h)\big|dv\int_{C'_{a,b}[0,T]}\big|A(\lambda;w)\big|d|f|(w) \\
&\le \|\psi\|_1 S(\lambda;h)M(|\lambda|;h)\int_{C'_{a,b}[0,T]}k(q_0;w)d|f|(w) \\
&< +\infty,
\end{aligned} \tag{4.20}$$

where $S(\lambda;h)$ is given by equation (4.12). Clearly $K_\lambda(F;h) : L_1(\mathbb{R}) \to L_\infty(\mathbb{R})$ is linear. Thus, for all $\lambda \in \mathrm{Int}(\Gamma_{q_0})$,

$$\|K_\lambda(F;h)\|_o \le S(\lambda;h)M(|\lambda|;h)\int_{C'_{a,b}[0,T]}k(q_0;w)d|f|(w)$$

and the lemma is proved. □

Lemma 4.5. *Let q_0, F, h, Γ_{q_0} and $(K_\lambda(F;h)\psi)(\xi)$ be as in Lemma 4.4. Then $(K_\lambda(F;h)\psi)(\xi)$ is an analytic function of λ on* $\mathrm{Int}(\Gamma_{q_0})$.

Proof. Let $\lambda \in \mathrm{Int}(\Gamma_{q_0})$ be given and let $\{\lambda_l\}_{l=1}^\infty$ be a sequence in $\mathbb{C}_+$ such that $\lambda_l \to \lambda$. Clearly, $0 \le |\mathrm{Arg}(\lambda)| < \pi/2$. Thus there exist $\theta_0 \in (\mathrm{Arg}(\lambda), \pi/2)$ and $n_0 \in \mathbb{N}$ such that $\lambda_l \in \mathrm{Int}(\Gamma_{q_0})$ and $0 < |\mathrm{Arg}(\lambda_l)| < \theta_0$ for all $l > n_0$. We first note that for each $l > n_0$,

$$\frac{[\mathrm{Re}(\sqrt{\lambda_l})]^2}{\mathrm{Re}(\lambda_l)} = \frac{1}{2}\Big(\frac{|\lambda_l|}{\mathrm{Re}(\lambda_l)} + 1\Big) = \frac{1}{2}(\sec\mathrm{Arg}(\lambda_l) + 1) < \frac{1}{2}(\sec\theta_0 + 1).$$

Using this and the Cauchy–Schwartz inequality, it follows that for all $l > n_0$ and $\psi \in L^1(\mathbb{R})$,

$$\begin{aligned}
&\big|\psi(v)\big|\big|V(\lambda_l;\xi,v;h,w)L(\lambda_l;\xi,v;h)H(\lambda_l;\xi,v;h)A(\lambda_l;w)\big| \\
&= \big|\psi(v)\big|\exp\bigg\{-\frac{\mathrm{Re}(\lambda_l)(v-\xi)^2}{2\|h\|^2_{C'_{a,b}}} - \frac{\mathrm{Re}(\lambda_l)}{2|\lambda_l|^2\|h\|^2_{C'_{a,b}}}\Big[\|h\|^2_{C'_{a,b}}\|w\|^2_{C'_{a,b}} - (h,w)^2_{C'_{a,b}}\Big] \\
&\qquad + \frac{\mathrm{Re}(\sqrt{\lambda_l})(v-\xi)(h,a)_{C'_{a,b}}}{\|h\|^2_{C'_{a,b}}} - \frac{(h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}} - \mathrm{Im}(\lambda_l^{-1/2})\Big[\|w\|^2_{C'_{a,b}} - (w,e_1)^2_{C'_{a,b}}\Big]^{1/2}(e_2(w),a)_{C'_{a,b}}\bigg\}
\end{aligned} \tag{4.21}$$

$$\le \big|\psi(v)\big| \exp\bigg\{ -\frac{\mathrm{Re}(\lambda_l)(v-\xi)^2}{2\|h\|^2_{C'_{a,b}}} + \frac{\mathrm{Re}(\sqrt{\lambda_l})(v-\xi)(h,a)_{C'_{a,b}}}{\|h\|^2_{C'_{a,b}}} - \frac{(h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}} - \mathrm{Im}(\lambda^{-1/2})\Big[\|w\|^2_{C'_{a,b}} - (w,e_1)^2_{C'_{a,b}}\Big]^{1/2}(e_2(w),a)_{C'_{a,b}}\bigg\}$$

$$= \big|\psi(v)\big| \exp\bigg\{ -\frac{\mathrm{Re}(\lambda_l)}{2\|h\|^2_{C'_{a,b}}}\Big[(v-\xi) - \frac{\mathrm{Re}(\sqrt{\lambda_l})}{\mathrm{Re}(\lambda_l)}(h,a)_{C'_{a,b}}\Big]^2 + \frac{[\mathrm{Re}(\sqrt{\lambda_l})]^2}{2\|h\|^2_{C'_{a,b}}\mathrm{Re}(\lambda_l)}(h,a)^2_{C'_{a,b}} - \frac{(h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}} - \mathrm{Im}(\lambda^{-1/2})\Big[\|w\|^2_{C'_{a,b}} - (w,e_1)^2_{C'_{a,b}}\Big]^{1/2}(e_2(w),a)_{C'_{a,b}}\bigg\}$$

$$\le \big|\psi(v)\big| \exp\bigg\{ \frac{(h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}}\frac{[\mathrm{Re}(\sqrt{\lambda_l})]^2}{\mathrm{Re}(\lambda_l)} + \big|\mathrm{Im}(\lambda^{-1/2})\big|\Big[\|w\|^2_{C'_{a,b}} - (w,e_1)^2_{C'_{a,b}}\Big]^{1/2}|(e_2(w),a)_{C'_{a,b}}|\bigg\}$$

$$= \big|\psi(v)\big| \exp\bigg\{ \frac{(h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}}\frac{[\mathrm{Re}(\sqrt{\lambda_l})]^2}{\mathrm{Re}(\lambda_l)} + \big|\mathrm{Im}(\lambda^{-1/2})\big|\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\bigg\}$$

$$< \big|\psi(v)\big| \exp\bigg\{ \frac{(h,a)^2_{C'_{a,b}}}{4\|h\|^2_{C'_{a,b}}}(\sec\theta_0+1)\bigg\}k(q_0;w)$$

where $e_2(w)$ and $k(q_0;w)$ are given by (4.15) and (4.16), respectively. Since $\psi \in L^1(\mathbb{R})$, and f, the corresponding measure of F by (4.1), satisfies condition (4.18), the last expression of (4.21) is integrable on the product space $(\mathbb{R}\times C'_{a,b}[0,T], \mathrm{m}_L\times f)$, as a function of (v,w), where m_L denotes the Lebesgue measure on $\mathbb{R}$. Hence by the dominated convergence theorem, we see that the right-hand side of equation (4.8) is a continuous function of λ on $\mathrm{Int}(\Gamma_{q_0})$. Next we note that for all $(\xi,v,h,w)\in\mathbb{R}^2\times(C'_{a,b}[0,T]\backslash\{0\})\times C'_{a,b}[0,T]$,

$$V(\lambda;\xi,v;h,w)L(\lambda;\xi,v;h)H(\lambda;\xi,v;h)A(\lambda;w)$$

is an analytic function of λ throughout the domain $\mathrm{Int}(\Gamma_{q_0})$. Thus using (4.8), the Fubini theorem, and the Morera theorem, it follows that for every rectifiable simple closed curve Δ in $\mathrm{Int}(\Gamma_{q_0})$,

$$\begin{aligned}&\int_\Delta K_\lambda(F;h)\psi)(\xi)d\lambda\\ &= M(\lambda;h)\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}}\psi(v)\Big(\int_\Delta V(\lambda;\xi,v;h,w)L(\lambda;\xi,v;h)H(\lambda;\xi,v;h)A(\lambda;w)d\lambda\Big)dvdf(w)\\ &= 0.\end{aligned}$$

Therefore for all $(\xi,h,\psi)\in\mathbb{R}\times(C'_{a,b}[0,T]\backslash\{0\})\times L^1(\mathbb{R})$, $(K_\lambda(F;h)\psi)(\xi)$ is an analytic function of λ throughout the domain $\mathrm{Int}(\Gamma_{q_0})$. □

Theorem 4.6. *Let q_0, F, h and Γ_{q_0} be as in Lemma 4.4. Then for each $\lambda\in\mathrm{Int}(\Gamma_{q_0})$, the AOVGFSI $I_\lambda^{\mathrm{an}}(F;h)$ exists and is given by the right-hand side of equation (4.8). Thus, $I_\lambda^{\mathrm{an}}(F;h)$ is an element of $\mathcal{L}(L^1(\mathbb{R}),L^\infty(\mathbb{R}))$ for each $\lambda\in\mathrm{Int}(\Gamma_{q_0})$.*

Proof. Let $(\lambda,\xi,\psi)\in(0,+\infty)\times\mathbb{R}\times L^1(\mathbb{R})$. We begin by evaluating the function space integral

$$\begin{aligned}(I_\lambda(F;h)\psi)(\xi) &= \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}(h,x)^\sim+\xi)d\mu(x)\\ &= \int_{C_{a,b}[0,T]}\int_{C'_{a,b}[0,T]}\exp\{i\lambda^{-1/2}(w,x)^\sim\}\psi(\lambda^{-1/2}(h,x)^\sim+\xi)df(w)d\mu(x).\end{aligned} \tag{4.22}$$

Using the Fubini theorem, we can change the order of integration in (4.22). Since $\psi \in L^1(\mathbb{R})$, $f \in \mathcal{M}(C'_{a,b}[0,T])$, and $(h,x)^\sim$ is a Gaussian random variable with mean $(h,a)_{C'_{a,b}}$ and variance $\|h\|^2_{C'_{a,b}}$, it follows that for $\lambda > 0$,

$$\begin{aligned}\big|(I_\lambda(F;h)\psi)(\xi)\big| &\le \int_{C'_{a,b}[0,T]}\int_{C_{a,b}[0,T]} \big|\psi(\lambda^{-1/2}(h,x)^\sim + \xi)\big| d\mu(x)d|f|(w)\\ &\le M(|\lambda|;h)\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}} |\psi(v)|H(\lambda;\xi,v;h)dvd|f|(w)\\ &\le M(|\lambda|;h)\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}} |\psi(v)|dud|f|(w)\\ &= M(|\lambda|;h)\|\psi\|_1\|f\|\\ &< +\infty.\end{aligned}$$

Next, using (4.22), the Fubini theorem, (4.2), (2.3), (2.4), (4.3), (4.4), (4.5), (4.6), and (4.7), it follows that

$$\begin{aligned}&(I_\lambda(F;h)\psi)(\xi)\\ &= \int_{C'_{a,b}[0,T]}\int_{C_{a,b}[0,T]} \psi(\lambda^{-1/2}\|h\|_{C'_{a,b}}(e_1,x)^\sim + \xi)\\ &\qquad \times \exp\Big\{i\lambda^{-1/2}(w,e_1)_{C'_{a,b}}(e_1,x)^\sim + i\lambda^{-1/2}\beta_w(e_2(w),x)^\sim\Big\}d\mu(x)df(w)\\ &= \Big(\frac{\lambda}{2\pi}\Big)\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}^2}\psi(\|h\|_{C'_{a,b}}u_1+\xi)\\ &\qquad \times \exp\Big\{i(w,,e_1)_{C'_{a,b}}u_1 + i\beta_w u_2 - \frac{(\sqrt{\lambda}u_1 - (e_1,a)_{C'_{a,b}})^2}{2} - \frac{(\sqrt{\lambda}u_2 - (e_2(w),a)_{C'_{a,b}})^2}{2}\Big\}du_1du_2df(w)\\ &= \Big(\frac{\lambda}{2\pi}\Big)^{1/2}\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}}\psi(\|h\|_{C'_{a,b}}u_1+\xi)\exp\Big\{i(w,e_1)_{C'_{a,b}}u_1 - \frac{(\sqrt{\lambda}u_1 - (e_1,a)_{C'_{a,b}})^2}{2}\Big\}du_1\\ &\qquad \times \exp\Big\{-\frac{1}{2\lambda}\beta_w^2 + \frac{i}{\sqrt{\lambda}}\beta_w(e_2(w),a)_{C'_{a,b}}\Big\}df(w)\\ &= M(\lambda;h)\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}}\psi(v)\exp\Big\{i\frac{(w,e_1)_{C'_{a,b}}}{\|h\|_{C'_{a,b}}}(v-\xi) - \frac{(\sqrt{\lambda}(v-\xi) - \|h\|_{C'_{a,b}}(e_1,a)_{C'_{a,b}})^2}{2\|h\|^2_{C'_{a,b}}}\Big\}dv\\ &\qquad \times \exp\Big\{-\frac{1}{2\lambda}\beta_w^2 + \frac{i}{\sqrt{\lambda}}\beta_w(e_2(w),a)_{C'_{a,b}}\Big\}df(w)\\ &= M(\lambda;h)\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}}\psi(v)V(\lambda;\xi,v;h,w)L(\lambda;\xi,v;h)H(\lambda;\xi,v;h)A(\lambda;w)dvdf(w)\\ &= (K_\lambda(F;h)\psi)(\xi).\end{aligned}$$

Hence we see that the OVGFSI $I_\lambda(F;h)$ exists for all $(\lambda,h) \in (0,+\infty)\times(C'_{a,b}[0,T]\backslash\{0\})$.

Let $I_\lambda^{\mathrm{an}}(F;h)\psi = K_\lambda(F;h)\psi$ for all $\lambda \in \mathrm{Int}(\Gamma_{q_0})$. Then by Lemmas 4.4 and 4.5, we obtain the desired result. □

5. The analytic operator-valued generalized Feynman integral

In this section we study the AOVG'Feynman'I $J_q^{\mathrm{an}}(F;h)$ for functionals F in $\mathcal{F}(C_{a,b}[0,T])$. First of all, we note that for any $q \in \mathbb{R}\setminus\{0\}$ and any $(\xi,v,h,w) \in \mathbb{R}^2\times(C'_{a,b}[0,T]\backslash\{0\})\times C'_{a,b}[0,T]$,

$$\big|V(-iq;\xi,v;h,w)L(-iq;\xi,v;h)\big| = 1.$$

Let $\lambda = -iq \in \widetilde{\mathbb{C}}_+ - \mathbb{C}_+$. Then

$$\sqrt{\lambda} = \sqrt{-iq} = \sqrt{|q|/2} - i\operatorname{sign}(q)\sqrt{|q|/2}.$$

Hence for $\lambda = -iq$ with $q \in \mathbb{R} \setminus \{0\}$, $[\mathrm{Re}(\sqrt{-iq})]^2 - [\mathrm{Im}(\sqrt{-iq})]^2 = 0$, and so

$$\left|H(-iq;\xi,v;h)\right| = \exp\left\{\frac{\sqrt{2|q|}(h,a)_{C'_{a,b}}(v-\xi) - (h,a)^2_{C'_{a,b}}}{2\|h\|^2_{C'_{a,b}}}\right\}$$

which is not necessarily in $L^p(\mathbb{R})$, as a function of v, for any $p \in [1,+\infty]$. Hence $K_{-iq}(F;h)$ might not exist as an element of $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$.

Let $q = -1$ and let h be an element of $C'_{a,b}[0,T]$ with $\|h\|_{C'_{a,b}} = 1$ and with $(h,a)_{a,b} > 0$ (we can choose h to be $a/\|a\|_{C'_{a,b}}$ in $C'_{a,b}[0,T]$). Let $\psi : \mathbb{R} \to \mathbb{C}$ be defined by the formula

$$\psi(v) = v\chi_{[0,+\infty)}(v)\exp\left\{\frac{iv^2}{2} - \frac{i\sqrt{2}(h,a)_{C'_{a,b}}v}{2} + \frac{(h,a)^2_{C'_{a,b}}}{2} - \frac{\sqrt{2}(h,a)_{C'_{a,b}}v}{4}\right\}.$$

We note that

$$|\psi(v)| = v\chi_{[0,+\infty)}(v)\exp\left\{\frac{(h,a)^2_{C'_{a,b}}}{2} - \frac{\sqrt{2}(h,a)_{C'_{a,b}}v}{4}\right\},$$

and hence $\psi \in L^p(\mathbb{R})$ for all $p \in [1,+\infty]$. In fact, ψ is also an element of $C_0(\mathbb{R})$, the space of bounded continuous functions on $\mathbb{R}$ that vanish at infinity.

Let $F(x) \equiv 1$. Then F is an element of $\mathcal{F}^{q_0}$ for all $q_0 \in (0,+\infty)$, and $(K_{-iq}(F;h)\psi)(\xi)$ with $q = -1$ is given by

$$(K_i(1;h)\psi)(\xi) = \left(\frac{i}{2\pi}\right)^{1/2}\int_{\mathbb{R}}\psi(v)H(i;\xi,v;h)dv. \tag{5.1}$$

Next, using equation (4.9) with $\lambda = i$ and $\sqrt{\lambda} = \sqrt{i} = (1+i)/\sqrt{2}$, we observe that

$$H(i;\xi,v;h) = \exp\left\{-i\frac{(v-\xi)^2}{2} + \frac{(h,a)_{C'_{a,b}}(v-\xi)}{\sqrt{2}} + \frac{i(h,a)_{C'_{a,b}}(v-\xi)}{\sqrt{2}} - \frac{(h,a)^2_{C'_{a,b}}}{2}\right\},$$

and hence,

$$\psi(v)H(i;\xi,v;h) = v\chi_{[0,+\infty)}(v)\exp\left\{\frac{\sqrt{2}(h,a)_{C'_{a,b}}v}{4} + i\xi v - \frac{i\xi^2}{2} - \left(\frac{1+i}{\sqrt{2}}\right)(h,a)_{C'_{a,b}}\xi\right\} \tag{5.2}$$

which is not an element of $L^p(\mathbb{R})$, as a function of v, for any $p \in [1,+\infty]$.

Then, using equations (5.1) and (5.2), we see that

$$(K_i(1;h)\psi)(\xi) = \left(\frac{i}{2\pi}\right)^{1/2}\exp\left\{-\frac{i\xi^2}{2} - \left(\frac{1+i}{\sqrt{2}}\right)(h,a)_{C'_{a,b}}\xi\right\}\int_{\mathbb{R}}v\chi_{[0,+\infty)}(v)\exp\left\{\frac{\sqrt{2}(h,a)_{C'_{a,b}}v}{4} + i\xi v\right\}dv.$$

Hence, choosing $\xi = 0$, and using the fact that $(h,a)_{C'_{a,b}}$ is positive, we see that

$$\left|(K_i(1;h)\psi)(0)\right| = (2\pi)^{-1/2}\int_0^{+\infty}v\exp\left\{\frac{\sqrt{2}(h,a)_{C'_{a,b}}v}{4}\right\}dv = +\infty.$$

In fact, for each fixed $\xi\in\mathbb{R}$, we observe that

$$\Big|(K_i(1;h)\psi)(\xi)\Big| = (2\pi)^{-1/2}\exp\Big\{-\frac{1}{\sqrt{2}}(h,a)_{C'_{a,b}}\xi\Big\}\Big|\int_{\mathbb{R}} v\chi_{[0,+\infty)}(v)\exp\Big\{\frac{\sqrt{2}(h,a)_{C'_{a,b}}v}{4}+i\xi v\Big\}dv\Big| = +\infty,$$

and so $(K_i(1;h)\psi)$ is not an element of $L^\infty(\mathbb{R})$ even though ψ was an element of $L^1(\mathbb{R})$. Hence $K_{-iq}(F;h)\psi\equiv K_i(1;h)\psi$ is not in $\mathcal{L}(L^1(\mathbb{R}),L^\infty(\mathbb{R}))$.

In this section, we thus clearly need to impose additional restrictions on ψ for the existence of our AOVG'Feynman'I.

For any positive real number δ, let $\nu_{\delta,a}$ be a measure on $\mathbb{R}$ with

$$d\nu_{\delta,a}=\exp\{\delta\mathrm{Var}(a)u^2\}du$$

where $\mathrm{Var}(a)=|a|(T)$ denotes the total variation of a, the mean function of the GBMP, on $[0,T]$ and let $L^1(\mathbb{R},\nu_{\delta,a})$ be the space of $\mathbb{C}$-valued Lebesgue measurable functions ψ on $\mathbb{R}$ such that ψ is integrable with respect to the measure $\nu_{\delta,a}$ on $\mathbb{R}$. Let $\|\cdot\|_{1,\delta}$ denote the $L^1(\mathbb{R},\nu_{\delta,a})$-norm. Then for all $\delta>0$, we have the following inclusion

$$L^1(\mathbb{R},\nu_{\delta,a})\subsetneq L^1(\mathbb{R}) \tag{5.3}$$

as sets, because $\|\psi\|_1\le\|\psi\|_{1,\delta}$ for all $\psi\in L^1(\mathbb{R})$.

Let $\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R}))$ be the space of continuous linear operators form $L^1(\mathbb{R},\nu_{\delta,a})$ to $L^\infty(\mathbb{R})$. In Theorem 4.6, we proved that for all $\psi\in L^1(\mathbb{R})$, $I_\lambda^{\mathrm{an}}(F;h)\psi$ is in $L^\infty(\mathbb{R})$. From the inclusion (5.3), we see that for all $\psi\in L^1(\mathbb{R},\nu_{\delta,a})$, $I_\lambda^{\mathrm{an}}(F;h)\psi$ is in $L^\infty(\mathbb{R})$. Furthermore, for all $\delta>0$,

$$\mathcal{L}(L^1(\mathbb{R}),L^\infty(\mathbb{R}))\subset\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R})), \tag{5.4}$$

as sets.

Now, the notation $\|\cdot\|_{o,\delta}$ will be used for the norm on $\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R}))$. We already shown in (4.20) that for all $(\lambda,\xi,\psi)\in\mathrm{Int}(\Gamma_{q_0})\times\mathbb{R}\times L^1(\mathbb{R})$,

$$\Big|(K_\lambda(F;h)\psi)(\xi)\Big|\le M(|\lambda|;h)\int_{\mathbb{R}}\Big|\psi(v)\Big|\Big|H(\lambda;\xi,v;h)\Big|dv\int_{C'_{a,b}[0,T]}\Big|A(\lambda;w)\Big|d|f|(w).$$

But, by the same method, (4.13), and (4.19), it also follows that for any $\delta>0$ and all $(\lambda,\xi,\psi)\in\mathrm{Int}(\Gamma_{q_0})\times\mathbb{R}\times L^1(\mathbb{R},\nu_{\delta,a})$,

$$\begin{aligned}
&\Big|(K_\lambda(F;h)\psi)(\xi)\Big|\\
&\le M(|\lambda|;h)\int_{\mathbb{R}}\Big|\psi(v)\Big|\Big|H(\lambda;\xi,v;h)\Big|dv\int_{C'_{a,b}[0,T]}\Big|A(\lambda;w)\Big|d|f|(w)\\
&\le M(|\lambda|;h)\int_{\mathbb{R}}\Big|\psi(v)\exp\{\delta\mathrm{Var}(a)v^2\}\Big|\Big|H(\lambda;\xi,v;h)\Big|dv\int_{C'_{a,b}[0,T]}\Big|A(\lambda;w)\Big|d|f|(w)\\
&\le M(|\lambda|;h)S(\lambda;h)\int_{\mathbb{R}}\Big|\psi(v)\exp\{\delta\mathrm{Var}(a)v^2\}\Big|dv\int_{C'_{a,b}[0,T]}k(q_0;w)d|f|(w)\\
&\le\|\psi\|_{1,\delta}\Big(S(\lambda;h)M(|\lambda|;h)\int_{C'_{a,b}[0,T]}k(q_0;w)d|f|(w)\Big)
\end{aligned} \tag{5.5}$$

and so

$$\|K_\lambda(F;h)\|_{o,\delta}\le S(\lambda;h)M(|\lambda|;h)\int_{C'_{a,b}[0,T]}k(q_0;w)d|f|(w).$$

Thus we have the following definition.

Definition 5.1. *Given a non-zero real number q, let Γ_q be a connected neighborhood of $-iq$ in $\widetilde{\mathbb{C}}_+$ such that* $\mathrm{Int}(\Gamma_q)$ *satisfies the conditions stated in Definition 3.1. If there exists an operator $J_q^{\mathrm{an}}(F;h)$ in $\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R}))$ for some $\delta>0$ such that for every ψ in $L^1(\mathbb{R},\nu_{\delta,a})$,*

$$\Big\|J_q^{\mathrm{an}}(F;h)\psi - I_\lambda^{\mathrm{an}}(F;h)\psi\Big\|_\infty \to 0$$

as $\lambda\to -iq$ through $\mathrm{Int}(\Gamma_q)$*, then $J_\lambda^{\mathrm{an}}(F;h)$ is called the AOVG'Feynman'I of F with parameter q.*

Theorem 5.2. *Let q_0, F, h and Γ_{q_0} be as in Lemma 4.4. Then for all real q with $|q|>q_0$, the AOVG'Feynman'I of F, $J_q^{\mathrm{an}}(F;h)$, exists as an element of $\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R}))$ for any $\delta>0$, and is given by the right-hand side of equation* (4.8) *with $\lambda=-iq$.*

Proof. First, we will show that $K_{-iq}(F;h)$ is an element of $\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R}))$. Note that for every $\delta>0$, $|H(-iq;\xi,v;h)|\exp\{-\delta\mathrm{Var}(a)u^2\}$ is bounded by 1. Hence for any $\delta\in(0,+\infty)$ and every $\psi\in L^1(\mathbb{R},\nu_\delta)$,

$$\begin{aligned}
&\int_{\mathbb{R}}\Big|\psi(v)\Big|\Big|H(-iq;\xi,v;h)\Big|dv\\
&=\int_{\mathbb{R}}\Big|\psi(v)\Big|\exp\Big\{\delta\mathrm{Var}(a)u^2\Big\}\Big|H(-iq;\xi,v;h)\Big|\exp\Big\{-\delta\mathrm{Var}(a)u^2\Big\}dv\\
&\le\|\psi\|_{1,\delta}.
\end{aligned}$$

Also, by a simple calculation, it follows that

$$|V(-iq;\xi,v;h,w)|\Big|L(-iq;\xi,v;h)\Big|=1.$$

Thus, using these and (4.19), it also follows that for all real q with $|q|>q_0$,

$$\begin{aligned}
&\Big|(K_{-iq}(F;h)\psi)(\xi)\Big|\\
&\le M(|q|;h)\int_{C'_{a,b}[0,T]}\int_{\mathbb{R}}\Big|\psi(v)\Big|\Big|V(-iq;\xi,v;h,w)\Big|\Big|L(-iq;\xi,v;h)\Big|\Big|H(-iq;\xi,v;h)\Big|\Big|A(-iq;w)\Big|dvd|f|(w)\\
&=M(|q|;h)\int_{\mathbb{R}}\Big|\psi(v)\Big|\Big|H(-iq;\xi,v;h)\Big|dv\int_{C'_{a,b}[0,T]}\Big|A(-iq;w)\Big|d|f|(w)\\
&\le\|\psi\|_{1,\delta}\Big(M(|q|;h)\int_{C'_{a,b}[0,T]}k(q_0;w)d|f|(w)\Big).
\end{aligned}\tag{5.6}$$

Therefore we have that

$$\Big\|K_{-iq}(F;h)\psi\Big\|_\infty\le\|\psi\|_{1,\delta}\Big(M(|q|;h)\int_{C'_{a,b}[0,T]}k(q_0;w)d|f|(w)\Big)$$

and

$$\Big\|K_{-iq}(F;h)\Big\|_{o,\delta}\le M(|q|;h)\int_{C'_{a,b}[0,T]}k(q_0;w)d|f|(w),$$

and implies that $K_{-iq}(F;h)\in\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R}))$.

We now want to show that the AOVG'Feynman'I $J_q^{\mathrm{an}}(F;h)$ of F exists and is given by the right-hand side of (4.8) with $\lambda=-iq$. To do this, it suffices to show that for every ψ in $L^1(\mathbb{R},\nu_{\delta,a})$

$$\Big\|K_{-iq}(F;h)\psi - I_\lambda^{\mathrm{an}}(F;h)\psi\Big\|_\infty\to 0$$

as $\lambda \to -iq$ through $\mathrm{Int}(\Gamma_{q_0})$, where Γ_{q_0} is given by equation (4.17). But, in view of Lemmas 4.4, 4.5, Theorem 4.6, and equation (5.4), we already proved that $I_\lambda^{\mathrm{an}}(F;h) = K_\lambda(F;h)$ for all $\lambda \in \mathrm{Int}(\Gamma_{q_0})$ and that $K_\lambda(F;h)$ is an element of $\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R}))$. Next, by (5.5) and (5.6), we obtain that for all $(\lambda,\xi,\psi) \in \Gamma_{q_0}\times\mathbb{R}\times L^1(\mathbb{R},\nu_\delta)$,

$$
\begin{aligned}
&\big|(K_\lambda(F;h)\psi)(\xi)\big| \\
&\le \begin{cases} \|\psi\|_{1,\delta}\Big\{S(\lambda;h)M(|\lambda|;h)\int_{C'_{a,b}[0,T]} k(q_0;w)d|f|(w)\Big\}, & \lambda \in \mathrm{Int}(\Gamma_{q_0}) \\ \|\psi\|_{1,\delta}\Big\{M(|q|;h)\int_{C'_{a,b}[0,T]} k(q_0;w)d|f|(w)\Big\}, & \lambda = -iq,\ q \in \mathbb{R}\backslash\{0\} \end{cases} \\
&< +\infty.
\end{aligned}
$$

Moreover, using the techniques similar to those used in the proof of Lemma 4.5, one can easily verify that there exists a sufficiently small $\varepsilon_0 > 0$ satisfying the inequality:

$$
\begin{aligned}
&\big|(K_\lambda(F;h)\psi)(\xi)\big| \\
&\le \|\psi\|_{1,\delta}\Bigg(\exp\Bigg\{\frac{(h,a)^2_{C'_{a,b}}}{4\|h\|^2_{C'_{a,b}}}\Big(\frac{q_0}{\varepsilon_0}+1\Big)\Bigg\}M(1+|q|;h)\int_{C'_{a,b}[0,T]} k(q_0;w)d|f|(w)\Bigg) \\
&< +\infty
\end{aligned}
$$

for all $\lambda \in \Gamma_{q_0}\cap\{\lambda \in \widetilde{\mathbb{C}} : |\lambda-(-iq)| < \varepsilon_0\}$ (we have already commented in Remark 4.3 that Γ_{q_0} is a simple connected neighborhood of $-iq$ in $\widetilde{\mathbb{C}}_+$). Hence by the dominated convergence theorem, we have

$$
\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathrm{Int}(\Gamma_{q_0})}} (I_\lambda^{\mathrm{an}}(F;h)\psi)(\xi) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathrm{Int}(\Gamma_{q_0})}} (K_\lambda(F;h)\psi)(\xi) = (K_{-iq}(F;h)\psi)(\xi)
$$

for each $\xi \in \mathbb{R}$. Thus $J_q^{\mathrm{an}}(F;h)$ exists as an element of $\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R}))$ and is given by the right-hand side of equation (4.8) with $\lambda = -iq$. □

It is clear that given two positive real number δ_1 and δ_2 with $\delta_1 < \delta_2$,

$$
L^1(\mathbb{R},\nu_{\delta_2,a}) \subsetneq L^1(\mathbb{R},\nu_{\delta_1,a}) \subsetneq L^1(\mathbb{R}).
$$

Thus it follows that

$$
\mathcal{L}(L^1(\mathbb{R}),L^\infty(\mathbb{R})) \subsetneq \mathcal{L}(L^1(\mathbb{R},\nu_{\delta_1,a}),L^\infty(\mathbb{R})) \subsetneq \mathcal{L}(L^1(\mathbb{R},\nu_{\delta_2,a}),L^\infty(\mathbb{R})).
$$

Let

$$
L^{1,a}(\mathbb{R}) = \bigcup_{\delta>0} L^1(\mathbb{R},\nu_{\delta,a})
$$

and let

$$
\mathfrak{B}(L^{1,a}(\mathbb{R}),L^\infty(\mathbb{R})) = \bigcap_{\delta>0} \mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R})).
$$

We note that $L^{1,a}(\mathbb{R})$ and $\mathfrak{B}(L^{1,a}(\mathbb{R}),L^\infty(\mathbb{R}))$ are not normed spaces. However we can suggest set theoretic structures between them as follows: since $L^1(\mathbb{R},\nu_{\delta,a}) \subset L^{1,a}(\mathbb{R}) \subset L^1(\mathbb{R})$ for any $\delta > 0$, it follows that

$$
\mathcal{L}(L^1(\mathbb{R}),L^\infty(\mathbb{R})) \subset \mathfrak{B}(L^{1,a}(\mathbb{R}),L^\infty(\mathbb{R})) \subset \mathcal{L}(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R})).
$$

From this observation and Theorem 5.2, we can obtain the following assertion.

Theorem 5.3. *Let q_0, F, h and Γ_{q_0} be as in Lemma 4.4. Then for all real q with $|q| > q_0$, the AOVG'Feynman'I $J_q^{\mathrm{an}}(F;h)$ exists as an element of $\mathfrak{B}(L^{1,a}(\mathbb{R}),L^\infty(\mathbb{R}))$.*

Remark 5.4. *If $b(t) = t$ and $a(t) \equiv 0$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. In this case, the three linear spaces $L^1(\mathbb{R})$, $L^1(\mathbb{R},\nu_{\delta,0})$ and $L^{1,0}(\mathbb{R})$ coincide each other. Furthermore, the three classes $\mathcal{L}(L^1(\mathbb{R}),L^\infty(\mathbb{R}))$, $\mathfrak{B}(L^{1,0}(\mathbb{R}),L^\infty(\mathbb{R}))$, and $\mathcal{L}(L^1(\mathbb{R},\nu_{\delta,0}),L^\infty(\mathbb{R}))$ also coincide.*

6. Examples

In this section, we present interesting examples to which our results in previous sections can be applied.

Let $\mathcal{M}(\mathbb{R})$ be the class of complex-valued, countably additive Borel measures on $\mathcal{B}(\mathbb{R})$. For $\eta \in \mathcal{M}(\mathbb{R})$, the Fourier transform $\widehat{\eta}$ of η is a $\mathbb{C}$-valued function defined on $\mathbb{R}$, given by the formula

$$\widehat{\eta}(u) = \int_{\mathbb{R}} \exp\{iuv\} d\eta(v).$$

(1) Let $w_0 \in C'_{a,b}[0,T]$ and let $\eta \in \mathcal{M}(\mathbb{R})$. Define $F_1 : C_{a,b}[0,T] \to \mathbb{C}$ by

$$F_1(x) = \widehat{\eta}((w_0, x)^{\sim}).$$

Define a function $\phi : \mathbb{R} \to C'_{a,b}[0,T]$ by $\phi(v) = vw_0$. Let $f = \eta \circ \phi^{-1}$. It is quite clear that f is in $\mathcal{M}(C'_{a,b}[0,T])$ and is supported by $[w_0]$, the subspace of $C'_{a,b}[0,T]$ spanned by $\{w_0\}$. Now for s-a.e. $x \in C_{a,b}[0,T]$,

$$\begin{aligned}\int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\} df(w) &= \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^{\sim}\} d(\eta \circ \phi^{-1})(w) \\ &= \int_{\mathbb{R}} \exp\{i(\phi(v),x)^{\sim}\} d\eta(v) \\ &= \int_{\mathbb{R}} \exp\{i(w_0,x)^{\sim} v\} d\eta(v) \\ &= \widehat{\eta}((w_0,x)^{\sim}).\end{aligned}$$

Thus F_1 is an element of $\mathcal{F}(C_{a,b}[0,T])$.

Suppose that for a fixed positive real number $q_0 > 0$,

$$\int_{\mathbb{R}} \exp\Big\{(2q_0)^{-1/2} \|w_0\|_{C'_{a,b}} \|a\|_{C'_{a,b}} |v|\Big\} d|\eta|(v) < +\infty. \tag{6.1}$$

It is easy to show that condition (6.1) is equivalent to condition (4.18) with $f = \eta \circ \phi^{-1}$. Thus, under condition (6.1), F_1 is an element of $\mathcal{F}^{q_0}$ and so, by Theorem 5.2, $J_q^{\text{an}}(F_1; h)$ exists as an element of $\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^{\infty}(\mathbb{R}))$ for all real q with $|q| > q_0$, all $h \in C'_{a,b}[0,T] \setminus \{0\}$, and any $\delta > 0$. Moreover $J_q^{\text{an}}(F_1; h)$ is an element of the space $\mathfrak{B}(L^{1,a}(\mathbb{R}), L^{\infty}(\mathbb{R}))$ by Theorem 5.3.

Next, we present more explicit examples of functionals in $\mathcal{F}(C_{a,b}[0,T])$ whose associated measures satisfy condition (6.1).

(2) Let $S : C'_{a,b}[0,T] \to C'_{a,b}[0,T]$ be the linear operator defined by $Sw(t) = \int_0^t w(s) db(s)$. Then the adjoint operator S^* of S is given by

$$S^* w(t) = \int_0^t \Big(w(T) - w(s)\Big) db(s)$$

and for $x \in C_{a,b}[0,T]$, $(S^*b, x)^{\sim} = \int_0^T x(t) db(t)$ by an integration by parts formula.

Given m and σ^2 in $\mathbb{R}$ with $\sigma^2 > 0$, let $\eta_{\mathrm{m},\sigma^2}$ be the Gaussian measure given by

$$\eta_{\mathrm{m},\sigma^2}(B) = (2\pi\sigma^2)^{-1/2} \int_B \exp\Big\{-\frac{(v-\mathrm{m})^2}{2\sigma^2}\Big\} dv, \quad B \in \mathcal{B}(\mathbb{R}). \tag{6.2}$$

Then $\eta_{\mathrm{m},\sigma^2} \in \mathcal{M}(\mathbb{R})$ and

$$\widehat{\eta_{\mathrm{m},\sigma^2}}(u) = \int_{\mathbb{R}} \exp\{iuv\} d\eta_{\mathrm{m},\sigma^2}(v) = \exp\Big\{-\frac{1}{2}\sigma^2 u^2 + i\mathrm{m}u\Big\}.$$

The complex measure η_{m,σ^2} given by equation (6.2) satisfies condition (6.1) for all $q_0 > 0$. Thus we can apply the results in argument (1) to the functional $F_2 : C_{a,b}[0,T] \to \mathbb{C}$ given by

$$F_2(x) = \widehat{\eta_{\mathrm{m},\sigma^2}}((w_0, x)^\sim) = \exp\Big\{ -\frac{1}{2}\sigma^2[(w_0, x)^\sim]^2 + i\mathrm{m}(w_0, x)^\sim\Big\}. \tag{6.3}$$

For example, if we choose $w_0 = S^*b$, m = 0 and $\sigma^2 = 2$ in (6.3), we have

$$F_3(x) = \exp\Big\{ -[(S^*b, x)^\sim]^2\Big\} = \exp\Big\{ -\Big(\int_0^T x(t)db(t)\Big)^2\Big\}$$

for $x \in C_{a,b}[0,T]$.

We note that the functional F_3 is in $\cap_{q_0>0}\mathcal{F}^{q_0}$, and so that for every nonzero real number q, the AOVG'Feynman'I $J_q^{\mathrm{an}}(F_3; h)$ exists as an element of $\mathfrak{B}(L^{1,\alpha}(\mathbb{R}), L^\infty(\mathbb{R}))$.

(3) Let $F_4 : C_{a,b}[0,T] \to \mathbb{C}$ be given by

$$F_4(x) = \exp\Big\{i\int_0^T x(t)db(t)\Big\}.$$

Then F_4 is a functional in $\mathcal{F}(C_{a,b}[0,T])$, because

$$F_4(x) = \exp\{i(S^*b, x)^\sim\} = \int_{C'_{a,b}[0,T]} \exp\{i(w, x)^\sim\}d\zeta(w)$$

for s-a.e. $x \in C_{a,b}[0,T]$, where ζ is the Dirac measure concentrated at S^*b in $C'_{a,b}[0,T]$. The Dirac measure ζ also satisfies condition (4.18) with f replaced with ζ for all $q_0 > 0$; that is, $F_4 \in \cap_{q_0>0}\mathcal{F}^{q_0}$, and so that for every nonzero real number q, the AOVG'Feynman'I $J_q^{\mathrm{an}}(F_4; h)$ exists as an element of $\mathfrak{B}(L^{1,\alpha}(\mathbb{R}), L^\infty(\mathbb{R}))$.

Acknowledgements

The author would like to express his gratitude to the editor and the referees for their valuable comments and suggestions which have improved the original paper.

References

[1] R. H. Cameron and D. A. Storvick, An operator valued function space integral and a related integral equation, J. Math. Mech. **18** (1968) 517–552.
[2] R. H. Cameron and D. A. Storvick, An integral equation related to the Schroedinger equation with an application to integration in function space, In: Gunning, R.C. (ed.) Problems in Analysis (A Symposium in Honor of Salomon Bochner, pp. 175–193. Princeton Univ. Press, Princeton, New Jersey, 1970.
[3] R. H. Cameron and D. A. Storvick, An operator valued function space integral applied to integrals of functions of class L_2, J. Math. Anal. Appl. **42** (1973) 330–372.
[4] R. H. Cameron and D. A. Storvick, An operator-valued function space integral applied to integrals of functions of class L_1, Proc. London Math. Soc. **27** (1973) 345–360.
[5] R. H. Cameron and D. A. Storvick, An operator valued function space integral applied to multiple integrals of functions of class L_1, Nagoya Math. J. **51** (1973) 91–122.
[6] S. J. Chang and J. G. Choi, Effect of drift of the generalized Brownian motion process: an example for the analytic Feynman integral, Arch. Math. **106** (2016) 591–600.
[7] S. J. Chang, J. G. Choi and A. Y. Ko, Multiple generalized analytic Fourier–Feynman transform via rotation of Gaussian paths on function space, Banach J. Math. Anal. **9** (2015) 58–80.
[8] S. J. Chang, J. G. Choi and S. D. Lee, A Fresnel type class on function space, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **16** (2009) 107–119.
[9] S. J. Chang, J. G. Choi and D. Skoug, Integration by parts formulas involving generalized Fourier–Feynman transforms on function space, Trans. Amer. Math. Soc. **355** (2003) 2925–2948.
[10] S. J. Chang and D. Skoug, Generalized Fourier–Feynman transforms and a first variation on function space, Integral Transforms Spec. Funct. **14** (2003) 375–393.

[11] J. G. Choi, H. S. Chung and S. J. Chang, Sequential generalized transforms on function space, Abstr. Appl. Anal. **2013** (2013) Article ID: 565832.
[12] J. G. Choi and D. Skoug, Further results involving the Hilbert space $L^2_{a,b}[0,T]$, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **27** (2020) 1–11.
[13] J. G. Choi, D. Skoug and S. J. Chang, Analytic operator-valued Feynman integrals of certain finite-dimensional functionals on function space, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **108** (2014) 907–916.
[14] D. M. Chung, C. Park and D. Skoug, Operator-valued Feynman integrals via conditional Feynman integrals, Pacific J. Math. **146** (1990) 21–42.
[15] G. W. Johnson and D.L. Skoug, Operator-valued Feynman integrals of certain finite-dimensional functionals, Proc. Amer. Math. Soc. **24** (1970) 774–780.
[16] G. W. Johnson and D.L. Skoug, Operator-valued Feynman integrals of finite-dimensional functionals, Pacific J. Math. **34** (1970) 415–425.
[17] G. W. Johnson and D.L. Skoug, An operator valued function space integral: A sequel to Cameron and Storvick's paper, Proc. Amer. Math. Soc. **27** (1971) 514–518.
[18] G. W. Johnson and D.L. Skoug, A Banach algebra of Feynman integrable functionals with application to an integral equation formally equivalent to Schroedinger's equation, J. Funct. Anal. **12** (1973) 129–152.
[19] G. W. Johnson and D.L. Skoug, Feynman integrals of non-factorable finite-dimensional functionals, Pacific J. Math. **45** (1973) 257–267.
[20] G. W. Johnson and D.L. Skoug, Cameron and Storvick's function space integral for certain Banach spaces of functionals, J. London Math. Soc. **9** (1974) 103–117.
[21] G. W. Johnson and D.L. Skoug, A function space integral for a Banach space of functionals on Wiener space, Proc. Amer. Math. Soc. **43** (1974) 141–148.
[22] G. W. Johnson and D.L. Skoug, Cameron and Storvick's function space integral for a Banach space of functionals generated by finite-dimensional functionals, Annali di Matematica Pura ed Applicata **104** (1975) 67–83.
[23] G. W. Johnson and D.L. Skoug, The Cameron–Storvick function space integral: The L_1 theory, J. Math. Anal. Appl. **50** (1975) 647–667.
[24] G. W. Johnson and D.L. Skoug, The Cameron–Storvick function space integral: An $\mathcal{L}(L_p, L_{p'})$ theory, Nagoya Math. J. **60** (1976) 93–137.
[25] J. Yeh, Singularity of Gaussian measures on function spaces induced by Brownian motion processes with non-stationary increments, Illinois J. Math. **15** (1971) 37–46.
[26] J. Yeh, Stochastic Processes and the Wiener Integral, Marcel Dekker, New York, 1973.