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A completely hyperexpansive completion problem for weighted shifts on directed trees with one branching vertex

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Abstract. Let $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^n$ be given a finite sequence of positive real numbers. The completely hyperexpansive completion problem seeks equivalence conditions for the existence of a completely hyperexpansive weighted shift $W_{\hat{\boldsymbol{\alpha}}}$ such that $\boldsymbol{\alpha} \subseteq \hat{\boldsymbol{\alpha}}$. Let $\mathscr{T}_{\eta,\kappa}$ be a directed tree consisting of one branching vertex, η branches and a trunk of length κ, and let $\mathscr{T}_{\eta,\kappa,p}$ be a subtree of $\mathscr{T}_{\eta,\kappa}$ whose members consist of the p-generation family from branching vertex. Suppose S_λ is the weighted shift acting on the tree $\mathscr{T}_{\eta,\kappa}$. This object S_λ on the tree $\mathscr{T}_{\eta,\kappa}$ has been applied to the several topics. Recently, Exner-Jung-Stochel-Yun studied the subnormal completion problem for weighted shifts on $\mathscr{T}_{\eta,\kappa}$ in 2018. In this paper we discuss the completely hyperexpansive completion problem for weighted shifts on $\mathscr{T}_{\eta,\kappa}$ as a counterpart of the subnormal completion problem for S_λ.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\boldsymbol{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \boldsymbol{B}(\mathcal{H})$ is *subnormal* if T is unitarily equivalent to the restriction of a normal operator to an invariant subspace. There are several characterizations for the subnormality of operators in $\boldsymbol{B}(\mathcal{H})$. In [1], J. Agler proved that $T \in \boldsymbol{B}(\mathcal{H})$ is a subnormal contractive operator if and only if $\Theta_n(T) \geq 0$ for all $n \in \mathbb{N}$ ($\mathbb{N}$ stands for the set of positive integers), where

$$\Theta_n(T) := \sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*n} T^n, \quad n \in \mathbb{N}.$$

An operator $T \in \boldsymbol{B}(\mathcal{H})$ is *completely hyperexpansive* if $\Theta_n(T) \leq 0$ for all $n \in \mathbb{N}$. In some sense, the notion of complete hyperexpansivity of $T \in \boldsymbol{B}(\mathcal{H})$ can be considered as a counterpart of subnormality in a subclass of $\boldsymbol{B}(\mathcal{H})$. Let $W_{\boldsymbol{\alpha}}$ be the unilateral weighted shift on ℓ^2 with a bounded weight sequence $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^\infty \subseteq (0,\infty)$, i.e., $W_{\boldsymbol{\alpha}}$ is the bounded linear operator on ℓ^2 defined by $W_{\boldsymbol{\alpha}} e_n = \alpha_n e_{n+1}$, $n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, where $\{e_n\}_{n=0}^\infty$ is the standard orthonormal basis of ℓ^2. Let $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^n \subseteq (0,\infty)$ be a given finite sequence. The *subnormal*

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completion problem of $\boldsymbol{\alpha}$ is finding equivalence conditions for the existence of a subnormal weighted shift $W_{\hat{\boldsymbol{\alpha}}}$ with weight sequence $\hat{\boldsymbol{\alpha}} = \{\hat{\alpha}_k\}_{k=0}^{\infty}$ such that $\boldsymbol{\alpha} \subseteq \hat{\boldsymbol{\alpha}}$, i.e., $\hat{\alpha}_k = \alpha_k$ for $k = 0, \cdots, n$. Since J. Stampfli ([21]) provided a construction of the subnormal completion of three weights in 1966, this topic has been studied and developed until now by several operator theorists ([7–9], etc.). It is well known that the subnormal completion problem is solved by solving the truncated Stieltjes moment problem which is an important topic in both pure and applied mathematics ([20]). Hence it should be worth studying the completely hyperexpansive completion problem as a counterpart of the subnormal completion problem: the *completely hyperexpansive completion problem* of $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^{n}$ seeks equivalence conditions for the existence of a completely hyperexpansive weighted shift $W_{\hat{\boldsymbol{\alpha}}}$ such that $\boldsymbol{\alpha} \subseteq \hat{\boldsymbol{\alpha}}$ ([11, 14–16], etc.). The object of weighted shifts acting on directed trees is a generalization of the classical weighted shifts and has been studied by several operator theorists at present ([17, 19], etc.). In particular, the weighted shift on directed trees with one branching vertex is a good model which provides several exotic examples to solve some open problems in operator theory ([5, 18, 19], etc.). Recently Exner-Jung-Stochel-Yun studied the subnormal completion problem for weighted shifts on directed trees in [12] and [13]. Hence it is natural to study the completely hyperexpansive completion problem for weighted shifts on directed trees as a counterpart of Exner-Jung-Stochel-Yun's subnormal completion problem. In this paper, we discuss the completely hyperexpansive completion problem for weighted shifts on directed trees with one branching vertex.

This paper consists of four sections. In Section 2, we introduce the basic notions for the completely hyperexpansive completion problem and the weighted shifts $S_{\boldsymbol{\lambda}}$ on directed trees. Also we give some criteria for the complete hyperexpansivity of $S_{\boldsymbol{\lambda}}$. We construct the completely hyperexpansive completion similarly as in the notion of subnormal completion on directed trees. Under the consideration of the tree $\mathscr{T}_{\eta,\kappa}$ which has η branches, a trunk of length κ, and one branching vertex and its subtree $\mathscr{T}_{\eta,\kappa,p}$ of $\mathscr{T}_{\eta,\kappa}$ with p generation, we discuss the p-generation completely hyperexpansive problem on $\mathscr{T}_{\eta,\kappa}$. In Section 3, we study the 1-generation completely hyperexpansive completion on $\mathscr{T}_{\eta,\kappa}$. In addition, we analyze the conditions for the completely hyperexpansive completion when $\kappa = 1$ and $p = 1$ in detail, and describe their representing measures. In Section 4, we discuss the p-generation flat completely hyperexpansive completion on $\mathscr{T}_{\eta,\kappa}$ for arbitrary p-generation. Finally, we provide some relationships between the completely hyperexpansive completion on the tree $\mathscr{T}_{\eta,\kappa}$ and classical unilateral weighted shifts which are obtained by slicing the tree $\mathscr{T}_{\eta,\kappa}$.

2. Preliminaries

2.1. Basic notions

We write $\mathbb{Z}$ and $\mathbb{Z}_+$ for the sets of integers and nonnegative integers, respectively. Denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}$, $\mathbb{N}_\iota = \{k \in \mathbb{N} : k \geq \iota\}$ with $\iota \geq 2$, $\bar{\mathbb{N}}_\iota = \mathbb{N}_\iota \cup \{\infty\}$, and $J_\iota = \{k \in \mathbb{N} : k \leq \iota\}$, $\iota \in \bar{\mathbb{Z}}_+$. We write $\mathscr{B}([0,1])$ for the family of all Borel subsets of the closed interval $[0,1]$.

Recall that a sequence $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ is *completely alternating* if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} a_{m+j} \leq 0, \quad m \in \mathbb{Z}_+,\ n \in \mathbb{N},$$

where $\binom{n}{j} = \frac{n!}{(n-j)!j!}$ is the binomial coefficient. It follows from [4, Proposition 4.6.12] (see also [17, p.75]) that

a sequence $\{a_n\}_{n=0}^{\infty}$ is completely alternating if and only if there exists a unique finite positive Borel measure τ on $[0,1]$ such that

$$a_n = a_0 + \int_{[0,1]} (1 + \cdots + s^{n-1}) d\tau(s), \quad n \in \mathbb{N}.$$

Let $W_{\boldsymbol{\alpha}}$ be the unilateral weighted shift with a weight sequence $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{\infty}$ of positive real numbers. We define the moment sequence $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^{\infty} \subseteq (0, \infty)$ of W_α by $\gamma_0 = 1$ and $\gamma_n = \alpha_0^2 \cdots \alpha_{n-1}^2$ ($n \in \mathbb{N}$). This sequence

$\{\gamma_n\}_{n=0}^{\infty}$ is sometimes said to be *moments*. Alternatively, given a moment sequence $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^{\infty} \subseteq (0,\infty)$, we obtain the corresponding sequence $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{\infty}$ by $\alpha_n = \sqrt{\gamma_{n+1}/\gamma_n}$ for $n \in \mathbb{Z}_+$.

Let $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{p}$ be a finite sequence of positive real numbers with $p \in \mathbb{Z}_+$. A weighted shift $W_{\hat{\boldsymbol{\alpha}}}$ with positive weights $\hat{\boldsymbol{\alpha}}$ is called a *completely hyperexpansive completion* (say: *CH-completion*) of $\boldsymbol{\alpha}$ if $W_{\hat{\boldsymbol{\alpha}}}$ is completely hyperexpansive and $\boldsymbol{\alpha} \subseteq \hat{\boldsymbol{\alpha}}$ (See [15, Definition 4.3]).

Proposition 2.1 ([15, Lemma 4.5]). *Suppose $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{p}$ is a sequence of positive real numbers with $p \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) *$\boldsymbol{\alpha}$ admits a CH-completion,*
(ii) *there exists a completely alternating sequence $\hat{\boldsymbol{\gamma}} = \{\hat{\gamma}_n\}_{n=0}^{\infty}$ of real numbers such that*

$$\hat{\gamma}_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=0}^{n-1} \alpha_j^2 & \text{if } n \in J_{p+1}. \end{cases}$$

2.2. Weighted shifts on directed trees

In this subsection we define the main objects of our study. Let $\mathscr{T} = (V, E)$ be a directed tree with the set of vertices V and the set of edges E. A vertex $v \in V$ is said to be the *parent* of u if there exists at most one vertex $v \in V$ such that $(v, u) \in E$, and denoted by $\mathsf{par}(u)$. A vertex of $\mathscr{T}$ which has no parent is called a *root* of $\mathscr{T}$. If $\mathscr{T}$ has a root, we denote it by root and write $V^\circ = V \setminus \{\mathsf{root}\}$. Put $\mathsf{Chi}\,(u) = \{v \in V \colon (u, v) \in E\}$ for $u \in V$. We call a member of $\mathsf{Chi}\,(u)$ a *child* of u. We write $V' = \{u \in V \colon \mathsf{Chi}\,(u) \neq \varnothing\}$ and call a member of $V \setminus V'$ a *leaf* of $\mathscr{T}$. A vertex $v \in V$ is called a *descendant* of $u \in V$ if there exist $v_0, \ldots, v_n \in V$ with $n \in \mathbb{Z}_+$ such that $v_0 = v$, $v_n = u$ and $v_{j+1} = \mathsf{par}(v_j)$ for $j = 0, \ldots, n-1$ (provided $n \in \mathbb{N}$). The set of all descendants of $u \in V$ is denoted by $\mathsf{Des}_{\mathscr{T}}(u)$, i.e., $\mathsf{Des}(u) = \cup_{n=0}^{\infty} \mathsf{Chi}^{<n>}(u)$, where $\mathsf{Chi}^{<0>}(u) = \{u\}$, $\mathsf{Chi}(W) = \cup_{v \in W} \mathsf{Chi}(v)$, $W \subseteq V$, and $\mathsf{Chi}^{<n+1>}(u) = \mathsf{Chi}(\mathsf{Chi}^{<n>}(u))$, $n \in \mathbb{Z}_+$. If W is a nonempty subset of V such that $\mathscr{T}_W := (W, (W \times W) \cap E)$ is a directed tree, then we say that $\mathscr{T}_W$ is a (*directed*) *subtree* of $\mathscr{T}$.

For a directed graph $\mathscr{T} = (V, E)$, we write $\ell^2(V)$ for the usual Hardy space on V with the standard basis $\{e_u\}_{u \in V}$ of $\ell^2(V)$ (cf. [6, Example 1.1.7]). For a system $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, define the map $\Lambda_{\mathscr{T}}$ defined on functions $f \colon V \to \mathbb{C}$ by

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}, \end{cases}$$

and define the operator $S_{\boldsymbol{\lambda}}$ on $\ell^2(V)$ with domain $\mathcal{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) \colon \Lambda_{\mathscr{T}} f \in \ell^2(V)\}$ by $S_{\boldsymbol{\lambda}} f = \Lambda_{\mathscr{T}} f$, $f \in \mathcal{D}(S_{\boldsymbol{\lambda}})$ ([17, Definition 3.1.1]). The operator $S_{\boldsymbol{\lambda}}$ is called the *weighted shift on the directed tree $\mathscr{T}$ with weights $\{\lambda_v\}_{v \in V^\circ}$*. In particular, if $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$,

$$S_{\boldsymbol{\lambda}} e_u = \sum_{v \in \mathsf{Chi}(u)} \lambda_v e_v \quad \text{and} \quad \|S_{\boldsymbol{\lambda}}\| = \sup_{u \in V} \left(\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \right)^{1/2}; \tag{1}$$

see [17] for more information of the theory of weighted shifts on directed trees.

We now give some technical lemmas which are connected to the completely alternating sequences on a directed tree.

Lemma 2.2 ([17, Lemma 7.1.8]). *Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, and let $u \in V'$ be such that $\{\|S_{\boldsymbol{\lambda}}^n e_v\|^2\}_{n=0}^{\infty}$ is completely alternating for every $v \in \mathsf{Chi}\,(u)$. Then the following conditions are equivalent:*

(i) *the sequence $\{\|S_{\boldsymbol{\lambda}}^n e_u\|^2\}_{n=0}^{\infty}$ is completely alternating,*

(ii) *$S_{\boldsymbol{\lambda}}$ satisfies the consistency condition at u, i.e.,*[1)]

$$\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2 \geq 1+\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2\int_0^1\frac{1}{s}d\tau_v(s).$$

If (i) *holds, then $\tau_v(\{0\})=0$ for every $v\in\mathsf{Chi}(u)$ such that $\lambda_v\neq 0$, and the representing measure τ_u of $\{\|S_{\boldsymbol{\lambda}}^n e_u\|^2\}_{n=0}^{\infty}$ is given by*

$$\tau_u(\sigma)=\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2\int_\sigma\frac{1}{s}d\tau_v(s)+\left(\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2-1-\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2\int_0^1\frac{1}{s}d\tau_v(s)\right)\delta_0(\sigma), \tag{2}$$

for $\sigma\in\mathscr{B}([0,1])$. Moreover, $\tau_u(\{0\})=0$ if and only if $S_{\boldsymbol{\lambda}}$ satisfies the strong consistency condition at u, i.e.,

$$\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2 = 1+\sum_{v\in\mathsf{Chi}(u)}|\lambda_v|^2\int_0^1\frac{1}{s}d\tau_v(s). \tag{3}$$

Lemma 2.3 ([17, Theorem 7.1.4]). *Let $S_{\boldsymbol{\lambda}}\in\boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V^\circ}$. Then the following conditions are equivalent:*

(i) *$S_{\boldsymbol{\lambda}}$ is completely hyperexpansive,*

(ii) *for any $u\in V$, there exists the (unique finite) positive Borel measure τ_u on $[0,1]$ such that*

$$\|S_{\boldsymbol{\lambda}}^n e_u\|^2=1+\int_0^1(1+\cdots+s^{n-1})d\tau_u(s),\quad n\in\mathbb{N}. \tag{4}$$

Notation. We write $\tau_u^{\boldsymbol{\lambda}}$ for the unique finite positive Borel measure τ_u corresponded by $\boldsymbol{\lambda}$ and u which are appeared in Lemma 2.3.

We now introduce a particular directed tree with one branching vertex, η branches and a trunk of length κ; see Definition 2.4 below and Figure 1. This is the main object of this paper, which has provided several interesting results and exotic examples related to subnormality since 2012 (see e.g., [2, 5, 18, 19], etc.).

Definition 2.4 ([17]). *Given $\eta\in\bar{\mathbb{N}}_2$ and $\kappa\in\bar{\mathbb{Z}}_+$, we define the directed tree $\mathscr{T}_{\eta,\kappa}=(V_{\eta,\kappa},E_{\eta,\kappa})$ by (see Figure 1)*

$$\begin{aligned}
V_{\eta,\kappa}&=\{-k\colon k\in J_\kappa\}\cup\{0\}\cup\{(i,j)\colon i\in J_\eta,\ j\in\mathbb{N}\},\\
E_{\eta,\kappa}&=E_\kappa\cup\{(0,(i,1))\colon i\in J_\eta\}\cup\{((i,j),(i,j+1))\colon i\in J_\eta,\ j\in\mathbb{N}\},\\
E_\kappa&=\{(-k,-k+1)\colon k\in J_\kappa\}.
\end{aligned}$$

Recall that a weighted shift $S_{\boldsymbol{\lambda}}$ satisfies the unitary equivalence property ([17, Theorem 3.2.1]), and that if $\lambda_u=0$ for some $u\in V^\circ$, then $S_{\boldsymbol{\lambda}}$ can be decomposed into two weighted shifts on subtrees of $\mathscr{T}$ ([17, Theorem 3.1.6]). Thus, to study the structure of $S_{\boldsymbol{\lambda}}$, we usually consider positive real values for the weights $\{\lambda_v\}_{v\in V^\circ}$ of $S_{\boldsymbol{\lambda}}$ (cf. [10, 12, 13], etc.). Thus

> *we only deal with bounded weighted shifts $S_{\boldsymbol{\lambda}}$ on directed trees $\mathscr{T}_{\eta,\kappa}=(V_{\eta,\kappa},E_{\eta,\kappa})$ with positive weights $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V^\circ}$, where $\eta\in\bar{\mathbb{N}}_2$ and $\kappa\in\mathbb{Z}_+$, unless we specify otherwise,*

throughout this paper. The following lemma is the counterpart of the version of subnormality in Corollary 6.2.2 in [17] for the completely hyperexpansivity of weighted shift on a directed tree.

Lemma 2.5 ([17, Corollary 7.2.3]). *Suppose $\eta\in\bar{\mathbb{N}}_2$ and $\kappa\in\bar{\mathbb{Z}}_+$ are given. Let $S_{\boldsymbol{\lambda}}\in\boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ be as usual. Then the following assertions hold*[2)]*:*

[1)]We consider the convention that $\frac{1}{0}=\infty$.

[2)]From now on, for brevity we write $\lambda_{i,j}$, $e_{i,j}$ and $\tau_{i,j}^{\boldsymbol{\lambda}}$ instead of $\lambda_{(i,j)}$, $e_{(i,j)}$ and $\tau_{(i,j)}^{\boldsymbol{\lambda}}$, respectively.

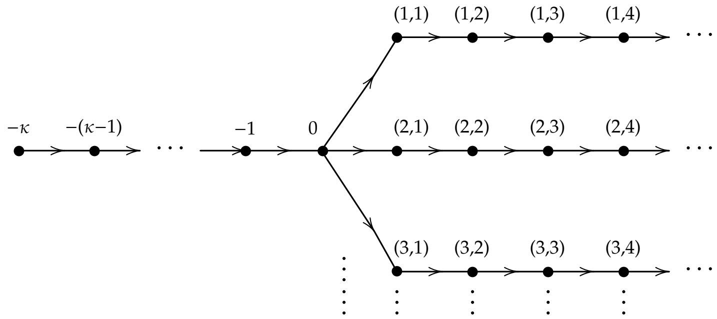

Figure 1: An illustration of the directed tree $\mathscr{T}_{\eta,\kappa}$ for $\kappa < \infty$.

(i) *If $\kappa = 0$, then S_λ is completely hyperexpansive if and only if there exist positive Borel measures $\{\tau_i\}_{i=1}^{\eta}$ on $[0,1]$ such that*

$$1 + \int_0^1 (1 + \cdots + s^{n-1}) d\tau_i(s) = \prod_{j=2}^{n+1} \lambda_{i,j}^2, \quad n \in \mathbb{N},\ i \in J_\eta, \tag{5}$$

$$\sum_{i=1}^{\eta} \lambda_{i,1}^2 \geq 1 + \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s} d\tau_i(s).$$

(ii) *If $0 < \kappa < \infty$, then S_λ is completely hyperexpansive if and only if one of the following two equivalent conditions holds*:

(ii-a) *there exist positive Borel measures $\{\tau_i\}_{i=1}^{\eta}$ on $[0,1]$ which satisfy* (5) *and the following requirements*:

$$\sum_{i=1}^{\eta} \lambda_{i,1}^2 = 1 + \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s} d\tau_i(s), \tag{6}$$

$$\lambda_{-(k-1)}^2 = 1 + \prod_{j=0}^{k-1} \lambda_{-j}^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s^{k+1}} d\tau_i(s), \quad k \in J_{\kappa-1}, \tag{7}$$

$$\lambda_{-(\kappa-1)}^2 \geq 1 + \prod_{j=0}^{\kappa-1} \lambda_{-j}^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s^{\kappa+1}} d\tau_i(s); \tag{8}$$

(ii-b) *there exist positive Borel measures $\{\tau_i\}_{i=1}^{\eta}$ and ν on $[0,1]$ which satisfy* (5) *and the equations below*

$$1 + \int_0^1 (1 + \cdots + s^{n-1}) d\nu(s) = \begin{cases} \prod_{j=\kappa-n}^{\kappa-1} \lambda_{-j}^2 & \text{if } n \in J_\kappa, \\ (\prod_{j=0}^{\kappa-1} \lambda_{-j}^2)(\sum_{i=1}^{\eta} \prod_{j=1}^{n-\kappa} \lambda_{i,j}^2) & \text{if } n \in \mathbb{N}_{\kappa+1}. \end{cases} \tag{9}$$

(iii) *If $\kappa = \infty$, then S_λ is completely hyperexpansive if and only if S_λ is an isometry.*

Moreover, if S_λ is completely hyperexpansive and $\{\tau_i\}_{i=1}^{\eta}$ are positive Borel measures on $[0,1]$ satisfying (5)*, then $\tau_i = \tau_{i,1}^{\lambda}$ for all $i \in J_\eta$. If $0 < \kappa < \infty$ and* (ii-b) *holds, then $\nu = \tau_{-\kappa}^{\lambda}$.*

Before closing this subsection, we give a statement which will be used in the subsequent sections of this paper:

$\|S_\lambda^n e_{\mathrm{root}}\|^2$ *coincides the right side of* (9) *and* $\|S_\lambda^n e_{i,1}\|^2$ *coincides the right side of* (5) *for* $i \in J_\eta$.

2.3. CH-completion problem

The CH-completion can be defined similarly as in the notion of subnormal completion on directed trees.

Definition 2.6. *Let $\mathscr{T} = (V, E)$ be a subtree of a directed tree $\hat{\mathscr{T}} = (\hat{V}, \hat{E})$ and let $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq (0, \infty)$. A weighted shift $S_{\hat{\boldsymbol{\lambda}}}$ on $\hat{\mathscr{T}}$ with weights $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_v\}_{v\in\hat{V}^\circ} \subseteq (0, \infty)$ is said to be a CH-completion of $\boldsymbol{\lambda}$ on $\hat{\mathscr{T}}$ if $S_{\hat{\boldsymbol{\lambda}}} \in \boldsymbol{B}(\ell^2(\hat{V}))$, $\boldsymbol{\lambda} \subseteq \hat{\boldsymbol{\lambda}}$, i.e., $\lambda_v = \hat{\lambda}_v$ for all $v \in V^\circ$, and $S_{\hat{\boldsymbol{\lambda}}}$ is completely hyperexpansive. If such a completion exists, we also say that $\boldsymbol{\lambda}$ admits a CH-completion on $\hat{\mathscr{T}}$.*

CH-completion problem. In terms of Definition 2.6, the *CH-completion problem* for $(\mathscr{T}, \hat{\mathscr{T}})$ is finding equivalent conditions for $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq (0, \infty)$ to have a CH-completion on $\hat{\mathscr{T}}$.

We take $\mathscr{T}_{\eta,\kappa}$ as the background tree and define a subtree $\mathscr{T}_{\eta,\kappa,p}$ of $\mathscr{T}_{\eta,\kappa}$ as below.

Definition 2.7 ([12, Definition 4.6]). *Given $\eta \in \bar{\mathbb{N}}_2$, $\kappa \in \bar{\mathbb{Z}}_+$ and $p \in \mathbb{N}$, we define the directed tree $\mathscr{T}_{\eta,\kappa,p} = (V_{\eta,\kappa,p}, E_{\eta,\kappa,p})$ by (see Figure 2)*

$$V_{\eta,\kappa,p} = \{-k\colon k \in J_\kappa\} \cup \{0\} \cup \{(i, j)\colon i \in J_\eta, j \in J_p\},$$
$$E_{\eta,\kappa,p} = E_\kappa \cup \{(0, (i, 1))\colon i \in J_\eta\} \cup \{((i, j), (i, j + 1))\colon i \in J_\eta, j \in J_{p-1}\}.$$

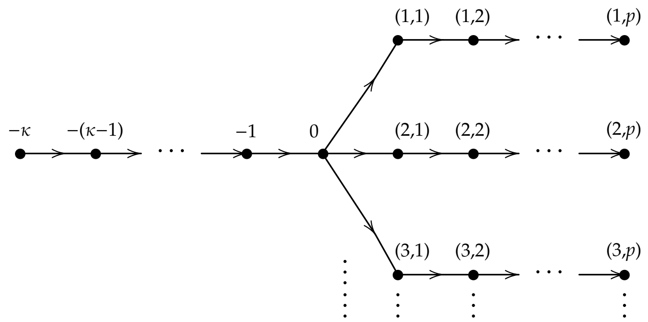

Figure 2: An illustration of the directed tree $\mathscr{T}_{\eta,\kappa,p}$ for $\kappa < \infty$.

In this paper we will discuss the completely hyperexpansive completion problem for $(\mathscr{T}_{\eta,\kappa,p}, \mathscr{T}_{\eta,\kappa})$ with $p \in \mathbb{N}$ as following.

***p*-Generation CH-completion problem.** Suppose $\eta \in \bar{\mathbb{N}}_2$, $\kappa \in \bar{\mathbb{Z}}_+$ and $p \in \mathbb{N}$. Let $(\mathscr{T}_{\eta,\kappa,p}, \mathscr{T}_{\eta,\kappa})$ be as above. The *p-generation CH-completion problem* on $\mathscr{T}_{\eta,\kappa}$ is CH-completion problem of the initial data $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ_{\eta,\kappa,p}}$.

Firstly we list below some properties which can be obtained easily by mimicking the ideas in [12].

- **1°** The sequence $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ_{\eta,0,1}}$ admits a CH-completion on $\mathscr{T}_{\eta,0}$ if and only if $1 \leq \sum_{i=1}^{\eta} \lambda_{i,1}^2 < \infty$ (see [17, Propositions 7.4.1 and 7.4.2]).
- **2°** $\sum_{i=1}^{\eta} \lambda_{i,1}^2 = 1$ if and only if $\|S_{\hat{\boldsymbol{\lambda}}} e_v\| = 1$ for all $v \in V^\circ \setminus \{0\}$, i.e., $S_{\hat{\boldsymbol{\lambda}}_{\to(0)}}$ is an isometry[3)] (see [17, Proposition 7.4.1]).
- **3°** For each $p \in \mathbb{N}$, the p-generation CH-completion problem on $\mathscr{T}_{\eta,\infty}$ is solved by Lemma 2.5(iii), i.e., $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V_{\eta,\infty,p}}$ admits a CH-completion on $\mathscr{T}_{\eta,\infty}$ if and only if $\sum_{i=1}^{\eta} \lambda_{i,1}^2 = 1$ and $\lambda_v = 1$, $v \in V_{\eta,\infty,p} \setminus \{(i, 1)\}_{i\in J_\eta}$.
- **4°** If $\mathscr{T} = (V, E)$ is a subtree of a directed tree $\hat{\mathscr{T}} = (\hat{V}, \hat{E})$, $S_{\hat{\boldsymbol{\lambda}}} \in \boldsymbol{B}(\ell^2(\hat{V}))$ is a CH-completion of $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq (0, \infty)$ on $\hat{\mathscr{T}}$, and $S_{\boldsymbol{\lambda}}$ is the weighted shift on $\mathscr{T}$, then $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ and $\|S_{\boldsymbol{\lambda}}\| \leq \|S_{\hat{\boldsymbol{\lambda}}}\|$ (see [12, Proposition 4.3]).

[3)]For $u \in V^\circ$, $S_{\hat{\lambda}_{\to(u)}}$ denotes the weighted shift on the directed tree $\mathscr{T}_{\mathsf{Des(u)}}$ with weights $\lambda_{\to(u)} := \{\lambda_v\}_{v\in\mathsf{Des}(u)\setminus\{u\}}$ ([17, Notation 3.1.5]).

5° Let $\mathscr{T} = (V, E)$ be a subtree of a directed tree $\hat{\mathscr{T}} = (\hat{V}, \hat{E})$ such that, for some $w \in V \setminus \mathsf{Root}(\mathscr{T})$, $\mathsf{Chi}_{\mathscr{T}}(w) \neq \mathsf{Chi}_{\hat{\mathscr{T}}}(w)$, $\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w)) = \mathsf{Chi}_{\hat{\mathscr{T}}}(\mathsf{par}(w))$, and $\mathsf{Des}_{\mathscr{T}}(v) = \mathsf{Des}_{\hat{\mathscr{T}}}(v)$ for all $v \in \mathsf{Chi}_{\mathscr{T}}(w) \cup (\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w)) \setminus \{w\})$. Assume that $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a completely hyperexpansive weighted shift on $\mathscr{T}$ with nonzero weights $\boldsymbol{\lambda} = \{\lambda_u\}_{v \in V^\circ}$. If $S_{\boldsymbol{\lambda}}$ satisfies one of the following conditions:
  i) $w \in V \setminus (\mathsf{Root}(\mathscr{T}) \cup \mathsf{Chi}(\mathsf{Root}(\mathscr{T})))$,
  ii) $S_{\boldsymbol{\lambda}}$ satisfies the strong consistency condition at $u = \mathsf{par}(w)$, i.e., (3) is valid for $u = \mathsf{par}(w)$,

then $\boldsymbol{\lambda}$ does not admit a CH-completion on $\hat{\mathscr{T}}$ (see [17, Proposition 7.5.1]).

6° If $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ is completely hyperexpansive where $\kappa \in \mathbb{N}$, then $\tau_{i,1}^{\boldsymbol{\lambda}}(\{0\}) = 0$ for every $i \in J_\eta$ and the supports of the measures $\tau_{-\kappa}^{\boldsymbol{\lambda}}$ and $\{\tau_{i,1}^{\boldsymbol{\lambda}}\}_{i \in J_\eta}$ satisfy the equation

$$\operatorname{supp} \tau_{-\kappa}^{\boldsymbol{\lambda}} = \begin{cases} (\cup_{i \in J_\eta} \operatorname{supp} \tau_{i,1}^{\boldsymbol{\lambda}})^- & \text{if } \tau_{-\kappa}^{\boldsymbol{\lambda}}(\{0\}) = 0, \\ \{0\} \cup (\cup_{i \in J_\eta} \operatorname{supp} \tau_{i,1}^{\boldsymbol{\lambda}})^- & \text{otherwise,} \end{cases}$$

(see [12, Theorem 3.5]).

Now we give a general solution of p-generation CH-completion problem for $0 < \kappa < \infty$. The following proposition is the counterpart of [12, Lemma 4.7] for CH-completion problem.

Proposition 2.8. *Let $\eta \in \bar{\mathbb{N}}_2$, $\kappa, p \in \mathbb{N}$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ_{\eta,\kappa,p}} \subseteq (0, \infty)$ be given. Then $\boldsymbol{\lambda}$ admits a CH-completion on $\mathscr{T}_{\eta,\kappa}$ if and only if there exist positive Borel measures $\{\tau_i\}_{i=1}^{\eta}$ on $[0, 1]$ satisfying the conditions* (6), (7), (8),

$$M_{\boldsymbol{\tau}} := \sup_{i \in J_\eta} \tau_i([0, 1]) < \infty, \tag{10}$$

and

$$1 + \int_0^1 (1 + \cdots + s^{n-1}) d\tau_i(s) = \prod_{j=2}^{n+1} \lambda_{i,j}^2, \quad n \in J_{p-1},\ i \in J_\eta. \tag{11}$$

Proof. Since the proof of "sufficiency" is similar to the proof of [12, Lemma 4.7], we will show only the "necessity". To do so, we consider a system $\{\tau_i\}_{i=1}^{\eta}$ of positive Borel measures on $[0, 1]$ which satisfy (6), (7), (8), (10) and (11). Since each τ_i is a positive measure for $i \in J_\eta$ and using (10), it holds that $0 \leq \int_0^1 s^n d\tau_i(s) < \infty$ ($n \in \mathbb{Z}_+$). For $p \geq 2$, we define the system $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_v\}_{v \in V^\circ_{\eta,\kappa}} \subseteq (0, \infty)$ by

$$\hat{\lambda}_v = \begin{cases} \lambda_v & \text{if } v \in V^\circ_{\eta,\kappa,p}, \\ \sqrt{\frac{1 + \int_0^1 (1 + \cdots + s^{j-2}) d\tau_i(s)}{1 + \int_0^1 (1 + \cdots + s^{j-3}) d\tau_i(s)}} & \text{if } v = (i, j) \in J_\eta \times \mathbb{N}_{p+1}, \end{cases} \tag{12}$$

and for $p = 1$, we define again the system $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_v\}_{v \in V^\circ_{\eta,\kappa}} \subseteq (0, \infty)$ by

$$\hat{\lambda}_v = \begin{cases} \lambda_v & \text{if } v \in V^\circ_{\eta,\kappa,1}, \\ \sqrt{1 + \tau_i([0, 1])} & \text{if } v = (i, 2),\ i \in J_\eta, \\ \sqrt{\frac{1 + \int_0^1 (1 + \cdots + s^{j-2}) d\tau_i(s)}{1 + \int_0^1 (1 + \cdots + s^{j-3}) d\tau_i(s)}} & \text{if } v = (i, j) \in J_\eta \times \mathbb{N}_3. \end{cases} \tag{13}$$

Combining (11), (12) and (13), it follows that the condition (5) with $\hat{\boldsymbol{\lambda}}$ in place of $\boldsymbol{\lambda}$ holds. It is easy to check that $\{\tau_i\}_{i=1}^{\eta}$ and $\hat{\boldsymbol{\lambda}}$ satisfy the conditions (5)-(8). Finally, we will claim that $S_{\hat{\boldsymbol{\lambda}}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$. The condition (8) implies $\tau_i(\{0\}) = 0$ for all $i \in J_\eta$. Since $\frac{1}{s^2} \geq \frac{1}{s}$ on $[0, 1]$,

$$\int_0^1 \frac{1}{s^2} d\tau_i(s) \geq \int_0^1 \frac{1}{s} d\tau_i(s), \quad i \in J_\eta. \tag{14}$$

If $\kappa = 1$, then by applying the inequality (8) we get

$$\begin{aligned}\lambda_0^2 &\geq 1 + \lambda_0^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s^2} d\tau_i(s) \\ &\overset{(14)}{\geq} 1 + \lambda_0^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s} d\tau_i(s) \overset{(6)}{=} 1 + \lambda_0^2 (\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1),\end{aligned} \tag{15}$$

which implies that

$$\sum_{i=1}^{\eta} \lambda_{i,1}^2 < \infty. \tag{16}$$

Suppose now that $\kappa \in \mathbb{N}_2$. Using (7) with $k = 1$ in place of (8) and arguing as in (15), we also obtain (16). It follows from (11), (12), (13) and [3, Proposition 4], $\{\hat{\lambda}_{i,j}\}_{j=2}^{\infty}$ is monotonically decreasing for all $i \in J_\eta$. As a consequence, we have

$$\sup_{i \in J_\eta} \sup_{j \geq 2} \hat{\lambda}_{i,j}^2 = \sup_{i \in J_\eta} \hat{\lambda}_{i,2}^2 \overset{(11)}{=} \sup_{i \in J_\eta} (1 + \tau_i([0,1])) \overset{(10)}{<} \infty.$$

Combining this with (16) and $\kappa < \infty$, we see that $\sup_{u \in V_{\eta,\kappa}} \sum_{v \in \mathsf{Chi}(u)} \hat{\lambda}_v^2 < \infty$. Hence, by (1), $S_{\hat{\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$. Since $\boldsymbol{\lambda} \subseteq \hat{\boldsymbol{\lambda}}$, we infer from Lemma 2.5(ii-a) that $S_{\hat{\lambda}}$ is a CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,\kappa}$. This completes the proof. □

3. The 1-generation CH-completion problem

3.1. The general case of trunk $\kappa \geq 1$

Suppose $\eta \in \bar{\mathbb{N}}_2$, $\kappa \in \bar{\mathbb{Z}}_+$, $r \in \mathbb{N}$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ} \subseteq (0,\infty)$ are given. Assume that $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$. Recall that $S_{\boldsymbol{\lambda}}$ is *r-generation flat* if $\lambda_{i,j} = \lambda_{1,j}$ for all $i \in J_\eta$ and $j \geq r$ ([12, Definition 6.1]). The following theorem is the counterpart of [12, Theorem 6.2] for CH-completion problem.

Theorem 3.1. *Suppose $\eta \in \bar{\mathbb{N}}_2$, $\kappa \in \mathbb{N}$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa,1}^\circ} \subseteq (0,\infty)$ are given. Then $\boldsymbol{\lambda}$ admits a CH-completion on $\mathscr{T}_{\eta,\kappa}$ if and only if $\boldsymbol{\lambda}$ admits a 2-generation flat CH-completion on $\mathscr{T}_{\eta,\kappa}$. Moreover, if $S_{\tilde{\lambda}}$ is a CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,\kappa}$, then there is a 2-generation flat CH-completion $S_{\hat{\lambda}}$ on $\mathscr{T}_{\eta,\kappa}$ such that $\tau_{1,1}^{\hat{\lambda}} = \frac{1}{c}\sum_{k=1}^{\eta} \lambda_{k,1}^2 \tau_{k,1}^{\tilde{\lambda}} = \tau_{i,1}^{\hat{\lambda}}$ for all $i \in J_\eta$, where $c := \sum_{i=1}^{\eta} \lambda_{i,1}^2$.*

Proof. The idea of this proof comes from [12, Theorem 6.2]. Since the necessity is trivial, we will sketch only the proof of the "moreover" part. Assume that $S_{\tilde{\lambda}}$ is a CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,\kappa}$. Set $\tau_{-\kappa} = \tau_{-\kappa}^{\tilde{\lambda}}$, $\tau_i = \tau_{i,1}^{\tilde{\lambda}}$ for all $i \in J_\eta$ and $\rho = \frac{1}{c}\sum_{i=1}^{\eta} \lambda_{i,1}^2 \tau_i$. By [17, Lemma 7.3.1], ρ is a finite positive Borel measure on $[0,1]$. And $0 < c < \infty$ by $\|S_{\tilde{\lambda}} e_0\|^2 = c$. Since $\tau_{-\kappa}$ (resp., τ_i) is the representing measure of $\|S_{\tilde{\lambda}}^n e_{-\kappa}\|^2$(resp., $\|S_{\tilde{\lambda}}^n e_{i,1}\|^2$), by Lemmas 2.3 and 2.5 with $\tilde{\lambda}$ in place of $\boldsymbol{\lambda}$, we see that the integral parts "$\int_0^\infty s^n d\mu_{-\kappa}(s)$ [resp., $\int_0^\infty s^{n-\kappa-1} d\mu_i(s)$]" in (6.2) of [12, Theorem 6.2] is changed to "$1 + \int_0^1 (1 + \cdots + s^{n-1}) d\tau_{-\kappa}(s)$ [resp., $1 + \int_0^1 (1 + \cdots + s^{n-\kappa-2}) d\tau_i(s)$]". Thus, by using (5) and (9), it holds that

$$1 + \int_0^1 (1 + \cdots + s^{n-1}) d\tau_{-\kappa}(s) = c \prod_{j=0}^{\kappa-1} \lambda_{-j}^2 \left(1 + \int_0^1 (1 + \cdots + s^{n-\kappa-2}) d\rho(s)\right),\ n \in \mathbb{N}_{\kappa+2}.$$

Consider the system $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_v\}_{v \in V_{\eta,\kappa}^\circ}$ of positive real numbers which is defined as the equation where τ_i is replaced by ρ in (13); note that this equation is well-defined clearly. As in the proof of [12, Theorem 6.2], we can prove routinely the remaining part of the proof by using Lemma 2.5(ii-b) and we conclude that $S_{\hat{\lambda}}$ is a CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ such that $\tau_{i,1}^{\hat{\lambda}} = \rho$ for all $i \in J_\eta$ and $\tau_{-\kappa}^{\hat{\lambda}} = \tau_{-\kappa}^{\tilde{\lambda}}$. That $S_{\hat{\lambda}}$ is 2-generation flat follows from the definition of $\hat{\boldsymbol{\lambda}}$. This completes the proof. □

We now consider the special case of trunk $\kappa = 1$ and $p = 1$. In this case, we can describe specifically the data getting from the CH-completion. The following is the counterpart of [12, Theorem 5.1] for CH-completion problem.

Theorem 3.2. *Suppose $\eta \in \bar{\mathbb{N}}_2$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ_{\eta,1,1}} \subseteq (0,\infty)$ are given. Then $\boldsymbol{\lambda}$ admits a CH-completion on $\mathscr{T}_{\eta,1}$ if and only if*

$$1 \le \sum_{i=1}^{\eta} \lambda_{i,1}^2 \quad \textit{and} \quad \lambda_0^{-2} + \sum_{i=1}^{\eta} \lambda_{i,1}^2 \le 2. \tag{17}$$

In this case, there exists a CH-completion $S_{\hat{\lambda}}$ of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$ such that

$$\tau_{i,1}^{\hat{\lambda}} = \left(\frac{a-1}{a}\right)\delta_1, \quad i \in J_\eta, \tag{18}$$

where $a = \sum_{i=1}^{\eta} \lambda_{i,1}^2$. In this case, $\hat{\lambda}_{i,j} = \sqrt{\frac{a+(a-1)(j-3)}{a+(a-1)(j-2)}}$ for all $i \in J_\eta$ and $j \in \mathbb{N}_2$.

Proof. Let $S_{\hat{\lambda}}$ be a CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$. Setting $\tau_i = \tau_{i,1}^{\hat{\lambda}}$ for $i \in J_\eta$. The first inequality of (17) is a direct consequence by [17, Proposition 7.4.1(ii)]. The second inequality of (17) follows from (15) in the proof of Proposition 2.8.

Conversely, assume that (17) holds. Set $a = \sum_{i=1}^{\eta} \lambda_{i,1}^2$ and $\tau_i = (\frac{a-1}{a})\delta_1$ for all $i \in J_\eta$. By (17), we see that $0 \le \frac{a-1}{a} < 1$, which implies that the measure τ_i, $i \in J_\eta$ is a finite positive Borel measure and (10) holds. Also,

$$1 + \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s} d\tau_i(s) = 1 + \sum_{i=1}^{\eta} \lambda_{i,1}^2 \left(\frac{a-1}{a}\right) = \sum_{i=1}^{\eta} \lambda_{i,1}^2,$$

and using the second inequality of (17)

$$1 + \lambda_0^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s^2} d\tau_i(s) = 1 + \lambda_0^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2 \left(\frac{a-1}{a}\right) = 1 + \lambda_0^2 (\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1) \le \lambda_0^2.$$

As above, we conclude that (6) and (8) are valid as well. Of course, (7) and (11) vanish when $\kappa = p = 1$. By Proposition 2.8, $\boldsymbol{\lambda}$ admits a CH-completion on $\mathscr{T}_{\eta,1}$. Now we consider a system $\hat{\boldsymbol{\lambda}} = \{\lambda_v\}_{v \in V^\circ_{\eta,1}} \subseteq (0,\infty)$ defined as (13) with $\kappa = 1$. Then $S_{\hat{\lambda}}$ is a CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$, which satisfies the conditions (18) and $\hat{\lambda}_{i,j} = \sqrt{\frac{a+(a-1)(j-3)}{a+(a-1)(j-2)}}$ for $(i,j) \in J_\eta \times \mathbb{N}_2$. This completes the proof. □

3.2. *The explicit case of trunk $\kappa = 1$*

Recall that if $\eta \in \bar{\mathbb{N}}_2$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ_{\eta,1,1}} \subseteq (0,\infty)$, then it follows from that $\boldsymbol{\lambda}$ admits a CH-completion on $\mathscr{T}_{\eta,1}$ if and only if (17) holds. Observe that the condition (17) for the CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$ can be divided by the following four cases:

C1. $1 = \sum_{i=1}^{\eta} \lambda_{i,1}^2$ and $\lambda_0^{-2} + \sum_{i=1}^{\eta} \lambda_{i,1}^2 = 2$,

C2. $1 = \sum_{i=1}^{\eta} \lambda_{i,1}^2$ and $\lambda_0^{-2} + \sum_{i=1}^{\eta} \lambda_{i,1}^2 < 2$,

C3. $1 < \sum_{i=1}^{\eta} \lambda_{i,1}^2$ and $\lambda_0^{-2} + \sum_{i=1}^{\eta} \lambda_{i,1}^2 = 2$, C4. $1<\sum_{i=1}^{\eta}\lambda_{i,1}^2$ and $\lambda_0^{-2}+\sum_{i=1}^{\eta}\lambda_{i,1}^2<2$.

In this subsection we discuss how the above conditions affect the CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$. Before doing this we recall the extremality of the complete hyperexpansivity of $S_{\boldsymbol{\lambda}}$. For $\eta\in\bar{\mathbb{N}}_2$ and $\kappa\in\mathbb{N}$, a completely hyperexpansive weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{\eta,\kappa}$ with $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V_{\eta,\kappa}^{\circ}}$ is said to be *extremal* if $\|S_{\boldsymbol{\lambda}}e_{\mathrm{root}}\|=\min\|S_{\tilde{\boldsymbol{\lambda}}}e_{\mathrm{root}}\|$, where the minimum is taken over all completely hyperexpansive weighted shifts $S_{\tilde{\boldsymbol{\lambda}}}$ on $\mathscr{T}_{\eta,\kappa}$ with $\tilde{\boldsymbol{\lambda}}=\{\tilde{\lambda}_v\}_{v\in V_{\eta,\kappa}^{\circ}}$ such that $\lambda_v=\tilde{\lambda}_v$ for all $v\neq-\kappa+1$. It follows from [17, Remark 7.3.3] that if $S_{\boldsymbol{\lambda}}$ is a completely hyperexpansive weighted shift on $\mathscr{T}_{\eta,\kappa}$ with $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V_{\eta,\kappa}^{\circ}}$, then $S_{\boldsymbol{\lambda}}$ is extremal if and only if it satisfies the condition

$$\lambda_{-(\kappa-1)}^2=1+\prod_{j=0}^{\kappa-1}\lambda_{-j}^2\sum_{i=1}^{\eta}\lambda_{i,1}^2\int_0^1\frac{1}{s^{\kappa+1}}d\tau_i(s);\tag{19}$$

note that this equality (19) comes from the inequality in (8).

Definition 3.3. *Suppose $\eta\in\bar{\mathbb{N}}_2$, $\kappa,p\in\mathbb{N}$ and $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V_{\eta,\kappa,p}^{\circ}}\subseteq(0,\infty)$ are given. A CH-completion $S_{\hat{\boldsymbol{\lambda}}}$ of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ which satisfies* (19) *is called the extremal CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,\kappa}$.*

It follows from Property **2**° in Subsection 2.3 that

if $S_{\hat{\boldsymbol{\lambda}}}$ is a CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$, then $\sum_{i=1}^{\eta}\lambda_{i,1}^2=1$ if and only if $S_{\hat{\boldsymbol{\lambda}}_{\to(0)}}$ is an isometry,

which is related to the conditions C1 and C2. In particular, it follows from [17, Proposition 7.4.1] that

if $S_{\hat{\boldsymbol{\lambda}}}$ is an extremal CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$, then $\sum_{i=1}^{\eta}\lambda_{i,1}^2=1$ if and only if $S_{\hat{\boldsymbol{\lambda}}}$ is an isometry,

which is related to C2. The condition C3 is related to the extremal CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$ as following.

Theorem 3.4. *Let $\eta\in\bar{\mathbb{N}}_2$ and $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V_{\eta,1,1}^{\circ}}\subseteq(0,\infty)$. If one of the disjunction conditions C3, and C4 holds, then $\boldsymbol{\lambda}$ admits an extremal CH-completion $S_{\hat{\boldsymbol{\lambda}}}$ on $\mathscr{T}_{\eta,1}$ such that*

$$\tau_{-1}^{\hat{\boldsymbol{\lambda}}}=(\lambda_0^2-1)\delta_t,\tag{20}$$

$$\tau_0^{\hat{\boldsymbol{\lambda}}}=\left(\sum_{i=1}^{\eta}\lambda_{i,1}^2-1\right)\delta_t,\tag{21}$$

$$\tau_{i,1}^{\hat{\boldsymbol{\lambda}}}=\alpha_i\delta_t\ \text{ for some } \alpha_i\in\mathbb{R}_+\ \text{ and all } i\in J_\eta,\tag{22}$$

where t and α_i's satisfy the equalities

$$t=\frac{\lambda_0^2(\sum_{i=1}^{\eta}\lambda_{i,1}^2-1)}{\lambda_0^2-1}\quad\text{and}\quad\sum_{i=1}^{\eta}\lambda_{i,1}^2\alpha_i=\frac{\lambda_0^2(\sum_{i=1}^{\eta}\lambda_{i,1}^2-1)^2}{\lambda_0^2-1}.\tag{23}$$

Proof. We consider a system $\{\tau_i\}_{i=1}^{\eta}$ such that (22), (23) and $\sup_{i\in J_\eta}\alpha_i<\infty$ hold. Since the condition C3 (resp., C4) implies $t=1$ (resp., $t\in(0,1)$), each τ_i is positive Borel measure on $[0,1]$ for $i\in J_\eta$. The simplest solution of the second equality in (23) is $\alpha_1=\frac{\lambda_0^2(\sum_{i=1}^{\eta}\lambda_{i,1}^2-1)^2}{\lambda_{1,1}^2(\lambda_0^2-1)}$ and $\alpha_i=0$ otherwise. Define the system $\hat{\boldsymbol{\lambda}}=\{\hat{\lambda}_v\}_{v\in V_{\eta,1}^{\circ}}\subseteq(0,\infty)$ by (13). It is easy to check that $\{\tau_i\}_{i=1}^{\eta}$ and $\hat{\boldsymbol{\lambda}}$ satisfy (5), (6), (10) and (19). Therefore $S_{\hat{\boldsymbol{\lambda}}}$ is an extremal CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$ where $\tau_{i,1}^{\hat{\boldsymbol{\lambda}}}=\tau_i$ for all $i\in J_\eta$. Using (23) and (2) with $u=0$ (resp., $u=-1$), it follows that (21) (resp., (20)) holds. This completes the proof. □

Proposition 3.5. *Let $\eta \in \bar{\mathbb{N}}_2$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ_{\eta,1,1}} \subseteq (0,\infty)$. If $S_{\hat{\lambda}}$ is CH-completion of $\boldsymbol{\lambda}$ which satisfies the condition C3, then it is an extremal CH-completion such that*

$$\begin{aligned} \tau^{\hat{\lambda}}_{-1} &= (\lambda_0^2 - 1)\delta_1, && (24)\\ \tau^{\hat{\lambda}}_{0} &= (1 - \lambda_0^{-2})\delta_1, && (25)\\ \tau^{\hat{\lambda}}_{i,1} &= \alpha_i\delta_1 \ \textit{for all } i \in J_\eta, \end{aligned}$$

where α_i's satisfy the equality

$$\sum_{i=1}^{\eta} \lambda_{i,1}^2 = 1 + \sum_{i=1}^{\eta} \lambda_{i,1}^2 \alpha_i. \tag{26}$$

Proof. By Theorem 3.2, $\boldsymbol{\lambda}$ admits a CH-completion on $\mathscr{T}_{\eta,1}$. Let $S_{\hat{\lambda}}$ be any CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$. We show that $S_{\hat{\lambda}}$ is extremal. Set $\tau_{-1} = \tau^{\hat{\lambda}}_{-1}$, $\tau_0 = \tau^{\hat{\lambda}}_0$ and $\tau_i = \tau^{\hat{\lambda}}_{i,1}$ for $i \in J_\eta$. By (6), (8) and C3, we have

$$\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s} d\tau_i(s) \geq \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s^2} d\tau_i(s). \tag{27}$$

By (14), the converse inequality of (27) holds, and so we get

$$\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s^2} d\tau_i(s) = \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s} d\tau_i(s). \tag{28}$$

As a consequence, using (6), we obtain

$$1 + \lambda_0^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s^2} d\tau_i(s) = 1 + \lambda_0^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^1 \frac{1}{s} d\tau_i(s) = 1 + \lambda_0^2 (\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1) = \lambda_0^2,$$

which means that (19) holds. Thus $S_{\hat{\lambda}}$ is extremal. Since $\lambda_{i,1}^2 > 0$ for all $i \in J_\eta$, by (14) and (28), we get $\int_0^1 (\frac{1}{s^2} - \frac{1}{s}) d\tau_i(s) = 0$ for all $i \in J_\eta$, which implies that $\tau_i = \alpha_i\delta_1$ for some α_i for each $i \in J_\eta$. Applying (6) with $\tau_i = \alpha_i\delta_1$, we obtain the equation (22). Applying the expression (2) with $u = 0$ and using (6), we get

$$\tau_0(\sigma) = \sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_\sigma \frac{1}{s} d\alpha_i\delta_1(s), \quad \sigma \in \mathscr{B}([0,1]).$$

Combining this equation with C3 and (26), we obtain (25). Applying (2) with $u = -1$ again, we have that

$$\tau_{-1}(\sigma) = \lambda_0^2 \int_\sigma \frac{1}{s} d\tau_0(s) + (\lambda_0^2 - 1 - \lambda_0^2 \int_0^1 \frac{1}{s} d\tau_0(s))\delta_0(\sigma), \quad \sigma \in \mathscr{B}([0,1]). \tag{29}$$

Substituting (25) into (29), we obtain (24). This completes the proof. □

In follows from Theorem 3.2 that the conditions C1-C4 provide the representing measures for CH-completion $S_{\hat{\lambda}}$. We describe the expressions of their measures in detail below.

Suppose $\eta \in \bar{\mathbb{N}}_2$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ_{\eta,1,1}} \subseteq (0,\infty)$ with the conditions (17) are given. If $S_{\hat{\lambda}}$ is a CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,1}$, then a finite sequence $(\lambda_0^2, \sum_{i=1}^{\eta} \lambda_{i,1}^2)$ admits a CH-completion. And, the moment sequence $\{\|S^n_{\hat{\lambda}} e_{-1}\|\}_{n=0}^\infty$ is completely alternating with associated measure $\tau^{\hat{\lambda}}_{-1}$ such that $\{1, \lambda_0^2, \lambda_0^2(\sum_{i=1}^{\eta} \lambda_{i,1}^2)\} \subseteq \{\|S^n_{\hat{\lambda}} e_{-1}\|\}_{n=0}^\infty$. By [15], the measure $\tau^{\hat{\lambda}}_{-1}$ has at most two atoms. Thus we may consider seven cases as following:

(i) $a\delta_0$,
(ii) $a\delta_t$ with $t \in (0,1)$,
(iii) $a\delta_1$,
(iv) $a\delta_0 + b\delta_1$,
(v) $a\delta_0 + b\delta_t$ with $t \in (0,1)$,
(vi) $a\delta_s + b\delta_1$ with $s \in (0,1)$,
(vii) $a\delta_s + b\delta_t$ with $0 < s < t < 1$.

By Property **6**° in Subsection 2.3, $\mathrm{supp}\tau_{i,1}^{\hat{\lambda}} \subseteq \mathrm{supp}\tau_{-1}^{\hat{\lambda}}$ and $\tau_{i,1}^{\hat{\lambda}}(\{0\}) = 0$ for $i \in J_\eta$. Using Lemma 2.2 with $u = 0$, we can see that $\mathrm{supp}\tau_0^{\hat{\lambda}} \subseteq \mathrm{supp}\tau_{-1}^{\hat{\lambda}}$ and $\tau_0^{\hat{\lambda}}(\{0\}) = 0$. We can classify the measure $\tau_{-1}^{\hat{\lambda}}$ all cases using (6) and (8) with $k = 1$. We describe the summary for its representing measures below without some cases detail proof.

Description of representing measures. Suppose $\eta \in \bar{\mathbb{N}}_2$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ_{\eta,1,1}} \subseteq (0,\infty)$ with (17) are given. Then the moment measures of each cases can be represented as following:

(i) If $\boldsymbol{\lambda}$ satisfies C1, then $\boldsymbol{\lambda}$ admits an isometry CH-completion (and thus extremal) $S_{\hat{\lambda}}$ on $\mathscr{T}_{\eta,1}$. In this case, its representing measures $\tau_{-1}^{\hat{\lambda}}$, $\tau_0^{\hat{\lambda}}$ and $\tau_{i,1}^{\hat{\lambda}}$ are trivial for all $i \in J_\eta$, i.e., $\tau_{-1}^{\hat{\lambda}}(\sigma) = \tau_0^{\hat{\lambda}}(\sigma) = \tau_{i,1}^{\hat{\lambda}}(\sigma) = 0$ for all $\sigma \in \mathscr{B}([0,1])$.

(ii) If $\boldsymbol{\lambda}$ satisfies C2, then $\boldsymbol{\lambda}$ admits a CH-completion $S_{\hat{\lambda}}$ on $\mathscr{T}_{\eta,1}$. In this case, its representing measures $\tau_0^{\hat{\lambda}}$ and $\tau_{i,1}^{\hat{\lambda}}$ are trivial for all $i \in J_\eta$, and $\tau_{-1}^{\hat{\lambda}} = (\lambda_0^2 - 1)\delta_0$.

(iii) If $\boldsymbol{\lambda}$ satisfies C3, then $\boldsymbol{\lambda}$ admits an extremal CH-completion $S_{\hat{\lambda}}$ on $\mathscr{T}_{\eta,1}$. In this case, its representing measures $\tau_{-1}^{\hat{\lambda}}$, $\tau_0^{\hat{\lambda}}$, and $\tau_{i,1}^{\hat{\lambda}}$ are given by $\tau_{-1}^{\hat{\lambda}} = (\lambda_0^2 - 1)\delta_1$, $\tau_0^{\hat{\lambda}} = (1 - \frac{1}{\lambda_0^2})\delta_1$, and $\tau_{i,1}^{\hat{\lambda}} = \alpha_i\delta_1$ for all $i \in J_\eta$, resp., where α_i's satisfy the equality $\sum_{i=1}^{\eta} \lambda_{i,1}^2 = 1 + \sum_{i=1}^{\eta} \lambda_{i,1}^2\alpha_i$.

(iv) If $\boldsymbol{\lambda}$ satisfies C4, then $\boldsymbol{\lambda}$ admits a CH-completion $S_{\hat{\lambda}}$ on $\mathscr{T}_{\eta,1}$. In this case, its representing measures are described as three cases below:

(iv-a) If $S_{\hat{\lambda}}$ is not extremal and each $\tau_{i,1}^{\hat{\lambda}}$ has one atom for all $i \in J_\eta$, then its representing measures $\tau_{-1}^{\hat{\lambda}}$, $\tau_0^{\hat{\lambda}}$, and $\tau_{i,1}^{\hat{\lambda}}$ are given by

$$\tau_{-1}^{\hat{\lambda}} = \frac{\lambda_0^2}{t}\left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1\right)\delta_t + \left(\lambda_0^2 - 1 - \frac{\lambda_0^2}{t}\left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1\right)\right)\delta_0,$$

$$\tau_0^{\hat{\lambda}} = \left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1\right)\delta_t, \quad \tau_{i,1}^{\hat{\lambda}} = \alpha_i\delta_t \text{ for all } i \in J_\eta,$$

where t and α_i's satisfy the equalities

$$t \in \left(\frac{\lambda_0^2(\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1)}{\lambda_0^2 - 1}, 1\right] \quad \text{and} \quad \sum_{i=1}^{\eta} \lambda_{i,1}^2\alpha_i = \left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1\right)t.$$

(iv-b) If $S_{\hat{\lambda}}$ is an extremal and each $\tau_{i,1}^{\hat{\lambda}}$ has one atom for all $i \in J_\eta$, then its representing measures $\tau_{-1}^{\hat{\lambda}}$, $\tau_0^{\hat{\lambda}}$, and $\tau_{i,1}^{\hat{\lambda}}$ are given by

$$\tau_{-1}^{\hat{\lambda}} = (\lambda_0^2 - 1)\delta_t, \quad \tau_0^{\hat{\lambda}} = \left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1\right)\delta_t,$$
$$\tau_{i,1}^{\hat{\lambda}} = \alpha_i\delta_t \text{ for all } i \in J_\eta,$$

where t and α_i's satisfy the equalities

$$t = \frac{\lambda_0^2(\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1)}{\lambda_0^2 - 1} \in (0,1) \quad \text{and} \quad \sum_{i=1}^{\eta} \lambda_{i,1}^2\alpha_i = \frac{\lambda_0^2(\sum_{i=1}^{\eta} \lambda_{i,1}^2 - 1)^2}{\lambda_0^2 - 1}.$$

(iv-c) If $S_{\hat{\lambda}}$ is an extremal and each $\tau_{i,1}^{\hat{\lambda}}$ has two atom for all $i \in J_\eta$, then its representing measures $\tau_{-1}^{\hat{\lambda}}$, $\tau_0^{\hat{\lambda}}$, and $\tau_{i,1}^{\hat{\lambda}}$ are given by

$$
\begin{aligned}
\tau_{-1}^{\hat{\lambda}} &= \frac{(\lambda_0^2-1)s - \lambda_0^2(\sum_{i=1}^{\eta}\lambda_{i,1}^2 - 1)}{s-t}\delta_t + \frac{\lambda_0^2(\sum_{i=1}^{\eta}\lambda_{i,1}^2-1) - (\lambda_0^2-1)t}{s-t}\delta_s \\
&= \lambda_0^2\left(\sum_{i=1}^{\eta}\frac{\lambda_{i,1}^2\alpha_i}{t^2}\right)\delta_t + \lambda_0^2\left(\sum_{i=1}^{\eta}\frac{\lambda_{i,1}^2\beta_i}{s^2}\right)\delta_s, \\
\tau_0^{\hat{\lambda}} &= \frac{(\lambda_0^2-1)st - \lambda_0^2(\sum_{i=1}^{\eta}\lambda_{i,1}^2-1)t}{\lambda_0^2(s-t)}\delta_t + \frac{\lambda_0^2(\sum_{i=1}^{\eta}\lambda_{i,1}^2-1)s - (\lambda_0^2-1)ts}{\lambda_0^2(s-t)}\delta_s \\
&= \left(\sum_{i=1}^{\eta}\frac{\lambda_{i,1}^2\alpha_i}{t}\right)\delta_t + \left(\sum_{i=1}^{\eta}\frac{\lambda_{i,1}^2\beta_i}{s}\right)\delta_s, \\
\tau_{i,1}^{\hat{\lambda}} &= \alpha_i\delta_t + \beta_i\delta_s \;\text{ for all } i \in J_\eta,
\end{aligned}
$$

where s, t, and α_i's satisfy the equalities

$$
0 < t < \frac{\lambda_0^2(\sum_{i=1}^{\eta}\lambda_{i,1}^2 - 1)}{\lambda_0^2 - 1} < s \le 1,
$$

$$
\sum_{i=1}^{\eta}\lambda_{i,1}^2 = 1 + \sum_{i=1}^{\eta}\lambda_{i,1}^2\left(\frac{\alpha_i}{t} + \frac{\beta_i}{s}\right) \quad \text{and} \quad \lambda_0^2 = 1 + \lambda_0^2\sum_{i=1}^{\eta}\lambda_{i,1}^2\left(\frac{\alpha_i}{t^2} + \frac{\beta_i}{s^2}\right).
$$

4. CH-completion and classical weighted shifts

4.1. *p-generation CH-completion problem for $p \ge 2$*

In [12, Theorem 7.1], Exner-Jung-Stochel-Yun obtained equivalent conditions for the p-generation subnormal completion of $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa,p}^{\circ}}$ for $p \ge 2$. The relationship for the counterpart of the p-generation subnormal completion for the CH-completion of λ provides only the sufficiency, not the equivalent condition; see Theorem 4.1 and Example 4.2.

Theorem 4.1. *Let $\eta \in \mathbb{N}_2$, $p \in \mathbb{N}_2$, $\kappa \in \mathbb{N}$ and $\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa,p}^{\circ}} \subseteq (0,\infty)$ be given. If there exists a system $\{\pi_{i,n}\}_{n=0}^{\kappa+p} \subseteq (0,\infty)$, $i \in J_\eta$, such that*

$\{\pi_{i,n}\}_{n=0}^{\kappa+p}$ is a truncated completely alternating sequence for all $i \in J_\eta$,

$$
\lambda_{-(\kappa-1)}^2 \ge 1 + \prod_{j=0}^{\kappa-1}\lambda_{-j}^2\sum_{i=1}^{\eta}\lambda_{i,1}^2(\pi_{i,1} - \pi_{i,0}), \tag{30}
$$

$$
\lambda_{-\kappa+(k+1)}^2 = 1 + \prod_{j=0}^{\kappa-(k+1)}\lambda_{-j}^2\sum_{i=1}^{\eta}\lambda_{i,1}^2(\pi_{i,k+1} - \pi_{i,k}), \;\; k \in J_{\kappa-1}, \tag{31}
$$

$$
\sum_{i=1}^{\eta}\lambda_{i,1}^2 = 1 + \sum_{i=1}^{\eta}\lambda_{i,1}^2(\pi_{i,\kappa+1} - \pi_{i,\kappa}), \tag{32}
$$

$$
\pi_{i,\kappa+1} = 1, \quad i \in J_\eta, \tag{33}
$$

$$
\pi_{i,\kappa+k} = \prod_{j=2}^{k}\lambda_{i,j}^2, \quad i \in J_\eta,\; k \in \{2,\cdots,p\}, \tag{34}
$$

then λ admits a CH-completion on $\mathscr{T}_{\eta,\kappa}$.

Example 4.2. *Consider $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ_{2,2,5}} \subseteq (0,\infty)$ such that*

$$\lambda_{-1} = \sqrt{x}, \lambda_0 = \sqrt{y}, \lambda_{2,1} = \sqrt{z}, \lambda_{1,j} = 1 \ (j \in J_5);\ \lambda_{2,j} = \sqrt{\frac{j}{j-1}} \ (2 \le j \le 5),$$

where x, y, z are any positive real numbers. Taking the trivial measure τ_1 and $\tau_2 = \delta_1$, we can see that $\{\tau_1, \tau_2\}$ satisfy (6)-(8), (10) *and* (11) *in Proposition 2.8, and so $\boldsymbol{\lambda}$ admits a CH-completion on $\mathscr{T}_{\eta,\kappa}$. We claim that there does not exist a truncated completely alternating sequence $\{x_{i,n}\}_{n=0}^{7} \subseteq (0,\infty)$ for $i = 1,2$ such that* (30)-(34) *hold. For a contradiction, we suppose that there exists a completely alternating sequence $\gamma_i = \{\gamma_{i,n}\}_{n=0}^{\infty} \subseteq (0,\infty)$ for $i = 1,2$ such that* (30)-(34) *hold. Let $\alpha_i = \{\alpha_{i,n}\}_{n=0}^{\infty}$ be the corresponding sequence to γ_i for $i = 1,2$ (see Proposition 2.1). Since*

$$\alpha_{i,4} = \lambda_{i,2}, \alpha_{i,5} = \lambda_{i,3}, \alpha_{i,6} = \lambda_{i,4}, \alpha_{i,7} = \lambda_{i,5}, \quad i = 1,2,$$

we get

$$\gamma_1 = \{\gamma_{1,0}, \gamma_{1,1}, \gamma_{1,2}, 1, 1, 1, 1, 1, ...\},$$
$$\gamma_2 = \{\gamma_{2,0}, \gamma_{2,1}, \gamma_{2,2}, 1, 2, 3, 4, 5, ...\}.$$

Recall that the sequence $\beta := \{1, 2, 3, ...\}$ induces the Dirichet shift W which is completely hyperexpansive. However, it follows from [17, (7.2.1)] *W is not backward extendable, which contradicts to the fact that γ_2 is a completely alternating sequence.*

We now prove Theorem 4.1.

Proof. Suppose that there exist completely alternating sequences $\widehat{\pi}_i = \{\widehat{\pi}_{i,n}\}_{n=0}^{\infty}$ such that $\{\pi_{i,n}\}_{n=0}^{\kappa+p} \subseteq \widehat{\pi}_i$ for each $i \in J_\eta$. By [4, Proposition 4.6.12](see also [3, Remark 1]), there exist positive Borel measures $\{\rho_i\}_{i \in J_\eta}$ on $[0,1]$ such that

$$\widehat{\pi}_{i,n} = \widehat{\pi}_{i,0} + \int_0^1 (1 + \cdots + s^{n-1}) d\rho_i(s), \quad n \in \mathbb{N},\ i \in J_\eta.$$

Define the Borel measures τ_i on $[0,1]$ by

$$\tau_i(\sigma) = \int_\sigma s^{\kappa+1} d\rho_i(s), \quad \sigma \in \mathscr{B}([0,1]),\ i \in J_\eta.$$

Clearly, each τ_i is a finite positive Borel measure on $[0,1]$ with $\tau_i(\{0\}) = 0$. Combining this with $\eta \in \mathbb{N}_2$, (10) holds. Since $\widehat{\pi}_{i,n} = \pi_{i,n}$ for $0 \le n \le \kappa + p$, $i \in J_\eta$, we get

$$\pi_{i,k} - \pi_{i,k-1} = \int_0^1 s^{k-1} d\rho_i(s) = \int_0^1 \frac{1}{s^{\kappa-k+2}} d\tau_i(s), \quad k \in J_{\kappa+1},\ i \in J_\eta \tag{35}$$

Using (35), we deduce from (30), (31) and (32) that (6)-(8) hold. Using (33) and (34), we obtain that for $n \in J_{p-1}$, $i \in J_\eta$,

$$\begin{aligned}
1 + \int_0^1 (1 + \cdots + s^{n-1}) d\tau_i(s) &= \widehat{\pi}_{i,\kappa+1} + \int_0^1 (1 + \cdots + s^{n-1}) d\tau_i(s) \\
&= \widehat{\pi}_{i,0} + \int_0^1 (1 + \cdots + s^{\kappa}) d\rho_i(s) + \int_0^1 (1 + \cdots + s^{n-1}) s^{\kappa+1} d\rho_i(s) \\
&= \widehat{\pi}_{i,0} + \int_0^1 (1 + \cdots + s^{\kappa+n}) d\rho_i(s) \\
&= \widehat{\pi}_{i,\kappa+n+1} = \prod_{j=2}^{n+1} \lambda_{i,j}^2,
\end{aligned}$$

which means that (11) holds. According to Proposition 2.8, $\boldsymbol{\lambda}$ admits a CH-completion on $\mathscr{T}_{\eta,\kappa}$. So the proof is complete. □

In particular, if $\eta = \infty$ in Theorem 4.1, the condition $\sup_{i\in J_\eta}(\pi_{i,\kappa+2} - \pi_{i,\kappa+1}) < \infty$ is necessary to hold Theorem 4.1.

We now consider the counterpart of [12, Corollary 8.2] for r -generation flat CH-completion. We begin a lemma whose idea comes from [12, Proposition 8.1].

Lemma 4.3. *Let $\eta \in \mathbb{N}_2$, $\kappa \in \mathbb{N}$ and $r \in \mathbb{N}_2$ be given. If $S_\lambda \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ is a r-generation flat completely hyperexpansive weighted shift on $\mathscr{T}_{\eta,\kappa}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}} \subseteq (0,\infty)$, then S_λ is 2-generation flat.*

Proof. Set $\tau_i = \tau^\lambda_{i,1}$, $\gamma^{(i)}_0 = 1$, and $\gamma^{(i)}_n := \|S^n_\lambda e_{i,1}\|^2$ for $i \in J_\eta$ as in Lemma 2.5. In fact the idea of proof is similar with the proof of [12, Proposition 8.1]. For reader's convenience we provide the outline of the proof here. We replace μ_i and $\int_0^\infty s^n d\mu_i(s)$ in the proof of [12, Proposition 8.1] by τ_i and $1 + \int_0^1 (1 + \cdots + s^{n-1})d\tau_i(s)$, respectively. Then, $\{\frac{\gamma^{(i)}_{n+r-2}}{\gamma^{(i)}_{r-2}}\}^\infty_{n=0}$ is a completely alternating sequence with its representing measure $\frac{s^{r-2}}{\gamma^{(i)}_{r-2}}d\tau_i(s)$, for $i \in J_\eta$. By the uniqueness of the representing measure and $\frac{\gamma^{(i)}_{n+r-2}}{\gamma^{(i)}_{r-2}} = \prod^{n+r-1}_{j=r} \lambda^2_{1,j}(n \in \mathbb{N},\ i \in J_\eta)$, we have

$$\frac{s^{r-2}}{\gamma^{(i)}_{r-2}}d\tau_i(s) = \frac{s^{r-2}}{\gamma^{(1)}_{r-2}}d\tau_1(s).$$

Since the sequence $\{\|S^n_\lambda e_0\|^2\}$ is also completely alternating, by Lemma 2.2, we get that $\tau_i(\{0\}) = 0$ for all $i \in J_\eta$. So, we deduce that

$$\tau_i(\sigma) = \frac{\gamma^{(i)}_{r-2}}{\gamma^{(1)}_{r-2}}\tau_1(\sigma), \quad \sigma \in \mathscr{B}([0,1]),\ i \in J_\eta.$$

Put $a_i = \frac{\gamma^{(i)}_{r-2}}{\gamma^{(1)}_{r-2}}$, for $i \in J_\eta$. Then $\tau_i = a_i\tau_1$, for $i \in J_\eta$. Combining this with (4), we have

$$a_i = \frac{1 + a_i \int_0^1 (1 + \cdots + s^{r-3})d\tau_1(s)}{1 + \int_0^1 (1 + \cdots + s^{r-3})d\tau_1(s)}, \quad i \in J_\eta,\ r \in \mathbb{N}_3,$$

which yields $a_i = 1$ for all $i \in J_\eta$. (Clearly, $a_i = 1$ if $r = 2$.) Thus $\tau_i = \tau_1$ for all $i \in J_\eta$. This and (5) imply that $\lambda^2_{i,j} = \lambda^2_{1,j}$ for $j \in \mathbb{N}_2$. Hence the proof is complete. □

The following proposition comes immediately from Lemma 4.3.

Proposition 4.4. *Let $\eta \in \mathbb{N}_2$, $\kappa \in \mathbb{N}$ and $p \in \mathbb{N}_2$ be given. If $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ_{\eta,\kappa,p}} \subseteq (0,\infty)$ admits a r-generation flat CH-completion $S_{\hat\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ for some $r \in \mathbb{N}_2$, then $\lambda_{i,j} = \lambda_{1,j}$ for all $i \in J_\eta$ and $j \in \{2,\cdots,p\}$.*

4.2. CH-completion and branching shifts

In [10] Exner-Jung-Lee introduced a branching (classical) weighted shift on ℓ^2 to discuss relationships between the classical weighted shifts and the weighted shifts S_λ acting on the directed tree $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$.

Definition 4.5 ([10, Definition 1.2]). *Let $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ be the directed tree and let S_λ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$. In what follows we assume $\kappa \in \mathbb{Z}_+$ and $\eta \in \mathbb{N}_2$. We consider the i-th branching shifts $W^{(i)}$ which are sliced from the weighted shift S_λ on $\mathscr{T}_{\eta,\kappa}$ as following: let $W^{(i)}$ be the classical weighted shift with the weight sequence $\boldsymbol{\alpha}^{(i)}$ of positive real numbers given by*

$$\boldsymbol{\alpha}^{(i)}:\ \lambda_{i,2},\ \lambda_{i,3},\ \lambda_{i,4},\ \lambda_{i,5}, \ldots, \quad i \in J_\eta, \tag{36}$$

under the order of branches as in Figure 3. As well, let $W^{(0)}$ be the classical weighted shift with the weight sequence $\boldsymbol{\alpha}^{(0)} = \{\alpha_i^{(0)}\}_{i=-\kappa+1}^{\infty}$ of positive real numbers given by

$$\alpha_i^{(0)} = \lambda_i, \ i \in (-J_{\kappa-1}) \cup \{0\}, \text{ provided } \kappa \in \mathbb{N}, \tag{37}$$

$$\alpha_1^{(0)} := \left(\sum_{i \in J_\eta} \lambda_{i,1}^2 \right)^{1/2}, \ \alpha_{j+1}^{(0)} := \left(\frac{\sum\limits_{i \in J_\eta} \prod\limits_{k \in J_{j+1}} \lambda_{i,k}^2}{\sum\limits_{i \in J_\eta} \prod\limits_{k \in J_j} \lambda_{i,k}^2} \right)^{1/2}, \quad j \in \mathbb{N}. \tag{38}$$

We say that $W^{(0)}$ is the basic (sliced) branching shift of S_λ. For our convenience, we say that "$W^{(j)}$ is the j-th (sliced) branching shift of S_λ for $j \in J_\eta \cup \{0\}$".

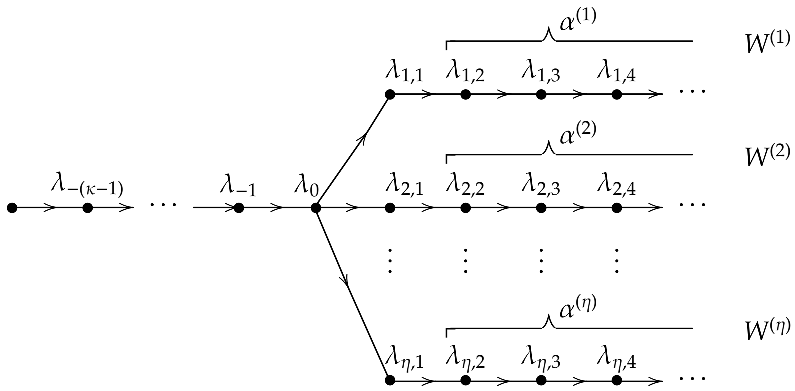

Figure 3: The illustration of $W^{(j)}$ of S_λ for $j \in J_\eta$.

The following statement is well-known (see [10, p.807] or [17, Corollay 6.2.2 (ii-b)]).

S_λ is subnormal if and only if every i -th branching shift is subnormal for all $i \in J_\eta \cup \{0\}$.

This statement for subnormality can be preserved in the case of completely hyperexpansivity of S_λ as following.

Lemma 4.6. *Let S_λ be a weighted shift on $\mathscr{T}_{\eta,\kappa}$. Then S_λ is completely hyperexpansive if and only if every j-th branching shift $W^{(j)}$ is completely hyperexpansive for $j \in J_\eta \cup \{0\}$.*

Proof. Since each moment sequence of $W^{(i)}$ for $i \in \eta$ (resp., $W^{(0)}$) is exactly the sequence $\{\|S_\lambda^n e_{i,1}\|^2\}_{n=0}^{\infty}$ (resp., $\{\|S_\lambda^n e_{\mathsf{root}}\|^2\}_{n=0}^{\infty}$), this lemma is clear by Lemma 2.5. □

The following is the counterpart of [12, Proposition 7.6] for CH-completion problem. This is provides a relationship among the CH-completion problem for $(\mathscr{T}_{\eta,\kappa,1}, \mathscr{T}_{\eta,\kappa})$, the classical CH-completion problem for $\kappa + 1$ weights, and j-th branching shift $W^{(j)}$.

Proposition 4.7. *Let $\eta \in \mathbb{N}_2$, $\kappa \in \mathbb{N}$, $p \in \mathbb{N}_2$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ_{\eta,\kappa,p}} \subseteq (0,\infty)$ be given. Suppose $\boldsymbol{\lambda}$ admits a CH-completion $S_{\hat{\lambda}}$ on $T_{\eta,\kappa}$. Let*

$$\boldsymbol{\alpha} := \left(\lambda_{-\kappa+1}, \ldots, \lambda_0, \sqrt{\sum_{i=1}^{\eta} \lambda_{i,1}^2}, \sqrt{\frac{\sum_{i=1}^{\eta} \lambda_{i,1}^2 \lambda_{i,2}^2}{\sum_{i=1}^{\eta} \lambda_{i,1}^2}}, \ldots, \sqrt{\frac{\sum_{i=1}^{\eta} \prod_{j=1}^{p} \lambda_{i,j}^2}{\sum_{i=1}^{\eta} \prod_{j=1}^{p-1} \lambda_{i,j}^2}} \right) \tag{39}$$

and

$$\boldsymbol{\alpha}^{(i)} := (\lambda_{i,2}, \lambda_{i,3}, \ldots, \lambda_{i,p}), \quad i \in J_\eta.$$

Then the following assertions hold:

(i) $\boldsymbol{\alpha}$ [*resp.*, $\boldsymbol{\alpha}^{(i)}$] *admits a completion to a completely hyperexpansive unilateral weighted shift* $W_{\hat{\boldsymbol{\alpha}}}$ [*resp.*, $W_{\hat{\boldsymbol{\alpha}}^{(i)}}$],

(ii) *the j-th branching shift* $W^{(j)}$ *of* $S_{\hat{\boldsymbol{\lambda}}}$ *is completely hyperexpansive completion of* $\boldsymbol{\alpha}^{(j)}$ *for all* $j \in J_\eta \cup \{0\}$, *where* $\boldsymbol{\alpha}^{(0)} := \boldsymbol{\alpha}$.

Proof. (i) Define the sequence $\hat{\boldsymbol{\gamma}} = \{\hat{\gamma}_n\}_{n=0}^\infty$ by $\hat{\gamma}_n = \|S_{\hat{\boldsymbol{\lambda}}}^n e_{-\kappa}\|^2$ for $n \in \mathbb{Z}_+$. Since $S_{\hat{\boldsymbol{\lambda}}}$ is completely hyperexpansive, $\hat{\boldsymbol{\gamma}}$ is a completely alternating sequence with the representing measure $\nu := \tau_{-\kappa}^{\hat{\boldsymbol{\lambda}}}$. It follows from (9) that $\boldsymbol{\gamma} \subseteq \hat{\boldsymbol{\gamma}}$, where

$$\begin{aligned}\boldsymbol{\gamma} := \Bigg(&1, \lambda_{-\kappa+1}^2, \ldots, \prod_{k=0}^{\kappa-1}\lambda_{-k}^2, \left(\prod_{k=0}^{\kappa-1}\lambda_{-k}^2\right)\sum_{i=1}^{\eta}\lambda_{i,1}^2, \left(\prod_{k=0}^{\kappa-1}\lambda_{-k}^2\right)\sum_{i=1}^{\eta}\prod_{j=1}^{2}\lambda_{i,j}^2,\\ &\ldots, \left(\prod_{k=0}^{\kappa-1}\lambda_{-k}^2\right)\sum_{i=1}^{\eta}\prod_{j=1}^{p}\lambda_{i,j}^2\Bigg). \end{aligned} \tag{40}$$

Let $\hat{\boldsymbol{\alpha}} = \{\hat{\alpha}_n\}_{n=0}^\infty$ be the sequence of weights induced by the completely alternating sequence $\hat{\boldsymbol{\gamma}}$ as in Proposition 2.1. Then, according to the assumption of this proposition, $W_{\hat{\boldsymbol{\alpha}}}$ is a CH-completion of $\boldsymbol{\alpha}$ with the associated measure ν. A similar argument can be made to obtain results for the sequence $\{\alpha^{(i)}\}_{i\in J_\eta}$.

(ii) Comparing (37), (38) and (39), we can see that the basic branching shift $W^{(0)}$ of $S_{\hat{\boldsymbol{\lambda}}}$ is completely hyperexpansive completion of $\boldsymbol{\alpha}$ in the above proof. Since the i-branching shift $W^{(i)}$ of $S_{\hat{\boldsymbol{\lambda}}}$ has the weights as in (36) with $\hat{\lambda}_{i,j}$ in place of $\lambda_{i,j}$, ($j \in \mathbb{N}_{p+1}$) and its corresponding moment sequence coincides $\{\|S_{\hat{\boldsymbol{\lambda}}}^n e_{i,1}\|^2\}_{n=0}^\infty$, it follows that the i-branching shift $W^{(i)}$ of $S_{\hat{\boldsymbol{\lambda}}}$ is completely hyperexpansive completion of $\boldsymbol{\alpha}^{(i)}$ for $i \in J_\eta$. □

The following is the counterpart of [12, Theorem 8.3] for CH-completion problem.

Theorem 4.8. *Let* $\eta \in \mathbb{N}_2$, $\kappa \in \mathbb{N}$, $p \in \mathbb{N}_2$ *and* $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V_{\eta,\kappa,p}^\circ} \subseteq (0,\infty)$ *be given. Suppose that* $\lambda_{i,j} = \lambda_{1,j}$ *for all* $i \in J_\eta$ *and* $j \in \{2, ..., p\}$. *Then the following conditions are equivalent*:

(i) $\boldsymbol{\lambda}$ *admits a CH-completion on* $\mathscr{T}_{\eta,\kappa}$,

(ii) $\boldsymbol{\lambda}$ *admits a* 2*-generation flat CH-completion on* $\mathscr{T}_{\eta,\kappa}$,

(iii) *the sequence* $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{\kappa+p-1})$ *given by* (39) *admits a completion to a completely hyperexpansive unilateral weighted shift* $W_{\hat{\boldsymbol{\alpha}}}$,

(iv) *the sequence* $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{\kappa+p})$ *given by* (40) *has an extension to a completely alternating sequence* $\hat{\boldsymbol{\gamma}}$.

Moreover, if (i) *holds, then the basic branching shift* $W^{(0)}$ *of* $S_{\hat{\boldsymbol{\lambda}}}$ *is completely hyperexpansive completion of* $\boldsymbol{\alpha}$, *where* $S_{\hat{\boldsymbol{\lambda}}}$ *is any CH-completion.*

Proof. (i)⇒(iii) This is done by Proposition 4.7 (and is true without assuming that $\lambda_{i,j} = \lambda_{1,j}$ for all $i \in J_\eta$ and $j \in \{2, \ldots, p\}$). The "moreover" part also proved by Proposition 4.7.

(iii)⇔(iv) In view of the above discussion, this equivalence follows directly from Proposition 2.1.

(ii)⇒(i) This implication is obvious.

(iv)⇒(ii) We sketch the proof of this implication (cf. [12, Theorem 8.3]). Let $\hat{\boldsymbol{\gamma}} = \{\hat{\gamma}_n\}_{n=0}^\infty$ be an extension of $\boldsymbol{\gamma}$ to a completely alternating sequence with its representing measure ω. Consider a positive Borel measure τ given by

$$\tau(\sigma) = \frac{1}{\gamma_{\kappa+1}}\int_\sigma s^{\kappa+1}d\omega(s), \quad \sigma \in \mathscr{B}([0,1]).$$

And we define the system $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_v\}_{v\in V_{\eta,\kappa}^\circ} \subseteq (0,\infty)$ as in (12) with τ in place of τ_i. Then (5) holds with $\hat{\boldsymbol{\lambda}}$ in place of $\boldsymbol{\lambda}$ and with $\tau_i = \tau$ for each $i \in J_\eta$. (Here we use the condition $\lambda_{i,j} = \lambda_{1,j}$ for all $i \in J_\eta$ to see (5).) Also, (9) holds with $\hat{\boldsymbol{\lambda}}$ in place of $\boldsymbol{\lambda}$ and with ω in place of ν. It follows from the similar proof for the boundedness of $S_{\hat{\boldsymbol{\lambda}}}$ in the proof of Proposition 2.8 that $S_{\hat{\boldsymbol{\lambda}}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$. Using Lemma 2.5(ii-b) with $\tau_i = \tau$ for every $i \in J_\eta$ and $\nu = \omega$, we deduce that $S_{\hat{\boldsymbol{\lambda}}}$ is completely hyperexpansive. Since $\boldsymbol{\lambda} \subseteq \hat{\boldsymbol{\lambda}}$ and (12), $S_{\hat{\boldsymbol{\lambda}}}$ is a 2-generation flat CH-completion of $\boldsymbol{\lambda}$ on $\mathscr{T}_{\eta,\kappa}$. □

The following corollary comes from Theorem 4.8 immediately.

Corollary 4.9. *Suppose* $\eta \in \mathbb{N}_2$, $\kappa \in \mathbb{N}$ *and* $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ_{\eta,\kappa,1}} \subseteq (0, \infty)$ *are given. Then the following conditions are equivalent*:

(i) $\boldsymbol{\lambda}$ *admits a CH-completion on* $\mathscr{T}_{\eta,\kappa}$,

(ii) *the finite real sequence* $\boldsymbol{\alpha} = \left(\lambda_{-\kappa+1}, \ldots, \lambda_0, \sqrt{\sum_{i=1}^{\eta} \lambda_{i,1}^2}\right)$ *admits a completion to a completely hyperexpansive unilateral weighted shift* $W_{\hat{\boldsymbol{\alpha}}}$,

(iii) *the finite real sequence*

$$\boldsymbol{\gamma} = \left(1, \lambda_{-\kappa+1}^2, \lambda_{-\kappa+1}^2 \lambda_{-\kappa+2}^2, \ldots, \lambda_{-\kappa+1}^2 \cdots \lambda_0^2, \lambda_{-\kappa+1}^2 \cdots \lambda_0^2 \sum_{i=1}^{\eta} \lambda_{i,1}^2\right)$$

has an extension to a completely alternating sequence.

Moreover, if (i) *holds, then the basic branching shift* $W^{(0)}$ *of* $S_{\hat{\lambda}}$ *is completely hyperexpansive completion of* $\boldsymbol{\alpha}$, *where* $S_{\hat{\lambda}}$ *is any CH-completion.*

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