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Regular integral transformations on time scales and generalized statistical convergence

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Abstract. In this work, using regular integral transformations on time scales, we generalize the concept of statistical convergence. This enables us not only to unify discrete and continuous cases known in the literature but also to derive new convergence methods with choices of appropriate transformations and time scales. This is a continuation of our earlier work and includes many new methods. We obtain sufficient conditions for regularity of kernel functions on time scales and also we prove a characterization theorem for the generalized statistical convergence. At the end of the paper we display some applications and special cases of our results.

1. Introduction

In 1990, Hilger [34] introduced the calculus of measure chains in order to unify continuous and discrete analysis. Later this idea of Hilger was called the *time scale calculus* (see [12, 13] for details). In recent years, time scale calculus has taken great attentions from different fields such as population dynamics, economics, physics, control theory and so on (see, i.e., [1, 5, 7–9, 35, 40]). The reason why time scale calculus is so popular is that it allows the opportunity to work on both continuous and discrete cases at the same time. Although the theory of time scales has been studied in so many different areas, the idea of using time scales in summability theory was first introduced by Turan and Duman (see [46]). In that work, the authors presented the notion of *statistical convergence on time scales* and investigated its fundamental properties. Some recent developments in this direction may be found in the papers [47–49].

The aim of the present paper is to extend the concept of A-statistical convergence known for number sequences to functions defined on time scales, which enables us not only to unify discrete and continuous cases known in the literature but also to derive new convergence methods with choices of appropriate transformations and time scales. To achieve this, we first obtain sufficient conditions for the regularity of integral transformations on time scales. Later, we prove a characterization theorem for our general convergence method on time scales.

We now recall some definitions and notations from the summability theory and also give some fundamental concepts from the theory of time scales.

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As usual, the concept of statistical convergence for number sequences can be generalized by a nonnegative regular summability matrix A. Then, this generalized limit is called *A-statistical convergence*. This idea was first mentioned by Buck [15] in 1953 and has been further studied by Freedman and Sember [29, 43], Connor [17, 18], Miller [37] and Demirci [22, 23, 25]. Recall that a summability matrix $A = (a_{nk})$ is called *regular* if it preserves the usual convergence, that is, for a given convergent sequence (u_k),

$$\lim_{n\to\infty} (Au)_n = L \text{ whenever } \lim_{k\to\infty} u_k = L,$$

where $((Au)_n)$ is the *transformed sequence* of (u_k) and defined by

$$(Au)_n = \sum_{k=1}^{\infty} a_{nk}u_k.$$

We say that $A = (a_{nk})$ is *nonnegative* if all entries of A are nonnegative. A sequence (u_k) is said to be *statistically convergent* to a number L if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\#\{k \le n : |u_k - L| \ge \varepsilon\}}{n} = 0, \tag{1}$$

where $\#\{B\}$ denotes the *cardinality* of a subset $B \subset \mathbb{N}$. It is well-known that every convergent sequence is statistically convergent to the same value, however the converse is not always true. Also, we say that a sequence (u_k) is said to be *A-statistically convergent* to a number L if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{k\in\mathbb{N}\,:\,|u_k-L|\ge\varepsilon} a_{nk} = 0. \tag{2}$$

It is easy to check that if we take $A = C_1 = (c_{nk})$, the *Cesàro matrix* given by

$$c_{nk} = \begin{cases} \frac{1}{n}, & \text{if } k \le n \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

then (2) is reduced to (1).

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers (see [13] for details). Throughout the paper we are interested in time scales such that

$$\inf \mathbb{T} = t_0 \ (t_0 > 0) \quad \text{and} \quad \sup \mathbb{T} = \infty. \tag{4}$$

Then, the forward jump operator σ on $\mathbb{T}$ is defined by

$$\sigma : \mathbb{T} \to \mathbb{T}, \sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

Similarly, the backward jump operator ρ on $\mathbb{T}$ is given by

$$\rho : \mathbb{T} \to \mathbb{T}, \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

Another frequently used function on time scales is the graininess function μ given by

$$\mu : \mathbb{T} \to [0, \infty), \mu(t) = \sigma(t) - t.$$

By the notation $[a, b]_{\mathbb{T}}$ we denote an interval entirely in $\mathbb{T}$ such that $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$. Obviously, the intervals $[a, b)_{\mathbb{T}}$ and $(a, b]_{\mathbb{T}}$ can be defined similarly.

As in our previous works (see [46–49]), in this paper, we will use the Lebesgue Δ-measure which is denoted by μ_Δ and introduced by Guseinov in [32]. To obtain this measure, Guseinov needed to define a set function on all intervals having the form $[a, b)_{\mathbb{T}}$. Then the set function became countably additive measure and the Carathéodory extension of the set function gave us the Lebesgue Δ-measure (for more detailed information, see [3, 32]). In these papers, Guseinov also calculated the Lebesgue Δ-measure of all form of the intervals on $\mathbb{T}$ as follows:

Let $a, b \in \mathbb{T}$ and $a \le b$. Then

- $\mu_{\Delta}\left([a,b)_{\mathbb{T}}\right) = b - a,$
- $\mu_{\Delta}\left((a,b)_{\mathbb{T}}\right) = b - \sigma(a),$
- $\mu_{\Delta}\left((a,b]_{\mathbb{T}}\right) = \sigma(b) - \sigma(a),$
- $\mu_{\Delta}\left([a,b]_{\mathbb{T}}\right) = \sigma(b) - a.$

Now, we will recall the main density definitions and convergence methods on time scales.
Let Ω be a Δ-measurable subset of $\mathbb{T}$ and consider the following set:

$$\Omega(x) = \{y \in [t_0, x]_{\mathbb{T}} : y \in \Omega\}. \tag{5}$$

Then, the density of Ω is given by

$$\delta_{\mathbb{T}}(\Omega) := \lim_{x\to\infty} \frac{\mu_{\Delta}(\Omega(x))}{\mu_{\Delta}\left([t_0, x]_{\mathbb{T}}\right)}$$

provided that the above limit exists. Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. We say that f is statistically convergent on $\mathbb{T}$ to a number L if, for every $\varepsilon > 0$,

$$\delta_{\mathbb{T}}\left(\left\{x \in \mathbb{T} : \left|f(x) - L\right| \geqslant \varepsilon\right\}\right) = 0 \tag{6}$$

holds (see [46] for details). We denote this statistical limit on a time scale $\mathbb{T}$ by

$$st_{\mathbb{T}} - \lim f = L.$$

Then, it is not hard to see that (6) is equivalent to following limit:

$$\lim_{x\to\infty} \frac{\mu_{\Delta}\left(\left\{y \in [t_0, x]_{\mathbb{T}} : \left|f(y) - L\right| \geqslant \varepsilon\right\}\right)}{\mu_{\Delta}\left([t_0, x]_{\mathbb{T}}\right)} = 0, \tag{7}$$

which generalizes the (discrete) statistical convergence in (1).

Assume that $\theta = \{k_r\}_{r=0}^{\infty}$ is an increasing sequence in $\mathbb{T}$ for which

$$\sigma(k_r) - \sigma(k_{r-1}) \to \infty$$

as $r \to \infty$ with $k_0 = 0$. In this case, θ is called a *lacunary sequence* in $\mathbb{T}$. We also say that a Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ is *lacunary statistically convergent* to a number L if, for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{\mu_{\Delta}\left(\left\{y \in (k_{r-1}, k_r]_{\mathbb{T}} : \left|f(y) - L\right| \geq \varepsilon\right\}\right)}{\mu_{\Delta}\left((k_{r-1}, k_r]_{\mathbb{T}}\right)} = 0, \tag{8}$$

which is denoted by $st_{\mathbb{T}-\theta} - \lim f = L$ (see [48]).

2. Regular Integral Transformations on Time Scales

It is well-known that the regularity of a summability matrix $A = (a_{nk})$ is characterized by *Silverman-Toeplitz Theorem* as follows (see, for instance, [14]).

Theorem 2.1 (Silverman-Toeplitz Theorem). *A matrix $A = (a_{nk})$ is regular if and only if it satisfies all of the following properties:*

- $\lim_{n\to\infty} a_{nk} = 0$ *for every* $k \in \mathbb{N}$,

- $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$,
- $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

It is natural to replace a discrete variable by a continuous variable. Then, we replace the sequence (u_k) by a function $f(y)$, and the auxiliary matrix (a_{nk}) by $k(x, y)$, a *kernel function* of two continuous variables. In this case, we use integral transformations instead of sequence transformations. We should note that regularity conditions for the continuous case is more complicated than the discrete case. For example, consider the following special integral transformation:

$$\nu(x) := \int_a^x k(x, y) f(y) dy, \tag{9}$$

where $f(x)$ is bounded and integrable, $0 \le x \le x_1$, and $k(x, y)$ is defined, $a < x$, $a \le y \le x$, and integrable in y for each x. Then, Silverman proved that (see Theorem I in [45]), sufficient conditions that $k(x, y)$ correspond to a regular transformation, that is, $\lim_{x\to\infty} \nu(x) = L$ whenever $\lim_{y\to\infty} f(y) = L$, are the following:

- $\lim_{x\to\infty} k(x, y) = 0$ uniformly in y, $a \le y \le q$,
- $\lim_{x\to\infty} \int_a^x k(x, y) dy = 1$,
- $\int_a^x \left| k(x, y) \right| dy < M$, $a < x$,

where q is an arbitrary constant and M is a positive constant. Note that the last three conditions stated above are only sufficient conditions for regularity. So, we need additional assumptions so that they are also necessary conditions. Indeed, according to Theorem II in [45], if we also assume that $k(x, y)$ is continuous in y uniformly with respect to x, $x > h > a$ and that the zeros of $k(x, y)$ for each x consist of a set of segments and a set of points of measure zero, then $k(x, y)$ correspond a regular transformation "if and only if" the last three conditions hold.

Let us now consider the following integral transformation, which is more general than (9):

$$\alpha(x) := \int_0^\infty k(x, y) f(y) dy, \tag{10}$$

where $k(x, y)$ is a measurable function on $\mathbb{R}^2$. Since the kernel function $k(x, y)$ may behave in a more complex way for finite x and y than a function of discrete variables, Hardy proved the following sufficient conditions for regularity of $k(x, y)$, which are little less symmetrical with respect to the above conditions (see [33]). We should note that this integral transformation in (10) was also examined by Connor and Swardson in [19].

Theorem 2.2 (see [33]). *A kernel function $k(x, y)$ in (10) is regular for bounded functions (that is, $\lim_{x\to\infty} \alpha(x) = L$ whenever $\lim_{y\to\infty} f(y) = L$ for every bounded function f) if it is sufficient that the following conditions hold:*

- $\lim_{x\to\infty} \int_0^Y \left| k(x, y) \right| dy = 0$ *for every finite* Y,
- $\lim_{x\to\infty} \int_0^\infty k(x, y) dy = 1$,

- $\sup_{x\in[0,\infty)} \int_0^\infty \left|k(x,y)\right| dy < \infty.$

The main purpose of this section is to study the integral transformation given by (2) on a time scale and then obtain a time scale version of Theorem 2.2, which gives sufficient conditions for regularity of a kernel function on time scales.

Now let $\mathbb{T}$ be a time scale satisfying (4) and let $k : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ be a $\Delta\times\Delta$-measurable function on the product time scale $\mathbb{T}\times\mathbb{T}$. Throughout this paper we also assume that, for every $y \in \mathbb{T}$, $k(\cdot, y)$ is a Lebesgue Δ-integrable function on $\mathbb{T}$ (see [16, 32] for details about the Lebesgue Δ-integration on time scales). As in the continuous case (10), the Δ-integral transformation (or, k-transformation) of a given measurable and Δ-integrable function f on $\mathbb{T}$ will be as follows:

$$\alpha_{\mathbb{T}}(x) := \int_{\mathbb{T}} k(x,y) f(y) \Delta y. \tag{11}$$

As in the discrete and continuous cases, we say that a kernel function $k(x, y)$ is regular for bounded functions on a time scale $\mathbb{T}$ provided that

$$\lim_{x\to\infty} \alpha_{\mathbb{T}}(x) = L \ \text{ whenever } \ \lim_{y\to\infty} f(y) = L$$

Then, we get the next result.

Theorem 2.3. *A kernel function $k(x, y)$ is regular for bounded functions on a time scale $\mathbb{T}$ if the following conditions hold:*

(i) $\lim_{x\to\infty} \int_{[t_0,Y]_{\mathbb{T}}} \left|k(x,y)\right| \Delta y = 0$ *for every finite* $Y \in \mathbb{T}$,

(ii) $\lim_{x\to\infty} \int_{\mathbb{T}} k(x,y)\Delta y = 1,$

(iii) $\sup_{x\in\mathbb{T}} \int_{\mathbb{T}} \left|k(x,y)\right| \Delta y < \infty.$

Proof. Assume that k satisfies the conditions (i), (ii) and (iii). By a direct computation, we get from (11) that

$$|\alpha_{\mathbb{T}}(x) - L| \le |L| \left| \int_{\mathbb{T}} k(x,y)\Delta y - 1 \right| + \int_{\mathbb{T}} \left|k(x,y)\right| \left|f(y) - L\right| \Delta y. \tag{12}$$

From (ii), the first term on the right hand side of (12) goes to zero as $x \to \infty$. Now assuming $\lim_{y\to\infty} f(y) = L$, we may write that, for every $\varepsilon > 0$, there exists a $Y \in \mathbb{T}$ such that $\left|f(y) - L\right| < \varepsilon$ for every $y > Y$ with $y \in \mathbb{T}$. Then, we observe that

$$\begin{aligned}
\int_{\mathbb{T}} \left|k(x,y)\right| \left|f(y) - L\right| \Delta y &= \int_{[t_0,Y]_{\mathbb{T}}} \left|k(x,y)\right| \left|f(y) - L\right| \Delta y \\
&\quad + \int_{\mathbb{T}\setminus[t_0,Y]_{\mathbb{T}}} \left|k(x,y)\right| \left|f(y) - L\right| \Delta y \\
&\le \int_{[t_0,Y]_{\mathbb{T}}} \left|k(x,y)\right| \left|f(y) - L\right| \Delta y + \varepsilon \int_{\mathbb{T}} \left|k(x,y)\right| \Delta s.
\end{aligned}$$

From the condition (*iii*) and the boundedness of f on $\mathbb{T}$, there exist positive constants B and M such that

$$\int_{\mathbb{T}} \left|k(x,y)\right| \left|f(y)-L\right| \Delta y \leq B \int_{[t_0,Y]_{\mathbb{T}}} \left|k(x,y)\right| \Delta y + M\varepsilon.$$

Now, using (*i*) we immediately see that the second term on the right hand side of (12) also goes to zero as $x \to \infty$. Therefore, we get

$$\lim_{x\to\infty} \alpha_{\mathbb{T}}(x) = L,$$

which completes the proof. □

We should remark that unfortunately we are unable to give necessary and sufficient conditions for regularity on time scales due to the complexity of Δ-integral. Hence, at this stage, if and only if conditions for regularity on time scales remain as an open problem. However, conditions (*i*), (*ii*) and (*iii*) in Theorem 2.3 are enough for us to define a general convergence method on time scales and to get its characterization.

3. Generalized Statistical Convergence and its Characterization

By Ψ we denote the family of all nonnegative kernel functions k satisfying the conditions (*i*), (*ii*) and (*iii*) in Theorem 2.3. Then, it follows from Theorem 2.3 that if $k \in \Psi$, then k is regular for bounded functions on $\mathbb{T}$.

Now we are ready to give our definitions.

Definition 3.1. *Let $k \in \Psi$ and Ω be a Δ-measurable subset of $\mathbb{T}$. Then, the k-density of Ω on $\mathbb{T}$, which is denoted by $\delta_{k-\mathbb{T}}(\Omega)$, is defined by*

$$\delta_{k-\mathbb{T}}(\Omega) := \lim_{x\to\infty} \int_{y\in\Omega} k(x,y)\,\Delta y$$

provided that the above limit exists.

We should note that the special case of Definition 3.1 was examined by Connor and Swardson in [19] when $\mathbb{T} = [0,\infty)$.

From Definition 3.1, we observe the following facts.

(*a*) $\delta_{k-\mathbb{T}}(\mathbb{T}) = 1$.
(*b*) For any Δ-measurable subset Ω of $\mathbb{T}$, $0 \leq \delta_{k-\mathbb{T}}(\Omega) \leq 1$.
(*c*) If A and B are Δ-measurable subsets of $\mathbb{T}$ and $\delta_{k-\mathbb{T}}(A)$, $\delta_{k-\mathbb{T}}(B)$ exist, then:

- $\delta_{k-\mathbb{T}}(A \cup B) \leq \delta_{k-\mathbb{T}}(A) + \delta_{k-\mathbb{T}}(B)$,
- if $A \cap B = \emptyset$, $\delta_{k-\mathbb{T}}(A \cup B) = \delta_{k-\mathbb{T}}(A) + \delta_{k-\mathbb{T}}(B)$,
- if $A \subset B$, then $\delta_{k-\mathbb{T}}(A) \leq \delta_{k-\mathbb{T}}(B)$.

Now we can give the definition of k-statistical convergence on time scales, which generalizes (1), (2), (7) and (8).

Definition 3.2. *Let $k \in \Psi$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. We say that f is k-statistically convergent to a number L if, for every $\varepsilon > 0$,*

$$\delta_{k-\mathbb{T}}\left(\left\{y \in \mathbb{T} : \left|f(y)-L\right| \geq \varepsilon\right\}\right) = 0$$

holds. Then, this limit is denoted by

$$st_{k-\mathbb{T}} - \lim f = L.$$

From Definition 3.2, observe that $st_{k-\mathbb{T}} - \lim f = L$ if and only if

$$\lim_{x\to\infty} \int\limits_{y\in\mathbb{T}\,:\,\left|f(y)-L\right|\geq\varepsilon} k(x,y)\,\Delta y = 0.$$

The next properties immediately follow from Definition 3.2:

- k-statistical limit is unique,
- k-statistical convergence is a linear method.

Observe that if we fix $\mathbb{T} = \mathbb{N}$ in Definition 3.2, the kernel function $k = k(x, y)$ becomes a matrix $A = (a_{nk})$, and hence the Lebesgue Δ-integration reduces to the classical summation. In this case, the k-statistical convergence coincides with the concept of A-statistical convergence in (2).

Some significant applications of k-density and k-statistical convergence will be discussed in the next section.

We now give a characterization for k-statistical convergence on time scales. To achieve this we first need the following two lemmas.

Lemma 3.3. *Let $k \in \Psi$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. Assume that $st_{k-\mathbb{T}} - \lim f = L$. If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function at L, then*

$$st_{k-\mathbb{T}} - \lim h \circ f = h(L), \tag{13}$$

which means continuous functions preserve k-statistical limit on time scales.

Proof. Since h is continuous at L, for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|s - L| < \delta$ implies $|h(s) - h(L)| < \varepsilon$. Hence, the inequality $|h(s) - h(L)| \geq \varepsilon$ implies $|s - L| \geq \delta$. In particular taking $s = f(y)$, we see that $\left|h(f(y)) - h(L)\right| \geq \varepsilon$ implies $\left|f(y) - L\right| \geq \delta$. Then, we obtain the next inclusion:

$$\left\{y \in \mathbb{T} : \left|h(f(y)) - h(L)\right| \geq \varepsilon\right\} \subseteq \left\{y \in \mathbb{T} : \left|f(y) - L\right| \geq \delta\right\}.$$

From the properties of k-density, we may write that

$$\delta_{k-\mathbb{T}}\left(\left\{y \in \mathbb{T} : \left|h(f(y)) - h(L)\right| \geq \varepsilon\right\}\right) \leq \delta_{k-\mathbb{T}}\left(\left\{y \in \mathbb{T} : \left|f(y) - L\right| \geq \delta\right\}\right).$$

Since $st_{k-\mathbb{T}} - \lim f = L$, the right hand side of the last inequality must be zero. Then, we get

$$\delta_{k-\mathbb{T}}\left(\left\{y \in \mathbb{T} : \left|h(f(y)) - h(L)\right| \geq \varepsilon\right\}\right) = 0$$

for every $\varepsilon > 0$, which gives (13). Therefore, the proof is completed. □

Lemma 3.4. *Let $k \in \Psi$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable and bounded function on $\mathbb{T}$. If $st_{k-\mathbb{T}} - \lim f = L$, then we have*

$$\lim_{x\to\infty} \int\limits_{\mathbb{T}} k(x,y)\,f(y)\,\Delta y = L.$$

Proof. Without loss of generality, we may assume that $L = 0$. Then, we observe that

$$\begin{aligned}\left|\int\limits_{\mathbb{T}} k(x,y)\,f(y)\,\Delta y\right| &\leq \int\limits_{y\in\mathbb{T}:\,\left|f(y)\right|\geq\varepsilon} \left|f(y)\right| k(x,y)\,\Delta y + \int\limits_{y\in\mathbb{T}:\,\left|f(y)\right|<\varepsilon} \left|f(y)\right| k(x,y)\,\Delta y \\ &\leq C \int\limits_{y\in\mathbb{T}:\,\left|f(y)\right|\geq\varepsilon} k(x,y)\,\Delta y + \varepsilon \int\limits_{\mathbb{T}} k(x,y)\,\Delta y,\end{aligned}$$

where the positive constant C comes from the boundedness of f. Since f is k-statistically convergent to 0 and also $k \in \Psi$, the right hand side of the last inequality tends to zero as $x \to \infty$, which completes the proof. □

Now, we can give our characterization theorem, which generalizes the discrete case studied by Demirci in [21].

Theorem 3.5. *Let $k \in \Psi$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. Then,*

$$st_{k-\mathbb{T}} - \lim f = L \tag{14}$$

if and only if, for every $\beta \in \mathbb{R}$,

$$\lim_{x\to\infty} \int_{\mathbb{T}} k(x,y)\, e^{i\beta f(y)} \Delta y = e^{i\beta L} \tag{15}$$

holds.

Proof. First assume that (14) holds. Since, for any fixed $\beta \in \mathbb{R}$, $e^{i\beta s}$ is a continuous function, Lemma 3.3 implies that

$$st_{k-\mathbb{T}} - \lim e^{i\beta f(y)} = e^{i\beta L}.$$

Also since $e^{i\beta f(y)}$ is a bounded function and $k \in \Psi$, we immediately obtain (15) from (3.4), which completes the proof of the necessity part.

Assume now that (15) holds for every fixed $\beta \in \mathbb{R}$. Then, using same manner in [42] (see also [21, 46]) we first define a continuous function B by

$$B(t) = \begin{cases} 0, & \text{if } t \le -1 \\ 1+t, & \text{if } -1 < t < 0 \\ 1-t, & \text{if } 0 \le t < 1 \\ 0, & \text{if } t \ge 1. \end{cases}$$

Then by the inverse Fourier transformation (see, i.e., [28, 42]) $B(t)$ has the following integral representation:

$$B(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}} \right)^2 e^{it\beta} d\beta \quad \text{for } t \in \mathbb{R}. \tag{16}$$

In order to prove (14), without loss of generality, we may assume $L = 0$. Hence, from (15),

$$\lim_{x\to\infty} \int_{\mathbb{T}} k(x,y)\, e^{i\beta f(y)} \Delta y = 1 \tag{17}$$

holds for every $\beta \in \mathbb{R}$. Using appropriate changing variables in (16) one can write that

$$B\left(\frac{f(y)}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin \frac{\beta\varepsilon}{2}}{\frac{\beta\varepsilon}{2}} \right)^2 e^{if(y)\beta} d\beta \tag{18}$$

and hence

$$\int_{\mathbb{T}} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right) \Delta y = \frac{\varepsilon}{2\pi} \int_{\mathbb{T}} k(x,y) \left\{ \int_{\mathbb{R}} \left(\frac{\sin \frac{\beta\varepsilon}{2}}{\frac{\beta\varepsilon}{2}} \right)^2 e^{if(y)\beta} d\beta \right\} \Delta y.$$

Since the integral in (18) is absolutely convergent, by the Fubini theorem on time scales (see [10, 11]) we have

$$\int_{\mathbb{T}} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y = \frac{\varepsilon}{2\pi}\int_{\mathbb{R}}\left(\frac{\sin\frac{\beta\varepsilon}{2}}{\frac{\beta\varepsilon}{2}}\right)^2 \left\{\int_{\mathbb{T}} k(x,y)\, e^{if(y)\beta}\Delta y\right\} d\beta \tag{19}$$

Since $k \in \Psi$, there exists a finite constant M such that

$$\left|\int_{\mathbb{T}} k(x,y)\, e^{if(y)\beta}\Delta y\right| \le \int_{\mathbb{T}} k(x,y)\,\Delta y \le M$$

holds for all $\beta \in \mathbb{R}$ and $x \in \mathbb{T}$. Hence, using (17), (19) and the Lebesgue Dominated Convergence Theorem on time scales (see [11]), we have

$$\begin{aligned}\lim_{x\to\infty}\int_{\mathbb{T}} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y &= \frac{\varepsilon}{2\pi}\int_{\mathbb{R}}\left(\frac{\sin\frac{\beta\varepsilon}{2}}{\frac{\beta\varepsilon}{2}}\right)^2 \left\{\lim_{x\to\infty}\int_{\mathbb{T}} k(x,y)\, e^{if(y)\beta}\Delta y\right\} d\beta \\ &= \frac{\varepsilon}{2\pi}\int_{\mathbb{R}}\left(\frac{\sin\frac{\beta\varepsilon}{2}}{\frac{\beta\varepsilon}{2}}\right)^2 d\beta.\end{aligned}$$

From (16) we get

$$\lim_{x\to\infty}\int_{\mathbb{T}} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y = B(0) = 1. \tag{20}$$

Also, we may write that

$$\begin{aligned}\int_{\mathbb{T}} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y &= \int_{y\in\mathbb{T}\,:\,|f(y)|\ge\varepsilon} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y \\ &\quad + \int_{y\in\mathbb{T}\,:\,|f(y)|<\varepsilon} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y.\end{aligned}$$

Considering the definition of the function B, if $|f(y)| \ge \varepsilon$, then $B\left(\frac{f(y)}{\varepsilon}\right) = 0$, which implies

$$\begin{aligned}\int_{\mathbb{T}} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y &= \int_{y\in\mathbb{T}\,:\,|f(y)|<\varepsilon} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y \\ &\le \int_{y\in\mathbb{T}\,:\,|f(y)|<\varepsilon} k(x,y)\,\Delta y \\ &= 1 - \int_{y\in\mathbb{T}\,:\,|f(y)|\ge\varepsilon} k(x,y)\,\Delta y.\end{aligned}$$

Then, we get

$$\int_{y\in\mathbb{T}\,:\,|f(y)|\ge\varepsilon} k(x,y)\,\Delta y \le 1 - \int_{\mathbb{T}} k(x,y)\, B\left(\frac{f(y)}{\varepsilon}\right)\Delta y.$$

Now taking limit as $x \to \infty$ and also using (20), we observe that

$$\lim_{x\to\infty} \int\limits_{y\in\mathbb{T}\,:\,|f(y)|\geq\varepsilon} k(x,y)\,\Delta y = 0$$

holds for every $\varepsilon > 0$. This means that $st_{k-\mathbb{T}} - \lim f = 0$, which completes the proof. □

4. Applications and Concluding Remarks

Now we examine some applications of k-density and k-statistical convergence. Here we need to pay attention that Definition 3.1 and Definition 3.2 depend on both a kernel function k and a time scale $\mathbb{T}$. So, with their appropriate choices, it is possible to derive many density functions and convergence methods from these definitions.

Example 4.1. *Consider the following kernel function:*

$$k(x,y) = \begin{cases} \frac{1}{\mu_\Delta([t_0,x]_\mathbb{T})}, & \text{if } y \in [t_0,x]_\mathbb{T}, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

which corresponds a time scale version of the classical Cesàro matrix in (3)*. Then, we immediately get the concept of density on time scales in [46]. Indeed,*

$$\begin{aligned} \delta_{k-\mathbb{T}}(\Omega) &= \lim_{x\to\infty} \int\limits_{y\in\Omega\cap[t_0,x]_\mathbb{T}} \frac{1}{\mu_\Delta([t_0,x]_\mathbb{T})}\Delta y \\ &= \lim_{x\to\infty} \frac{1}{\mu_\Delta([t_0,x]_\mathbb{T})} \int\limits_{s\in\Omega\cap[t_0,x]_\mathbb{T}} \Delta y \\ &= \lim_{x\to\infty} \frac{\mu_\Delta(\Omega(x))}{\mu_\Delta([t_0,x]_\mathbb{T})}, \end{aligned}$$

where $\Omega(x)$ is the same as in (5)*. Now, specializing some well-known time scales, one can obtain the asymptotic density in [39] for the discrete case and the approximate density in [26]* (*see also [38]*) *for the continuous case. It is also possible to derive different density definitions on time scales by appropriately changing the k-transformation. Also, using the kernel function $k(x,y)$ in* (21)*, we also get the statistical convergence on time scales in* (7) (*see [46] for details*)*. In this case, observe that this function k belongs to the family Ψ* (*see [49]*)*. We also know that the discrete case gives the usual definition of statistical convergence given by Fast [27] while the continuous case reduces to the statistical convergence for measurable functions introduced by Móricz [38]. Furthermore, taking the time scale $\mathbb{T} = q^{\mathbb{N}}$ $(q > 1)$, we get the concept of q-statistical convergence given by Aktuglu and Bekar [2].*

We should note that the (discrete) Cesàro summability matrix in (3) and its continuous case have also been used in sequence spaces and function spaces, respectively. The basic properties of Cesàro sequence spaces (corresponding to the discrete case) and Cesàro function spaces (corresponding to the continuous case), such as *duality*, *reflexivity* and *fixed point property*, have been examined frequently in the literature (see, for instance, [4, 6, 20, 36, 41, 44]). In fact, such works first appeared in 1968 in connection with the problem of the Dutch Mathematical Society to find the duals of these spaces. Hence, as future works, the time scale version of Cesàro summability given by (21) may also be interesting for such studies since it unifies both continuous and discrete cases.

Example 4.2. *As an other example, consider the following kernel function:*

$$k(k_r,y) = \begin{cases} \frac{1}{\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)}, & \text{if } y \in (k_{r-1},k_r]_\mathbb{T}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta = \{k_r\}$ *is a lacunary sequence in* $\mathbb{T}$. *In this case, Definition* 3.2 *turns out to be the lacunary statistical convergence on time scales in* (8) (*see [47, 48] for details*). *Of course, the case of* $\mathbb{T} = \mathbb{N}$ *is well-known in the literature* (*see [24, 30, 31]*).

Finally, fixing $k(x, y)$ or $\mathbb{T}$ in Theorem 3.5 one can arrive known results or derive new ones.

- If we consider the kernel k defined by (21) in Theorem 3.5, we get an our earlier result (see Theorem 3.12 in [46]). Then, the case of $\mathbb{T} = \mathbb{N}$ coincides with the classical result by Schoenberg in [42]; and in the case of $T = [a, \infty)$ $(a > 0)$ we get Theorem 1 in [28].
- If we fix $\mathbb{T} = \mathbb{N}$ in Theorem 3.5, then the function $k = k(x, y)$ becomes a matrix $A = (a_{nk})$, and hence our theorem coincides with Demirci's result in [21].
- Any other suitable kernel or time scale generates new density functions and convergence methods.

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