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Comparison between the two Huang-Kotz FGM types by some information measures in generalized order statistics and their concomitants

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Abstract. In this paper, we study the concomitants of $m-$dual generalized order statistics (and consequently $m-$generalized order statistics) from Huang–Kotz Farlie–Gumble–Morgenstern bivariate distribution (HK–FGM) of the types 1 and 2. Some important information measures are derived and studied for ordinary order statistics and sequential order statistics with a comparison. Specifically, the Shannon entropy, inaccuracy measure, and Fisher information number. Moreover, a comparison between the two types of HK–FGM distribution is carried out.

1. Introduction

Morgenstern [34] proposed the Farlie-Gumbel-Morgenstern (FGM) distribution for Cauchy marginals. The same structure was examined by Gumbel [22] for exponential marginals. In connection with his studies of the correlation coefficient, Farlie [19] proposed a new generic form of a bivariate distribution for given arbitrary marginals, based on Morgenstern's and Gumbel's work [22]. Johnson and Kotz [28], [29] applied the proposed bivariate distribution to the multivariate case, coining the known name "FGM" distribution function (DF). The FGM distribution is defined by $F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\theta(1-F_X(x))(1-F_Y(y))]$, $-1 \leq \theta \leq 1$, where F_X and F_Y are the marginal DFs of some random variables (RVs) X and Y. While the classical FGM distribution is a versatile family with many applications, one of its most well-known drawbacks is the low level of dependence it allows between RVs, with a maximum positive correlation coefficient of 0.33. As a result, FGM distribution is only applicable to data with low correlation. One of the successful attempts to overcome this drawback is due to the Huang and Kotz [26] who employed multiple iterations of the FGM distribution to boost the correlation between the components, demonstrating that just one iteration can triple the covariance for particular marginals. Alawady et al. [6], Barakat and Husseiny [10], and Barakat et al. [11] later studied this model in depth. Huang and Kotz [27] are responsible for one of the most effective and well-known attempts to broaden the range of correlation and give the FGM distribution more flexibility. Huang and Kotz [27] offered the following two comparable extended DFs $F^{(1)}_{X,Y}(x,y)$ (denoted by

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HK-FGM1(θ_1, p_1)) and $F^{(2)}_{X,Y}(x,y)$ (denoted by HK-FGM2(θ_2, p_2))

$$F^{(1)}_{X,Y}(x,y) = F_X(x)F_Y(y)\left[1+\theta_1(1-F^{p_1}_X(x))(1-F^{p_1}_Y(y))\right],\ p_1 \geq 1, \tag{1}$$

with the probability density function (PDF)

$$f^{(1)}_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1+\theta_1((1+p_1)F^{p_1}_X(x)-1)((1+p_1)F^{p_1}_Y(y)-1)\right],$$

and

$$F^{(2)}_{X,Y}(x,y) = F_X(x)F_Y(y)\left[1+\theta_2(1-F_X(x))^{p_2}(1-F_Y(y))^{p_2}\right],\ p_2 \geq 1, \tag{2}$$

with the PDF

$$f^{(2)}_{X,Y}(x,y) = f_X(x)f_Y(y)\left[1+\theta_2(1-F_X(x))^{p_2-1}(1-F_Y(y))^{p_2-1}\right.$$
$$\left.\times\, ((1+p_2)F_X(x)-1)((1+p_2)F_Y(y)-1)\right].$$

The admissible range of the shape-parameter vectors (θ_1, p_1) and (θ_2, p_2) are $\Omega_1 = \{(\theta_1, p_1) : p_1^{-2} \leq \theta_1 \leq p_1^{-1},\ p_1 \geq 1\}$, and $\Omega_2 = \{(\theta_2, p_2) : -1 \leq \theta_2 \leq \left(\frac{p_2+1}{p_2-1}\right)^{p_2-1},\ p_2 > 1 \text{ or } -1 \leq \theta_2 \leq +1,\ p_2 = 1\}$, respectively. The maximum positive correlations of the models (1) and (2) are 0.375 and 0.391, respectively, which are achieved at $p_1 = 2$ and $p_2 = 1.1877$. There is a small difference between the two maximal positive correlations produced by the comparable models (1) and (2). As a result, determining the trade-off between the models (1) and (2) is difficult. The only work that treated this issue is Barakat et al. [14], where the focus was only on the Fisher information (FI) measure. The majority of works on the extensions (1) and (2), on the other hand, are concerned with the family (1). Abd Elgawad et al. [4], Bairamov and Kotz [9], Barakat et al. [12], [13], and Fisher and Klein [20] are among them. One of the main aims of this work will be to conduct an analysis of type (2) in light of various significant information measures (specifically, the Shannon entropy, inaccuracy measure, and Fisher information number (FIN)). Moreover, another purpose of this study is to compare the DFs (1) and (2) utilizing these information measures. Huang and Kotz [27] demonstrated that using the model (1), the positive correlation between marginal distributions may be enhanced to ≈ 0.39, while the maximum negative correlation remains $-\frac{1}{3}$. Furthermore, when uniform marginals are used in the model (1), the range $0 < p_1 < 1$ results in a fast-declining positive correlation, while the admissible range quickly widens. Furthermore, at $p_1 = 1$, the highest negative correlation is found. As a result, we'll only deal with the case $p_1 \geq 1$. Many researchers have expressed interest in using and generalizing the simple analytical form of the HK–FGM families in various aspects of science, including Amblard and Girard [8], Bairamov and Kotz [9], Barakat et al. [12], [13], Domma and Giordano [18], Fischer and Klein [20], and Mokhlis and Khames [32], [33], among others.

Kamps [30] introduced the concept of generalized order statistics (GOSs) as a unified approach to a variety of models of ascendingly ordered RVs. The concept of dual GOSs, denoted by DGOSs, was introduced by Burkschat et al. [15] as a parallel concept of GOSs to enable a common approach to descendingly ordered RVs.

The subclasses $m-$GOSs and $m-$DGOSs of GOSs and DGOSs, respectively, contain many important models of ordered RVs such as ordinary order statistics (OOSs), lower and upper record values, $k-$records, sequential order statistics (SOSs), and type II censored OOSs. Let $F(.)$ be an arbitrary continuous DF, with PDF $f(.)$ and survival function $\overline{F}(.) = 1 - F(.)$. Then the RVs $X(1,n,m,k) \leq X(2,n,m,k) \leq ... \leq X(n,n,m,k)$ $(k > 0,\ m \geq -1)$ are said to be $m-$GOSs, if their joint PDF (JPDF) is given by (cf. Kamps [30])

$$f^{(m,k)}_{1,2,...,n:n}(x_1, x_2, ..., x_n) = \left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}\overline{F}^{m}(x_j)f(x_j)\right)\overline{F}^{k-1}(x_n)f(x_n),$$

where $F^{-1}(1) \geq x_n \geq ... \geq x_1 \geq F^{-1}(0)$ and $\gamma_j = k + (n-j)(m+1) > 0, j = 1, 2, ..., n$ (note that $\gamma_n = k$). On the other hand, the RVs $X_d(1,n,m,k) \geq X_d(2,n,m,k) \geq ... \geq X_d(n,n,m,k)$ $(k > 0,\ m \geq -1)$ are said to be

$m-$DGOSs, if their JPDF is given by (cf. Burkschat et al. [15])

$$f_{1,2,...,n:n}^{d(m,k)}(x_1, x_2, ..., x_n) = \left(\prod_{j=1}^{n} \gamma_j\right)\left(\prod_{j=1}^{n-1} F^m(x_j) f(x_j)\right) F^{k-1}(x_n) f(x_n),$$

where $F^{-1}(1) \geq x_1 \geq ... \geq x_n \geq F^{-1}(0)$. Clearly, $f_{1,2,...,n:n}^{d(m,k)}(x_1, x_2, ..., x_n)$ is obtained just by replacing $\overline{F}$ by F. Thus, any obtained result for DGOSs can be easily deduced for GOSs and vice versa. For this reason, in this paper we consider only the model of $m-$DGOSs.

The marginal PDF of rth $m-$DGOS, $1 \leq r \leq n$, is given by (cf. Burkschat et al. [15])

$$f_{X_d(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} F^{\gamma_r - 1}(x) f(x) g_m^{r-1}(F(x)), \tag{3}$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i, r = 1, 2, ..., n, g_m(x) = h_m(x) - h_m(1), x \in (0, 1)$ and

$$h_m(x) = \begin{cases} -\frac{x^{m+1}}{(m+1)}, & m \neq -1, \\ -\ln(x), & m = -1. \end{cases}$$

David [16] was the first to introduce the concept of concomitants of OOSs. In problems of selection and prediction, the concomitants are of interest. David and Nagaraja [17] provided a comprehensive review of concomitants of OOSs. Let $(X_i, Y_i), i = 1, 2, ..., n$, be a random sample from a bivariate DF $F_{X,Y}(x, y)$. If the X–variates are arranged in descending order as $X_d(1, n, m, k) \geq X_d(2, n, m, k) \geq .... \geq X_d(n, n, m, k)$, for the X sample, then Y-variates paired with these $m-$DGOSs are called the concomitants of $m-$DGOSs and denoted by $Y_{[r,n,m,k]}, r = 1, 2, ..., n$. The PDF of the concomitant of rth $m-$DGOS is given by

$$f_{[r,n,m,k]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{X_d(r,n,m,k)}(x) dx, \tag{4}$$

where $f_{X_d(r,n,m,k)}(x)$ is the PDF of $X_d(r, n, m, k)$ which is given by (3) and $f_{Y|X}(y|x)$ is the conditional PDF of Y given X. The concomitants of GOSs and DGOSs have been explored by a number of authors. Among them are Abd Elgawad and Alawady [1], Abd Elgawad et al. [2], Alawady et al. [5], Haseeb and Nayabuddin [23], [24], Haseeb et al. [25], Nayabuddin [35], Saman and Muhammad [38], and Tahmasebi et al. [40].

In this paper, we consider three information measures, which are Shannon entropy, inaccuracy measure, and FIN. These information measures can be briefly explained as follows:

1. The Shannon entropy is a statistical measure of information that determines how much uncertainty or variability an RV reduces on average. This measure is maximal for uniform distribution, additive for independent events, rising in the number of outcomes with non-zero probabilities, continuous, non-negative, and permutation-invariant as the number of outcomes with non-zero probabilities increases. See Abd Elgawad et al. [3], [4], Alawady et al. [7], Barakat and Husseiny [10], and Pathria and Beale [37] for further information on this measure. The Shannon entropy of a continuous RV X having PDF $f_X(x)$ is defined by

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x)\, dx. \tag{5}$$

2. Let X and Y be two non-negative RVs with DFs $F_X(x)$ and $G_Y(x)$, respectively. If $f_X(x)$ is the actual PDF corresponding to the observation and $g_Y(x)$ is the density assigned by the experimenter, then the inaccuracy measure of X and Y is defined by Kerridge [31] as

$$I(X, Y) = -\int_{0}^{\infty} f_X(x) \log g_Y(x)\, dx. \tag{6}$$

3. The FIN is the second moment of the "score function" where the derivative is with respect to x in a given PDF $f_X(\theta, x)$, rather than the parameter θ. It is a FI for a location parameter, for this reason, it is also called shift-invariant FI. Recently, FIN is frequently used in different aspects of science. For example, the FIN is intimately related to many of the fundamental equations of theoretical physics, cf. Frieden and Gatenby [21]. Moreover, the FIN matrix is used to define a class of measures of multivariate dependence between the components of a random vector, see Zografos [41]. For some recent works about this measure, see Abd Elgawad et al. [3], [4], Tahmasebi and Jafari [39], and the references therein. The FIN of the RV X having PDF $f_X(x)$ is defined by (cf. Papaioannou and Ferentinos [36])

$$\mathrm{FIN}(X) = \mathrm{E}\left(\frac{\partial \log f_X(x)}{\partial x}\bigg|_{x=X}\right)^2 = \int_{-\infty}^{\infty}\left(\frac{\partial f_X(x)}{\partial x}\right)^2 \frac{1}{f_X(x)}dx. \tag{7}$$

The rest of the paper is unfolded as follows: In Section 2, we study the rth concomitant of $m-$DGOSs in HK–FGM2 with general marginals. Moreover, the Shannon entropy, inaccuracy measure, and FIN are derived. It is worth mentioning that Abd Elgawad et al. [4] derived the Shannon entropy and FIN for HK–FGM1, but no computational study was conducted for their theoretical results. In Section 3, we carry out a computational study for the three information measures related to the two types HK–FGM1 and HK–FGM2. Moreover, comparisons between the two types and between OOSs and SOSs are carried out based on these information measures.

2. Concomitants of m-DGOS based on HK–FGM2

In this section, we derive the marginal DF of the rth concomitant of DGOSs based on HK–FGM2, as well as the Shannon entropy, inaccuracy measure, and FIN. Moreover, the inaccuracy measure is also derived for HK–FGM1, since this measure was not discussed in Abd Elgawad et al. [4].

Lemma 2.1. *Let $X \sim F_X$ and $Y \sim F_Y$. Furthermore, let p_2 is an integer number. Then*

$$f_{[r,n,m,k]}(y) = f_Y(y)\left[1 + (1 - F_Y(y))^{p_2-1}(1 - (1 + p_2)F_Y(y))\Delta_{r,n,m,k;p_2}\right], \tag{8}$$

where

$$\Delta_{r,n,m,k;p_2} = \theta_2 \sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i \prod_{j=1}^{r}\gamma_j\left[\frac{1}{\gamma_j + i} - \frac{(1+p_2)}{\gamma_j + i + 1}\right], \tag{9}$$

and $\gamma_j = k + (n - j)(m + 1)$.

Proof. In view of (4), the PDF of $Y_{[r,n,m,k]}$ is given by

$$\begin{aligned} f_{[r,n,m,k]}(y) &= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{X_{d(r,n,m,k)}}(x)dx \\ &= \int_{-\infty}^{\infty} f_Y(y)[1 + \theta_2(1 - F_X(x))^{p_2-1}(1 - (1 + p_2)F_X(x))(1 - F_Y(y))^{p_2-1}(1 - (1 + p_2)F_Y(y))] \\ &\quad \times \frac{C_{r-1}}{(r-1)!}F_X^{\gamma_r-1}(x)g_m^{r-1}(F_X(x))f_X(x)dx. \end{aligned}$$

Thus, we can write

$$f_{[r,n,m,k]}(y) = f_Y(y)[1 + (1 - F_Y(y))^{p_2-1}(1 - (1 + p_2)F_Y(y))\Delta_{r,n,m,k;p_2}], \tag{10}$$

where

$$\Delta_{r,n,m,k;p_2} = \frac{\theta_2 C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}(1 - F_X(x))^{p_2-1}(1 - (1 + p_2)F_X(x))F_X^{\gamma_r-1}(x)g_m^{r-1}(F_X(x))f_X(x)dx$$

$$= \frac{\theta_2 C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} (1-F_X(x))^{p_2-1}(1-(1+p_2)F_X(x))F_X^{\gamma_r-1}(x)\left(\frac{1-F_X^{m+1}(x)}{m+1}\right)^{r-1} f_X(x)dx. \tag{11}$$

Apply the binomial theorem and take the transformation $u = \frac{1-F_X^{m+1}(x)}{m+1}$ in (11), then

$$\Delta_{r,n,m,k;p_2} = \frac{\theta_2 C_{r-1}}{(r-1)!} \sum_{i=0}^{p_2-1} \binom{p_2-1}{i} (-1)^i [\int_0^{\frac{1}{m+1}} u^{r-1}(1-(1+m)u)^{\frac{\gamma_r+i}{m+1}-1} du$$

$$-(1+p_2)\int_0^{\frac{1}{m+1}} u^{r-1}(1-(1+m)u)^{\frac{\gamma_r+i+1}{m+1}-1} du]$$

$$= \frac{\theta_2 C_{r-1}}{(r-1)!(1+m)^r} \sum_{i=0}^{p_2-1} \binom{p_2-1}{i} (-1)^i \left[\beta(r, \frac{\gamma_r+i}{m+1}) - (1+p_2)\beta(r, \frac{\gamma_r+i+1}{m+1})\right]$$

$$= \frac{\theta_2 C_{r-1}}{(r-1)!(1+m)^r} \sum_{i=0}^{p_2-1} \binom{p_2-1}{i} (-1)^i \left[\frac{\Gamma(r)\Gamma(\frac{\gamma_r+i}{m+1})}{\Gamma(\frac{\gamma_r+i}{m+1}+r)} - (1+p_2)\frac{\Gamma(r)\Gamma(\frac{\gamma_r+i+1}{m+1})}{\Gamma(\frac{\gamma_r+i+1}{m+1}+r)}\right]. \tag{12}$$

Simplifying (12), we get (9) and by combining (9) and (10), we get the required result. □

Now, we introduce Theorems 2.2, 2.4, and 2.5 to get an explicit form of the Shannon entropy, inaccuracy measure and FIN for concomitants of $m-$DGOS of HK–FGM2, respectively, while Theorem 2.3 gives an explicit form of the inaccuracy measure for concomitants of $m-$DGOS of HK–FGM1.

Theorem 2.2. *Let $Y_{[r,n,m,k]}$, $1 \le r \le n$, be the concomitant of the rth $m-$DGOS based on HK–FGM2. Then, the Shannon entropy of $Y_{[r,n,m,k]}$ is given by*

$$H(Y_{[r,n,m,k]}) = H(Y) - \Delta_{r,n,m,k;p_2} \sum_{i=0}^{p_2-1} \binom{p_2-1}{i} (-1)^i \left[(1+p_2)\Phi_Y(i+1) - \Phi_Y(i)\right] - \Omega_{r,n,m,k;p_2},$$

where $\Phi_Y(i) = \int_{-\infty}^{\infty} f_Y(y)F_Y^i(y)\log f_Y(y)dy$, and

$$\Omega_{r,n,m,k;p_2} = \frac{-2\Delta_{r,n,m,k;p_2} - (\Delta_{r,n,m,k;p_2}-1)^2\log(1-\Delta_{r,n,m,k;p_2}) + (1+\Delta_{r,n,m,k;p_2})^2\log(1+\Delta_{r,n,m,k;p_2})}{4\Delta_{r,n,m,k;p_2}}.$$

Proof. For simplicity, write $\Delta_r = \Delta_{r,n,m,k;p_2}$. Using (5) and (8), the Shannon entropy for $Y_{[r,n,m,k]}$ is given by

$$H(Y_{[r,n,m,k]}) = -\int_{-\infty}^{\infty} f_{[r,n,m,k]}(y)\log f_{[r,n,m,k]}(y)dy$$

$$= -\int_{-\infty}^{\infty} f_Y(y)[1+(1-F_Y(y))^{p_2-1}(1-(1+p_2)F_Y(y))\Delta_r]$$

$$\times[1+(1-F_Y(y))^{p_2-1}(1-(1+p_2)F_Y(y))\Delta_r]\log f_Y(y)dy$$

$$= -E[\log f_Y(Y_{[r,n,m,k]})] - E[\log(1+(1-F_Y(Y_{[r,n,m,k]})))^{p_2-1}(1-(1+p_2)F_Y(Y_{[r,n,m,k]}))\Delta_r] = J_1 + J_2,$$

where $J_1 = -E[\log f_Y(Y_{[r,n,m,k]})]$, and $J_2 = -E[\log(1+(1-F_Y(Y_{[r,n,m,k]})))^{p_2-1}(1-(1+p_2)F_Y(Y_{[r,n,m,k]}))\,\Delta_r]$. Now,

$$J_1 = -E[\log f_Y(Y_{[r,n,m,k]})] = -\int_{-\infty}^{\infty}(\log f_Y(y))[f_Y(y)(1+(1-F_Y(y))^{p_2-1}(1-(1+p_2)F_Y(y))\Delta_r)]dy$$

$$= -\int_{-\infty}^{\infty}(\log f_Y(y))[f_Y(y) + \Delta_r \sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i[f_Y(y)F_Y^i(y) - \Delta_r(1+p_2)f_Y(y)F_Y^{i+1}(y)]]dy$$

$$= -\int_{-\infty}^{\infty} f_Y(y)\log f_Y(y)dy - \Delta_r \sum_{i=0}^{p_2-1}(-1)^i \int_{-\infty}^{\infty} f_Y(y)F_Y^i(y)\log f_Y(y)dy$$

$$+ \Delta_r(1+p_2)\sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i \int_{-\infty}^{\infty} f_Y(y)F_Y^{i+1}(y)\log f_Y(y)dy$$

$$= H(Y) - \Delta_r \sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i \int_{-\infty}^{\infty} f_Y(y)F_Y^i(y)\log f_Y(y)dy$$

$$+ \Delta_r(1+p_2)\sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i \int_{-\infty}^{\infty} f_Y(y)F_Y^{i+1}(y)\log f_Y(y)dy$$

$$= H(Y) - \Delta_r \sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i[(1+p_2)\Phi_Y(i+1) - \Phi_Y(i)].$$

Furthermore, by using the integrating by parts we get $J_2 = \frac{\Delta_r}{p_2}\log(1+\Delta_r) - \int_{-\infty}^{\infty} VdU$, where $U = \log[(1 + (1-F_Y(y))^{p_2-1}(1-(1+p_2)F_Y(y))\Delta_r)]$, and $dV = f_Y(y) + f_Y(y)(1 + (1-F_Y(y))^{p_2-1}(1-(1+p_2)F_Y(y))\Delta_r)$. On the other hand, by using Mathematica ver. 11.3, we can easily prove that $J_2 = \Omega_{r,n,m,k;p_2}$, which proves the theorem. □

Theorem 2.3. *Let $Y_{[r,n,m,k]}$ be the concomitant of rth m−DGOS in HK–FGM1 family. Then, the inaccuracy measure is given by*

$$I^{\star}(Y_{[r,n,m,k]}, Y) = (1 + \Delta^{\star}_{r,n,m,k;p_1})H(y) + \Delta^{\star}_{r,n,m,k;p_1}(1+p_1)\Phi_Y(p_1),$$

where

$$\Delta^{\star}_{r,n,m,k;p_1} = \theta_1(1 - (1+p_1)\frac{\beta(r+p_1, n-r+1)}{\beta(r, n-r+1)}).$$

Proof. By using (6) and the result of Abd Elgawad et al. [4] concerning the marginal PDF of concomitants of DGOS based on HK–FGM1, the inaccuracy measure is given by

$$I^{\star}(Y_{[r,n,m,k]}, Y) = -\int_{-\infty}^{\infty} f_{r,n,m,k:n}(y)\log f_Y(y)dy$$

$$= -\int_{-\infty}^{\infty} f_Y(y)(1 + \Delta^{\star}_{r,n,m,k;p_1}(1-(1+p_1)F_Y^{p_1}(y)))\log f_Y(y)dy$$

$$= -\int_{-\infty}^{\infty} f(y)\log f(y)dy - \Delta^{\star}_{r,n,m,k;p_1}\int_{-\infty}^{\infty} f_Y(y)(1-(1+p_1)F_Y^{p_1}(y))\log f_Y(y)dy$$

$$= -(1+\Delta^{\star}_{r,n,m,k;p_1})\int_{-\infty}^{\infty} f_Y(y)\log f_Y(y)dy + \Delta^{\star}_{r,n,m,k;p_1}(1+p_1)\int_{-\infty}^{\infty} f_Y(y)F_Y^{p_1}(y)\log f_Y(y)dy.$$

Hence, $I^{\star}(Y_{[r,n,m,k]}, Y) = (1+\Delta^{\star}_{r,n,m,k;p_1})H(y) + \Delta^{\star}_{r,n,m,k;p_1}(1+p_1)\Phi_Y(p_1)$, as required to prove. □

Theorem 2.4. *Let $Y_{[r,n,m,k]}$ be the concomitant of rth m−DGOS in HK–FGM2 family. Then, the inaccuracy measure is given by*

$$I(Y_{[r,n,m,k]}, Y) = H(y) - \Delta_{r,n,m,k;p_2}\sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i[\Phi_Y(i) - (1+p_2)\Phi_Y(i+1)].$$

Proof. For simplicity, write $\Delta_r = \Delta_{r,n,m,k;p_2}$. By using (6) and (8), the inaccuracy is given by

$$\begin{aligned}
I(Y_{[r,n,m,k]}, Y) &= -\int_{-\infty}^{\infty} f_{r,n,m,k:n}(y)\log f_Y(y)dy \\
&= -\int_{-\infty}^{\infty} f_Y(y)[1+(1-F_Y(y))^{p_2-1}(1-(1+p_2)F_Y(y))\Delta_r]\log f_Y(y)dy \\
&= -\int_{-\infty}^{\infty} f_Y(y)\log f_Y(y)dy - \Delta_r\int_{-\infty}^{\infty} f_Y(y)(1-F_Y(y))^{p_2-1}(1-(1+p_2)F_Y(y))\log f_Y(y)dy \\
&= -\int_{-\infty}^{\infty} f_Y(y)\log f_Y(y)dy - \Delta_r\sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i[\int_{-\infty}^{\infty} f_Y(y)F_Y^i(y)\log f_Y(y)dy \\
&\quad -(1+p_2)\int_{-\infty}^{\infty} f_Y(y)F_Y^{i+1}(y)\log f_Y(y)dy].
\end{aligned}$$

Hence, $I(Y_{[r,n,m,k]}, Y) = H(y) - \Delta_r\sum_{i=0}^{p_2-1}\binom{p_2-1}{i}(-1)^i[\Phi_Y(i)-(1+p_2)\Phi_Y(i+1)]$. □

Theorem 2.5. *Let $Y_{[r,n,m,k]}$, $1\le r\le n$, be the concomitant of rth $m-$DGOS in HK–FGM2 family. Then, the FIN of $Y_{[r,n,m,k]}$ is given by*

$$FIN(Y_{[r,n,m,k]}) = FIN(Y) + \tau(p_2) - \Delta_{r,n,m,k;p_2}\delta(p_2) - 2\Delta_{r,n,m,k;p_2}\Psi(p_2), \tag{13}$$

where

$$\tau(p_2) = \int_0^1\left(\frac{f'_Y(F_Y^{-1}(u))}{f_Y(F_Y^{-1}(u))}\right)^2\left(1+\Delta_{r,n,m,k;p_2}(1-(1+p_2)u)(1-u)^{p_2-1}\right)du,$$

$$\delta(p_2) = \int_0^1\frac{((1+p_2)(1-u)^{p_2-1}+(p_2-1)(1-u)^{p_2-2})(1-(1+p_2)u)}{1+\Delta_{r,n,m,k;p_2}(1-(1+p_2)u)(1-u)^{p_2-1}}f_Y(F_Y^{-1}(u))du,$$

and

$$\Psi(p_2) = \int_0^1((1+p_2)(1-u)^{p_2-1}+(p_2-1)(1-u)^{p_2-2})(1-(1+p_2)u)f_Y(F_Y^{-1}(u))du.$$

Proof. Again for simplicity, write $\Delta_r = \Delta_{r,n,m,k;p_2}$. By using (7) and (8), then the FIN is given by

$$\begin{aligned}
\mathrm{FIN}(Y_{[r,n,m,k]}) &= E\left(\frac{\partial\log f_{[r,n,m,k]}(y)}{\partial y}\right)^2_{y=Y_{[r,n,m,k]}} = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial y}\log f_{[r,n,m,k]}(y)\right)^2 f_{[r,n,m,k]}(y)dy \\
&= \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial y}(\log f_Y(y)) + \log(1+\Delta_r(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-1})\right]^2 \\
&\quad\times\left[f_Y(y)(1+\Delta_r(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-1})\right]dy \\
&= \mathrm{FIN}(Y) + \int_{-\infty}^{\infty}\left[\frac{f'_Y(y)}{f_Y(y)}\right]^2 f_Y(y)(1+\Delta_r(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-1})dy \\
&+\int_{-\infty}^{\infty}\left[\frac{-(1+p_2)\Delta_r f_Y(y)(1-F_Y(y))^{p_2-1}-(p_2-1)\Delta_r f_Y(y)(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-2}}{(1+(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-1})\Delta_r}\right]^2 \\
&\quad\times[f_Y(y)(1+\Delta_r(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-1})]dy \\
&+2\int_{-\infty}^{\infty}\frac{-(1+p_2)\Delta_r f_Y(y)(1-F_Y(y))^{p_2-1}-(p_2-1)\Delta_r f_Y(y)(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-2}}{(1+(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-1})\Delta_r} \\
&\quad\times\frac{f'_Y(y)}{f_Y(y)}\left[f_Y(y)(1+\Delta_r(1-(1+p_2)F_Y(y))(1-F_Y(y))^{p_2-1})\right]dy.
\end{aligned}$$

Upon using the transformation $u = F_Y(y)$ and after some algebra, the required result (13) follows. □

3. Computational Study and Comparison between HK–FGM1 and HK–FGM2

We start this section with a comparison between HK–FGM1 and HK–FGM2 according to some selected values of the correlation coefficient versus the different values of the parameter values of the two types.

Table 1: Correlation Coefficient ρ versus the parameters in HK–FGM1 and HK–FGM2

ρ	HK-FGM1 (θ_1, p_1)	HK-FGM2 (θ_2, p_2)
-0.37	(-0.67,1.5), (-0.493,2), (-0.206,3)	(-1.452,1.18), (-1.528,1.2), (-1.649,1.25)
-0.3	(-0.544,1.5), (-0.4,2), (-0.167,3)	(-1.201,1.18), (-1.237,1.2), (-1.337,1.25)
-0.2	(-0.363,1.5), (-0.267,2), (-0.185,3)	(-0.801,1.18), (-0.826,1.2), (-0.891,1.25)
0.2	(0.363,1.5), (0.267,2), (0.185,3)	(0.801,1.18), (0.826,1.2), (0.891,1.25)
0.3	(0.544,1.5), (0.4,2), (0.167,3)	(1.201,1.18), (1.237,1.2),(1.337,1.25)
0.37	(0.67,1.5), (0.493,2), (0.206,3)	(1.452,1.18), (1.528,1.2), (1.649,1.25)

3.1. Comparison between HK–FGM1 and HK–FGM2, as well as OOSs and SOSs, according to Shannon entropy

Tables 2 and 3 introduce the Shannon entropy of HK–FGM1 with uniform marginals at $p_1 = 1, 2$, and different values of θ_1, according to OOSs (for which we have $(m, k) = (0, 1)$, cf. Kamps [30]) and SOSs (for which we have $(m, k) = (1, 1)$, cf. Kamps [30]). The calculations were carried out based on the result of Abd Elgwad et al. [4] and by using the MATHEMATICA Ver.11.3. Tables 2 and 3 uncovered the following features:

1. The maximum value of $H(Y_{[r,n,0,1]})$ attains at $(r, n) = (2, 3)$ and $(\theta_1, p_1) = (\pm 0.15, 2)$.
2. The maximum value of $H(Y_{[r,n,1,1]})$ attains at $(r, n) = (4, 5)$ and $(\theta_1, p_1) = (-0.15, 2)$.

Tables 4 and 5 display the Shannon entropy of HK–FGM2 with uniform marginals at $p_2 = 1, 2$, and different values of θ_2, according to OOSs and SOSs. The calculations were carried out based on the results of Theorem 2.2 and by using the MATHEMATICA Ver.11.3. Tables 4 and 5 uncovered the following features:

1. The maximum value of $H(Y_{[r,n,0,1]})$ attains at $(r, n) = (4, 5)$ and $(\theta_2, p_2) = (\pm 0.05, 2)$.
2. The greatest value of $H(Y_{[r,n,1,1]})$ attains at $(r, n) = (6, 7)$ and $(\theta_2, p_2) = (\pm 0.05, 2)$.

Moreover, Tables 2-5 reveal an interesting fact that $H(Y_{[r,n,0,1]})$ based on HK–FGM2 is greater than $H(Y_{[r,n,0,1]})$ based on HK–FGM1, Also, $H(Y_{[r,n,1,1]})$ based on HK–FGM2 is greater than $H(Y_{[r,n,1,1]})$ based on HK–FGM1. Finally, for the two types HK–FGM1 and HK–FGM2, we could not recognize any general trend based on r or n for which one can decide $H(Y_{[r,n,1,1]}) < H(Y_{[r,n,0,1]})$, or $H(Y_{[r,n,1,1]}) > H(Y_{[r,n,0,1]})$.

Table 2: Shannon entropy in $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$ for the copula model of HK–FGM1

$H(Y_{[r,n,0,1]})$		$p_1 = 1$				$H(Y_{[r,n,1,1]})$		$p_1 = 1$			
		θ_1						θ_1			
n	**r**	0.25	0.5	0.75	0.99	**n**	**r**	0.25	0.5	0.75	0.99
3	1	-0.00261	-0.01071	-0.11575	-12.3841	3	1	-0.00464	-0.03320	-15.075	-9372.87
3	2	0	0	0	0	3	2	-0.00065	-0.00261	-0.00588	-0.01047
3	3	-0.00261	-0.01048	-0.02363	-0.04017	3	3	-0.00147	-0.00588	-0.01328	-0.023166
5	1	-0.00464	-0.03320	-15.075	-9372.87	5	1	-0.00670	-0.29298	-786.172	$-5.420*10^{-6}$
5	2	-0.00116	-0.00464	-0.01071	-0.03076	5	2	-0.00345	-0.01550	-0.99699	-230.26
5	3	0	0	0	0	5	3	-0.00102	-0.00408	-0.00930	-0.02082
5	4	-0.00116	-0.00464	-0.01048	-0.018301	5	4	$-2.543*10^{-6}$	-0.00001	-0.00002	-0.00004
5	5	-0.00464	-0.01868	-0.04089	-0.05456	5	5	-0.00269	-0.01081	-0.02437	-0.041278
7	1	-0.00588	-0.11575	-176.254	-400072	7	1	-0.00772	-0.8967	-458.65	$-1.960*10^{8}$
7	2	-0.00261	-0.01071	-0.115746	-12.3841	7	2	-0.00516	-0.05323	-43.1767	-43877.1
7	3	-0.00065	-0.00261	-0.00588	-0.010468	7	3	-0.0030	-0.01242	-0.28310	-43.2373
7	4	0	0	0	0	7	4	-0.00121	-0.00485	-0.011252	-0.036452
7	5	-0.00065	-0.00261	-0.00588	-0.01027	7	5	-0.00014	-0.00057	-0.001288	-0.00225
7	6	-0.00261	-0.01048	-0.02363	-0.04017	7	6	-0.00027	-0.00110	-0.002467	-0.00430
7	7	-0.00588	-0.02363	-0.04924	-0.03554	7	7	-0.0035	-0.01418	-0.031723	-0.050686

Table 3: Shannon entropy in $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$ for the copula model of HK–FGM1

$H(Y_{[r,n,0,1]})$		$p_1 = 2$				$H(Y_{[r,n,1,1]})$		$p_1 = 2$			
		θ_1						θ_1			
n	**r**	-0.25	-0.15	0.15	0.25	**n**	**r**	-0.25	-0.15	0.15	0.25
3	1	-0.01677	-0.00592	-0.005652	-0.01671	3	1	-0.03318	-0.012240	-0.011621	-0.17804
3	2	-0.000249	-0.00009	-0.00009	-0.000251	3	2	-0.00207	-0.000741	-0.00073	-0.002016
3	3	-0.01215	-0.00434	-0.004513	-0.01277	3	3	-0.00800	-0.002897	-0.00299	-0.008437
5	1	-0.03318	-0.012240	-0.011621	-0.17804	5	1	-0.032781	-0.019998	-0.02211	-5.44399
5	2	-0.004702	-0.00168	-0.001635	-0.004512	5	2	-0.02170	-0.00767	-0.0073	-0.02628
5	3	-0.00051	-0.000183	-0.000184	-0.00051	5	3	-0.00337	-0.0012035	-0.00118	-0.00326
5	4	-0.00800	-0.00290	-0.00299	-0.00847	5	4	-0.000820	-0.000296	-0.000299	-0.000834
5	5	-0.02069	-0.00648	-0.00681	-0.01928	5	5	-0.013217	-0.004677	-0.00488	-0.01380
7	1	-0.03847	-0.01677	-0.01671	-1.4507	7	1	0.001957	-0.024176	-0.03532	-25.2189
7	2	-0.01470	-0.00519	-0.004972	-0.0141907	7	2	-0.03562	-0.013523	-0.01292	-0.33037
7	3	-0.001583	-0.000576	-0.00056	-0.001546	7	3	-0.016765	-0.005917	-0.00565	-0.01671
7	4	-0.000689	-0.00025	-0.000251	-0.00070	7	4	-0.00409	-0.001458	-0.001425	-0.00393
7	5	-0.00613	-0.00222	-0.00229	-0.00643	7	5	0	0	0	0
7	6	-0.01419	-0.00497	-0.00519	-0.01470	7	6	-0.00393	-0.001425	-0.001458	-0.004089
7	7	-0.02735	-0.00741	-0.00781	-0.02207	7	7	-0.01671	-0.005652	-0.00592	-0.01677

Table 4: Shannon entropy in $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$ for the copula model of HK–FGM2

$H(Y_{[r,n,0,1]})$		$p_2 = 1$				$H(Y_{[r,n,1,1]})$		$p_2 = 1$			
		θ_2						θ_2			
n	**r**	0.25	0.5	0.75	0.99	**n**	**r**	0.25	0.5	0.75	0.99
3	1	-0.00261	-0.010483	-0.02378	-0.04192	3	1	-0.00464	-0.018731	-0.04279	-0.07625
3	2	0	0	0	0	3	2	-0.000651	-0.00261	-0.00588	-0.010273
3	3	-0.002608	-0.010483	-0.02378	-0.04192	3	3	-0.001466	-0.00588	-0.01329	-0.023301
5	1	-0.00464	-0.018731	-0.04279	-0.07625	5	1	-0.00609	-0.027114	-0.06242	-0.11274
5	2	-0.001158	-0.00464	0.010483	-0.01835	5	2	-0.00345	-0.013893	-0.03161	-0.05595
5	3	0	0	0	0	5	3	-0.00102	-0.004079	-0.00921	-0.01611
5	4	-0.001158	-0.00464	-0.010483	-0.01835	5	4	$-2.543*10^{-6}$	-0.000010	-0.000023	-0.000040
5	5	-0.00464	-0.01873	-0.04279	-0.07625	5	5	-0.002691	-0.010815	-0.02454	-0.04327
7	1	-0.00588	-0.02378	-0.05458	-0.09801	7	1	-0.007689	-0.031207	-0.072134	-0.13132
7	2	-0.00261	-0.010483	-0.023781	-0.04192	7	2	-0.005155	-0.020819	-0.04765	-0.08517
7	3	-0.000651	-0.00261	-0.00588	-0.010273	7	3	-0.002955	-0.011884	-0.02699	-0.04766
7	4	0	0	0	0	7	4	-0.0010	-0.0006	-0.0029	-0.0081
7	5	-0.000651	-0.00261	-0.00588	-0.010273	7	5	-0.000143	-0.000572	-0.001288	-0.002246
7	6	-0.00261	-0010483	-0.02378	-0.04192	7	6	-0.000274	-0.001096	-0.002467	-0.004304
7	7	-0.00588	-0.02378	-0.054578	-0.09801	7	7	-0.003524	-0.014189-0.03229	-0.057177	

Table 5: Shannon entropy in $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$ for the copula model of HK–FGM2

$H(Y_{[r,n,0,1]})$		$p_2 = 2$				$H(Y_{[r,n,1,1]})$		$p_2 = 2$			
		θ_2						θ_2			
n	**r**	±0.25	±0.15	±0.10	±0.05	**n**	**r**	±0.25	±0.15	±0.10	±0.05
3	1	-0.00042	-0.00015	-0.00007	-0.00002	3	1	-0.00378	-0.00014	-0.00006	-0.00002
3	2	-0.00010	-0.00004	-0.00002	-4.16710^{-6}	3	2	-0.00048	-0.00017	-0.00008	-0.00002
3	3	-0.00094	-0.00034	-0.00015	-0.00004	3	3	-0.00033	-0.00012	-0.00005	-0.00001
5	1	-0.00038	-0.00014	-0.00006	-0.00002	5	1	-0.00022	-0.00008	-0.00004	$-8.8155*10^{-6}$
5	2	-0.00059	-0.000213	-0.0009	-0.00002	5	2	-0.00060	-0.00022	-0.00010	-0.00002
5	3	-0.000213	-0.00008	-0.00003	$-8.503*10^{-6}$	5	3	-0.00071	-0.00026	-0.00011	-0.00003
5	4	-0.00009	-0.00003	-0.00002	$-3.779*10^{-6}$	5	4	-0.00024	-0.00009	-0.00004	$-9.4462*10^{-6}$
5	5	-0.00237	-0.00085	-0.00038	-0.00009	5	5	-0.00087	-0.00031	-0.00014	-0.00003
7	1	-0.00029	-0.00010	-0.00005	-0.00001	7	1	-0.00014	-0.00005	-0.00002	$-5.4222*10^{-6}$
7	2	-0.00065	-0.00023	-0.00010	-0.00003	7	2	-0.00044	-0.00016	-0.00007	-0.00002
7	3	-0.00065	-0.00023	-0.00010	-0.00003	7	3	-0.00073	-0.00026	-0.00012	-0.00002
7	4	-0.00029	-0.00010	-0.00005	-0.00001	7	4	-0.00082	-0.00030	-0.00013	-0.00003
7	5	0	0	0	0	7	5	-0.00057	-0.00021	-0.00010	-0.00002
7	6	-0.00065	-0.00023	-0.00010	-0.00003	7	6	-0.00006	-0.00002	$-9.671*10^{-6}$	$-2.393*10^{-6}$
7	7	-0.00355	-0.00128	-0.00057	-0.00014	7	7	-0.00137	-0.00049	-0.00022	-0.00005

3.2. Comparison between HK–FGM1 and HK–FGM2, as well as OOSs and SOSs, according to inaccuracy measure

Table 6 displays the inaccuracy measure of HK–FGM1 with exponential marginals with mean = 1, at $p_1 = 1,2,3,4$, and $\theta_1 = 0.25$, according to OOSs and SOSs. The calculations were carried out based on Theorem 2.3 and by using the MATHEMATICA Ver.11.3. Table 6 uncovered the following features:

1. In general, $I^\star(Y_{[r,n,0,1]}, Y) < I^\star(Y_{[r,n,1,1]}, Y)$, if $r < \frac{n+1}{2}$, and $I^\star(Y_{[r,n,0,1]}, Y) > I^\star(Y_{[r,n,1,1]}, Y)$, if $r > \frac{n+1}{2}$.
2. The maximum value of $I^\star(Y_{[r,n,0,1]}, Y)$ attains at $(r, n) = (1, 5)$ and $(\theta_1, p_1) = (0.25, 4)$.
3. The maximum value of $I^\star(Y_{[r,n,1,1]}, Y)$ attains at $(r, n) = (1, 5)$ and $(\theta_1, p_1) = (0.25, 4)$.

Table 7 displays the inaccuracy measure of HK–FGM2 with exponential marginals with mean = 1, at $p_2 = 1,2,3,4$, and $\theta_2 = 0.25$, according to OOSs and SOSs. The calculations were carried out based on Theorem 2.4 and by using the MATHEMATICA Ver.11.3. Table 7 uncovered the following features:

1. In general, $I(Y_{[r,n,0,1]}, Y) < I(Y_{[r,n,1,1]}, Y)$, if $r < \frac{n+1}{2}$, and $I(Y_{[r,n,0,1]}, Y) > I(Y_{[r,n,1,1]}, Y)$, if $r > \frac{n+1}{2}$.
2. The maximum value of $I(Y_{[r,n,0,1]}, Y)$ attains at $(r, n) = (2, 5)$ and $(\theta_2, p_2) = (0.25, 2)$.
3. The maximum value of $I(Y_{[r,n,1,1]}, Y)$ attains at $(r, n) = (2, 5)$ and $(\theta_2, p_2) = (0.25, 2)$.

Moreover, Tables 6 and 7 reveal a fact that $I(Y_{[r,n,0,1]}, Y) > I^\star(Y_{[r,n,0,1]}, Y)$, and $I(Y_{[r,n,1,1]}, Y) > I^\star(Y_{[r,n,1,1]}, Y)$.

Table 6: Inaccuracy measure for HK–FGM1 in $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$

$I^\star(Y_{[r,n,0,1]}, Y)$		$\theta_1 = 0.25$				$I^\star(Y_{[r,n,1,1]}, Y)$		$\theta_1 = 0.25$			
		p_1						p_1			
n	**r**	1	2	3	4	**n**	**r**	1	2	3	4
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1.08333	1.10417	1.1625	1.21389	2	1	1.125	1.16667	1.27083	1.36667
3	1	1.125	1.16667	1.27083	1.36667	3	1	1.16667	1.2381	1.40625	1.57037
3	2	1	0.979167	0.945833	0.908333	3	2	1.0625	1.05952	1.06771	1.06111
5	1	1.16667	1.2381	1.40625	1.57037	5	1	1.2	1.30303	1.54167	1.78974
5	2	1.08333	1.08929	1.11607	1.12477	5	2	1.14375	1.18939	1.29792	1.3859
5	3	1	1.01786	0.922619	0.870139	5	3	1.07813	1.07576	1.08464	1.07179

Table 7: Inaccuracy measure for HK–FGM2 in $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$

$I(Y_{[r,n,0,1]}, Y)$		$\theta_2 = 0.25$				$I(Y_{[r,n,1,1]}, Y)$		$\theta_2 = 0.25$			
		p_2						p_2			
n	**r**	1	2	3	4	**n**	**r**	1	2	3	4
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1.08333	1.02083	1.00833	1.00417	2	1	1.125	1.025	1.00833	1.00357
3	1	1.125	1.025	1.00833	1.00357	3	1	1.16667	1.02381	1.00595	1.00198
3	2	1	.0125	1.00833	1.00536	3	2	1.0625	1.02679	1.0119	1.00595
5	1	1.16667	1.02381	1.00595	1.00198	5	1	1.2	1.01818	1.00303	1.0007
5	2	1.08333	1.02976	1.0119	1.00546	5	2	1.14375	1.03011	1.00814	1.00267
5	3	1	1.01786	1.0119	1.00744	5	3	1.07813	1.03267	1.01326	1.0059

3.3. Comparison between HK–FGM1 and HK–FGM2 according to FIN

Table 8 displays the FIN of HK–FGM1 with exponential marginals with mean = 1, at $p_1 = 1, 2, 3, 4$, and $\theta_1 = 0.25$, according to OOSs and SOSs. The calculations were carried out based on the result of Abd Elgwad et al. [4] and by using the MATHEMATICA Ver.11.3. Table 8 revealed the following properties:

1. In general, $\mathrm{FIN}(Y_{[r,n,0,1]}) < \mathrm{FIN}(Y_{[r,n,1,1]})$.
2. The maximum value of $\mathrm{FIN}(Y_{[r,n,0,1]})$ attains at $(r, n) = (3, 10)$ and $(\theta_1, p_1) = (0.25, 3)$.
3. The maximum value of $\mathrm{FIN}(Y_{[r,n,1,1]})$ attains at $(r, n) = (3, 10)$ and $(\theta_1, p_1) = (0.25, 4)$.

Table 9 displays the inaccuracy measure of HK–FGM2 with exponential marginals with mean = 1, at $p_2 = 1, 2, 3, 4$, and $\theta_2 = 0.25$, according to OOSs and SOSs. The calculations were carried out based on the result of Theorem 2.5 and by using the MATHEMATICA Ver.11.3. Table 9 revealed the following properties:

1. In general, we could not recognize any general trend based on r or n for which one can decide $\mathrm{FIN}(Y_{[r,n,0,1]}) < \mathrm{FIN}(Y_{[r,n,1,1]})$, or $\mathrm{FIN}(Y_{[r,n,0,1]}) > \mathrm{FIN}(Y_{[r,n,1,1]})$.
2. The maximum value of $\mathrm{FIN}(Y_{[r,n,0,1]})$ attains at $(r, n) = (8, 10)$ and $(\theta_2, p_2) = (0.25, 2)$.
3. The maximum value of $\mathrm{FIN}(Y_{[r,n,1,1]})$ attains at $(r, n) = (3, 10)$ and $(\theta_2, p_2) = (0.25, 4)$.

Moreover, Tables 8 and 9 reveal a fact that $\mathrm{FIN}(Y_{[r,n,0,1]})$ according to HK-FGM2 is greater than $\mathrm{FIN}(Y_{[r,n,0,1]})$ according to HK-FGM1 and $\mathrm{FIN}(Y_{[r,n,1,1]})$ according to HK–FGM2 is greater than $\mathrm{FIN}(Y_{[r,n,1,1]})$ according to HK-FGM1.

Table 8: FIN of $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$ for the copula model of HK–FGM1

FIN($Y_{[r,n,0,1]}$)		$\theta_1 = 0.25$				FIN($Y_{[r,n,1,1]}$)		$\theta_1 = 0.25$			
		p_1						p_1			
n	**r**	1	2	3	4	**n**	**r**	1	2	3	4
6	3	1.07316	1.0361	0.976385	0.919482	6	3	1.23928	1.33696	1.36076	1.34337
6	4	0.930243	0.8309	0.762424	0.724453	6	4	1.10509	1.07871	1.01571	0.949775
8	3	1.17634	1.21221	1.18963	1.14354	8	3	1.33009	1.54063	1.67244	1.74288
8	4	1.0566	1	0.926224	0.862401	8	4	1.23501	1.32082	1.32806	1.2931
8	5	0.945459	0.841588	0.765936	0.723209	8	5	1.1341	1.12184	1.05977	0.987677
8	6	0.84223	0.726642	0.673399	0.661014	8	6	1.02387	0.94221	0.857054	0.793247
10	3	1.24557	1.34952	1.37811	1.36371	10	3	1.38377	1.67644	1.90413	2.06853
10	4	1.14279	1.14201	1.09002	1.02471	10	4	1.30948	1.4863	1.57703	1.60552
10	5	1.04615	0.977427	0.895681	0.829068	10	5	1.23244	1.31089	1.30781	1.26208
10	6	0.955226	0.848433	0.767834	0.721874	10	6	1.15159	1.14905	1.08831	1.01248
10	7	0.869634	0.749302	0.68682	0.667347	10	7	1.06521	1	0.912857	0.838703
10	8	0.789049	0.675483	0.638147	0.642186	10	8	0.970159	0.863214	0.777284	0.726245

Table 9: FIN of $Y_{[r,n,0,1]}$ and $Y_{[r,n,1,1]}$ for the copula model of HK–FGM2

FIN($Y_{[r,n,0,1]}$)		$\theta_2 = 0.25$			FIN($Y_{[r,n,1,1]}$)		$\theta_2 = 0.25$		
		p_2					p_2		
n	**r**	1	2	3	**n**	**r**	1	2	3
6	3	0.900356	0.923376	0.944998	6	3	0.87614	0.93684	0.96778
6	4	0.965102	0.943875	0.944998	6	4	0.884262	0.918218	0.945871
8	3	0.878371	0.927202	0.957171	8	3	0.884934	0.955843	0.982813
8	4	0.896963	0.917094	0.939855	8	4	0.871095	0.934531	0.967672
8	5	0.94643	0.929775	0.935694	8	5	0.873589	0.917096	0.949243
8	6	1.03403	0.979215	0.959861	8	6	0.901177	0.912422	0.933622
10	3	0.87589	0.937809	0.968867	10	3	0.897175	0.968377	0.990052
10	4	0.87589	0.919979	0.951055	10	4	0.878188	0.950985	0.98066
10	5	0.894816	0.912814	0.93633	10	5	0.867898	0.933123	0.967733
10	6	0.93482	0.921431	0.930667	10	6	0.868169	0.917546	0.952191
10	7	1	0.953341	0.942123	10	7	0.883051	0.908468	0.936707
10	8	1.09601	1.02139	0.985298	10	8	0.921847	0.914211	0.92777

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