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Injective edge coloring of product graphs and some complexity results

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Abstract. Three edges e_1, e_2 and e_3 in a graph G are consecutive if they form a cycle of length 3 or a path in this order. A k-injective edge coloring of a graph G is an edge coloring of G, (not necessarily proper), such that if edges e_1, e_2, e_3 are consecutive, then e_1 and e_3 receive distinct colors. The minimum k for which G has a k-injective edge coloring is called the injective edge chromatic index, denoted by $\chi_i'(G)$ [4]. In this article, the injective edge chromatic index of the resultant graphs by the operations union, join, Cartesian product and corona product of G and H are determined, where G and H are different classes of graphs. Also for any two arbitrary graphs G and H, bounds for $\chi_i'(G + H)$ and $\chi_i'(G \bigodot H)$ are obtained. Moreover the injective edge coloring problem restricted to $(2, 3, r)$-triregular graph, $(2, 4, r)$-triregular graph and $(2, r)$-biregular graph, $r \geq 3$ are also been demonstrated to be NP-complete.

1. Introduction

All graphs considered in this article are simple, finite and undirected. The sets V and E represent the vertex set and edge set of a graph G and the symbols $\Delta(G), \omega(G)$ and $N(u)$ denote the maximum degree, clique number of a graph and neighborhood set of a vertex $u \in V(G)$ respectively. For further graph-theoretic notations and terminologies refer [12] and [15].

An injective coloring of G is a coloring of the vertices of G such that for every vertex $v \in V(G)$, all the neighbors of v are assigned distinct colors, i.e., if x and y are two distinct neighbors of v, then $c(x) \neq c(y)$. The smallest integer k such that G has an injective k -coloring is the injective chromatic number of G, denoted by $\chi_i(G)$. Injective coloring of graphs was introduced by Hahn et al. in [11] and was originated from complexity theory on random access machines, and can be applied in the theory of error correcting codes [11]. In the same paper, they proved that, for $k \geq 3$, it is NP-complete to decide whether the injective chromatic number of a graph is at most k. Since then, many researchers studied on this coloring number and found many beautiful results.

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Similar to the injective coloring, an edge version of the injective coloring was introduced by Cardoso et al. in [3]. An *Injective edge coloring (i-edge coloring)* of a graph G is a coloring, $c : E(G) \to C$, such that if e_1, e_2 and e_3 are consecutive edges in G, then $c(e_1) \neq c(e_3)$. The injective edge coloring number or the injective edge chromatic index of a graph G, $\chi'_i(G)$, is the minimum number of colors permitted in an i-edge coloring. In the same paper, they gave the exact values of the injective edge coloring number for several classes of graphs, such as path, complete bipartite graph, complete graph and so on. And further, they also gave some bounds on injective edge coloring number of some graph and proved that checking whether $\chi'_i(G) = k$ is NP-complete.

A graph G is called an ω' edge injective colorable (or perfect EIC-) graph if $\chi'_i(G) = \omega'$, see [16]. In [16], Yue et al. constructed some perfect EIC-graphs, and gave a sharp bound of the injective coloring number of a 2-connected graph with some forbidden conditions. Also, they characterize some perfect EIC-graph classes. Moreover, Bu and Qi [1] and Ferdjallah [6] studied the injective edge coloring of sparse graphs in terms of the maximum average degree. Also, the injective edge coloring of subcubic graphs is well studied by Ferdjallah in [7] the authors also obtained the upper bounds for injective edge chromatic index and presented the relationships of the injective edge-coloring with other colorings of graphs.

In [13] Kostochka et al. provided, how large can be the injective edge chromatic index of G in terms of the maximum degree of G when there is a restriction on girth and/or chromatic number of G. They also compare the bounds with analogous bounds on the strong chromatic index. In the same year, Y Li and L Chen [14] gave the injective edge coloring numbers of generalized Petersen graphs $P(n, 1)$ and $P(n, 2)$. They determined the exact values of injective edge coloring numbers for $P(n, 1)$ with $n \geq 3$, and for $P(n, 2)$ with $4 \leq n \leq 7$. For $n \geq 8$, they gave that $4 \leq \chi'_i(P(n, 2)) \leq 5$. In [8], Foucaud et al. proved that injective 3-Edge-Coloring is NP-complete, even for triangle-free cubic graphs, planar subcubic graphs of arbitrarily large girth, and planar bipartite subcubic graphs of girth 6. Injective 4-Edge-Coloring remains NP-complete for cubic graphs. Also provided is that for any $k \geq 45$, injective k-Edge-Coloring remains NP-complete even for graphs of maximum degree at most $5\sqrt{3k}$. Further given that injective k-Edge-Coloring is linear-time solvable on graphs of bounded tree width. Moreover, they proved that all planar bipartite subcubic graphs of girth at least 16 are injectively 3-edge-colorable and any graph of maximum degree at most $\frac{k}{2}$ is injectively k-edge-colorable.

Some results which are useful in this article are given as follows.

Proposition 1.1 ([3]). *Let $P_n(C_n)$ be a path (cycle) of order n, $K_{m,n}$ be a complete bipartite graph, and W_n be a wheel graph on n vertices. Then*

i. $\chi'_i(P_n) = 2$, *for* $n \geq 4$,

ii. $\chi'_i(C_n) = \begin{cases} 2 \; if \; n \equiv 0 \bmod 4, \\ 3 \; otherwise \end{cases}$

iii. $\chi'_i(K_{m,n}) = min\{m, n\}$ *and*

iv. *For* $n \geq 4$, $\chi'_i(W_n) = \begin{cases} 6 \; if \; n \; is \; even, \\ 4 \; if \; n \; is \; odd \; and \; n - 1 \equiv 0 \bmod 4, \\ 5 \; if \; n \; is \; odd \; and \; n - 1 \not\equiv 0 \bmod 4. \end{cases}$

Proposition 1.2 ([3]). *If H is a subgraph of a connected graph G, then* $\chi'_i(H) \leq \chi'_i(G)$.

2. Results on injective edge coloring

The definition of the *bi-star graph* $B_{m,n}$ is the graph obtained from K_2 by joining m pendant edges to one end and n pendant edges to the other end of K_2. The *union* $G = G_1 \cup G_2$ of two graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Corollary 2.1. *For any bi-star graph* $G \cong B(m, n)$, $\chi'_i(G) = 2$.

Corollary 2.2. *Let* $G = \cup_{j=1}^{m}(G_j)$. *Then* $\chi_i'(G) = max\{\chi_i'(G_j) : j = 1, 2, 3, \cdots, m\}$.

In this section, the exact values of the injective edge chromatic index of the join of various kinds of graphs and a lower bound for the injective edge chromatic index of the join of two arbitrary graphs are discussed. In general, natural numbers are used as colors of edges. From [12] the *join* of G_1 and G_2, denoted by $G_1 + G_2$, has vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{xy : x \in V_1, y \in V_2\}$. Also we have $\overline{G_1 + G_2} = \overline{G_1} \cup \overline{G_2}$ [5]. Now moving to some results on $G + H$, let $u_1, u_2, \cdots, u_n$ be the vertices of G and $v_1, v_2, \cdots, v_n$ be the vertices of H. In Figure 2.1, u_iu_j, u_jv_k, v_kv_l form consecutive edges. Where u_iu_j, is an edge in G and v_kv_l is an edge in H. Thus we can say that no color of the edges in G can be the color of edges in H. Therefore the lower bound of injective edge chromatic index of $G + H$.

Proposition 2.3. $\chi_i'(G + H) \geq \chi_i'(G) + \chi_i'(H)$.

Proposition 2.4. $\chi_i'(\overline{G_1 + G_2}) = max\{\chi_i'(G_1), \chi_i'(G_2)\}$.

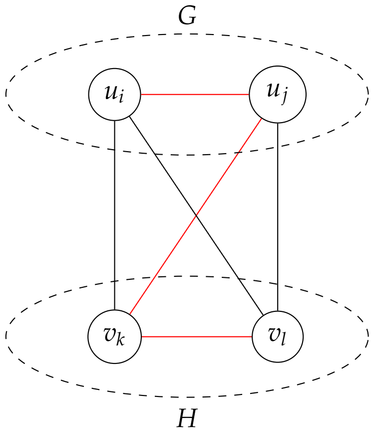

Figure 2.1

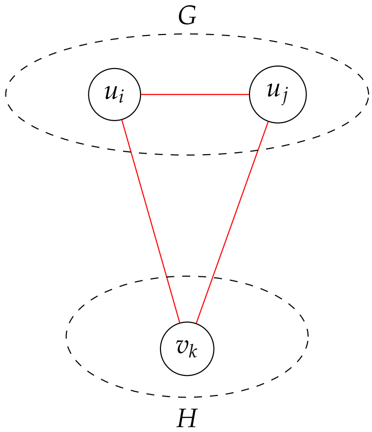

Figure 2.2

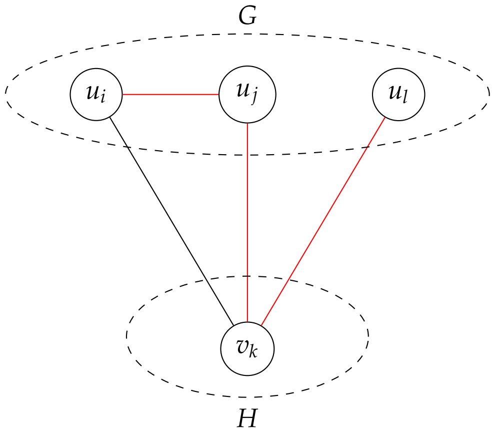

Figure 2.3

In particular we have, $K_n + K_m = K_{m+n}$ and $\overline{K_n} + \overline{K_m} = K_{m,n}$ [12]. In general a complete k-partite graph $K_{t_1,t_2,\cdots,t_k} = \overline{K_{t_1}} + \overline{K_{t_2}} + \cdots + \overline{K_{t_k}}$ [10].

Proposition 2.5. $\chi_i'(K_n + K_m) = \frac{(m+n)(m+n-1)}{2}$ *where* $m, n \geq 1$.

Proposition 2.6. $\chi_i'(\overline{K_n} + \overline{K_m}) = min\{m, n\}$ *where* $m, n \geq 1$.

Proposition 2.7. $\chi_i'(\overline{K_{t_1}} + \overline{K_{t_2}} + \cdots + \overline{K_{t_k}}) = min\{t_1, t_2, \cdots, t_k\}$ *where* $t_i \geq 1, 1 \leq i \leq k$.

A *fan graph* $F_{m,n}$ is defined as the graph join $\overline{K_m} + P_n$, where $\overline{K_m}$ is the empty graph on m nodes and P_n is the path graph on n nodes (see [10]). Next results are on the join of K_n, $\overline{K_n}$, P_n and C_n.

Theorem 2.8. $\chi_i'(K_n + \overline{K_m}) = n + \frac{n(n-1)}{2}$ *where* $m, n \geq 1$.

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of K_n and $v_1, v_2, \cdots, v_m$ be the vertices of $\overline{K_m}$. As the vertices $u_1, u_2, \cdots, u_n$ form an induced complete subgraph of $K_n + \overline{K_m}$, the edges $u_iu_j, i \neq j, i, j = 1, 2, \cdots, n$ are colored with distinct $\frac{n(n-1)}{2}$ colors. Now the edges u_iu_j, u_jv_k, v_ku_i in Figure 2.2 and u_iu_j, u_jv_k, v_ku_l in Figure 2.3 form consecutive edges and so no color of u_iu_j can be the color of u_kv_l, $i, j, l = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, m$. Next we can see that v_1u_i, u_iu_j, u_jv_1, $i, j = 1, 2, \cdots, n$ form consecutive edges. Thus the edges v_1u_i, $i = 1, 2, \cdots, n$ are colored with a new set of n colors. The same set of colors are used to color the edges v_ku_i, $k = 2, 3, \cdots, m$ and $i = 1, 2, \cdots, n$. That is for a fixed $k, 1 \leq k \leq m$, the edges v_ku_i is colored with color $\frac{n(n-1)}{2} + i$, $1 \leq i \leq n$. This gives the injective edge chromatic index of $K_n + \overline{K_m}$. □

Theorem 2.9. *$\chi'_i(K_n + P_m) = \frac{n^2+3n+4}{2}$ where $n \geq 1$, $m \geq 3$.*

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of K_n and $v_1, v_2, \cdots, v_m$ be the vertices of P_m. From Proposition 2.3, it is clear that $\chi'_i(K_n + P_m) \geq \chi'_i(K_n) + \chi'_i(P_m) = \frac{n(n-1)}{2} + 2$. First color the edges u_iu_j and v_kv_l, $i, j = 1, 2, \cdots, n$ and $k, l = 1, 2, \cdots, m$ with distinct $\frac{n(n-1)}{2} + 2$ colors. Now from Figure 2.2 and Figure 2.3 we can see that, no color the edges u_iu_j and v_kv_l can be the color of u_rv_s, $i, j, r = 1, 2, \cdots, n$ and $k, l, s = 1, 2, \cdots, m$. Also for a fixed k, the vertices v_k, u_i and u_j form an induced K_3 for any $i \neq j$, thus the edges v_ku_i and v_ku_j are colored with distinct colors. Further, the edges u_iv_k, v_kv_{k+1} and $v_{k+1}u_j$ form consecutive edges, thus no color of v_ku_i can be the color of $v_{k+1}u_j$. Now color the edges u_iv_j, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$ as follows.

- For an odd k, the edge v_ku_i is colored with color $\frac{n^2-n+4}{2} + i$.
- For an even k, the edge v_ku_i is colored with color $\frac{n^2+n+4}{2} + i$.

Thus distinct $2n$ colors are needed to color the edges u_iv_k. Hence $\chi'_i(K_n + P_m) = \frac{n(n-1)}{2} + 2 + 2n = \frac{n^2+3n+4}{2}$. □

Theorem 2.10. *For a fan graph $F_{m,n}$, the injective edge chromatic index is,* $\chi'_i(F_{m,n}) = \begin{cases} \chi'_i(P_n) + 2m \text{ if } 2m \leq n, \\ \chi'_i(P_n) + n \text{ if } n < 2m. \end{cases}$

Proof. We have $F_{m,n} = \overline{K_m} + P_n$. Let $u_1, u_2, \cdots, u_m$ be the vertices of $\overline{K_m}$ and $v_1, v_2, \cdots, v_n$ be the vertices of P_n. Since the edges u_iv_j, v_jv_{j+1} and $v_{j+1}u_i$ form consecutive edges, no color of u_iv_j can be the color of u_iv_{j+1}. Similarly, u_iv_j, v_ju_k, and u_kv_l form consecutive edges, no color of u_iv_j can be the color of u_kv_l. Also, no color of the edges v_jv_{j+1} (the edges of P_n) can be the color of u_iv_k. Since the vertices u_i, v_j and v_{j+1} form an induced K_3. With these arguments color the edges in each case.

Case 1. Assume that $2m \leq n$.

- For a fixed i color the edges u_iv_j with color $2i - 1$ for odd j and color $2i$ for even j. Thus $2m$ distinct colors are used to color the edges u_iv_j
- Now color the edges v_jv_{j+1} (the edges of P_n) with new set of $\chi'_i(P_n)$ colors.

Case 2. Assume that $n < 2m$.

- For a fixed j color the edges v_ju_i with the color j. Thus n distinct colors are used to color the edges u_iv_j.
- Now color the edges v_jv_{j+1} (the edges of P_n) with a new set of $\chi'_i(P_n)$ colors.

The above coloring procedure produces the injective edge chromatic index of the graph $F_{m,n}$. □

Illustration 2.11. *Injective edge coloring of $F_{2,5}$ and $F_{3,2}$.*

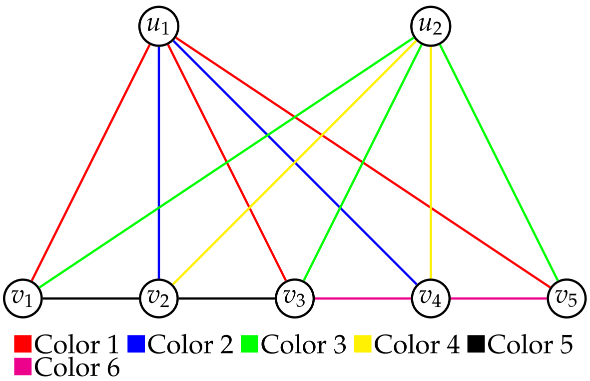

Figure 2.4: Injective Edge Coloring of $F_{2,5}$

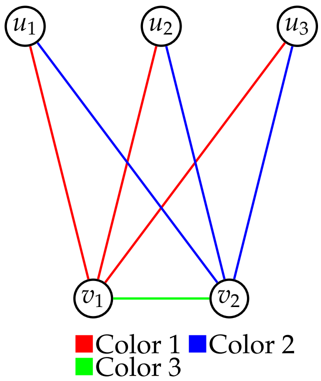

Figure 2.5: Injective Edge Coloring of $F_{3,2}$

Theorem 2.12. *For $n \geq 1$ and $m \geq 3$,* $\chi'_i(K_n + C_m) = \begin{cases} \frac{n(n-1)}{2} + \chi'_i(C_m) + 2n \text{ if } m \text{ even}, \\ \frac{n(n-1)}{2} + \chi'_i(C_m) + 3n \text{ if } m \text{ odd}. \end{cases}$

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of K_n and $v_1, v_2, \cdots, v_m$ be the vertices of C_m. First color the edges u_iu_j, $i, j = 1, 2, \cdots, n$ of K_n with distinct $\frac{n(n-1)}{2}$ colors and color the edges v_kv_l, $k, l = 1, 2, \cdots, m$ of C_m with $\chi'_i(C_m)$ new colors. From Figure 2.2 and Figure 2.3 we can see that no color of the edges u_iu_j (edges of K_n) and v_kv_l (edges of C_m) can be the color of the edges u_iv_k (the edges joining vertices of K_n and C_m). Now for a fixed i, the vertices v_i, u_j and u_k form an induced K_3, thus the edges v_iu_j, $j = 1, 2, \cdots, n$, colored with distinct n colors. Also for an edge v_iv_j of C_m, the edges v_iu_k and v_ju_l are colored with distinct colors, since $u_kv_i - v_iv_j - v_ju_l$ form consecutive edges. Now color the edges u_iv_k as follows.
Case 1. Assume that m is odd.

- For $i = 2k + 1$, $i < m$, the edges v_iu_j are colored with color j.
- For $i = 2k$, $i < m$, the edges v_iu_j are colored with color $n + j$.
- Color the edges v_mu_j with the colors $2n + j$

Case 2. Assume that m is even.

- For $i = 2k + 1$, $i < m$, the edges v_iu_j are colored with color j.
- For $i = 2k$, $i \leq m$, the edges v_iu_j are colored with color $n + j$.

The coloring described above produces the injective edge chromatic index of $K_n + C_m$. □

Theorem 2.13. $\chi'_i(P_n + P_m) = 2min\{m, n\} + \chi'_i(P_n) + \chi'_i(P_m)$ *where* $m, n \geq 2$.

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of P_n and $v_1, v_2, \cdots, v_m$ be the vertices of P_m. First color the edges u_iu_j, $i, j = 1, 2, \cdots, n$ with $\chi'_i(P_n)$ colors and the edges v_kv_l, $k, l = 1, 2, \cdots, m$ with $\chi'_i(P_m)$ colors.
Now let $m \leq n$. We start with the coloring of v_1u_j, $j = 1, 2, \cdots, n$. The vertices v_1, u_i and u_{i+1} form an induced K_3, thus the edges v_1u_i and v_1u_{i+1} are colored with two distinct colors. Similarly for a fixed k, the edges v_ku_i, $i = 1, 2, \cdots, n$ are colored with two distinct colors. Now v_ru_i, u_iv_l and v_lu_j form consecutive edges. Therefore no v_ru_i and v_lu_j, $r \neq l$, $r, l = 1, 2, \cdots, m$ and $i, j = 1, 2, \cdots, n$ have the same colors. Hence the edges u_iv_j are colored as follows.

- Color the edges v_1u_i with color 1 and 2 alternatively, for $i = 1, 2, \cdots, n$
- Color the edges v_2u_i with color 3 and 4 alternatively, for $i = 1, 2, \cdots, n$

$\vdots$

- Color the edges v_mu_i with color $2m - 1$ and $2m$ alternatively, for $i = 1, 2, \cdots, n$

The coloring described above produces the injective edge chromatic index of $P_n + P_m$. □

Illustration 2.14. *Consider the graph $P_n + P_m$ with $m \leq n$.*

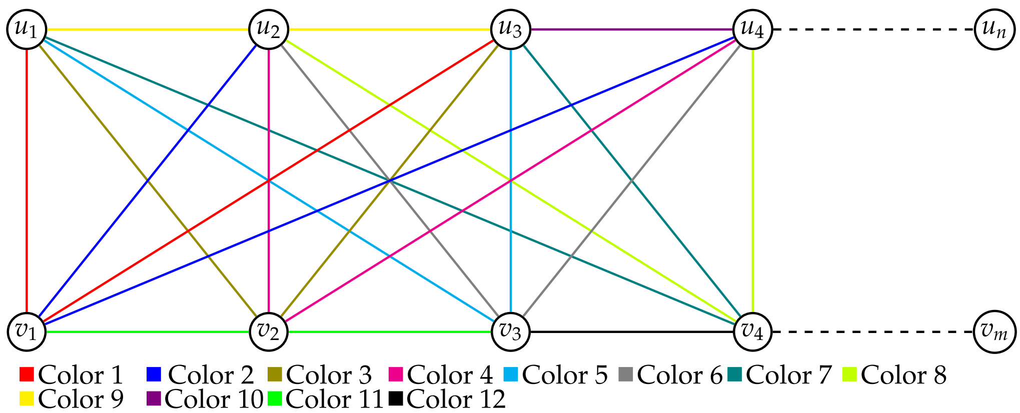

Figure 2.6: Injective edge coloring of $P_n + P_m$

Theorem 2.15. *For any $m, n \geq 3$,* $\chi'_i(C_n + C_m) = \begin{cases} \chi'_i(C_n) + \chi'_i(C_m) + 2\ min\{m,n\}\ if\ m \text{ and } n \text{ are even}, \\ \chi'_i(C_n) + \chi'_i(C_m) + 3\ min\{m,n\}\ if\ m \text{ and } n \text{ are odd}, \\ \chi'_i(C_n) + \chi'_i(C_m) + 2n\ if\ m \text{ even, } n \text{ odd and } 2n \leq 3m, \\ \chi'_i(C_n) + \chi'_i(C_m) + 3m\ if\ m \text{ even, } n \text{ odd and } 3m < 2n. \end{cases}$

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of C_n and $v_1, v_2, \cdots, v_m$ be the vertices of C_m. The edges u_iu_j, $i, j = 1, 2, \cdots, n$ are colored with $\chi'_i(C_n)$ colors and the edges v_kv_l, $k, l = 1, 2, \cdots, m$ are colored with $\chi'_i(C_m)$ colors. From Figure 2.2 and Figure 2.3, we can see that no color of the edges u_iu_j (edges of C_n) and v_kv_l (edges of C_m) can be the color of the edges u_iv_k (the edges joining vertices of C_n and C_m). For a fixed i, the vertices u_i, v_j and v_{j+1} form an induced K_3. So the edges $u_iv_j, 1 \leq j \leq m$, are colored with at least two colors. Also the edges v_ju_i, u_iu_{i+1} and $u_{i+1}v_k$ form consecutive edges. Thus no color of the edges u_iv_j is the color of the edges $u_{i+1}v_k$. With these arguments, the following cases describe the coloring of the edges u_iv_j.
Case 1. Assume that m and n are even and $m \leq n$.

- For odd $i, 1 \leq i \leq n$, color the edges u_iv_j with color $j, j = 1, 2, \cdots, m$.
- For even $i, 1 \leq i \leq n$, color the edges u_iv_j with color $m + j, j = 1, 2, \cdots, m$.

Case 2. Assume that m and n are odd and $m \leq n$.

- For odd $i, 1 \leq i < n$, color the edges u_iv_j with color $j, j = 1, 2, \cdots, m$.
- For even $i, 1 \leq i < n$, color the edges u_iv_j with color $m + j, j = 1, 2, \cdots, m$.
- For $i = n$, color the edges u_iv_j with color $2m + j, j = 1, 2, \cdots, m$.

Case 3. Assume that m even, n odd and $2n \leq 3m$.

- For odd $j, 1 \leq j \leq m$, color the edges v_ju_i with color $i, i = 1, 2, \cdots, n$.
- For even $j, 1 \leq j \leq m$, color the edges v_ju_i with color $n + i, i = 1, 2, \cdots, m$.

Case 4. Assume that m even, n odd and $3m < 2n$.

- For odd $i, 1 \leq i < n$, color the edges u_iv_j with color $j, j = 1, 2, \cdots, m$.
- For even $i, 1 \leq i < n$, color the edges u_iv_j with color $m + j, j = 1, 2, \cdots, m$.
- For $i = n$, color the edges u_iv_j with color $2m + j, j = 1, 2, \cdots, m$.

The coloring described above produces the injective edge chromatic index of $C_n + C_m$. □

Recall the definition of an *n-Ladder graph* [10] as $L_n = P_2 \Box P_n$, where P_n is a path of length n. Now the vertices of L_n be $u_1, u_2, \cdots, u_n$ for the first copy of P_n and $u_{n+1}, u_{n+2}, \cdots, u_{2n}$ for the second copy of P_n. The next theorem gives the injective edge chromatic index of join of any two ladder graphs L_n and L_m.

Proposition 2.16 ([4]). $\chi'_i(L_1) = 1$, $\chi'_i(L_2) = 2$ *and* $\chi'_i(L_n) = 3$ *for all* $n \geq 3$.

Theorem 2.17. $\chi'_i(L_n + L_m) = \chi'_i(L_n) + \chi'_i(L_m) + 4$ *for all* m, n.

Proof. Without loss of generality assume that $m \leq n$. Let $u_1, u_2, \cdots, u_n, u_{n+1}, u_{n+2}, \cdots, u_{2n}$ be the vertices of L_n and let $v_1, v_2, \cdots, v_m, v_{m+1}, v_{m+2}, \cdots, v_{2m}$ be the vertices of L_m. By Proposition 2.3, $\chi'_i(L_n + L_m) \geq \chi'_i(L_n) + \chi'_i(L_m)$. Now color the edges of L_n and L_m with $\chi'_i(L_n) + \chi'_i(L_m)$ colors.

Claim 1: No color of the edges u_iu_j (edges of L_n) is the color of the edges u_kv_l for $i, j, k = 1, 2, \cdots, 2n$ and $l = 1, 2, \cdots, 2m$.
For, let u_ru_s be an edge of L_n with color c_1 (say). Now the vertices u_r, u_s and v_l form an induced K_3, thus the color c_1 cannot be assigned as the color of u_rv_l or u_sv_l, for $l = 1, 2, \cdots, 2m$. Also, the edges u_ru_s, u_sv_l and v_lu_i form consecutive edges, thus the color c_1 cannot be assigned as the color of v_lu_i for $1 \leq i \leq 2n, i \neq r, s$ and $1 \leq l \leq 2m$.

Claim 2: For a fixed i, at least two colors are needed to color the edges u_iv_l, $1 \leq l \leq 2m$.

Let $v_l v_k$ be an edge of L_m. Then the vertices u_i, v_l and v_k form an induced K_3 in the graph $L_n + L_m$. Thus the edges $u_i v_l$ and $u_i v_k$ must receive distinct colors.

Also note that if there is an edge $u_i u_j$, then no color of the edges $u_i v_l$ can be the color of the edges $u_j v_t$ for $1 \leq l,\ t \leq 2m$, for, the edges $v_l u_i, u_i u_j$ and $u_j v_t$ form consecutive edges.

From the above statement, together with Claim 1 and 2, it can be concluded that at least four colors are needed to color the edges $u_k v_l$. Now providing an injective edge coloring using $\chi'_i(L_n) + \chi'_i(L_m) + 4$ colors shows that $\chi'_i(L_n + L_m) = \chi'_i(L_n) + \chi'_i(L_m) + 4$. The coloring is as follows.

- For $i = 1, 3, 5, \cdots, i \leq n$ and $i = n + 2, n + 4, n + 6, \cdots, i \leq 2n$.
  - Color the edges $u_i v_k$ with color 1 for $k = 1, 3, 5, \cdots, k \leq n$ and $k = n + 2, n + 4, n + 6, \cdots, k \leq 2n$.
  - Color the edges $u_i v_k$ with color 2, for $k = 2, 4, 6, \cdots, k \leq n$ and $k = n + 1, n + 3, n + 5, \cdots, k \leq 2n$.
- For $i = 2, 4, 6, \cdots, i \leq n$ and $i = n + 1, n + 3, n + 5, \cdots, i \leq 2n$.
  - Color the edges $u_i v_k$ with color 3 for $k = 1, 3, 5, \cdots, k \leq n$ and $k = n + 2, n + 4, n + 6, \cdots, k \leq 2n$.
  - Color the edges $u_i v_k$ with color 4, for $k = 2, 4, 6, \cdots, k \leq n$ and $k = n + 1, n + 3, n + 5, \cdots, k \leq 2n$.
- Color the edges $u_i u_j$ of L_n with $\chi'_i(L_n)$ colors.
- Color the edges $v_k v_l$ of L_m with $\chi'_i(L_m)$ colors.

□

In the next section, some results on injective edge chromatic index of Cartesian product of different classes of graphs are obtained. Recall from [12] that the *Cartesian product* of G_1 and G_2, denoted by $G_1 \times G_2$, has vertex set $V_1 \times V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and u_2 adjacent to v_2 or $u_2 = v_2$ and u_1 adjacent to v_1. Some results on the injective edge chromatic index of $P_n \square P_m$ are available in [4]. The following are few results on Cartesian product of P_n, C_n and K_n we have obtained.

Proposition 2.18 ([4]). $\chi'_i(P_n \square P_m) = \begin{cases} 3\ if\ n \geq 3,\ m = 2, \\ 4\ if\ m,\ n \geq 4. \end{cases}$

The *Prism graph* [10], denoted by Y_n is a graph corresponding to the skeleton of an n-prism and also Y_n is isomorphic to the graph Cartesian product $P_2 \square C_n$. Further $P_2 \square C_n$ is isomorphic to the generalized Petersen graph $P(n, 1)$. The injective edge chromatic index of the generalized Petersen graph $P(n, 1)$ is given below.

Proposition 2.19 ([14]). *If* $n \geq 6$, $\chi'_i(P(n, 1)) = \begin{cases} 3\ if\ n \equiv\ 0 \bmod 6, \\ 4\ otherwise. \end{cases}$ *Moreover,* $\chi'_i(P(3, 1)) = 6$, $\chi'_i(P(4, 1)) = 4$ *and* $\chi'_i(P(5, 1)) = 5$.

Theorem 2.20. *Injective edge chromatic index of* $P_m \square C_n$ *is obtained as follows*

1. For $n > 5$, $\chi'_i(P_2 \square C_n) = \begin{cases} 3\ if\ n \equiv\ 0 \bmod 6, \\ 4\ otherwise. \end{cases}$ Moreover, $\chi'_i(P_2 \square C_3) = 6$, $\chi'_i(P_2 \square C_4) = 4$ and $\chi'_i(P_2 \square C_5) = 5$.
2. For even n, $\chi'_i(P_3 \square C_n) = 4$. Moreover $\chi'_i(P_3 \square C_3) = 6$ and $\chi'_i(P_3 \square C_5) = 5$.
3. $\chi'_i(P_m \square C_3) = 6$ if $m \geq 2$.
4. $\chi'_i(P_m \square C_n) = 4$ if $n \equiv 0 \bmod 4$ and $m \geq 3$.

Proof.

1. First part of the theorem directly follows from Proposition 2.19.

2. In general the graph $P_3 \square C_n$ consists of 3 cycles C_n^i, $i = 1, 2, 3$, where C_n^i is the i^{th} copy of C_n (with C_n^1 has vertices $u_1, u_2, \cdots, u_n$, C_n^2 has vertices $v_1, v_2, \cdots, v_n$ and C_n^3 has vertices $w_1, w_2, \cdots, w_n$) and the paths $u_i - v_i - w_i$, $i = 1, 2, 3$.
Case 1. Assume that n is even.
Here the graph $P_3 \square P_4$ is a subgraph of $P_3 \square C_n$ with $\chi_i'(P_3 \square P_4) = 4$ (Proposition 2.18). Thus $\chi_i'(P_3 \square C_n) \geq 4$. Now providing an injective edge coloring of $P_3 \square C_n$ with 4 colors shows that $\chi_i'(P_3 \square C_n) = 4$. The coloring in each cases are given below.
Subcase i. $n \equiv 0 \bmod 4$.

   - Color the edges $u_1u_2, u_2u_3, \cdots, u_nu_1$ with colors 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$.
   - Color the edges $v_2v_3, v_3v_4, \cdots, v_nv_1, v_1v_2$, with colors 3 and 4 in the pattern $3, 3, 4, 4, 3, 3, \cdots$.
   - Color the edges $w_1w_2, w_2w_3, \cdots, w_nw_1$ with colors 1 and 2 in the pattern $2, 2, 1, 1, 2, 2, \cdots$.
   - For $i = 1, 5, 9, \cdots$, color the edges u_iv_i and v_iw_i with color 4.
   - For $i = 2, 6, 10, \cdots$, color the edges u_iv_i and v_iw_i with colors 1 and 2 respectively.
   - For $i = 3, 7, 11, \cdots$, color the edges u_iv_i and v_iw_i with color 3.
   - For $i = 4, 8, 12, \cdots$, color the edges u_iv_i and v_iw_i with colors 2 and 1 respectively.

   Subcase ii. $n \equiv 2 \bmod 4$.

   - Color the edges $u_{n-1}u_n$ and u_nu_1 with color 4 and for $i = 1, 2, \cdots, n-2$, color the edges u_iu_{i+1} with the colors 1 and 2 in the order $1, 1, 2, 2, \cdots$.
   - Color the edges $v_{n-2}v_{n-1}, v_{n-1}v_n, v_nv_1$ and v_1v_2 with color $1, 1, 2$ and 2 respectively and for $i = 2, 3, \cdots, n-3$, color the edges v_iv_{i+1} with the colors 3 and 4 in the order $3, 3, 4, 4, \cdots$.
   - Color the edges $w_{n-3}w_{n-2}, w_{n-2}w_{n-1}, w_{n-1}w_n, w_nw_1, w_1w_2$ and w_2w_3 with colors $4, 4, 3, 3, 4$ and 4 respectively and for $i = 3, 4, \cdots, n-4$, color the edges w_iw_{i+1} with the colors 1 and 2 in the order $1, 1, 2, 2, \cdots$.
   - If the adjacent edges u_iu_j and u_ju_k are of same color, assign this color to the edge u_jv_j.
   - If the adjacent edges v_iv_j and v_jv_k are of same color, assign this color to the edges u_jv_j and v_jw_j.
   - If the adjacent edges w_iw_j and w_jw_k are of same color, assign this color to the edge v_jw_j.

   Case 2. Assume that $n = 3, 5$.
The graph $P(3, 1)$ is a subgraph of $P_3 \square C_3$ and from Proposition 2.19, $\chi_i'(P_3 \square C_3) \geq 6$. Now Figure 2.7 provides an injective edge coloring of $P_3 \square C_3$ with 6 colors, which shows that $\chi_i'(P_3 \square C_3) = 6$. Similarly, from Proposition 11, $\chi_i'(P_3 \square C_5) \geq 5$ and Figure 2.8 provides an injective edge coloring of $P_3 \square C_5$ with 5 colors.

Color 1 Color 2 Color 3 Color 4 Color 5 Color 6

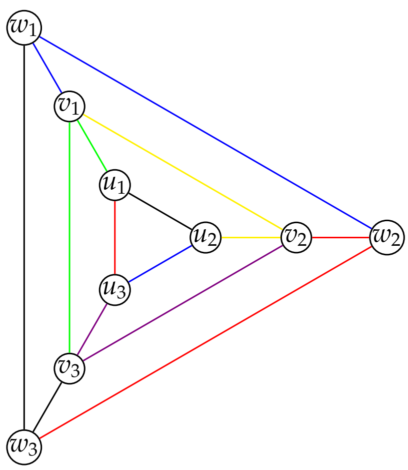

Figure 2.7: Injective edge coloring of $P_3 \square C_3$

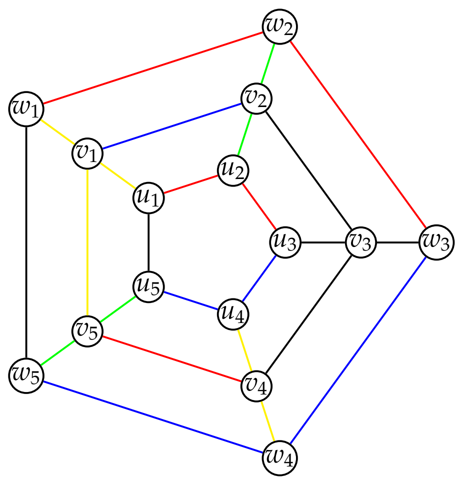

Figure 2.8: Injective edge coloring of $P_3 \square C_5$

3. Here the graph $P_m \square C_3$ consists of m cycles C_3^i, $i = 1, 2, \cdots, m$, where C_3^i is the i^{th} copy of C_3 (C_3^i has vertices u_1^i, u_2^i and u_3^i) and the paths $u_j^1 - u_j^2 - u_j^3 - \cdots - u_j^m$, $j = 1, 2, 3$. The Injective edge chromatic index of $P_2 \square C_3$ and $P_3 \square C_3$ follows from Theorem 2.20(1,2). Now for $m > 3$, the graph $P_3 \square C_3$ is a subgraph of $P_m \square C_3$ with $\chi_i'(P_3 \square C_3) = 6$ and by Proposition 1.2, $\chi_i'(P_m \square C_3) \geq 6$. Now providing an injective edge coloring of $P_m \square C_3$ with 6 colors shows that $\chi_i'(P_m \square C_3) = 6$. The coloring is as follows.

   - For $i = 1, 7, 13, \cdots$, color the edges $u_1^i u_2^i, u_2^i u_3^i$ and $u_3^i u_1^i$ with the colors $1, 2$ and 3 respectively.
   - For $i = 2, 8, 14, \cdots$, color the edges $u_1^i u_2^i, u_2^i u_3^i$ and $u_3^i u_1^i$ with the colors $4, 5$ and 6 respectively.
   - For $i = 3, 9, 15, \cdots$, color the edges $u_1^i u_2^i, u_2^i u_3^i$ and $u_3^i u_1^i$ with the colors $2, 3$ and 1 respectively.
   - For $i = 4, 10, 16, \cdots$, color the edges $u_1^i u_2^i, u_2^i u_3^i$ and $u_3^i u_1^i$ with the colors $5, 6$ and 4 respectively.
   - For $i = 5, 11, 17, \cdots$, color the edges $u_1^i u_2^i, u_2^i u_3^i$ and $u_3^i u_1^i$ with the colors $3, 1$ and 2 respectively.
   - For $i = 6, 12, 18, \cdots$, color the edges $u_1^i u_2^i, u_2^i u_3^i$ and $u_3^i u_1^i$ with the colors $6, 4$ and 5 respectively.
   - Color the edges $u_1^1 u_1^2$, $u_1^2 u_1^3$, $u_1^3 u_1^4, \cdots, u_1^{m-1} u_1^m$ with colors $1, 4, 2, 5, 3, 6$ up to $u_1^6 u_1^7$, repeat the same order of the colors after $u_1^6 u_1^7$ up to the remaining.
   - Color the edges $u_2^1 u_2^2$, $u_2^2 u_2^3$, $u_2^3 u_2^4, \cdots, u_2^{m-1} u_2^m$ with colors $2, 5, 3, 6, 1, 4$ up to $u_2^6 u_2^7$, repeat the same order of the colors after $u_2^6 u_2^7$ up to the remaining.
   - Color the edges $u_3^1 u_3^2$, $u_3^2 u_3^3$, $u_3^3 u_3^4, \cdots, u_3^{m-1} u_3^m$ with colors $3, 6, 1, 4, 2, 5$ up to $u_3^6 u_3^7$, repeat the same order of the colors after $u_3^6 u_3^7$ up to the remaining.

4. In general the graph $P_m \square C_n$ consists of m cycles C_n^i, $i = 1, 2, \cdots, m$, where C_n^i is the i^{th} copy of C_n (C_n^i has vertices $u_1^i, u_2^i, \cdots, u_n^i$) and the paths $u_j^1 - u_j^2 - u_j^3 - \cdots - u_j^m$, $j = 1, 2, 3, \cdots, n$. Here for $n \geq 3$, the graph $P_3 \square P_4$ is a subgraph of $P_m \square C_n$ with $\chi_i'(P_3 \square P_4) = 4$ (Proposition 2.18). Thus $\chi_i'(P_m \square C_n) \geq 4$. Now providing an injective edge coloring of $P_m \square C_n$ with 4 colors shows that $\chi_i'(P_m \square C_n) = 4$. The coloring is as follows.

   - For $i = 1, 5, 9, \cdots$, color the edges $u_1^i u_2^i, u_2^i u_3^i, \cdots, u_n^i u_1^i$ with colors 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$.
   - For $i = 2, 6, 10, \cdots$, color the edges $u_2^i u_3^i, u_3^i u_4^i, \cdots, u_n^i u_1^i, u_1^i u_2^i$, with colors 3 and 4 in the pattern $3, 3, 4, 4, 3, 3, \cdots$.
   - For $i = 3, 7, 11, \cdots$, color the edges $u_1^i u_2^i, u_2^i u_3^i, \cdots, u_n^i u_1^i$ with colors 1 and 2 in the pattern $2, 2, 1, 1, 2, 2, \cdots$.
   - For $i = 4, 8, 12, \cdots$, color the edges $u_2^i u_3^i, u_3^i u_4^i, \cdots, u_n^i u_1^i, u_1^i u_2^i$, with colors 3 and 4 in the pattern $4, 4, 3, 3, 4, 4, \cdots$.
   - For $j = 1, 5, 9, \cdots$, color the edges $u_j^1 u_j^2, u_j^2 u_j^3, \cdots, u_j^{m-1} u_j^m$ with colors 3 and 4 in the pattern $4, 4, 3, 3, 4, 4, \cdots$.
   - For $j = 2, 6, 10, \cdots$, color the edges $u_j^1 u_j^2, u_j^2 u_j^3, \cdots, u_j^{m-1} u_j^m$ with colors 1 and 2 in the pattern $1, 2, 2, 1, 1, 2, 2, \cdots$.
   - For $j = 3, 7, 11, \cdots$, color the edges $u_j^1 u_j^2, u_j^2 u_j^3, \cdots, u_j^{m-1} u_j^m$ with colors 3 and 4 in the pattern $3, 3, 4, 4, 3, 3, \cdots$.
   - For $j = 4, 8, 12, \cdots$, color the edges $u_j^1 u_j^2, u_j^2 u_j^3, \cdots, u_j^{m-1} u_j^m$ with colors 1 and 2 in the pattern $2, 1, 1, 2, 2, 1, 1, 2, 2, \cdots$.

□

Illustration 2.21. *Injective edge coloring of $P_4 \square C_4$ with four colors is illustrated below.*

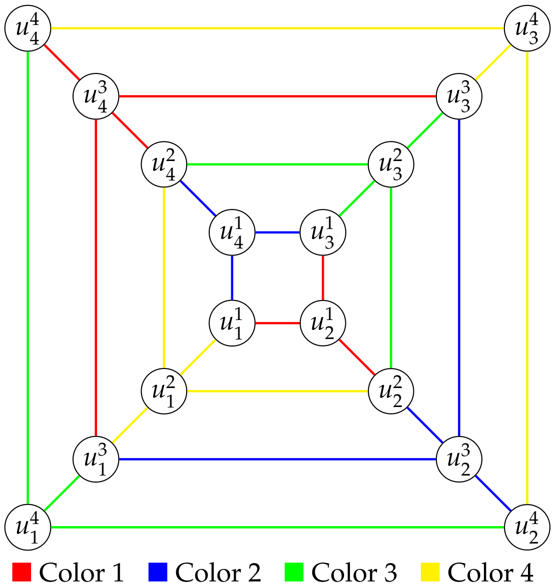

Figure 2.9: Injective edge coloring of $P_4 \square C_4$

From [9], we have the *corona* of two graphs G_1 and G_2 (where G_i has p_i vertices and q_i edges) as the graph $G = G_1 \odot G_2$ obtained by taking one copy of G_1 and p_1 copies of G_2, and then joining by an edge the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. Some results on the injective edge chromatic index of few classes of corona products are given as the following.

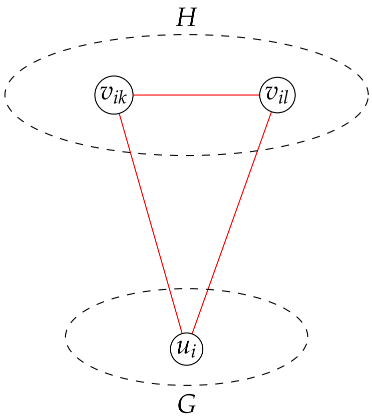

Figure 2.10

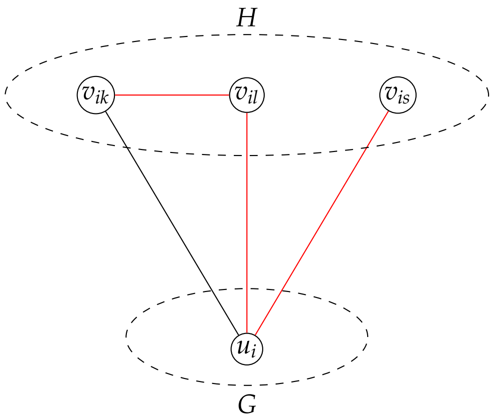

Figure 2.11

Theorem 2.22. *For any two connected nonempty graphs G and H,* $\chi_i'(G \odot H) \geq \chi_i'(H) + 2$.

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of G and $v_{i1}, v_{i2}, \cdots, v_{im}$ be the vertices of i^{th} copy of H for $i = 1, 2, \cdots, n$. Let $v_{ik}v_{il}$ be an arbitrary edge of H. Then the vertices u_i, v_{ik} and v_{il} form an induced K_3 (Figure 2.10). Also, $v_{ik} - v_{il} - u_i - v_{is}$ form paths of length 4 (Figure 2.11). Thus the color of $v_{ik}v_{il}$ cannot be the color of u_iv_{is} for $s = 1, 2, \cdots, m$. Also since the vertices u_i, v_{ik} and v_{il} form an induced K_3, the edges u_iv_{ik} and u_iv_{il} colored with distinct 2 colors other than $\chi_i'(H)$ colors. □

Theorem 2.23. *If* $m, n \geq 2$, *then* $\chi_i'(P_n \odot P_m) = \begin{cases} 4 \; if \; m, n = 2, 3, \\ 5 \; otherwise. \end{cases}$

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of P_n and $v_{i1}, v_{i2}, \cdots, v_{im}$ be the vertices of i^{th} copy of P_m for $i = 1, 2, \cdots, n$.

For Figure 2.12, Figure 2.13, Figure 2.14 and Figure 2.15

■ Color 1 ■ Color 2 ■ Color 3 ■ Color 4 ■ Color 4

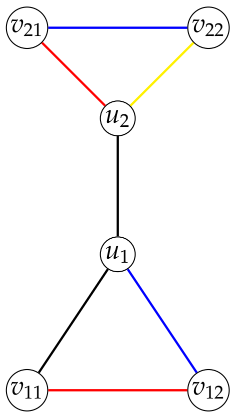

Figure 2.12: $P_2 \odot P_2$

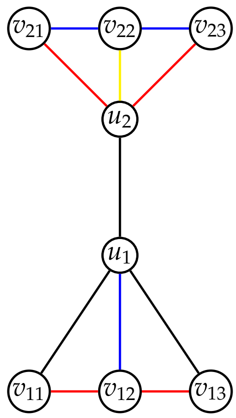

Figure 2.13: $P_2 \odot P_3$

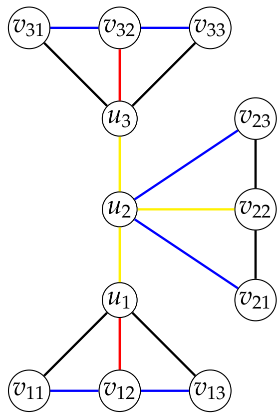

Figure 2.14: $P_3 \odot P_3$

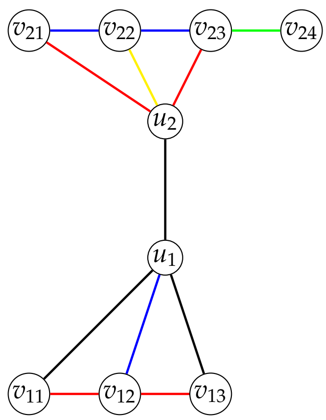

Figure 2.15: $\mathscr{H}$

Case 1. Assume that $m = n = 2$.
The vertices v_{11}, v_{12} and u_1 form an induced K_3 of $P_2 \odot P_2$. Thus the edges u_1v_{11}, u_1v_{12} and $v_{11}v_{12}$ are colored with the distinct colors $1, 2$ and 3 respectively. Now color the edge u_1u_2 and u_2v_{21} with color 1 and 3 respectively. Further $v_{22} - u_2 - u_1 - v_{11}$, $v_{22} - u_2 - u_1 - v_{12}$ and $v_{22} - v_{21} - u_2$ form paths of length 4. Therefore the edge u_2v_{22} cannot be colored with the colors $1, 2$ and 3 (the colors of the edges u_1v_{11}, u_1v_{12} and u_2v_{21}). Thus color 4 is given to the edge u_2v_{22}. Thus $\chi'_i(P_2 \odot P_2) \geq 4$ and the coloring in Figure 2.12 with 4 colors shows that $\chi'_i(P_2 \odot P_2) = 4$.
Case 2. Assume that $m = 3$ and $n = 2, 3$.
We have $P_2 \odot P_2$ as a subgraph of $P_2 \odot P_3$ and $P_3 \odot P_3$. Now using Proposition 1.2, we have $\chi'_i(P_2 \odot P_3) \geq 4$ and $\chi'_i(P_3 \odot P_3) \geq 4$. Also Figure 2.13 and Figure 2.14 provides an injective edge coloring with 4 colors. Therefore $\chi'_i(P_2 \odot P_3) = \chi'_i(P_3 \odot P_3) = 4$.

Case 3. Assume that $m, n \geq 4$.
Consider a subgraph $\mathscr{H}$ (Figure 2.15) of $P_n \odot P_m$. Since $P_2 \odot P_3$ forms a subgraph of $\mathscr{H}$ first color those edges in $\mathscr{H}$ as in $P_2 \odot P_3$. Next color the edge $v_{23}v_{24}$. Since $v_{24} - v_{23} - v_{22} - v_{21}$, $v_{24} - v_{23} - u_2 - v_{22}$, $v_{24} - v_{23} - u_2 - v_{21}$ and $v_{24} - v_{23} - u_2 - u_1$ form paths of length 4. Thus the edge $v_{23}v_{24}$ cannot be colored with the colors $1, 2, 3$ and 4 (colors of the edges $v_{22}v_{21}, u_2v_{22}, u_2v_{21}$ and u_2u_1). Thus the edge $v_{23}v_{24}$ is colored with color 5, $\chi'_i(\mathscr{H}) \geq 5$. Now the coloring depicted in Figure 2.15 is an injective edge coloring of $\mathscr{H}$ with 5 colors. Thus $\chi'_i(\mathscr{H}) = 5$. The graph $\mathscr{H}$ is the smallest subgraph of $P_n \odot P_m$ with injective edge chromatic index 5. Now the following is an injective edge coloring of $P_n \odot P_m$ with 5 colors.

- The edges u_1v_{1i} are colored with color 1 for odd i and color 2 for even i, $1 \leq i \leq m$.
- The edges $v_{11}v_{12}, v_{12}v_{13}, \cdots, v_{1(m-1)}v_{1m}$ with colors $3, 3, 4, 4, 3, 3, 4, 4, \cdots$ respectively.
- The edge u_1u_2 is colored with color 1.

- The edges u_2v_{2i} are colored with color 3 for odd i and color 4 for even i, $1 \leq i \leq m$.
- The edges $v_{21}v_{22}, v_{22}v_{23}, \cdots, v_{2(m-1)}v_{2m}$ with colors colors $2, 2, 5, 5, 2, 2, 5, 5, \cdots$ respectively.

□

Next moving to the injective edge coloring of $G \bigodot C_m$ where $G = P_n$ or C_n. Let $u_1, u_2, \cdots, u_n$ be the vertices of G and $v_{i1}, v_{i2}, \cdots, v_{im}$ be the vertices of i^{th} copy C_m^i of C_m for $i = 1, 2, \cdots, n$.

Lemma 2.24. *Let graph G be either the path P_n or the cycle C_n. Then for the graph $G \bigodot C_m$,*

i. No color of the edge $v_{ij}v_{i(j+1)}$ can be the color of the edges u_iv_{ik}, and vice versa, for $i = 1, 2, \cdots, n$ and $j, k = 1, 2, \cdots, m$.

ii. The edges u_iv_{ij}, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$ are colored with two distinct colors when m is even.

iii. The edges u_iv_{ij}, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$ are colored with three distinct colors when m is odd.

Proof.

i. Without loss of generality assume $j+1$ as 1 when $j = m$. The vertices $v_{ij}, v_{i(j+1)}$ and u_i forms an induced K_3. Also $v_{ij} - v_{i(j+1)} - u_i - v_{ik}$ form a path of length 4, $k \neq j, j+1$.

ii. Since u_i, v_{ij} and $v_{i(j+1)}$ form an induced K_3, the three edges are colored with distinct three colors. In particular, the edges u_iv_{ij} and $u_iv_{i(j+1)}$ are colored with 2 colors say color 1 and color 2. Now coloring the edges u_iv_{ij} with color 1 for odd j and coloring the edges u_iv_{ij} with color 2 for even j provides an injective edge coloring with 2 colors.

iii. Since u_i, v_{ij} and $v_{i(j+1)}$ form an induced K_3, the three edges are colored with distinct three colors. In particular, the edges u_iv_{ij} and $u_iv_{i(j+1)}$ are colored with two colors say color 1 and color 2. Now coloring the edges u_iv_{ij} with color 1 for odd j, $j \neq m$ and coloring the edges u_iv_{ij} with color 2 for even j. Now the vertices u_i, v_{im} and $v_{i(m-1)}$ form an induced K_3 and similarly the vertices u_i, v_{im} and v_{i1} also form an induces K_3. Thus the edge u_iv_{im} cannot be colored with color 1 or color 2 (colors of the edges $u_iv_{i(m-1)}$ and u_iv_{i1}). Thus the edge u_iv_{im} is colored with color 3.

□

Theorem 2.25. *If $n \geq 2$ and $m \geq 3$, then* $\chi_i'(P_n \bigodot C_m) = \begin{cases} \chi_i'(P_n) + 4 \; if \; m \equiv 0 \bmod 4, \\ \chi_i'(P_n) + 5 \; if \; m \equiv 2 \bmod 4, \\ \chi_i'(P_n) + 6 \; if \; m \; odd. \end{cases}$

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of P_n and $v_{i1}, v_{i2}, \cdots, v_{im}$ be the vertices of i^{th} copy C_m^i of C_m for $i = 1, 2, \cdots, n$.

Case 1. Assume that $m \equiv 0 \bmod 4$.

By Proposition 1.1(ii) $\chi_i'(C_m^i) = 2$. Therefore two colors are needed to color the edges of C_m^i and by Lemma 2.24(i) and Lemma 2.24(ii), new set of two colors are needed to color the edges u_iv_j. Color the edges u_iv_{ij} and $v_{ij}v_{ik}$ as follows.

For an odd i

- Color the edges $v_{ij}v_{i(j+1)}$, $j = 1, 2, \cdots, m$ with colors 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$.
- Color the edges u_iv_{ij} with color 3 when j is odd and with color 4 when j is even.

For an even i

- Color the edges $v_{ij}v_{i(j+1)}$, $j = 1, 2, \cdots, m$ with colors 3 and 4 in the pattern $3, 3, 4, 4, 3, 3, \cdots$.
- Color the edges u_iv_{ij} with color 1 when j is odd and with color 2 when j is even.

Now for any u_iu_{i+1}, the paths $u_{i+1} - u_i - v_{i3} - v_{i4}$, $u_{i+1} - u_i - v_{i3} - v_{i2}$, $u_i - u_{i+1} - v_{(i+1)3} - v_{(i+1)4}$ and $u_i - u_{i+1} - v_{(i+1)3} - v_{(i+1)2}$ form paths of length 4 and the edges $v_{i3}v_{i4}$, $v_{i3}v_{i2}$, $v_{(i+1)3}v_{(i+1)4}$ and $v_{(i+1)3}v_{(i+1)2}$ have colors 1,2,3 and 4. Now the edges u_iu_{i+1} of P_n are colored with $\chi_i'(P_n)$ new colors. Hence $\chi_i'(P_n + C_m) = 4 + \chi_i'(P_n)$.
Case 2. Assume that $m \equiv 2 \bmod 4$.
Here $\chi_i'(C_m) = 3$. Therefore color the edges $v_{ij}v_{i(j+1)}$ of C_m^i with 3 colors. Now by using Lemma 2.24(ii), the edges u_iv_{ij} are colored with new set of two colors.
For an odd i

- For $j = 1, 2, \cdots, m-2$, color the edges $v_{ij}v_{i(j+1)}$ with the colors 1 and 2 in a pattern $1, 1, 2, 2, 1, 1, \cdots$ and color the edges $v_{i(m-1)}v_{im}$ and $v_{im}v_{i1}$ with color 3.
- Color the edges u_iv_j with color 4 for odd j, with color 5 for even j.

For an even i

- For $j = 1, 2, \cdots, m-2$, color the edges $v_{ij}v_{i(j+1)}$ with the colors 4 and 5 in a pattern $4, 4, 5, 5, 4, 4, \cdots$ and color the edges $v_{i(m-1)}v_{im}$ and $v_{im}v_{i1}$ with color 3.
- Color the edges u_iv_j with color 1 for odd j, with color 2 for even j.

Now for any u_iu_{i+1}, the paths $u_{i+1}u_i - v_{i3} - v_{i4}$, $u_{i+1}u_i - v_{i3} - v_{i2}$ and $u_{i+1}u_i - v_{im} - v_{i1}$ form paths of length 4 and the edges $v_{i3}v_{i4}$, $v_{i3}v_{i2}$ and $v_{im}v_{i1}$ have colors 1 and 2 and 3. Similarly, $u_iu_{i+1} - v_{(i+1)3} - v_{(i+1)4}$ and $u_iu_{i+1} - v_{(i+1)3} - v_{(i+1)2}$ form paths of length 4 and the edges $v_{i3}v_{i4}$, $v_{i3}v_{i2}$ have colors 4 and 5. Thus the edges u_iu_{i+1} cannot be colored with colors 1,2,3, 4 and 5. Now the edges u_iu_{i+1} of P_n are colored with $\chi_i'(P_n)$ new colors. Hence $\chi_i'(P_n + C_m) = 5 + \chi_i'(P_n)$.

Case 3. Assume that m is odd.
Here $\chi_i'(C_m) = 3$. Therefore color the edges $v_{ij}v_{i(j+1)}$ of C_m^i with 3 colors. Now by using Lemma 2.24(iii), the edges u_iv_{ij} are colored with new set of three colors.
Subcase i. $m \equiv 1 \bmod 4$.
For an odd i

- For $j = 1, 2, \cdots, m-3$, color the edges $v_{ij}v_{i(j+1)}$ with the colors 1 and 2 in a pattern $1, 1, 2, 2, 1, 1, \cdots$ and color the edge $v_{i(m-2)}v_{i(m-1)}$, $v_{i(m-1)}v_{im}$, $v_{im}v_{i1}$ with colors 1, 3,2 respectively.
- Color the edges u_iv_j with color 4 for j odd and $j \neq m$, with color 5 for even j and with color 6 for $j = m$.

For an even i

- For $j = 1, 2, \cdots, m-3$, color the edges $v_{ij}v_{i(j+1)}$ with the colors 4 and 5 in a pattern $4, 4, 5, 5, 4, 4, \cdots$ and color the edge $v_{i(m-2)}v_{i(m-1)}$, $v_{i(m-1)}v_{im}$, $v_{im}v_{i1}$ with colors 4,6, 5 respectively.
- Color the edges u_iv_j with color 1 for j odd and $j \neq m$, with color 2 for even j and with color 3 for $j = m$.

Subcase ii. $m \equiv 3 \bmod 4$.
For an odd i

- For $j = 1, 2, \cdots, m-3$, color the edges $v_{ij}v_{i(j+1)}$ with the colors 1 and 2 in a pattern $1, 1, 2, 2, 1, 1, \cdots$ and color the edge $v_{im}v_{i1}$ with color 3.
- Color the edges u_iv_j with color 4 for j odd and $j \neq m$, with color 5 for even j and with color 6 for $j = m$.

For an even i

- For $j = 1, 2, \cdots, m-1$, color the edges $v_{ij}v_{i(j+1)}$ with the colors 4 and 5 in a pattern $4, 4, 5, 5, 4, 4, \cdots$ and color the edge $v_{im}v_{i1}$ with color 6.
- Color the edges u_iv_j with color 1 for j odd and $j \neq m$, with color 2 for even j and with color 3 for $j = m$.

Now for any u_iu_{i+1}, the paths $u_{i+1}u_i - v_{i3} - v_{i4}$, $u_{i+1}u_i - v_{i3} - v_{i2}$, $u_{i+1}u_i - v_{im} - v_{i1}$ $u_{i+1}u_i - v_{i(m-1)} - v_{im}$ form paths of length 4 and the edges $v_{i3}v_{i4}$, $v_{i3}v_{i2}$, $v_{im}v_{i1}$ and $v_{i(m-1)}v_{im}$ have colors 1, 2 and 3. Similarly, $u_iu_{i+1} - v_{(i+1)3} - v_{(i+1)4}$, $u_iu_{i+1} - v_{(i+1)3} - v_{(i+1)2}$ and $u_iu_{i+1} - v_{(i+1)(m-1)} - v_{(i+1)m}$ form paths of length 4 and the edges $v_{i3}v_{i4}$, $v_{i3}v_{i2}$ and $v_{(i+1)(m-1)}v_{(i+1)m}$ have colors 4, 5 and 6. Thus the edges u_iu_{i+1} cannot be colored with colors 1,2,3, 4, 5 and 6. Now the edges u_iu_{i+1} of P_n are colored with $\chi_i'(P_n)$ new colors. Hence $\chi_i'(P_n + C_m) = 6 + \chi_i'(P_n)$. □

Let $u_1, u_2, \cdots, u_n$ be the vertices of C_n and $v_{i1}, v_{i2}, \cdots, v_{im}$ be the vertices of i^{th} copy C_m^i of C_m for $i = 1, 2, \cdots, n$. The following Lemma is on $C_n \odot C_m$.

Lemma 2.26. *For the graph $C_n \odot C_m$,*

i. No color of the edges in the set $\{u_iv_{ij}, j = 1, 2, \cdots, m\}$ can be the color of the edges in the set $\{u_kv_{kj}, j = 1, 2, \cdots, m\}$ for $k = i-1$ or $k = i+1$.

ii. When m is even, either three or four distinct colors cannot be the color of $v_{ij}v_{i(j+1)}$ for each i.

iii. When m is odd, either four or five distinct colors cannot be the color of $v_{ij}v_{i(j+1)}$ for each i.

Proof. Without loss of generality assume $i+1$ as 1 when $i = n$ and $i-1$ as n when $i = 1$.

i. For any $j, l = 1, 2, \cdots, m$, $v_{ij} - u_i - u_{i+1} - v_{(i+1)l}$ forms paths of length 4. Thus no color of the edges in the set $\{u_iv_{ij}, j = 1, 2, \cdots, m\}$ can be the color of the edges in the set $\{u_{(i+1)}v_{(i+1)j}, j = 1, 2, \cdots, m\}$. Similarly $v_{ij} - u_i - u_{i-1} - v_{(i-1)l}$ forms paths of length 4. Thus no color of the edges in the set $\{u_iv_{ij}, j = 1, 2, \cdots, m\}$ can be the color of the edges in the set $\{u_{(i-1)}v_{(i-1)j}, j = 1, 2, \cdots, m\}$.

ii. The color of $u_{i-1}u_i$ and u_iu_{i+1} cannot be the color of $v_{ij}v_{i(j+1)}$, since $u_{i-1} - u_i - v_{ij} - v_{i(j+1)}$ and $u_{i+1} - u_i - v_{ij} - v_{i(j+1)}$ form paths of length 4. Also by Lemma 2.24(i) and Lemma 2.24(ii) the two colors of u_iv_{ij} cannot be the color of $v_{ij}v_{i(j+1)}$. Now if the edges $u_{i-1}u_i$ and u_iu_{i+1} are of same colors, then a total of three colors cannot be the color of $v_{ij}v_{i(j+1)}$. And if the edges $u_{i-1}u_i$ and u_iu_{i+1} are of different colors, then a total of four colors cannot be the color of $v_{ij}v_{i(j+1)}$.

iii. The color of $u_{i-1}u_i$ and u_iu_{i+1} cannot be the color of $v_{ij}v_{i(j+1)}$, since $u_{i-1} - u_i - v_{ij} - v_{i(j+1)}$ and $u_{i+1} - u_i - v_{ij} - v_{i(j+1)}$ form paths of length 4. Also by Lemma 2.24(i) and Lemma 2.24(iii) the three colors of u_iv_{ij} cannot be the color of $v_{ij}v_{i(j+1)}$. Now if the edges $u_{i-1}u_i$ and u_iu_{i+1} are of same colors, then a total of four colors cannot be the color of $v_{ij}v_{i(j+1)}$. And if the edges $u_{i-1}u_i$ and u_iu_{i+1} are of different colors, then a total of five colors cannot be the color of $v_{ij}v_{i(j+1)}$.

□

Theorem 2.27. *For $m, n \geq 3$,* $\chi_i'(C_n \odot C_m) = \begin{cases} 6 \; if \; m \equiv 0 \bmod 4, \\ 7 \; if \; m \equiv 2 \bmod 4, \\ 8 \; if \; m \text{ is odd and } n \neq 3, \\ 9 \; if \; m \text{ is odd and } n = 3. \end{cases}$

Proof. Let $u_1, u_2, \cdots, u_n$ be the vertices of C_n and $v_{i1}, v_{i2}, \cdots, v_{im}$ be the vertices of i^{th} copy C_m^i of C_m for $i = 1, 2, \cdots, n$.

1. $m \equiv 0 \bmod 4$.
Here $\chi_i'(C_m) = 2$. Now by Lemma 2.24(ii), Lemma 2.24(iii) and Lemma 2.26(ii), we can see that at least 6 colors are needed to color $C_n \odot C_m$. Now providing an injective edge coloring with 6 colors concludes.
Case 1. Assume that $n \equiv 0 \bmod 4$.

   - For odd i, color the edges $v_{ij}v_{i(j+1)}$ with colors 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$. Also color the edges u_iv_{ij} with colors 3 for odd j and with color 4 for even j.
   - For even i, color the edges $v_{ij}v_{i(j+1)}$ with colors 3 and 4 in the pattern $3, 3, 4, 4, 3, 3, \cdots$. Also color the edges u_iv_{ij} with colors 1 for odd j and with color 2 for even j.
   - For the edges u_iu_{i+1}, $i = 1, 2, \cdots, n$, without loss of generality assume $i+1 = 1$ when $i = n$. Color the edges u_iu_{i+1} with colors 5 and 6 in the pattern $5, 5, 6, 6, 5, 5, \cdots$.

Case 2. Assume that $n \equiv 1 \bmod 4$.

- For odd i and $i \neq n$, color the edges u_iv_{ij} with colors 1 for odd j and with color 2 for even j.
- For even i, color the edges u_iv_{ij} with colors 3 for odd j and with color 4 for even j.
- For $i = n$, color the edges u_iv_{ij} with colors 5 for odd j and with color 6 for even j.
- For $i = 2, 3, \cdots, n-3$, color the edges u_iu_{i+1} with colors 5 and 6 in the pattern $5, 5, 6, 6, 5, 5, \cdots$.
- Color the edges $u_2u_1, u_1u_n, u_nu_{n-1}$ and $u_{n-1}u_{n-2}$ with colors 3,2,4, and 1 respectively.
- By Lemma 2.26(ii), either three or four colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 2$ then the remaining two colors are used to color the edges $v_{ij}v_{i(j+1)}$, for each i.

Case 3. Assume that $n \equiv 2 \mod 4$.

- For $i = 1, 2, \cdots, n-2$, color the edges u_iu_{i+1} with the colors 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$.
- Color the edges $u_{n-1}u_n$ and u_nu_1 with color 3.
- Color the edges u_1v_{1j} with color 2 for odd j and with color 5 for even j.
- Color the edges u_2v_{2j} with color 1 for odd j and with color 4 for even j.
- For $i = 3, 5, 7, \cdots, n-3$, color the edges u_iv_{ij} with color 3 for odd j and with color 5 for even j.
- For $i = 4, 6, 8, \cdots, n-2$, color the edges u_iv_{ij} with color 2 for odd j and with color 4 for even j.
- Color the edges $u_{n-1}v_{(n-1)j}$ with color 1 for odd j and with color 5 for even j.
- Color the edges u_nv_{nj} with color 3 for odd j and with color 4 for even j.
- By Lemma 2.26(ii), either three or four colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 2$ then the remaining two colors are used to color the edges $v_{ij}v_{i(j+1)}$, for each i.

Case 4. Assume that $n \equiv 3 \mod 4$.

- For $i = 1, 2, \cdots, n-3$, color the edges u_iu_{i+1} with colors 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$.
- Color the edges $u_1u_n, u_nu_{n-1}, u_{n-1}u_{n-2}$ with colors 2,3,1 respectively.
- Color the edges u_1v_{1j} with color 5 for odd j and with color 6 for even j.
- Color the edges $u_{n-2}v_{(n-2)j}$ with color 1 for odd j and with color 4 for even j.
- Color the edges $u_{n-1}v_{(n-1)j}$ with color 3 for odd j and with color 5 for even j.
- Color the edges u_nv_{nj} with color 2 for odd j and with color 4 for even j.
- For $i = 2, 6, 10, \cdots, i < n-1$, color the edges u_iv_{ij} with color 1 for odd j and with color 3 for even j.
- For $i = 3, 5, 7, 9, \cdots, i < n-2$, color the edges u_iv_{ij} with color 4 for odd j and with color 5 for even j.
- For $i = 4, 8, 12, \cdots, i < n$, color the edges u_iv_{ij} with color 2 for odd j and with color 3 for even j.
- By Lemma 2.26(ii), either three or four colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 2$ then the remaining two colors are used to color the edges $v_{ij}v_{i(j+1)}$, for each i.

2. $m \equiv 2 \mod 4$.
For C_n at least two colors are needed for an injective edge coloring, therefore there are two edges say u_iu_{i+1} and $u_{i+1}u_{i+2}$ with distinct two colors. Then by Lemma 2.26(ii), four colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 3$. Thus a new set of three colors are needed to color the edges of C^i_m. Hence $\chi'_i(C_n \bigodot C_m) \geq 7$. Now providing an injective edge coloring with seven colors shows that $\chi'_i(C_n \bigodot C_m) = 7$. The coloring is given as follows.

Case 1. Assume that $n \equiv 0 \mod 4$.

- For $i = 1, 2, \cdots, n$, color the edges u_iu_{i+1} with color 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$. Without loss of generality assume $i+1$ as 1 when $i = n$.
- Let $i = 2, 6, 10, \cdots$.
  - Color the edges u_iv_{ij} with color 1 for odd j and with color 3 for even j.
  - For $j = 1, 2, \cdots, m-2$, color the edges $u_{ij}u_{i(j+1)}$ with color 2 and 4 in the pattern $2, 2, 4, 4, 2, 2, \cdots$ and color the edges $u_{i(m-1)}u_{im}$ and $u_{im}u_{i1}$ with color 5.

- Let $i = 4, 8, 12, \cdots$.
  - Color the edges u_iv_{ij} with color 2 for odd j and with color 3 for even j.
  - For $j = 1, 2, \cdots, m-2$, color the edges $u_{ij}u_{i(j+1)}$ with color 1 and 4 in the pattern $1, 1, 4, 4, 1, 1, \cdots$ and color the edges $u_{i(m-1)}u_{im}$ and $u_{im}u_{i1}$ with color 5.
- Let i be odd.
  - Color the edges u_iv_{ij} with color 4 for odd j and with color 5 for even j.
  - For $j = 1, 2, \cdots, m-2$, color the edges $u_{ij}u_{i(j+1)}$ with color 3 and 6 in the pattern $3, 3, 6, 6, 3, 3, \cdots$ and color the edges $u_{i(m-1)}u_{im}$ and $u_{im}u_{i1}$ with color 7.

Case 2. Assume that $n \equiv 1 \bmod 4$ and $n \equiv 3 \bmod 4$.

- For odd i and $i \neq n$, color the edges u_iv_{ij} with colors 1 for odd j and with color 2 for even j.
- For even i, color the edges u_iv_{ij} with colors 3 for odd j and with color 4 for even j.
- For $i = n$, color the edges u_iv_{ij} with colors 5 for odd j and with color 6 for even j.
- For $i = 2, 3, \cdots, n-3$, color the edges u_iu_{i+1} with colors 5 and 6 in the pattern $5, 5, 6, 6, 5, 5, \cdots$.
- color the edges $u_2u_1, u_1u_n, u_nu_{n-1}$ and $u_{n-1}u_{n-2}$ with colors 3,1,4, and 2 respectively.
- By Lemma 2.26(ii), either three or four colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 3$ then the remaining three colors are used to color the edges $v_{ij}v_{i(j+1)}$, for each i.

Case 3. Assume that $n \equiv 2 \bmod 4$.

- For $i = 1, 2, \cdots, n-2$, color the edges u_iu_{i+1} with the colors 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$.
- Color the edges $u_{n-1}u_n$ and u_nu_1 with color 3.
- Color the edges u_1v_{1j} with color 2 for odd j and with color 6 for even j.
- Color the edges u_2v_{2j} with color 1 for odd j and with color 4 for even j.
- Color the edges $u_{n-2}v_{(n-2)j}$ with color 2 for odd j and with color 4 for even j.
- Color the edges $u_{n-1}v_{(n-1)j}$ with color 1 for odd j and with color 6 for even j.
- Color the edges u_nv_{nj} with color 4 for odd j and with color 5 for even j.
- For $i = 3, 5, 7, \cdots, n-3$, color the edges u_iv_{ij} with color 5 for odd j and with color 6 for even j.
- For $i = 4, 6, 8, \cdots, n-4$, color the edges u_iv_{ij} with color 3 for odd j and with color 4 for even j.
- By Lemma 2.26(ii), either three or four colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 3$ then the remaining three colors are used to color the edges $v_{ij}v_{i(j+1)}$, for each i.

3. m odd and $n \neq 3$
   Here $\chi'_i(C_m) = 3$. Also by Lemma 2.24(i), Lemma 2.24(iii) and Lemma 2.26(iii) $\chi'_i(C_n \bigodot C_m) \geq 8$. Now providing an injective edge coloring with eight colors shows that $\chi'_i(C_n \bigodot C_m) = 8$. The coloring is given as follows.

   Case 1. Assume that $n \equiv 0 \bmod 4$.

   - For odd i, color the edges u_iv_{ij} with colors 1 when j is odd and $j \neq m$, with color 2 when j is even and with color 3 when $j = m$.
   - For even i, color the edges u_iv_{ij} with colors 4 when j is odd and $j \neq m$, with color 5 when j is even and with color 6 when $j = m$.
   - Color the edges u_iu_{i+1} with color 7 and 8 in the pattern $7, 7, 8, 8, 7, 7, \cdots$.
   - For odd i, color the edges of C_m^i with colors $4, 5$ and 6.
   - For even i, color the edges of C_m^i with colors $1, 2$ and 3.

   Case 2. Assume that $n \equiv 1 \bmod 4$.

   - For $i = 1, 2, \cdots, n-1$, color the edges u_iu_{i+1} with colors $1, 2, 3, 4, 1, 2, 3, 4$ and color the edge u_nu_1 with color 5.

- For $i = n$, color the edges u_iv_{ij} with colors 2 when j is odd and $j \neq m$, with color 5 when j is even and with color 7 when $j = m$.
- For $i = 1$, color the edges u_iv_{ij} with colors 1 when j is odd and $j \neq m$, with color 3 when j is even and with color 8 when $j = m$.
- For $i = 2$, color the edges u_iv_{ij} with colors 2 when j is odd and $j \neq m$, with color 4 when j is even and with color 6 when $j = m$.
- For $i = 3, 7, 11, \cdots$, color the edges u_iv_{ij} with colors 3 when j is odd and $j \neq m$, with color 5 when j is even and with color 7 when $j = m$.
- For $i = 4, 8, 12, \cdots$, color the edges u_iv_{ij} with colors 4 when j is odd and $j \neq m$, with color 6 when j is even and with color 8 when $j = m$.
- For $i = 5, 9, 13, \cdots$ and $i < n - 1$, color the edges u_iv_{ij} with colors 1 when j is odd and $j \neq m$, with color 5 when j is even and with color 7 when $j = m$.
- For $i = 6, 10, 14, \cdots$, color the edges u_iv_{ij} with colors 2 when j is odd and $j \neq m$, with color 6 when j is even and with color 8 when $j = m$.
- By Lemma 2.26(iii), either four or five colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 3$ thus the remaining three colors are used to color the edges of C_m^i, for each i.

Case 3. Assume that $n \equiv 2 \bmod 4$.

- For $i = 1, 2, \cdots, n-2$, color the edges u_iu_{i+1} with the colors 1 and 2 in the pattern $1, 1, 2, 2, 1, 1, \cdots$.
- Color the edges $u_{n-1}u_n$ and u_nu_1 with color 3.
- For $i = 1$, color the edges u_iv_{ij} with colors 2 when j is odd and $j \neq m$, with color 6 when j is even and with color 7 when $j = m$.
- For $i = n$, color the edges u_iv_{ij} with colors 3 when j is odd and $j \neq m$, with color 4 when j is even and with color 5 when $j = m$.
- For $i = n - 1$, color the edges u_iv_{ij} with colors 1 when j is odd and $j \neq m$, with color 6 when j is even and with color 7 when $j = m$.
- For $i = 2, 6, 10, \cdots, i < n - 1$, color the edges u_iv_{ij} with colors 1 when j is odd and $j \neq m$, with color 4 when j is even and with color 5 when $j = m$.
- For i odd and $3 \leq i \leq n-3$, color the edges u_iv_{ij} with colors 3 when j is odd and $j \neq m$, with color 6 when j is even and with color 7 when $j = m$.
- For $i = 4, 8, 12, \cdots, i < n - 1$, color the edges u_iv_{ij} with colors 2 when j is odd and $j \neq m$, with color 4 when j is even and with color 5 when $j = m$.
- By Lemma 2.26(iii), either four or five colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 3$ thus the remaining three colors are used to color the edges of C_m^i, for each i.

Case 4. Assume that $n \equiv 3 \bmod 4$.

- For $i = 1$, color the edges u_iv_{ij} with colors 1 when j is odd and $j \neq m$, with color 2 when j is even and with color 3 when $j = m$.
- For $i = 2$, color the edges u_iv_{ij} with colors 4 when j is odd and $j \neq m$, with color 5 when j is even and with color 6 when $j = m$.
- For $i = n$, color the edges u_iv_{ij} with colors 6 when j is odd and $j \neq m$, with color 7 when j is even and with color 8 when $j = m$.
- For $i = 3, 7, 11, \cdots, i < n$, color the edges u_iv_{ij} with colors 1 when j is odd and $j \neq m$, with color 2 when j is even and with color 7 when $j = m$.
- For even i, $4 < i < n$, $i < n$, color the edges u_iv_{ij} with colors 3 when j is odd and $j \neq m$, with color 4 when j is even and with color 5 when $j = m$.
- For $i = 5, 9, 13, \cdots, i < n$, color the edges u_iv_{ij} with colors 1 when j is odd and $j \neq m$, with color 2 when j is even and with color 6 when $j = m$.
- Color the edges $u_{n-2}u_{n-1}, u_{n-1}u_n, u_nu_1, u_1u_2$ with colors 2, 4, 1 and 3 respectively and for $i = 2, 3, 4, \cdots, n-3$, color the edges u_iu_{i+1} with the colors 7 and 6 in the pattern $7, 7, 6, 6, 7, 7, \cdots$.

- By Lemma 2.26(iii), either four or five colors cannot be the color of $v_{ij}v_{i(j+1)}$. Also $\chi'_i(C_m) = 3$ then the remaining three colors are used to color the edges of C_m^i, for each i.

4. m odd and $n = 3$
By Lemma 2.24(iii) and Lemma 2.26(i), nine distinct colors are used to color the edges u_iv_{ij}, $i = 1, 2, 3$ and $j = 1, 2, \cdots, m$. Therefore $\chi'_i(C_3 \bigodot C_m) \geq 9$. Now providing an injective edge coloring with nine colors shows that $\chi'_i(C_3 \bigodot C_m) = 9$. The coloring is given as follows.

    - For $i = 1$, color the edges u_iv_{ij} with color 1 when j is odd and $j \neq m$, with color 2 when j is even and with color 3 when $j = m$.
    - For $i = 2$, color the edges u_iv_{ij} with color 4 when j is odd and $j \neq m$, with color 5 when j is even and with color 6 when $j = m$.
    - For $i = 3$, color the edges u_iv_{ij} with color 7 when j is odd and $j \neq m$, with color 8 when j is even and with color 9 when $j = m$.
    - Color the edges of the cycle C_m^1 with colors 5, 6 and 7.
    - Color the edges of the cycle C_m^2 with colors 1, 2 and 3.
    - Color the edges of the cycle C_m^3 with colors 2, 3 and 4.
    - Color the edges u_1u_2, u_2u_3 and u_3u_1 with colors 1, 4 and 5.

□

3. On the complexity of Injective edge coloring

In the literature, few authors have studied the complexity of the injective edge coloring problem [4, 8]. The results are depicted as follows. First here describe the injective 3-edge coloring is NP-complete for some classes of graphs in Figure 3.1.

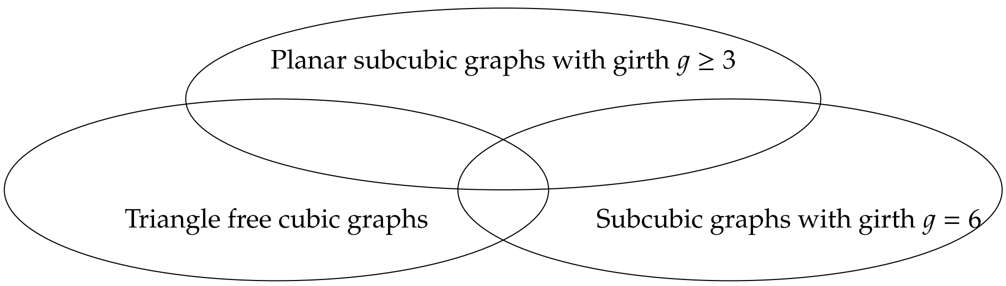

Figure 3.1: Injective 3-edge coloring is NP-complete

Also in [4, 8] the authors have proved that the injective k-edge coloring is NP-complete for the following graphs.

- Graphs with maximum degree atmost $5\sqrt{3k}$.
- Graphs with maximum degree $O(\sqrt{k})$.

And injective 4-edge coloring is NP-complete for cubic graphs. Further, the authors proved that injective k-edge coloring is polynomial-time solvable for outer planar graphs and K_4-minor free planar graphs.

Here CHRIND ($\mathcal{P}$) denotes the chromatic index problem restricted to graphs with property $\mathcal{P}$. A result on the complexity of proper edge coloring of regular graphs is given as follows.

Theorem 3.1 ([2]). *For each $r \geq 3$, CHRIND (r-regular graph) is NP-complete.*

By using Theorem 3.1, it is obtained that, the problem of checking whether the injective edge chromatic index of a $(2, 3, r)$-triregular graph is r is NP-complete.

Definition 3.2. *Let p, q and r be integers, $1 \le p < q < r$. A graph is said to be (p, q, r)-triregular graphs if its vertices assume exactly three different values p, q and r.*

Instance: A $(2, 3, r)$-triregular graph G.
Question: Is $\chi'_i(G) = r$?

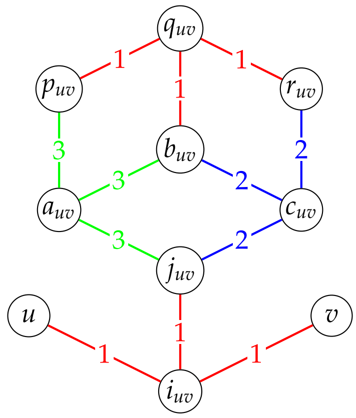

Figure 3.2: Edge gadget E with an injective 3 edge coloring

To prove Theorem 3.3, we use the gadget E in Figure 3.2 same as in [8].

Theorem 3.3. *For each $r \ge 3$, it is NP-complete to determine whether the injective edge chromatic index of a $(2, 3, r)-$triregular graph is r.*

Proof. Let G be the input r-regular graph. The proof will be proceeded by two steps: first create a $(2, 3, r)-$triregular graph H from G, then we show that H has an injective r-edge coloring if and only if G is properly r-edge colorable.
Create the graph H from G by removing all edges of G. For each edge uv of G, create a copy of a gadget E and connect it to u and v. Add eight new vertices $i_{uv}, j_{uv}, a_{uv}, b_{uv}, c_{uv}, p_{uv}, q_{uv}$ and r_{uv}. Also create the following edges $ui_{uv}, vi_{uv}, i_{uv}j_{uv}, j_{uv}a_{uv}, j_{uv}c_{uv}, a_{uv}b_{uv}, b_{uv}c_{uv}, a_{uv}p_{uv}, c_{uv}r_{uv}, b_{uv}q_{uv}, p_{uv}q_{uv}$ and $q_{uv}r_{uv}$.
Let G be a graph on n vertices and m edges. On creating the graph H, corresponding to each edge eight vertices are added, thus H has $8m + n$ vertices. In which n vertices have degree r, $6m$ vertices have degree 3 and $2m$ vertices have degree 2. Thus H becomes a $(2, 3, r)-$ triregular graph.
Further, it is clear from [8] that G is proper r-edge colorable if and only if H is injectively r-edge colorable. As $r \ge 3$, there are enough colors to color the edges of the edge gadget E added in place of each edge. □

Now by using the gadget $\mathcal{F}$ in Figure 3.3, here shows that it is NP-complete to determine the injective edge chromatic index of $(2, 4, r)$-triregular graph is r.

Theorem 3.4. *For each $r \ge 3$, it is NP-complete to determine whether the injective edge chromatic index of a $(2, 4, r)-$triregular graph is r.*

Proof. Let G be the input r-regular graph. It will be proceeded in two steps: first create a $(2, 4, r)-$ triregular graph H from G, then we show that H has an injective r-edge coloring if and only if G is properly r-edge

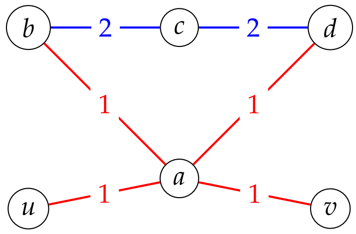

Figure 3.3: Edge gadget $\mathcal{F}$ with an injective 2 edge coloring

colorable.
Create the graph H from G by removing all edges of G. For each edge uv of G, create a copy of a gadget $\mathcal{F}$ and connect it to u and v. Add four new vertices a, b, c and d. Also create the following edges ua, va, ad, ab, cd and cb.
Let G be a graph on n vertices and m edges. On creating the graph H, corresponding to each edge four vertices are added, thus H has $4m + n$ vertices. In which n vertices have degree r, $3m$ vertices have degree 2 and m vertices have degree 4. Thus H becomes a $(2, 4, r)-$ triregular graph.
If G has an r-edge coloring c, then injectively r-edge color H by assigning to ua, va, ad and ab in H the color $c(uv)$; then extend the coloring to each gadget $\mathcal{F}$ corresponding to each edge, by assigning any one of the color from the remaining $r - 1$ colors to bc and cd.
Conversely, if H has an injective r-edge coloring, then color an edge uv of G with the color of the edge ua (or va) of H. The coloring is proper since the color of ua and va are the same. □

Similarly for an r-edge colorable graph, construct a graph G' by subdividing each edge uv to ux and xv by adding a vertex x and assigning the same color of uv to ux and xv gives an injective r-coloring of G'. The converse also follows similarly. The graph thus obtained is a $(2, r)$-biregular graph.

Corollary 3.5. *For each $r \geq 3$, it is NP-complete to determine whether the injective edge chromatic index of a $(2, r)-$biregular graph is r.*

4. Conclusions

In this article, the injective edge chromatic index of different graph products are obtained. In particular, the injective edge chromatic index of union of finite number of graphs, injective edge chromatic index of join of G and H, where $G, H = K_n, \bar{K}_n, P_n, C_n, L_n$ and the injective edge chromatic index of Cartesian product (or corona) of G and H are obtained for $G, H = P_n, C_n$. Also determined bounds for $\chi'_i(G)$ for the resultant graph G obtained by the operations join and corona. Furthermore, the injective edge colouring problem with $r \geq 3$ has been shown to be NP-complete for $(2, 3, r)$-triregular graphs, $(2, 4, r)$-triregular graphs, and $(2, r)$-biregular graphs. It is also open to compute the exact values of the injective chromatic index $\chi'_i(G \square H)$ and $\chi'_i(G \odot H)$ for any two arbitrary graphs G and H and the complexity of other classes of graphs.

References

[1] Y. Bu, C. Qi, Injective edge coloring of sparse graphs, Discrete Math. Algorithms Appl. 10 (2018): 1850022 (16 pages).
[2] L. Cai, J. A. Ellis, NP-completeness of edge-colouring some restricted graphs, Discrete Appl. Math. 30 (1991): 15—27.
[3] D. M. Cardoso, J. O. Cerdeira, C. Dominic and J. P. Cruz, Injective edge chromatic index of a graph, arXiv preprint arXiv:1510.02626, 2015.
[4] D. M. Cardoso, J. O. Cerdeira, C. Dominic and J. P. Cruz, Injective edge coloring of graphs, Filomat 33 (2019): 6411–6423.
[5] J. Clark, D. Holton, *A First Look at Graph Theory*, World Scientific, 1991.
[6] B. Ferdjallah, S. Kerdjoudj, A. Raspaud, Injective edge-coloring of sparse graphs, arXiv preprint arXiv:1907.09838, 2019.
[7] B. Ferdjallah, S. Kerdjoudj, A. Raspaud, Injective edge-coloring of subcubic graphs. Discrete Mathematics, Algorithms and Applications, 2250040, 2021.

[8] F. Foucaud, H. Hocquard, D. Lajou, Complexity and algorithms for injective edge-coloring in graphs, Inform. Process. Lett. 170 (2021) : 106121

[9] R. Frucht, F. Harary, On the corona of two graphs, Aequationes Math. 4 (1970): 322–325.

[10] J. A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 1 (2018) DS6.

[11] G. Hahn, J. Kratochvıl, J. Širáň, D. Sotteau, On the injective chromatic number of graphs, Discrete Math. 256 (2002): 179–192.

[12] F. Harary, *Graph theory*, Addison-Wesley, Reading, MA., 1969.

[13] A. Kostochka, A. Raspaud, J. Xu, Injective edge-coloring of graphs with given maximum degree, European J. Combin. 96 (2021): 103355.

[14] Y. Li, L. Chen, Injective edge coloring of generalized Petersen graphs, AIMS Mathematics 6 (2021): 7929–7943.

[15] D. B. West, *Introduction to graph theory*, Prentice hall upper saddle river, NJ, 2, 1996.

[16] J. Yue, S. Zhang, X. Zhang, Note on the perfect EIC-graphs, Appl. Math. Comput. 289 (2016): 481–485.