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Hamilton and Souplet-Zhang type gradient estimate along geometric flow

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Abstract. In this article we derive Hamilton type gradient estimate and Souplet-Zhang type gradient estimate for a generalized weighted parabolic heat equation with potential on a weighted Riemannian manifold evolving by a geometric flow in such a way that the Bakry-Émery Ricci tensor is bounded below.

1. Introduction

The study of gradient estimation is an important tool to understand the nature of the solution of a partial differential equation in the field of geometric analysis. There are different types of gradient estimations namely Li-Yau type, Hamilton type, Souplet-Zhang type etc. In [16], Yau proved gradient estimates for harmonic functions on complete manifolds with the Ricci curvature bounded below. In [12], Souplet and Zhang generalized Yau's result to the heat equation by adding a necessary logarithmic correction term. Several authors derived many results in this regard, see [8–10, 15] and the references therein. Abolarinwa [1] have studied gradient estimation on evolving manifold which involves Laplace-Beltrami operator. Recently, Azami [2] extended the Hamilton type and Souplet-Zhang type gradient estimations to the case of manifolds evolving under geometric flow involving weighted Laplacian. Now we furnish some primary informations to introduce our results for this article.

Let $(M^n, g, e^{-\phi}d\mu)$ be an n-dimensional complete weighted Riemannian manifold, where $e^{-\phi}d\mu$ is the weighted volume form, g is the Riemannian metric and $\phi \in C^2(M)$, where $C^2(M)$ is the space of all twice differentiable functions on M. Let $g(t)$ be an one parameter family of Riemannian metric evolving along the abstract geometric flow

$$\frac{\partial}{\partial t} g_{ij} = 2S_{ij}, \tag{1}$$

where $\mathcal{S}(e_i, e_j) := S_{ij}(t)$ is smooth symmetric 2-tensor on $(M, g(t))$, $t \in [0, T_0]$ and T_0 is the maximum time of existence. Define $S := tr(S_{ij}) = g^{ij}S_{ij}$. The weighted Laplacian operator is defined by

$$\Delta_\phi = \Delta - \nabla\phi\nabla,$$

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where Δ is the Laplace-Beltrami operator and $\phi \in C^2(M)$. Recently, Saha et al. [11] studied first eigenvalue of weighted p-Laplacian along cotton flow.

Inspired by the above works in this paper we derive both Hamilton type and Souplet-Zhang type gradient estimation for a positive solution of the generalized non-linear parabolic equation with potential

$$\Delta_\phi u = \frac{\partial u}{\partial t} + A(u)p(x,t) + B(u)q(x,t) + \xi(x,t)u \tag{2}$$

on the weighted Riemannian manifold $(M^n, g(t), e^{-\phi}d\mu)$ evolving under (1) such that the Bakry-Émery Ricci curvature is bounded below, where $p(x,t), q(x,t)$ and $\xi(x,t)$ are twice differentiable functions of x, t and $A(u), B(u)$ are differentiable functions of u.
In Section 3 we find a bound for the quantity $\frac{|\nabla u|}{\sqrt{u}}$ and in Section 4 we find a bound for the quantity $\frac{|\nabla u|}{u(1+\ln\kappa-\ln u)}$, where $u \leq \kappa$ and κ is a positive real number. These estimations are known as Hamilton type estimation and Souplet-Zhang type estimation respectively for a positive solution u of (2) along the flow (1).

2. Preliminaries

In this section we state some fundamental results and evolution equations that will be used later in the sequel.

Lemma 2.1. *(Weighted Bochner Formula) For any smooth function u, we have*

$$\frac{1}{2}\Delta_\phi|\nabla u|^2 = |Hess\ u|^2 + \langle\nabla\Delta_\phi u, \nabla u\rangle + Ric_\phi(\nabla u, \nabla u),$$

where $Ric_\phi := Ric + Hess\ \phi$, is called the Bakry-Émery tensor, Hess is the Hessian operator. For any integer $m > n$, an $(m-n)$-Bakry-Émery tensor is defined by

$$Ric_\phi^{m-n} := Ric + Hess\ \phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n}.$$

Lemma 2.2. *For any smooth function u under the geometric flow (1) we have*

$$\frac{\partial}{\partial t}|\nabla u|^2 = -2\mathcal{S}(\nabla u, \nabla u) + 2\langle\nabla u, \nabla u_t\rangle, \tag{3}$$

For any two points $x, y \in M$ and $T \in \mathbb{R}$, the quantity $d(x,y,t)$ denotes the geodesic distance between x and y under the metric $g(t)$ for any $t \in [0,T]$, $0 < T < T_0$.

For any fixed $x_0 \in M$ and $R > 0$ we define a compact set

$$Q_{2R,T} = \{(x,t) : d(x,x_0,t) \leq 2R, 0 \leq t \leq T\} \subset M^n \times (-\infty, +\infty).$$

For any smooth function f (depending on u) we define $\hat{A}(f) = \frac{A(u)}{u}$, $\hat{B}(f) = \frac{B(u)}{u}$, $\hat{A}_f = A'(u) - \frac{A(u)}{u}$ and $\hat{B}_f = B'(u) - \frac{B(u)}{u}$. For example: If $u = e^f$ and $A(u) = |u|^{\alpha-1}u$ then $\hat{A}(f) = \frac{A(u)}{u} = e^{(\alpha-1)f}$, which gives

1. $\hat{A}_f = (\alpha-1)e^{(\alpha-1)f}$
2. $\nabla\hat{A} = (\alpha-1)e^{(\alpha-1)f}\nabla f = \hat{A}_f\nabla f$.

In short, $\hat{A}_f$ in denotes the partial derivative of $\hat{A}$ with respect to f and the expression related to $\nabla\hat{A}$ gives nothing but the chain rule of differentiation.

For any smooth function $u > 0$, we define some non-negative real numbers

$$\lambda_1 := \sup_{Q_{2R,T}} |\hat{A}| \qquad B_1 := \sup_{M\times[0,T]} |\hat{B}| \qquad \gamma_2 := \sup_{Q_{2R,T}} |\nabla p|$$
$$\lambda_2 := \sup_{Q_{2R,T}} |\hat{A}_f| \qquad B_2 := \sup_{M\times[0,T]} |\hat{B}_f| \qquad \Gamma_1 := \sup_{M\times[0,T]} |p|$$
$$\Lambda_1 := \sup_{M\times[0,T]} |\hat{A}| \qquad \sigma_1 := \sup_{Q_{2R,T}} |q| \qquad \Gamma_2 := \sup_{M\times[0,T]} |\nabla p|$$
$$\Lambda_2 := \sup_{M\times[0,T]} |\hat{A}_f| \qquad \sigma_2 := \sup_{Q_{2R,T}} |\nabla q| \qquad m_1 := \sup_{Q_{2R,T}} |\xi|$$
$$b_1 := \sup_{Q_{2R,T}} |\hat{B}| \qquad \Sigma_1 := \sup_{M\times[0,T]} |q| \qquad m_2 := \sup_{Q_{2R,T}} |\nabla \xi|$$
$$b_2 := \sup_{Q_{2R,T}} |\hat{B}_f| \qquad \Sigma_2 := \sup_{M\times[0,T]} |\nabla q| \qquad M_1 := \sup_{M\times[0,T]} |\xi|$$
$$\gamma_1 := \sup_{Q_{2R,T}} |p| \qquad M_2 := \sup_{M\times[0,T]} |\nabla \xi|.$$

3. Hamilton type gradient estimate

Let $x_0 \in M$ and $R > 0$ be a positive real number.

Lemma 3.1. *If u is a positive solution to the equation (2) under the flow (1) and $v = u^{\frac{1}{3}}$ with*

$$Ric_\phi \geq -(n-1)k_1 g, \ \mathcal{S} \geq -k_2 g$$

on $Q_{2R,T}$ for some positive constants k_1, k_2, then the quantity $H := v|\nabla v|^2$ satisfies

$$(\Delta_\phi - \partial_t)H \geq -\Omega_1 H - 4v^{-1}\langle \nabla v, \nabla H\rangle + 4v^{-3}H^2 - \frac{2}{3}\Omega_2 H^{\frac{1}{2}} v^{\frac{3}{2}} - \frac{2}{3}\Omega_3 v H, \tag{4}$$

where

$$\Omega_1 = 2(n-1)k_1 + 2k_2 + (\lambda_1\gamma_1 + b_1\sigma_1 + m_1)$$
$$\Omega_2 = \lambda_1\gamma_2 + b_1\sigma_2 + m_2$$
$$\Omega_3 = \lambda_2\gamma_1 + b_2\sigma_1.$$

Proof. Putting $u = v^3$ in (2) we get

$$(\Delta_\phi - \partial_t)v = -2\frac{|\nabla v|^2}{v} + \frac{1}{3}\bar{f}, \tag{5}$$

where $\bar{f} = \hat{A}pv + \hat{B}qv + \xi v$.
Hence

$$\langle \nabla \bar{f}, \nabla v\rangle = (\hat{A}\langle \nabla p, \nabla v\rangle + \hat{B}\langle \nabla q, \nabla v\rangle + \langle \nabla \xi, \nabla v\rangle)v + (\hat{A}_v|\nabla v|^2 p + \hat{B}_v|\nabla v|^2 q)v + (\hat{A}p + \hat{B}q + \xi)|\nabla v|^2.$$

Thus using Cauchy-Schwarz inequality we infer

$$\bar{f} \leq (\lambda_1\gamma_1 + b_1\sigma_1 + m_1)v, \tag{6}$$
$$\langle \nabla \bar{f}, \nabla v\rangle \leq (\lambda_1\gamma_2 + b_1\sigma_2 + m_2)v|\nabla v| + (\lambda_2\gamma_1 + b_2\sigma_1)v|\nabla v|^2 + (\lambda_1\gamma_1 + b_1\sigma_1 + m_1)|\nabla v|^2. \tag{7}$$

Using Lemma 2.2 on $H = v|\nabla v|^2$ we have

$$\partial_t H = |\nabla v|^2\partial_t v + 2v\langle \nabla v, \nabla \Delta_\phi v\rangle + 2v\left\langle \nabla v, \nabla(\frac{2}{v}|\nabla v|^2)\right\rangle - \frac{2}{3}v\langle \nabla v, \nabla \bar{f}\rangle - 2v\mathcal{S}(\nabla v, \nabla v) \tag{8}$$

and

$$\Delta_\phi H = |\nabla v|^2\Delta_\phi v + 2v|\text{Hess } v|^2 + 2v\langle \nabla \Delta_\phi v, \nabla v\rangle + 2vRic_\phi(\nabla v, \nabla v) + 4\text{Hess } v(\nabla v, \nabla v). \tag{9}$$

A simple calculation yields

$$(\Delta_\phi - \partial_t)H = \frac{2|\nabla v|^4}{v} + 2|\text{Hess } v|^2 + 2vRic_\phi(\nabla v, \nabla v) + 2v\mathcal{S}(\nabla v, \nabla v) + \frac{2}{3}v\nabla\bar{f}\nabla v + \frac{1}{3}|\nabla v|^2\bar{f} + 4\text{Hess } v(\nabla v, \nabla v) - 8\text{Hess } v(\nabla v, \nabla v). \quad (10)$$

We see that

$$2v|\text{Hess } v|^2 + 4\text{Hess } v(\nabla v, \nabla v) + 2\frac{|\nabla v|^4}{v} = 2v\left|\text{Hess } v + \frac{\nabla v \otimes \nabla v}{v}\right|^2 \geq 0 \quad (11)$$

and

$$8\text{Hess } v(\nabla v, \nabla v) = 4v^{-1}\langle \nabla v, \nabla H\rangle - 4v^{-3}H^2. \quad (12)$$

Using (11), (12) in (10) we have

$$(\Delta_\phi - \partial_t)H \geq 2vRic_\phi(\nabla v, \nabla v) + 2v\mathcal{S}(\nabla v, \nabla v) + \frac{2}{3}v\nabla\bar{f}\nabla v + \frac{1}{3}|\nabla v|^2\bar{f} - 4v^{-1}\langle\nabla v, \nabla H\rangle + 4v^{-3}H^2. \quad (13)$$

Applying the bounds of Ric_ϕ, $\mathcal{S}$, $\bar{f}$ and $\langle\nabla\bar{f}, \nabla v\rangle$ on (13) we get our result (4). □

Theorem 3.2. *Let u be a positive solution to (2) along the flow (1) in $Q_{2R,T}$ satisfying $\tilde{\kappa}^3 \leq u \leq \kappa^3$ for some positive real numbers $\kappa, \tilde{\kappa}$. If there exist constants k_1, k_2 such that*

$$Ric_\phi \geq -(n-1)k_1 g, \ \mathcal{S} \geq -k_2 g$$

on $Q_{2R,T}$, then we have

$$\frac{|\nabla u|}{\sqrt{u}} \leq \frac{3\kappa^2}{\sqrt{\tilde{\kappa}}}\left\{\frac{1}{\sqrt{2}}(\Omega + \Omega_1 + \frac{2}{3}\Omega_3\kappa)^{\frac{1}{2}} + 3^{\frac{3}{4}}\frac{\sqrt{c_1}}{R} + \frac{1}{2^{\frac{5}{12}}3^{\frac{1}{12}}}\Omega_2^{\frac{1}{4}}\right\}, \quad (14)$$

where $\Omega = \frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2$ and $\Omega_i (i = 1, 2, 3)$ is defined in Lemma 3.1.

Proof. Consider a C^2-function ψ on $[0, \infty)$,

$$\psi(s) = \begin{cases} 1, & s \in [0, 1], \\ 0, & s \in [2, \infty), \end{cases}$$

and it satisfies $\psi(s) \in [0, 1]$, $-c_0 \leq \psi'(s) \leq 0$, $\psi''(s) \geq -c_1$ and $\frac{|\psi''(s)|^2}{\psi(s)} \leq c_1$, where c_0, c_1 are positive constants. Let $R \geq 1$ be a constant and define a function

$$\eta(x, t) = \psi\left(\frac{r(x, t)}{R}\right),$$

where $r(x, t) = d(x, x_0, t)$ denotes the geodesic distance between x, x_0 with respect to the metric $g(t)$. As $\psi(s)$ is Lipschitz, we may assume everywhere smoothness of $\eta(x, t)$ by using maximum principle and Calabi's trick [3] in the same argument of [7].

By generalized Laplacian comparison theorem we have

1. $\Delta_\phi r(x) \leq (n-1)\sqrt{k_1}\coth(\sqrt{k_1}r(x))$
2. $\Delta_\phi \eta \geq -\frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R}) - \frac{c_1}{R^2}$
3. $\frac{|\nabla\eta|^2}{\eta} \leq \frac{c_1}{R^2}$.

Let $G = \eta H$. Fix any $T_1 \in (0, T]$ and assume G achieves maximum at $(x_0, t_0) \in Q_{2R,T_1}$ and $G(x_0, t_0) \geq 0$ (if $G(x_0, t_0) < 0$ then the proof is trivial).

Hence at (x_0, t_0) we have

$$\nabla G = 0, \quad \Delta G \leq 0, \quad \partial_t G \geq 0.$$

Therefore,

$$\nabla H = -\frac{H}{\eta}\nabla \eta \tag{15}$$

and

$$0 \geq (\Delta_\phi - \partial_t)G = H(\Delta_\phi - \partial_t)\eta + \eta(\Delta_\phi - \partial_t)H + 2\langle \nabla\eta, \nabla H\rangle. \tag{16}$$

By [13], there is a constant c_2 such that

$$-H\eta_t \geq -c_2 k_2 H. \tag{17}$$

Using generalized Laplacian comparison theorem, (15) and (17) in (16) we get

$$0 \geq -\Big(\frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2 k_2\Big)H + \eta(\Delta_\phi - \partial_t)H. \tag{18}$$

From (15) we have

$$\eta\langle \nabla v, \nabla H\rangle \leq \frac{\sqrt{c_1}}{R}\eta^{\frac{1}{2}} H|\nabla v|. \tag{19}$$

Applying Lemma 3.1 and (19) on (18) we get

$$0 \;\; \geq \;\; -\Omega H - \Omega_1 G - 4v^{-1}\frac{\sqrt{c_1}}{R}\eta^{\frac{1}{2}} H|\nabla v| + 4\eta v^{-3}H^2 - \frac{2}{3}\eta\Omega_2 H^{\frac{1}{2}} v^{\frac{3}{2}} - \frac{2}{3}\Omega_3 v G, \tag{20}$$

where $\Omega = \frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2$. Multiplying both sides of (20) with ηv^3 we have

$$0 \;\; \geq \;\; -\Omega v^3 G - \Omega_1 \eta v^3 G - 4v^{\frac{3}{2}}\frac{\sqrt{c_1}}{R}G^{\frac{3}{2}} + 4G^2 - \frac{2}{3}\Omega_2 \eta^{\frac{3}{2}} G^{\frac{1}{2}} v^{\frac{9}{2}} - \frac{2}{3}\Omega_3 \eta v^4 G. \tag{21}$$

Let

$$\theta_1 = \Omega v^3 + \Omega_1 \eta v^3 + \frac{2}{3}\Omega_3 \eta v^4, \tag{22}$$

$$\theta_2 = \frac{4\sqrt{c_1}}{R} v^{\frac{3}{2}}, \tag{23}$$

$$\theta_3 = \frac{2}{3}\Omega_2 \eta^{\frac{3}{2}} v^{\frac{9}{2}}. \tag{24}$$

Hence (21) reduces to

$$0 \;\; \geq \;\; 4G^2 - \theta_1 G - \theta_2 G^{\frac{3}{2}} - \theta_3 G^{\frac{1}{2}}. \tag{25}$$

Applying Young's inequality we get

$$\theta_1 G \leq G^2 + \frac{\theta_1^2}{4}, \tag{26}$$

$$\theta_2 G^{\frac{3}{2}} \leq G^2 + \frac{27}{2^8}\theta_2^4, \tag{27}$$

$$\theta_3 G^{\frac{1}{2}} \leq G^2 + \frac{3}{4^{\frac{4}{3}}}\theta_3^{\frac{4}{3}}. \tag{28}$$

Using (26), (27) and (28) in (25) we get

$$G^2 \leq \frac{\theta_1^2}{4} + \frac{27}{2^8}\theta_2^4 + \frac{3}{4^{\frac{4}{3}}}\theta_3^{\frac{4}{3}}. \tag{29}$$

Putting the value of $\theta_1, \theta_2, \theta_3$ and multiplying the result with v^2 we get

$$v^2G^2 \leq \left\{\frac{1}{4}\left(\Omega + \Omega_1\eta + \frac{2}{3}\Omega_3 v\eta\right)^2 + \frac{27}{2^4}\left(\frac{2\sqrt{c_1}}{R}\right)^4 + \frac{1}{4^{\frac{5}{6}}3^{\frac{1}{3}}}\eta^2\Omega_2^{\frac{4}{3}}\right\}v^8. \tag{30}$$

Following [2], we have $\tilde{\kappa} \leq v \implies \tilde{\kappa}^2H^2 \leq v^2G^2$. Hence (30) reduces to

$$\tilde{\kappa}^2H^2 \leq \left\{\frac{1}{4}\left(\Omega + \Omega_1\eta + \frac{2}{3}\Omega_3\kappa\eta\right)^2 + \frac{27}{2^4}\left(\frac{2\sqrt{c_1}}{R}\right)^4 + \frac{1}{4^{\frac{5}{6}}3^{\frac{1}{3}}}\eta^2\Omega_2^{\frac{4}{3}}\right\}\kappa^8. \tag{31}$$

Putting $v = u^{\frac{1}{3}}$ in $H = v|\nabla v|^2$ we get $H = \frac{|\nabla u|^2}{9u}$.
Hence

$$\frac{|\nabla u|}{\sqrt{u}} \leq \frac{3\kappa^2}{\sqrt{\tilde{\kappa}}}\left\{\frac{1}{4}\left(\Omega + \Omega_1\eta + \frac{2}{3}\Omega_3\kappa\eta\right)^2 + \frac{27}{2^4}\left(\frac{2\sqrt{c_1}}{R} + \frac{1}{3}\eta^{\frac{1}{2}}\Omega_2\right)^4 + \frac{1}{4^{\frac{5}{6}}3^{\frac{1}{3}}}\eta^2\Omega_4^{\frac{4}{3}}\right\}^{\frac{1}{4}}. \tag{32}$$

Since $\eta = 1$ over $Q_{2R,T}$, so using the elementary inequality $\sqrt{xy} \leq \sqrt{x} + \sqrt{y}$ for positive x, y on (32) gives the result (14). □

4. Souplet-Zhang type gradient estimate

In this section we deduce a Souplet-Zhang type gradient estimate for a positive solution of (2) along the flow (1). Fix $x_0 \in M$ and $R > 0$ be a real number.

Lemma 4.1. *Let $u = e^f$ be a positive solution to the equation (2) under the flow (1) in $Q_{2R,T}$ satisfying $\tilde{\kappa} \leq u \leq \kappa$ for some positive real numbers $\kappa, \tilde{\kappa}$. If there exist positive constants k_1, k_2 such that*

$$Ric_\phi \geq -(n-1)k_1 g, \ \mathcal{S} \geq -k_2 g,$$

then the quantity $W := \frac{|\nabla f|^2}{(\rho - f)^2}$ satisfies

$$\begin{aligned}(\Delta_\phi - \partial_t)W \geq\ & \frac{2(f - \ln\kappa)}{\rho - f}\langle\nabla W, \nabla f\rangle + 2(\rho - f)W^2 + \frac{2\bar{f}}{\rho - f}W + 2(\hat{A}_f p + \hat{B}_f q)W \\ & - \frac{2}{\rho - f}(|\hat{A}||\nabla p| + |\hat{B}||\nabla q| + |\nabla\xi|)W^{\frac{1}{2}} - 2((n-1)k_1 + k_2)W,\end{aligned} \tag{33}$$

where $\rho := 1 + \ln\kappa$, $\bar{f} = \hat{A}p + \hat{B}q + \xi$.

Proof. Substituting $u = e^f$ in (2) we have

$$(\Delta_\phi - \partial_t)f = -|\nabla f|^2 + \bar{f}. \tag{34}$$

By virtue of $W := \frac{|\nabla f|^2}{(\rho - f)^2}$ we have

$$\begin{aligned}(\Delta_\phi - \partial_t)W =\ & \frac{6|\nabla f|^4}{(\rho - f)^4} + \frac{2|\nabla f|^2}{(\rho - f)^3}\Delta_\phi f + \frac{4}{(\rho - f)^3}\langle\nabla|\nabla f|^2, \nabla f\rangle \\ & + \frac{1}{(\rho - f)^2}\Delta_\phi|\nabla f|^2 + \frac{2}{(\rho - f)^2}\mathcal{S}(\nabla f, \nabla f) - \frac{2}{(\rho - f)^2}\langle\nabla f_t, \nabla f\rangle - \frac{2|\nabla f|^2}{(\rho - f)^3}\partial_t f.\end{aligned} \tag{35}$$

and using Lemma 2.2 and (34) on (35) we get

$$(\Delta_\phi - \partial_t)W = \frac{6|\nabla f|^4}{(\rho-f)^4} - \frac{2|\nabla f|^4}{(\rho-f)^3} + \frac{2|\nabla f|^2\bar{f}}{(\rho-f)^3} + \frac{8\text{Hess } f(\nabla f,\nabla f)}{(\rho-f)^3} + \frac{2|\text{Hess } f|^2}{(\rho-f)^2} + \frac{2}{(\rho-f)^2}(Ric_\phi + \mathcal{S})(\nabla f,\nabla f) - \frac{4\text{Hess } f(\nabla f,\nabla f)}{(\rho-f)^2} + \frac{2}{(\rho-f)^2}\langle\nabla\bar{f},\nabla f\rangle. \tag{36}$$

An elementary calculation yields

$$\frac{|\text{Hess } f|^2}{(\rho-f)^2} + \frac{2\text{Hess } f(\nabla f,\nabla f)}{(\rho-f)^3} + \frac{|\nabla f|^4}{(\rho-f)^4} = \frac{1}{(\rho-f)^2}\left|\text{Hess } f + \frac{\nabla f\otimes\nabla f}{\rho-f}\right|^2 \geq 0, \tag{37}$$

and

$$\frac{2\text{Hess } f(\nabla f,\nabla f)}{(\rho-f)^2} + \frac{2|\nabla f|^4}{(\rho-f)^3} = \langle\nabla W,\nabla f\rangle. \tag{38}$$

Using (37) and (38) in (36) we get

$$(\Delta_\phi - \partial_t)W \geq \frac{2\langle\nabla W,\nabla f\rangle}{\rho-f} - 2\langle\nabla W,\nabla f\rangle + \frac{|\nabla f|^4}{(\rho-f)^3} - \frac{2|\nabla f|^2\bar{f}}{(\rho-f)^3} + \frac{2}{(\rho-f)^2}\langle\nabla\bar{f},\nabla f\rangle + \frac{2}{(\rho-f)^2}(Ric_\phi + \mathcal{S})(\nabla f,\nabla f). \tag{39}$$

We have $\rho - f \geq 1$ and some basic analysis on $\bar{f}$ gives

$$\begin{aligned}\frac{2}{(\rho-f)^2}\langle\nabla\bar{f},\nabla f\rangle &= \frac{2}{(\rho-f)^2}\Big((\hat{A}_f p + \hat{B}_f q)\nabla f + \hat{A}\nabla p + \hat{B}\nabla q + \nabla\xi\Big)\nabla f\\ &\geq \frac{2}{(\rho-f)^2}\Big(\hat{A}_f p + \hat{B}_f q\Big)|\nabla f|^2 - \frac{2}{(\rho-f)^2}\Big(|\hat{A}||\nabla p| + |\hat{B}||\nabla q| + |\nabla\xi|\Big)|\nabla f|\\ &= 2\Big(\hat{A}_f p + \hat{B}_f q\Big)W - \frac{2}{(\rho-f)}\Big(|\hat{A}||\nabla p| + |\hat{B}||\nabla q| + |\nabla\xi|\Big)\sqrt{W}.\end{aligned} \tag{40}$$

Using (40) and bounds of Ric_ϕ and $\mathcal{S}$ on (39) we obtain the lemma. □

Theorem 4.2. *Let u be a positive solution of (2) along the flow (1) satisfying $\tilde{\kappa} \leq u \leq \kappa$ for two positive real numbers $\kappa, \tilde{\kappa}$. If there exist positive constants k_1, k_2 such that*

$$Ric_\phi \geq -(n-1)k_1 g,\ \mathcal{S} \geq -k_2 g$$

on $Q_{2R,T}$ then we have

$$\frac{|\nabla u|}{u} \leq \Big(1 + \ln(\frac{\kappa}{\tilde{\kappa}})\Big)\mathcal{R}, \tag{41}$$

where $\mathcal{R}$ is a positive real constant given by

$$\mathcal{R} = \left(\frac{2}{3\sqrt{3}}\right)^{\frac{1}{2}}\frac{\sqrt{c_1}}{R}\ln(\frac{\kappa}{\tilde{\kappa}}) + \sqrt{2}(\lambda_1\gamma_1 + b_1\sigma_1 + m_1 + \lambda_2\gamma_1 + b_2\sigma_1 + (n-1)k_1 + k_2)^{\frac{1}{2}} + \frac{\sqrt[4]{3}}{\sqrt{2}}(2\lambda_1\gamma_2 + 2b_1\sigma_2 + 2m_2)^{\frac{3}{16}} + \left(\frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2\right)^{\frac{1}{2}}.$$

Proof. Let $G = \eta W$, where η is defined in Theorem 3.2 along with its restrictions. Fix any $T_1 \in (0, T]$ and assume G achieves maximum at $(x_0, t_0) \in Q_{2R,T_1}$ and $G(x_0, t_0) \geq 0$.

Hence at (x_0, t_0) we have

$$\nabla G = 0, \quad \Delta G \leq 0, \quad \partial_t G \geq 0.$$

Therefore,

$$\nabla W = -\frac{W}{\eta}\nabla\eta \tag{42}$$

and

$$0 \geq (\Delta_\phi - \partial_t)G = W(\Delta_\phi - \partial_t)\eta + \eta(\Delta_\phi - \partial_t)W + \langle\nabla\eta, \nabla W\rangle. \tag{43}$$

By [13], there is a constant c_2 such that

$$-W\eta_t \geq -c_2k_2W. \tag{44}$$

Using generalized Laplacian comparison theorem, (42), (44) on (43) we get

$$0 \geq -\left(\frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2\right)W + \eta(\Delta_\phi - \partial_t)W. \tag{45}$$

Let $\Omega = \frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2$. Then (45) reduces to

$$0 \geq -\Omega W + \eta(\Delta_\phi - \partial_t)W. \tag{46}$$

Using Lemma 4.1 on the above equation we get

$$\begin{aligned} 0 \quad \geq \quad & -\Omega W + 2\eta\frac{(f - \ln\kappa)}{\rho - f}\langle\nabla W, \nabla f\rangle + 2\eta(\rho - f)W^2 + 2\eta\frac{\bar{f}}{\rho - f}W \\ & +2\eta(\hat{A}_f p + \hat{B}_f q)W - \frac{2\eta}{\rho - f}(|\hat{A}||\nabla p| + |\hat{B}||\nabla q| + |\nabla\xi|)W^{\frac{1}{2}} - 2\eta((n-1)k_1 + k_2)W. \end{aligned} \tag{47}$$

Multiplying both sides with η and some elementary calculation gives

$$\begin{aligned} 0 \quad \geq \quad & \frac{2(f - \ln\kappa)}{\rho - f}\eta^2\langle\nabla W, \nabla f\rangle + 2(\rho - f)G^2 - \left\{\frac{2|\bar{f}|}{\rho - f} + 2(n-1)k_1 + 2k_2 + 2(|\hat{A}_f||p| + |\hat{B}_f||q|)\right\}\eta G \\ - \quad & \frac{2}{\rho - f}\left(|\hat{A}||\nabla p| + |\hat{B}||\nabla q| + |\nabla\xi|\right)\eta^{\frac{3}{2}}G^{\frac{1}{2}} - \Omega G. \end{aligned} \tag{48}$$

Now we apply Young's inequality to each member of the right hand side of (48).

$$\begin{aligned} \frac{2(\ln\kappa - f)}{\rho - f}\eta^2\langle\nabla W, \nabla f\rangle &= \frac{2(f - \ln\kappa)}{\rho - f}G\langle\nabla\eta, \nabla f\rangle \\ &\leq 2\ln(\frac{\kappa}{\tilde{\kappa}})\frac{\sqrt{c_1}}{R}G^{\frac{3}{2}} \\ &\leq \frac{G^{\frac{3a}{2}}}{a}\epsilon^a + \frac{\left(2\ln(\frac{\kappa}{\tilde{\kappa}})\frac{\sqrt{c_1}}{R}\right)^b}{b\epsilon^b}. \end{aligned} \tag{49}$$

Set $a = \frac{4}{3}$, $b = 4$, $\epsilon^4 = 27$ we get

$$\frac{2(\ln\kappa - f)}{\rho - f}\eta^2\langle\nabla W, \nabla f\rangle \leq \frac{G^2}{4} + \frac{4c_1^2}{27R^4}\left(\ln\frac{\kappa}{\tilde{\kappa}}\right)^4. \tag{50}$$

Similarly setting $a = 4$, $b = \frac{4}{3}$ and $\epsilon = 1$ we have

$$\begin{aligned}\frac{2}{\rho - f}(|\hat{A}||\nabla p| + |\hat{B}||\nabla q| + |\nabla\xi|)\eta^{\frac{3}{2}}\sqrt{G} &\leq 2(\lambda_1\gamma_2 + b_1\sigma_2 + m_2)\eta^{\frac{3}{2}}\sqrt{G} \\ &\leq \frac{G^2}{4} + \frac{3}{4}(2\lambda_1\gamma_2 + 2b_1\sigma_2 + 2m_2)^{\frac{4}{3}}.\end{aligned} \tag{51}$$

Setting $a = 2$, $b = 2$ and apply Peter-Paul inequality we have

$$\begin{aligned}2\left(\frac{\bar{f}}{\rho - f} + (n-1)k_1 + k_2 + \hat{A}_f p + \hat{B}_f q\right)\eta G &\leq 2\nu\eta G \\ &\leq \frac{G^2}{4} + 4\nu^2,\end{aligned} \tag{52}$$

where $\nu = \lambda_1\gamma_1 + b_1\sigma_1 + m_1 + (n-1)k_1 + k_2 + \lambda_2\gamma_1 + b_2\sigma_1$ and the last term

$$\Omega G \leq \frac{G^2}{4} + \Omega^2. \tag{53}$$

Using (50), (51), (52), (53) in (48) and applying $\rho - f \geq 1$, we get

$$G^2 \leq \frac{4c_1^2}{27R^4}\left(\ln\frac{\kappa}{\tilde{\kappa}}\right)^4 + 4\nu^2 + \frac{3}{4}(2\lambda_1\gamma_2 + 2b_1\sigma_2 + 2m_2)^{\frac{3}{4}} + \Omega^2. \tag{54}$$

As $\eta(x,t) = 1$ in $Q_{2R,T}$, $f = \ln u$ and using the inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for positive real numbers x, y we have.

$$\begin{aligned}\frac{|\nabla u|}{u(\rho - \ln u)} \leq{}& \left(\frac{2}{3\sqrt{3}}\right)^{\frac{1}{2}}\frac{\sqrt{c_1}}{R}\ln(\frac{\kappa}{\tilde{\kappa}}) + \sqrt{2}(\lambda_1\gamma_1 + b_1\sigma_1 + m_1 + (n-1)k_1 + k_2 + \lambda_2\gamma_1 + b_2\sigma_1)^{\frac{1}{2}} \\ &+ \frac{\sqrt[4]{3}}{\sqrt{2}}(2\lambda_1\gamma_2 + 2b_1\sigma_2 + 2m_2)^{\frac{3}{16}} + \left(\frac{c_0}{R}(n-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{3c_1}{R^2} + c_2k_2\right)^{\frac{1}{2}}.\end{aligned} \tag{55}$$

We have $\rho - \ln u = 1 + \ln\kappa - \ln u = 1 + \ln(\frac{\kappa}{u})$. Since $\tilde{\kappa} \leq u \leq \kappa$, so $1 \leq \frac{\kappa}{u} \leq \frac{\kappa}{\tilde{\kappa}}$. Thus $\rho - f \leq 1 + \ln(\frac{\kappa}{\tilde{\kappa}})$. Apply this relation to the last deduced equation (55) to complete the proof. □

5. Applications

In this section we apply Theorem 3.2 and Theorem 4.2 to find the global gradient estimates of positive solution of the nonlinear parabolic heat type equation (2) on $M \times [0,T]$ under the flow (1) .

Corollary 5.1 (Hamilton type global gradient estimate). *Let u be a positive solution to (2) along the flow (1) in $M \times [0,T]$ satisfying $\tilde{\kappa}^3 \leq u \leq \kappa^3$ for some positive real numbers $\kappa, \tilde{\kappa}$. If there exist constants k_1, k_2 such that*

$$Ric_\phi \geq -(n-1)k_1 g,\ \mathcal{S} \geq -k_2 g$$

on $M \times [0,T]$, then we have

$$\frac{|\nabla u|}{\sqrt{u}} \leq \frac{3\kappa^2}{\sqrt{\tilde{\kappa}}}\left\{\frac{1}{\sqrt{2}}(\bar{\Omega} + \bar{\Omega}_1 + \frac{2}{3}\bar{\Omega}_3\kappa)^{\frac{1}{2}} + \frac{1}{2^{\frac{5}{12}}3^{\frac{1}{12}}}\bar{\Omega}_2^{\frac{1}{4}}\right\}, \tag{56}$$

where

$\bar{\Omega} = c_2k_2$
$\bar{\Omega}_1 = 2(n-1)k_1 + 2k_2 + \Lambda_1\Gamma_1 + B_1\Sigma_1 + M_1$
$\bar{\Omega}_2 = \Lambda_1\Gamma_2 + B_1\Sigma_2 + M_2$
$\bar{\Omega}_3 = \Lambda_2\Gamma_1 + B_2\Sigma_1$

Proof. From Theorem 3.2 we have the local Hamilton type estimation for a positive solution u of (2) evolving under (1). Letting R tends to infinity and using the global bounds in (14) we get (56). □

Corollary 5.2 (Souplet-Zhang type global gradient estimate). *Let u be a positive solution to (2) along the flow (1) in $M \times [0, T]$ satisfying $\tilde{\kappa} \leq u \leq \kappa$ for some positive real numbers $\kappa, \tilde{\kappa}$. If there exist constants k_1, k_2 such that*

$$Ric_\phi \geq -(n-1)k_1 g,\ \mathcal{S} \geq -k_2 g$$

on $M \times [0, T]$, then we have

$$\frac{\nabla u}{u} \leq \Big(1 + \ln(\frac{\kappa}{\tilde{\kappa}})\Big)\bar{\mathcal{R}}, \tag{57}$$

where

$$\bar{\mathcal{R}} = \sqrt{2}(\Lambda_1\Gamma_1 + B_1\Sigma_1 + M_1 + \Lambda_2\Gamma_1 + B_2\Sigma_1 + (n-1)k_1 + k_2)^{\frac{1}{2}} + \sqrt{c_2 k_2} + \frac{\sqrt[4]{3}}{\sqrt{2}}(2\Lambda_1\Gamma_2 + 2B_1\Sigma_2 + 2M_2)^{\frac{3}{16}}.$$

Proof. From Theorem 4.2 we have the local Souplet-Zhang type estimate. Letting $R \to +\infty$ and use the global bounds in (33) we get (57). □

6. Concluding remark

In modern geometric analysis gradient estimation is an active area of research. Several authors derived Hamilton type and Souplet-Zhang type estimations for maniolds without flow cite [4–6] and the references therein. Abimbola [1] derived gradient estimates for heat type equations

$$(\Delta - \partial_t)u = \mathcal{R}u$$

on evolving manifolds, which can be obtained by putting $\xi = \mathcal{R}$, $A = B = 0$ and treat ϕ as a constant map in (2). In [14], Wu established Li-Yau type estimates for a nonlinear parabolic equation

$$(\Delta_\phi - \partial_t)u = au\ln u + qu \tag{58}$$

on complete manifolds, which can be derived by setting $A(u) = u\ln u$, $p(x,t) = a$ (constant function), $B(u) = 0$, $\xi(x,t) = q$ (a real number) in (2). Azami [2] established the estimations for the equation

$$(\Delta_\phi - \partial_t)u(x,t) = q(x,t)u^{a+1} + p(x,t)A(u(x,t))$$

along a geometric flow, which can be obtained by putting $B(u) = u^{a+1}$ and $\xi(x,t) = 0$ in (2). Thus our work generalizes [2] and many other authors.

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